Working Paper

Geometric Ideas in Nonlinear and Multicriteria Optimization

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WP-90-46
September 1990

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FOREWORD

Some geometric properties of the solution set for nonlinear and multicriteria programming problems and the related numeric algorithms are considered. The author deals with necessary and sufficient conditions for nonlinear programming problem stability (in the nonconvex case), with Pareto set stability, Pareto set connectedness conditions, with weak efficiency, efficiency and proper efficiency criteria. A study of numerical algorithms based on geometric properties of the so-called convolutions function is also considered. Necessary and sufficient convergence conditions for large classes of algorithms are presented and easy to check sufficient conditions are given. Further results deal with problems of using local unconstrained minimization algorithms to solve quasi-convex problems and the problem of using some convolution functions for constructing decision making procedures. New classes of inverse nonlinear programming problems are discussed and software implementations of DISO/PC-MCNLP are presented.

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GEOMETRIC IDEAS IN NONLINEAR AND MULTICRITERIA OPTIMIZATION

Gregory G. Kotkin

Preface

Geometrical ideas in nonlinear programming (n.l.p.) are usually associated with various properties of goal and constraint functions mapping the argument space into the function values space (see Elster [1980]). We will interpret numerical nonlinear optimization methods with respect to functions values space (f.v.s.). This paper deals with some results based on the study of generalized sensitivity function (g.s.f.) and convolution function (c.f.) (see Kotkin [1988]). The g.s.f. is a dependance of the optimal value of some "main" function (goals or constraints) upon perturbations of other functions under the assumption that they are constraint functions.

It is easy to verify that the g.s.f. graph is a part of problem image bound. By problem image we mean the image of the argument space with respect to goal and constraint functions mapping into the f.v.s. Because of g.s.f. nature the optimal value of one goal n.l.p. problem is a value of g.s.f. at zero. The Pareto set is the intersection the g.s.f. graph and a plane which can be characterized by zero valued constraints (see Golikov and Kotkin [1986] and [1988]). A series of results devoted the connectedness of various optimal solution sets, Pareto set stability, efficiency conditions, etc. have been obtained in terms of g.s.f. properties. A brief review is given in Section 1.

The efficiency of numerical n.l.p. and m.c.o. algorithms is due to dual properties of g.s.f. and c.f. The c.f. is used to reduce n.l.p. or m.c.o. problem to a series of unconstrained minimization (u.c.m.) problems. C.f. arise in penalty functions methods, centers methods, dual Lagrange multiplier method, etc. (see Evtushenko [1985]). The main feature of c.f. is that it maps the f.v.s. into the space of values of function which arise in u.c.m. problem (we will call these functions u.f.). U.f. are constructed on the base of composition of c.f. and goal-constraint functions.

We obtain the necessary and sufficient convergence conditions of some classes of n.l.p. and m.c.o. problems in terms of c.f. properties (see Section 2). In Section 2 we also present easily to check convergence conditions. Section 3 is concerned with similar conditions for the m.c.o. problem and some other results. Inverse problems are considered in Section 4. Section 5 deals with the software implementation DISO/PC-MCNLP based on the ideas which are described in Sections 1 - 4.

1. Problem definition

Let us consider the following problem:

$$\min \; f(x), \qquad (1)$$
$$x \in X(0,0)$$

where $X(y,v) = \{x \in Q \subset R^n : g(x) \leq y, h(x)=v\}$; $Q = \{x \in R^n : a \leq x \leq b\}$ is rectangular constrained set, $a \in R^n$, $b \in R^n$, $a<b$, are given vectors; $f(x)$, $g(x)$, $h(x)$ are continuous vector-functions $f: Q \to R^{m+1}$, $g: Q \to R^p$, $h: Q \to R^s$.

The one goal n.l.p. problem is problem (1) in the case $m = 0$.

Let us denote the weak efficient, the efficient and the proper efficient (Geoffrion optimal) estimations set of problem (1) by $S(0,0)$, $P(0,0)$, and $G(0,0)$ respectively. Clearly, $G(0,0) \subset P(0,0) \subset S(0,0)$. Let us denote for any vector $a = (a^0,...,a^m)$ by $\hat{a}$ the following vector $(a^0,...,a^{j-1}, a^{j+1},...,a^m)$ for some j. Without loss of generality we will consider sometimes the case j=0 and will omit the superscript j=0 in notation, for example $F(\hat{z}, y,v) = F^0(\hat{z}, y,v)$.

We are concerned with the following objects in the space Z_+ of functions f, g,h values (see Fig.1):

1) problem image $\check{Z}$:

$$\check{Z} = \check{f}(Q), \text{ where } \check{f} = (f,g,h); \qquad (2)$$

2) generalized sensitivity functions (g.s.f.) $F^j : Y^j \to R^1$:

$$F^j(\hat{z},y,v) = \min \quad f^j(x), \qquad (2)$$
$$x \in X(\hat{z},y,v)$$

where

$$X(\hat{z},y,v) = \{x \in Q: f^i(x) \leq z^i, \; i=0,...,j-1,$$
$$j+1,...,m, g(x) \leq y, \; h(x) = v\}, \; j = 0,...,m;$$
$$Y=\{(\hat{z},y,v) : X(\hat{z},y,v) \neq \};$$

(3) weakly efficient estimations (w.e.e.) set $S(0,0)$:

$$S(0,0) = \{z \in R^{m+1} : \text{there are not exist } z_0 \in f(X(0,0)) \text{ such that } z^0 < z\};$$

(4) convolution functions (c.f.) $M_k(z,y,v)$:

$M_k: Z_+ \rightarrow R_1$; $M_k(z,y,v)$ are continuous nondecreasing with respect to z functions, defined on the domain $\tilde{Z} \subset Z_+$, $k=1,2,\cdots$

(5) the function value isolines $V_k(t)$ of c.f.:

$$V_k(t) = \{(z,y,v) \in Y: M_k(z,y,v) = t\}, \tag{4}$$

and the minimal function value isolines $V_k(t_*)$, where

$$t_* = \min_{(z,y,v) \in \check{Z} \cap \tilde{Z}} M_k(z,y,v) \tag{5}$$

The well known sensitivity function (see Elster [1980]) is a g.s.f. in one goal n.l.p. problem (problem (1) in the case $m=0$).

If the arguments y and v are fixed ($y=0$ and $v=0$) the g.s.f. transform into a function $F^j(\hat{z}) = F^j(\hat{z},0,0)$ which can be used for w.e.e. set parametrization in the following way.

Let us add vertical lines to the graph T^j of function $F^j(\hat{z})$ through the points where $F^j(\hat{z})$ is discontinuous. We obtain the set Tr^j:

$$Tr^j = \{(z^0,\dots,z^{j-1},b,z^{j+1},\dots,z^m) : \hat{z} \in Y^i, \exists\, \hat{z}_k \rightarrow \hat{z},\ \hat{z}_k < \hat{z}$$
$$\text{such that } F^j(\hat{z}) \le b \le F^j(\hat{z}_k)\},$$

under the assumption that $F^j(\hat{z}) = +\infty\ (\forall\, \hat{z} \notin Y^j)$.

Theorem 1 (see Kotkin [1988]).

1) $Tr^0 = Tr^1 = \dots = Tr^m$.

2) $S(0,0) = f(X(0,0)) \cap Tr^0$.

3) Under the assumption that g.s.f. $F^0(\hat{z},y,v)$ or function $F^0(\hat{z})$ is continuous function we have

$$S(0,0) = P(0,0) = T^0 \cap f(X(0,0)).$$

It is easy to show the efficient estimation set $P(0,0)$ is a set of points z_* of graph T^0 under the assumption that $F^0(\hat{z})$ "left hand side decreases" at $\hat{z}_*$, and the proper efficient estimation set $G(0,0)$ is the set of the points z_* of the graph T^0 under the assumption that derivatives of $F^0(\hat{z})$ at $\hat{z}_*$ are not equal to zero (see Golikov and Kotkin [1988], Kotkin [1988] and Fig. 2).

We will say the n.l.p. problem is stable if and only if the optimal value function $F(y,v)$ is continuous at (0,0). In this case the stability with respect to right hand side perturbations of the constraints is considered.

Let us consider the n.l.p. problem with inequality constraints (problem (1) in the case $m=0$, $s=0$) and denote

$$W(y) = \underset{g(x) \leq y,\ x \in Q}{\text{Argmin}} \quad f^0(x); \quad W_y(y) = g(W(y));$$

$$I(y) = \{x \in Q: g(x) < y\}.$$

Theorem 2 (see Kotkin [1988]).

Let us suppose that $f^0(x)$ and $g(x)$ are continuous functions, Q is a compact set and $I(y_0) \neq \emptyset$. Then in order that $F(y)$ be a continuous function at y_0 it is necessary and sufficient that

$$cl\, I(y_0) \cap W(y_0) \neq \emptyset. \qquad (6)$$

Theorem 2 and some similar results can be used to study the connectedness of various optimal solutions set and to obtain proper efficiency criteria (see Kotkin [1988]).

Theorem 3.

Let us suppose that $F^j(\hat{z})$, $j=0,\ldots,m$, are continuous functions and $X(0,0)$ is a connected (linear-wise connected) set. Then $S(0,0) = P(0,0)$ is a connected (linear-wise connected) set.

Let us denote $\tilde{f}^{-1}(z) = f^{-1}(z) \cap X(0,0)$.

Theorem 4.

Let us suppose that $F^j(\hat{z})$, $j=0,\ldots,m$, are continuous functions, $\tilde{f}^{-1}(z)$ is a connected set for any $z \in f(X(0,0))$ and $S(0,0)$, $P(0,0)$ or $S(0,0)=P(0,0)$ are a connected sets. Then $\tilde{f}^{-1}(S)$, $\tilde{f}^{-1}(P)$ or $\tilde{f}^{-1}(G)$ are the associated connected sets, respectively.

Theorem 5.

The sufficient conditions for proper efficiency of estimation $z_0 \in (F(\hat{z}_0), \hat{z}_0) \cap f(X(0,0)$ are the following:

$$\lim_{k \to \infty} \frac{f^0(X_k) - f^0(X_0)}{f^i(X_k) - f^i(X_0)} \neq 0, \text{ where } i=1,\ldots,m, \tag{1}$$

for any sequence $\{x_k\}$ which tends to any point

$$x_0 \in W(\hat{z}_0) = \operatorname*{Argmin}_{x \in X(\hat{z}_0,0,0)} f^0(x);$$

$F^j(\hat{z})$, $j=0,\ldots,m$, are continuous functions at the neighborhood of $z=(F^0(z_0), z_0^1,\ldots,z_0^{j-1}, z_0^{j+1},\ldots,z_0^m)$.

Let us consider the stability of the m.c.o. problem (1) with respect to right hand side perturbations of the constraints. We will say problem (1) is stable if and only if the w.e.e. set $S(y,v)$ of the perturbed problem is a continuous point-to-set mapping at (0,0). We use the Hausdorf metric in this definition.

We assume the following regularity condition holds: for any sequence $(y_k, v_k) \to (y_0, v_0)$ and any $\hat{z}_0((\hat{z}_0, y_0, v_0) \in Y^0)$ there exists a $\hat{z}_k \to \hat{z}_0$ such that $(\hat{z}_k, y_k, v_k) \in Y^0$.

Theorem 6 (see Kotkin [1988]).

If for any $\hat{z}_0$ g.s.f. $F^0(\hat{z}, y, v)$ is continuous function at $(\hat{z}_0, 0, 0)$ than the m.c.o. problem (1) is stable.

2. Geometric characteristics of the class of nonlinear programming methods

A great variety of numerical methods to solve n.l.p. and m.c.o. problem are based on their reduction to a sequence of unconstrained minimization (u.c.m.) problems. We will call these methods sequencial unconstrained minimization (s.u.m.) methods. In order to construct each u.c.m. problem, a so-called convolution function (c.f.) is used. This section is concerned with necessary and sufficient convergence conditions in terms of c.f. properties for class of s.u.m. methods for the problem with inequality constraints.

Let us consider, for example, the penalty functions method which reduce the n.l.p. problem to the following problem with rectangular constraints (we will called it the u.c.m. problem):

$$\min_{x \in Q} (f^0(x) + t_k \sum_i (g^i(x))_+^2 ,$$

where $(y)_+ = \max(y,0)$, for any $y \in R^1$.

In this case we have the following goal function in the u.c.m. problem (we will denote it by u.f.):

$$\bar{M}_k(x) = M_k(f^0(x), g(x)) = f^0(x) + t_k \sum_i (g^i(x))_+^2 ,$$

and the following c.f. is used for this method:

$$M_k(z,y) = z + t_k \sum_i (y^i)_+^2, \text{ where } z \in R^1, y \in R^p.$$

We have a sequence of u.c.m. problems and associated u.f. and c.f. when the penalty coefficient tends to infinity the following way: $t_1 = 10$, $t_{k+1} = 10\, t_k$.

Therefore the s.u.m. method wold be defined if we define a sequence of c.f. and choose a u.c.m. method. This way we are concerned with external iterative process which is changing the c.f. and the associated u.c.m. problem.

Because of the nature of s.u.m. methods we may consider the problem

$$\min_{(z,y) \in Z'} M_k(z,y), \text{ where } Z' = \check{Z} \cap \tilde{Z}, \tag{7}$$

instead of the u.c.m. problem

$$\min_{x \in Q} M_k(f^0(x), g(x)). \tag{8}$$

Therefore the efficiency of the s.u.m. method is determined by dual properties of the problem image $\check{Z}$ and c.f. $M_k(z,y)$ or minimal function value isolines $V_k(t_*)$ (see (5)).

It is easily to prove that $V_k(t_*)$ is situated not higher than problem image (see Kotkin [1988]).

Lemma 1. For any y, z_1, z_2 if

$$(z_1,y) \in V_k(t_*),\ (z_2,y) \in Z \setminus \operatorname*{Argmin}_{(z,y)\in Z} M_k(z,y)$$

then $z_1 < z_2$.

Therefore, if we would solve the equation

$$M_k(z,y) = t_* \tag{9}$$

with respect to z:

$$z = \tilde{M}_k(y), \tag{10}$$

then g.s.f. $F(y)$ greater than $\tilde{M}_k(y)$. It helps us to derive a convergence conditions for s.u.m. methods.

Similar conditions can be derived for m.c.o. methods if we would base on the fact that the w.e.e. set $S(0,0)$ is intersection of the g.s.f. $F^0(\hat{z},y,v)$ graph and the plane $\{(y,v): y=0,\ v=0\}$ (and the image $f(X(0,0)$, of course).

The theorems which are considered in this and the next sections permit us to prove convergence of s.u.m. methods and to construct m.c.o. methods on the base of well known n.l.p. methods.

For example, it has been proved that the following method converges in finite number of steps:

$$\min_{x\in Q} (f^0(x) - \mu_k)_+^2 + t_k \| g(x) \|\ q,$$

if $\mu_k \to f^0(x_*)$, $\mu_k < f^0(x_*)$, $t = \text{const}$;

where $\| y \|_1 = \sum_i |\ y^i\ |$; $\| y \|_q = \sqrt[q]{\sum_i | y^i |^q}$, for $1 < q < \infty$; $\| y \|_\infty = \max_i | y^i |$;

for any $y \in R^p$; x_* is the n.l.p. problem solution.

Let us define the optimal solution set W with respect to the space Z_+:

$$W = F(0) \times \{W_y(0)\} = f(W(0)) = f(\operatorname*{Argmin}_{g(x) \leq 0,\, x \in Q} f(x)).$$

Let us denote $y_- = (y_-^1,...,y_-^p)$; $y_+ = (y_+^1,...,y_-^p)$, for any $y \in R^p$.

Let us define the class of n.l.p. problems by the following conditions:

1) $F(y)$ is continuous function at zero;

2) $\exists\, \Theta > 0\ \forall\, y \in Y(y \nleq 0){:}\dfrac{F(y) - F(0)}{\| (y-0)_+ \|} \geq -\Theta$

3) $\exists\, y \in Y{:}y < 0$.

It is easy to show that the conditions 1) - 3) are satisfied in the case of stable problem with finite Lagrange function saddle point which satisfies the Slayter regularity condition.

We will suppose initially that we have the exact solutions of u.c.m. problem. We will call the c.f. exact if and only if the s.u.m. method converges in a finite number of steps. Otherwise we will call the c.f. smooth.

Let us define conditions that the minimum x_k of the u.c.m. problem is the exact (A) or approximate (B) solution of the n.l.p. problem:

A) $\exists\, k_0\ \forall k \geq k_0\ \forall\, x_k \in \operatorname*{Argmin}_{x \in q} M_k(f^0(x), g(x))$:

$g(x_k) \leq 0,\ f^0(x_k) = \min\limits_{x \in X(0)} f^0(x)$;

B) $\forall\, \varepsilon > 0\ \exists\, k_0\ \forall\, k \geq k_0\ \forall\, x \in \operatorname*{Argmin}_{X \in Q} M_k(f^0(x),\, g(x))$:

$g(x_k) \leq \varepsilon,\ f(x_k) - \min\limits_{x \in X(0)} f^0(x) \leq \varepsilon$;

where $\varepsilon = (\varepsilon,...,\varepsilon) \in R^p$.

We have the following conditions with respect to the space Z_+:

a) $\exists\, k_0\ \forall k \geq k_0\ \forall\, (z_k,y_k) \in \operatorname*{Argmin}_{(z,y) \in Z'} M_k(z,y){:}\ y_k \leq 0, z_k = F(0)$;

b) $\forall\, \varepsilon > 0\ \exists\, k_0\ \forall k \geq k_0\ \forall\, (z_k, y_k) \in \operatorname*{Argmin}_{(z,y) \in Z'} M_k(z,y)$:

$$y_k \leq \varepsilon, | z_k - F(0) | \leq \varepsilon;$$

where $Z' = \tilde{Z} \cap \check{Z}$.

There is a stronger condition with respect to the exact solution, namely the sets equality:

a') $\exists k_0 \forall k \geq k_0 \underset{(z,y)\in Z'}{\text{Argmin}}\, M_k(z,y) = \{F(0)\} \times W_y(0)$.

Let us define the following compact set:

$$K(a,b) = \{(z,y) : a^0 \leq z \leq b^0,\ \hat{a} \leq y \leq \hat{b},\ a \in R^{p+1},\ b \in R^{p+1},\ a < 0 < b\}.$$

We have to use sophisticated notation in order to take into account that the equality (9) can be solved with respect to z in the points $(z,y) \in \hat{Z}$ and cannot be solved in the points $(z,y) \in R^{p+1} \setminus \hat{Z}$. It is usually connected in nonlinear programming with using the following c.f. $M_k(z,y) = (z+\mu_k)_+^2 + t_k \sum_i (y^i)_+^2$. If one is not going to use such kind of c.f. then one can assume

$$M_k^0(Z,Y) = M_k(z,y),\ \tilde{M}_k^0(y,t_*) = \tilde{M}_k^0(y) = \tilde{M}_k(y),\ \hat{Z}_0 = \hat{Z} = \tilde{Z}_0 = \tilde{Z}$$

and continue from the Theorem 7.

Let us define the function

$$q(z_0, y_0) = \max z;$$

$$(z, y_0) \in \tilde{Z}$$

$$M(z, y_0) = M(z_0, y_0)$$

and consider the function $M_k^0(z,y)$ which is defined on the set $\tilde{Z}_0 = \{(q(z,y), y) : (z,y) \in \tilde{Z}\}$. Let us denote the function value isolines of the functions $M_k(z,y)$ and $M_k^0(z,y)$ by $\tilde{Z}(z_0, y_0)$ and $\tilde{Z}_0(z_0, y_0)$:

$$\tilde{Z}(z_0, y_0) = \{(z,y) \in \tilde{Z} : M_k(z,y) = M_k(z_0, y_0)\};$$

$$\tilde{Z}_0(z_0, y_0) = \{(z,y) \in \tilde{Z}_0 : M_k(z,y) = M_k(z_0, y_0)\}.$$

Let us denote

$$\hat{Z}(z_0, y_0) = \{(z,y) \in \hat{Z} : M_k(z,y) = M_k(z_0, y_0)\}.$$

Clearly, $Pr(\tilde{Z}(z_0, y_0)) = Pr(\tilde{Z}_0(z_0, y_0))$, where $Pr(\cdot)$ is a projection operator from Z_+ to $R^p \ni y_0$. Therefore we may consider function $M_k^0(z,y)$ instead of $M_k(z,y)$ because for any $(z,y) \in R^{p+1}$ we can solve the equation $M_k^0(z,y) = t_*$ with respect to $z : z = \tilde{M}_k^0(y)$ and conclude that the graph of the function $\tilde{M}_k^0(y)$ is located not lower than $V_k(t_*)$.

We will use also the notation $\tilde{M}_k^0(y, t_*) = \tilde{M}_k^0(y)$ in order to consider the relationship between the function $\tilde{M}_k^0(y,t)$ and parameter t.

It can be proved that the characteristic property of the s.u.m. methods, under the assumption that c.f. series does not depend on the n.l.p. problem which is solved, is convergence of $\tilde{M}_k^0(y)$ "global derivatives surface" to the negative orthant with the origin at $(F(0),0)$ (see Fig. 3). The exact c.f. differs from the smooth c.f. on that it has a breakpoint. To take this fact into account in the case of smooth c.f. we have to consider global derivatives anywhere but a ε – neighborhood $U(y,\varepsilon)$ of the points $y = (y^1,...,y^{j-1},0,y^{j+1},...,y^p)$.

Let us define the global derivatives $d^+ u(y,w,B)$ and $d^- u(y,w,B)$:

$$d^+ u(y,w,B) = \sup_{\{t \in R^1_+ : y + wt \in B\}} \frac{u(y+wt) - u(y)}{t} \in \bar{R}^1;$$

$$d^- u(y,w,B) = \inf_{\{t \in R^1_+ : y + wt \in B\}} \frac{u(y+wt) - u(y)}{t} \in \bar{R}^1.$$

Let us determine the class of s.u.m. methods with the following conditions:

1) $M_k(z,y)$ is nondecreasing function with respect to z;

2) $\forall (z_0,y_0) \in \tilde{Z}$: $\{y : y < y_0\} \subset Pr(\hat{Z}(z_0,y_0))$,
   $\forall (z_0,y_0) \in \tilde{Z}$: $\{y : y \leq y_0\} \subset Pr(\hat{Z}(z_0,y_0))$;

3) $\tilde{M}_k^0(y)$ is continuous function;

4) $\tilde{M}_k^0(y)$ has a derivative $\dfrac{d\tilde{M}_k^0}{dw}$ with respect to any direction w;

5) sequence of c.f. does not depend with respect to n.l.p. problem which is solved.

Theorem 7 (see Kotkin [1988]).

In order that condition b) holds for any n.l.p. problem it is necessary and sufficient that for any compact set $K(a,b)$

1) $R^1 \times \{y \in R^p : y < 0\} \subset \tilde{Z}$, minima $\min\limits_{(z,y)\in} M_k(z,y)$ exist for any $k \geq k_0$.

2) $\forall \delta > 0 \, \forall \, (z,y) \in K_0(a,b)$:

$d^+ \tilde{M}_k^0 (y_-, c, y_+, K_0(a,b,\delta)) \to -\infty$, if $k \to \infty$, where

$c = M_k(z,y), K_0(a,b,\delta) = \{y \in Pr(K_0(a,b)) : \exists i y^i \notin]-\delta, \delta[\}$;

3) $\forall \delta > 0 \, \forall (z,y) \in K_0(a,b) \, \forall \, w \in R^p \, (y \leq 0, w \leq 0)$:

$d^{(\mp)} \tilde{M}_k^0 (y,c,w,\hat{K}_0(a,b,\delta,y)) \to 0$, if $k \to \infty$, where

$c = M_k(z,y), \hat{K}_0(a,b,\delta,y) = K_0(a,b,\delta) \setminus U(y,\delta)$.

Let us assume for any compact set $K(a,b)$ that

$\exists k_0 \forall k \geq k_0 \forall (z,y) \in K_0(a,b) = \tilde{Z} \cap K(a,b) \forall w \in R^p (y \leq 0)$,

$w \leq 0$: $d^{(\mp)} \tilde{M}_k^0 (y,c,w, Pr(K_0(a,b))) = 0$, where $c = M_k(z,y)$.

In order that condition a') holds for any n.l.p. problem it is necessary and sufficient that for any compact set $K(a,b)$

1) $R' \times \{y \in R^p : y \leq 0\} \subset \tilde{Z}$;

2) $\forall \, (z,y) \in K_0(a,b)$:

$d^+ M_k(y_-, c, y_+, Pr(K_0(a,b))) \to -\infty$, if $k \to \infty$, where $c = M_k(z,y)$.

It should be noted that the conditions of Theorem 7 are sufficient convergence conditions if we exclude the assumption 5) about the relationship between the n.l.p. problem and the c.f. sequence.

Theorem 7 is interesting by itself but is hard to use. Easy to check convergence conditions in term of partial derivatives of the c.f. $M_k(z,y)$ are presented in the following theorem. It can be proved that a sufficient convergence condition for a wide range of s.u.m. methods is that the partial derivatives of $\tilde{M}_k(y)$ should tend to infinity if $y > 0$ and tend to zero if $y < 0$. We have to consider "right hand side" and "left hand side" semi-derivatives in order to study the exact c.f.

Let us consider these derivatives:

$$D^i_{k_{(\mp)}}(z,y) = -\frac{\partial_{(\mp)} M_k(z,y)}{\partial y^i} \Big/ \frac{\partial M_k(z,y)}{\partial z} , \; i = 1,\ldots,p.$$

The function $D^i_{k(\mp)}(z,y)$ has the following domain:

$$\hat{Z}_0 = \{(z,y) \in \tilde{Z}: \frac{\partial M_k(z,y)}{\partial z} \neq 0\} \subset \hat{Z}.$$

Let us denote

$$\hat{Z}_0(z_0, y_0) = \{(z,y) \in \hat{Z}_0: M_k(z,y) = M_k(z_0, y_0)\}.$$

To take into account the special case when c.f. $M(z,y)$ does not have semi-derivatives, let us denote

$$D^i_k(z, y, \Delta) = \frac{\tilde{M}_k(y^1,\dots,y^i + \Delta, \dots, y^p, M_k(z,y)) - \tilde{M}_k(y^1,\dots,y^p, M_k(z,y))}{\Delta}.$$

Let us denote the domain of $D^i_k(z, y, \Delta)$ by $\hat{Z}^i_\Delta$.

We will not write the assumptions with respect to c.f. $M_k(z,y)$. They are modification of the assumption 1) - 3) of theorem 7 (see Kotkin [1988]).

Theorem 8.

1. In order that condition a') holds for any n.l.p. problem it is necessary and sufficient that for any compact set $K(a,b)$

1) for any point $(z,y) \in \hat{Z}_0$ such that $y^i \geq 0$, $y^j \leq 0 \, (\forall j \neq i)$:

   $D^i_{k+}(z,y) \to -\infty$, if $k \to \infty, i = 1,\dots,p$;

2) for any big enough k and any point

   $(z,y) \in \hat{Z}_0 \cap K(a,b)$ such that $y \leq 0$:

   $D^i_{k-}(z,y) = 0$, $i = 1,\dots,p$;

2. In order that condition b) holds for any n.l.p. problem it is necessary and sufficient that for any compact set $K(a,b)$

1) for any point $(z,y) \in \hat{Z}_0$ such that $y^i > 0$, $y^j \leq 0 \, (\forall j \neq i)$:

$$D^i_k(z,y) \to -\infty, \text{ if } k \to \infty, i = 1,\dots,p;$$

2) for any big enough k and any point

$(z,y) \in \hat{Z}_0 \cap K(a,b)$ such that $y < 0$:

$D^i_{k-}(z,y) \leq 0,$

$D^i_{k-}(z,y) \to 0$, if $k \to$, $i = 1,\ldots,p$.

Theorem 8 is true if we would use $D^i_k(z, y, \Delta)$ instead of $D^i_{k+}(z,y)$ and $D^i_k(z, y, -\Delta)$ instead of $D^i_{k-}(z,y)$, where $\Delta \in R^1_+$ is little enough, and $\hat{Z}^i_\Delta$ instead of $\hat{Z}_0$.

3. Geometric properties of multicriteria optimization methods

Let us now consider the m.c.o. problem (1). The u.f. for this problem may be written as $M_k(f(x), g(x), h(x))$ and the c.f. may be written as $M_k(z, y, v)$, where $z \in R^{m+1}, y \in R^p, v \in R^s$ are the variables. We will consider the following two problems.

1) How to choose the type of c.f. with respect to variables $z^j, j = 0,\ldots,m$ in order to construct decision making procedure.

2) How to choose the type of c.f. with respect to variables $z \in R^{m+1}, y \in R^p, v \in R^s$ in order to provide convergence to some weakly efficient point (because the conditions may be out of order, for example). It can be shown that in this case it is not necessary to take into account the type of the c.f. $M_k(z,y,v)$ with respect to all $z^0,\ldots,z^m$, but z^0 (and y,v of course).

We consider first the second problem. From the formal definition point of view there is only one difference between the m.c.o. problem and the n.l.p. problem, namely the c.f. $M_k(z,y,v)$ and derivatives $D_{k\overline{(+)}}(z,y,v)$ of function $\tilde{M}_k(z,y,v)$ with respect to z^0 depend from $z \in R^{m+1}$.

To simplify the proofs, we suppose that derivatives of c.f. $M_k(z,y,v)$ with respect to $z^j, j = 0,\ldots,m$, are bounded by positive values. In addition to assumptions associated with the n.l.p. problem, we assume that all functions $F^j(\hat{z},y,v)$, $j = 0,\ldots,m$ are continuous. The Slayter condition should hold in a small neighborhood of a weak efficient solution.

Let us define conditions that the minimum x of the u.c.m. problem is the exact (a) or approximate (b) weak efficient solution of m.c.o. problem (1).

a) $\exists k_0 \forall k \geq k_0 \forall (z_k, y_k, v_k) \in \operatorname*{Argmin}_{(z,y,v) \in Z'} M_k(z,y,v)$:

$y_k \leq 0, v_k = 0, z_k^0 = F(\hat{z},0,0)$;

b) $\forall \varepsilon > 0 \exists k_0 \forall k \geq k_0 \forall (z_k, y_k, v_k) \in \operatorname*{Argmin}_{(z,y,v) \in Z'} M_k(z,y,v)$:

$y_k \leq \varepsilon, -\varepsilon \leq v \leq \varepsilon, \exists (\hat{z}_k 0,0) \in Y \mid z_k^0 - F(\hat{z},0,0) \mid \leq \varepsilon$;

where $Z' = \tilde{Z} \subset \hat{Z}, \varepsilon = (\varepsilon,\ldots,\varepsilon) \in R^l, l = p,s$.

a') $\exists k_0 \forall k \geq k_0$

$\operatorname*{Argmin}_{(z,y,v) \in Z'} M_k(z,y,v) = \{F(\hat{z}_k,0,0)\} \times W_y(\hat{z}_k,0,0)$.

Let us first consider the problem (1) with inequality constraint ($s = 0$).

Theorem 9 (see Kotkin [1988]).

The conditions of the Theorem 8 hold for the m.c.o. problem (1) if we use appropriate notation (D (z, y), z R ,

$$(D^i_{k(+)}(z,y), z \in R^{m+1}, y \in R^p;\ K(a,b), a, b \in R^{m+1+p};\ \hat{Z}_0,\ z \in R^{m+1}).$$

Theorem 9 can be modified for the case of the m.c.o. problem (1) with $s \neq 0$. It is easy to take into account approximate solutions of u.c.m. problem and to apply some considered ideas to study the s.u.m. methods which hardly depend on the properties of the n.l.p. problem and do not satisfy the conditions of Theorems 8 and 9 (for example, Dual Lagrange Multipliers Method, see Kotkin [1988]).

The following theorem describes the type of c.f. with respect to the decision making procedure. We consider the c.f. which is generalization of the reference point c.f.:

$M(z,y,v,z_0,y_0,v_0,u)$ is a continuous function which is increasing with respect to z^i,y^j,v^l if respectively $z^i > z_0^i, y^j > y_0^j, v^l > v_0^l$; constant with respect to z^i, y^j, if respectively $z^i \leq z_0^i, y^j \leq y_0^j$; decreasing with respect to v^l if $v^l \leq v_0^l$; and depends on the following parameters $z_0 = (z_0^0,\ldots,z_0^m)$, where $z_0^i \in R^1 \cup \{-\infty\}, y_0 \in R^p, v_0 \in R^s$, $u \in R^{m+1+p+s}, i = 0,\ldots,m, j = 1,\ldots,p, l = 1,\ldots,s$.

Let us define the associated u.f. $\bar{M}(x)$, minimal function value isoline $V(t)$, etc.:

$\bar{M}(x) = M(f(x), g(x), h(x), z_0, y_0, v_0, u)$;

$V(t) = \{(z,y,v) : M(z,y,v, z_0, y_0, v_0, u) = t\}$;

$C(t) = V(t) \cap \check{Z};$

$\eta_\varepsilon(A) = \{x \in Q : r(x,A) < \varepsilon\}$, where $r(x,A) = \inf_{x_0 \in A} \| x_0 - x \|$.

Let us suppose that $x_* \in Q$ is a solution of the following u.c.m. problem:

$$\min_{x \in Q} M(f(x), g(x), h(x), z_0, y_0, v_0, u).$$

Theorem 10 (see Kotkin [1988]).

If there exists a j such that $f^j(x_*) > z_0^j$, then

1) if $x_* \in X(0,0)$, then $f(x_*) \in S(0,0)$;

2) if in addition to the assumptions in 1) we have $f(x_*) > z_0$, then $f(x_*) \in P(0,0)$;

3) if in addition to the assumptions in 1) and 2) $M(z,y,v,z_0,y_0,v_0,u)$ is a differentiable function with respect to z and in the neighborhood of $(f(x_*), g(x_*), h(x_*))$ we have $\dfrac{\partial M(z,y,v,z_0,y_0,v_0,u)}{\partial z^i} \neq 0, i = 0,\ldots,m,$ then $f(x_*) \in G(0,0)$.

The following theorem allow to make a conclusion about quasi-convexity (unimodality) of the u.f. in the case of nonconvex (for example, quasi-convex) goals $f(x)$ or constraints $g(x)$ or $h(x)$. It is possible sometimes to use the u.c.m. local algorithms to investigate the quasi-convex problem (1). In this paper we consider only necessary conditions (see Kotkin [1988]).

Under the assumption that $\bar{M}^{-1}(t)$ is a continuous point-to-set mapping we have the following result (see Fig. 4).

Theorem 11.

If there exist a number t_* that the set $C(t_*)$ can be partitioned into the closed nonempty sets A and B, and for any $t < t_*$ we have $C(t) \cap \eta_\varepsilon(A) = \emptyset$ or $C(t) \cap \eta_\varepsilon(B) = \emptyset$ then $\bar{M}(x)$ is not unimodal.

4. Inverse nonlinear programming problems

Inverse nonlinear programming problem it is a new class of optimization problems which is erased in game theory, system optimization, multicriteria optimization, etc.

Inverse nonlinear programming problem may be formulated as follows. We have to find out the pair x_* and u_* which satisfy the following system.

$$x_* \operatorname*{Argmin}_{x \in X(u_*)} f^1(x, u_*);$$

$$G(x_*, u_*) \le 0, H(x_*, u_*) = 0,$$

where $X(u) = \{x \in R^n : g(x,u) \le 0, h(x,u) = 0\}; u \in R^m$;
$f^1(x,u), g(x,u), h(x,u), G(x,u), H(x,u)$ are continuous functions, $f^1 : R^{n+m} \to R^1, g : R^{n+m} \to R^p, h : R^{n+m} \to R^S, G : R^{n+m} \to R^l, H : R^{n+m} \to R^t$.

Let us consider the multicriteria optimization problem (1) and assume that we can write the decision maker's additional information about the solution $x_* \in S_x$ in the form of equality and inequality constraints.

For example, if the reference point z_* is given the additional constraints are

$$f^0(x_*) - z_*^0 = f^1(x_*) - z_*^1 = \dots = f^m(x_*) - z_*^m.$$

If the aspiration level z_* is given the additional constraints are $f^0(x_*) \le z_*$.

In common cases these constraints link the solution x_* and the parameter u_* :

$$G(x_*, u_*) \le 0; H(x_*, u_*) = 0,$$

where $G(x,u), H(x,u)$ are continuous vector-functions.

We have the following inverse problem:

$$\begin{cases} x_* \in \underset{x \in X(0,0)}{\text{Argmin}} < u_*, f(x) >; \\ G(x_*, u_*) \leq), H(x_*, u_*) = 0, \end{cases}$$

where $u_* \in R^{m+1}$; $f: R^n \rightarrow R^{m+1}$.

In this paper we present the generalized Newton method to solve the following inverse nonlinear programming problem:

$$\begin{cases} x_* \in \underset{x \in R^n}{\text{Argmin}}\ f^1(x, u_*); \\ x_* = A u_* + B, \end{cases}$$

where $f^1 : R^{n+m} \rightarrow R^1$ is sufficiently smooth strong convex function, A and B are given matrixes. It is based on the idea that we can calculate the derivatives $x_u(u)$ of a so-called solution function

$$x(u) = \underset{x \in R^n}{\text{argmin}}\ f^1(x, u),$$

if we consider the second order derivatives $f_{xx}(x,u)$ and $f_{xu}(x,u)$ of the function $f(x,u)$.

It can be proved that these derivatives $x(u)$ are the solutions of the following system

$$< [f_{xx}, f_{xu}], [< x_u(u_0), u - u_0 >, u - u_0] > = 0.$$

Therefore we can solve the system

$$x(u) - (Au + B) = 0,$$

with respect to u using usual Newton method.

This method converge to the solution u_* (and $x(u_*)$) of inverse nonlinear programming problem under the assumptions that the solution function $x(u)$ is sufficiently smooth and a function

$$q(u) = x(u) - (Au + B)$$

satisfy the usual assumptions of the Newton method. So we have the local convergence for this method.

5. Interactive system DISO/PC-MCNLP

The DISO/PC-MCNLP system is developed for IBM-PC/XT compatible computers in the Computer Center of the USSR Academy of Sciences for multicriteria nonlinear programming solving.

DISO/PC-MCNLP is based on ideas of multicriteria and nonlinear optimization associated with parametric optimization, sensitivity analysis and inverse optimization problems. The great variety of numerical algorithms, interactive procedures, parametric and inverse study possibilities and flexible control are the main features of the DISO/PC-MCNLP system. DISO/PC-MCNLP reduces any optimization problem (nonlinear programming or multicriteria optimization) to a parametric or inverse optimization problem. Therefore all functions such as goals, constraints or parametric constraints are equal on the base level of DISO/PC-MCNLP system program. Some geometric parametric ideas form on the base of DISO/PC-MCNLP programm. The function identification is a property of some environmental level. Some protection against incorrect data is the essence of the next level (see Fig. 5). The flexible control system Field Manager and analytical differentiation language DIFALG are the user level. One can use the "C" language to make a problem definition if one prefers "C" to DIFALG.

The unconstrained optimization algorithm, nonlinear programming algorithm, interactive procedure of multicriteria search, any parameters of numerical method and some parameters of the applied problem can be changed assynchronically with respect to the calculation process. Field Manager allows one to adjust the interface to one's own applied problem. The beginning of the possible adjustments is choosing the numerical algorithm and its parameters, followed by preparing (if so one needs) windows, the form of presentation of the system and problem objects (numeric, histograms, graphs), defining the applied problem objects, names, etc. One can change the values of parameters which lie in the basis of one's own applied problem, write and read these parameters and other information from a floppy disk assynchronically to the calculation process. All these features of DISO/PC-MCNLP allow the easy construction of the interactive system for applied optimization problems.

Such interactive systems have been constructed for water resources distribution problem (see Kotkin and Mironov [1989]), metalworking production and other applications.

A great variety of numerical algorithms and interactive procedures are available in DISO/PC-MCNLP system. They are need to choose the appropriate algorithm for solving the problem. The DISO/PC-MCNLP has a multi level structure with respect to numerical methods (see Fig. 6). Several unconstrained minimization methods are at the base level. They are the result of the long time experience of a group of scientists from the Computer Center of the USSR Academy of Sciences (see Evtushenko [1985]).

The next level consist of a number of nonlinear programming techniques because any multicriteria programming problem is usually reduced to one goal programming problem. The last level consist of a series of decision making procedures.

The DISO/PC-MCNLP includes the following methods.

Unconstrained minimization methods:

1u) coordinate descent;

2u) direct search (two modifications);

3u) random search method;

4u) conjugate gradient;

5u) Newton method.

Nonlinear programming methods:

1n) center method modifications;

2n) penalty functions method modifications;

3n) barrier methods;

4n) exact penalty function method modifications.

Decision making methods:

1d) gradient method (Geoffrion);

2d) parametric programming method (Guddat);

3d) reference point method modifications;

4d) scalarization method modifications;

5d) nonlinear parametric programming method.

In order to construct nonlinear programming method and decision making procedure we used a convolution wich satisfying conditions of Theorem 9 and in special case Theorem 10:

$$\sum_{i=0}^{m+p+s+l} (f^i(x) - v^i)^{\begin{Bmatrix} p_-^i, \text{if } f^i(x) \le v^i \\ p_+^i, \text{if } f^i(x) > v^i \end{Bmatrix}} \; x \begin{Bmatrix} u^i, \text{if } f^i(x) < v^i \\ u^i, \text{if } f^i(x) > v^i \end{Bmatrix}$$

We can construct the methods 1n) - 4n) and 1d) - 5d) if we choose the parameters $v^i, u\_l^i, u\_r^i \in R^1$; $p\_l^i, p\_r^i \in \{-1,1,2,4\}, i=0,\ldots,m+1+p+s$, using Table 1.

Table 1

Parameters of c.f.															
Method	With respect to					With respect to					With respect to				
—	v^i	u^i	u_+^i	p_-^i	p_+^i	v^i	u_-^i	u_+^i	p_-^i	p_+^i	v^i	u_-^i	u_+^i	p_-^i	p_+^i
$2n+$ $4d$	0	1	1	1	1	0	0	t1 →∞	2	2	0	t2 →∞	t2 →∞	2	2
$3n+$ $4d$	0	1	1	1	1	0	-t1 →0	t1 →∞	-1	2	0	t2 →	t2 →∞	2	2
$1n+$ $3d$	z^i var	0	1	2	2	0	0	t1 →∞	2	2	0	t2 →∞	t2 →∞	2	2
$3n+$ $3d$	z^i var	0	1	2	2	0	0 →0	t1 →∞	1	1	0	t2 →∞	t2 →∞	2	2
$4n+$ $3d$	z^i var	0	1	2	2	0	0	t1 →∞	1	1	0	t2 →∞	t2 →∞	1	1
$1n+$ $5d$	z^i var	0	1	2	2	$v1^i$ var	0	t1 →∞	2	2	$v2^i$ var	t2 →∞	t2 →∞	2	2

The interface possibilities of DISO/PC-MCNLP system are provided by the Field Manager system (see Mazourik [1988]). Field Manager allows to construct a number of windows and to receive any information in these windows. The main idea of the Field Manager system is as follows. We may look out and correct a set of data which have been marked in our program. There are the following kind of data to be marked (they are called objects): scalars, vectors, matrices, functions with respect to real, integer, character or string values. When running the program, we can choose the following form of presentations for the objects: numeric, histograms, graphs. We can also link to some reaction with special user actions. For example, the control menu is just the vector of strings with some reaction to pushing the "enter" key (running some process for instance).

We need not think about all these things when writing the program, but only mark the objects we are interested in and define how often we want to redraw them on the display.
We may construct a number of windows with any objects on it in any form of presentation with any names (labels) after running the program. This way we may prepare some standard windows and make other windows when we wish.

The set of DISO/PC-MCNLP system objects is fixed. These objects allow to choose various numerical algorithms and interactive multicriteria procedures, change its parameters, to run and stop processes and correct some problem definition parameters. This way we may see and correct the following problem definition parameters:

1) functions type (goal, inequality or equality constraint);
2) right hand sides of inequality or equality constraints;
3) lower and upper bounds of rectangular constraints;
4) vectors of parameters introduced by the user if one wishes to correct the applied problem parameters during the numerical method calculations.

Therefore, there are not only universal decision making procedures in DISO/PC-MCNLP system but special procedures which can be prepared by the user. The desired "applied" parameters are the control parameters in this procedure.

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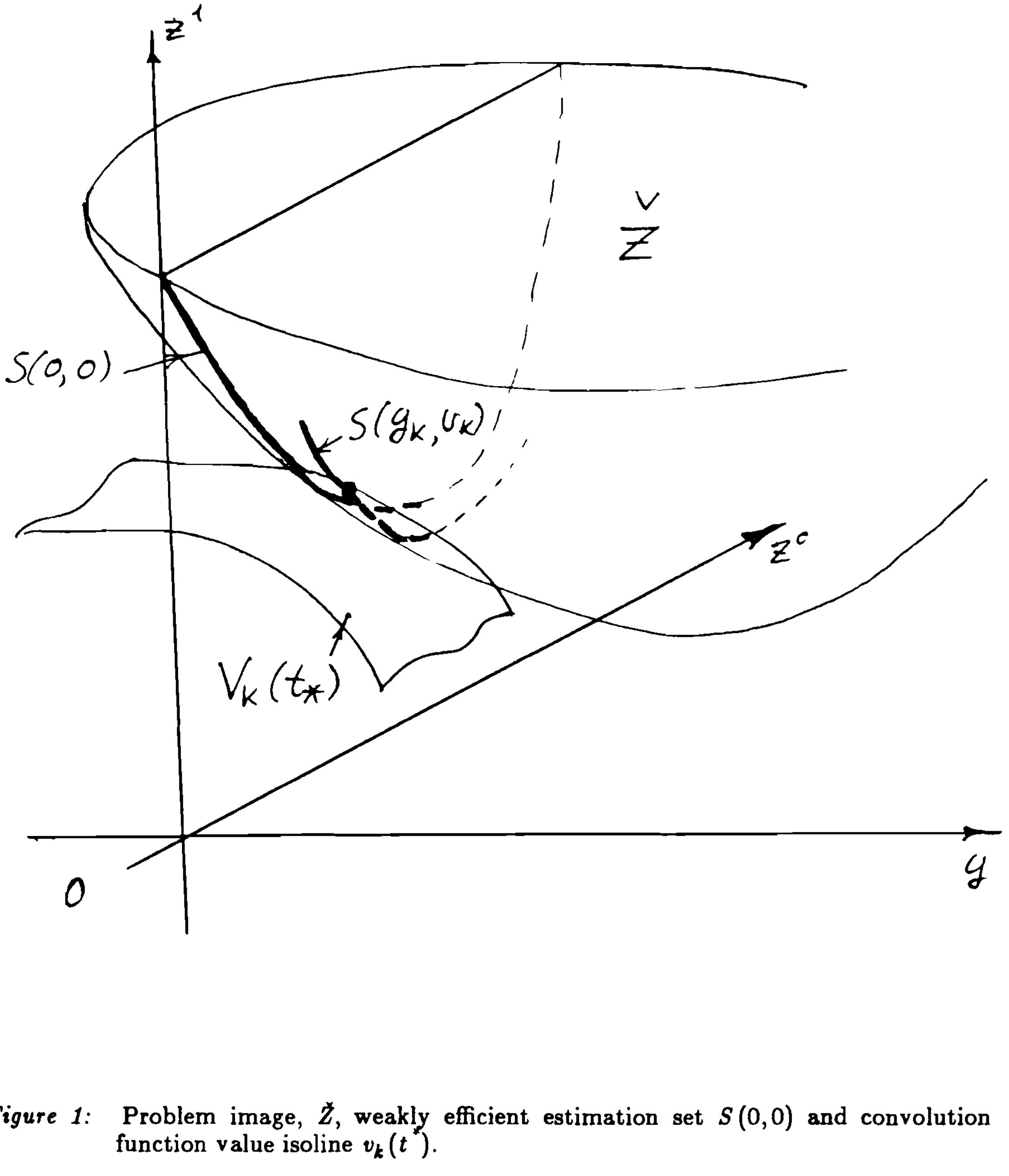
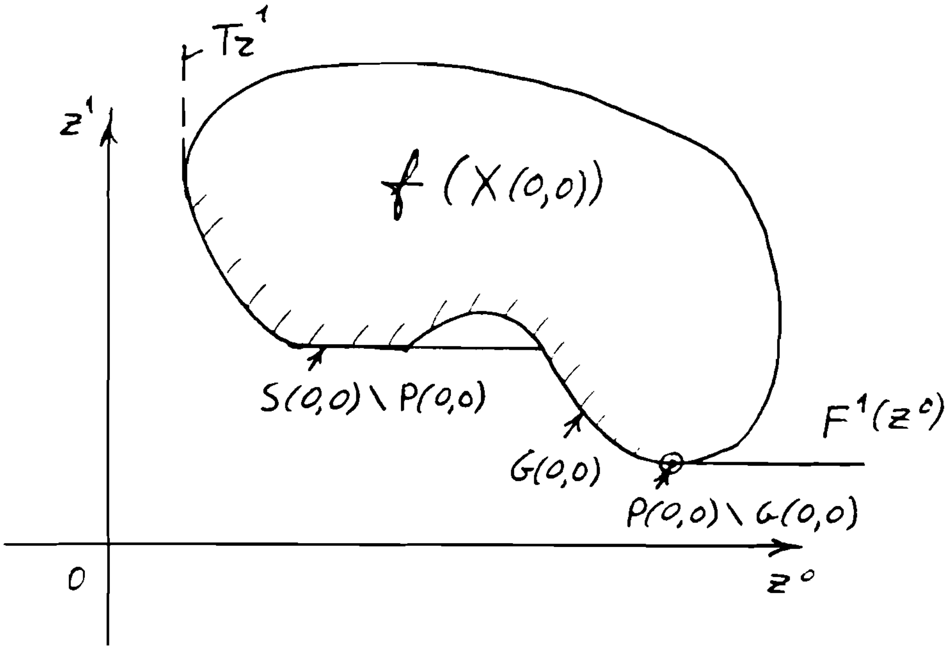


Figure 1: Problem image, $\check{Z}$, weakly efficient estimation set $S(0,0)$ and convolution function value isoline $v_k(t^*)$.



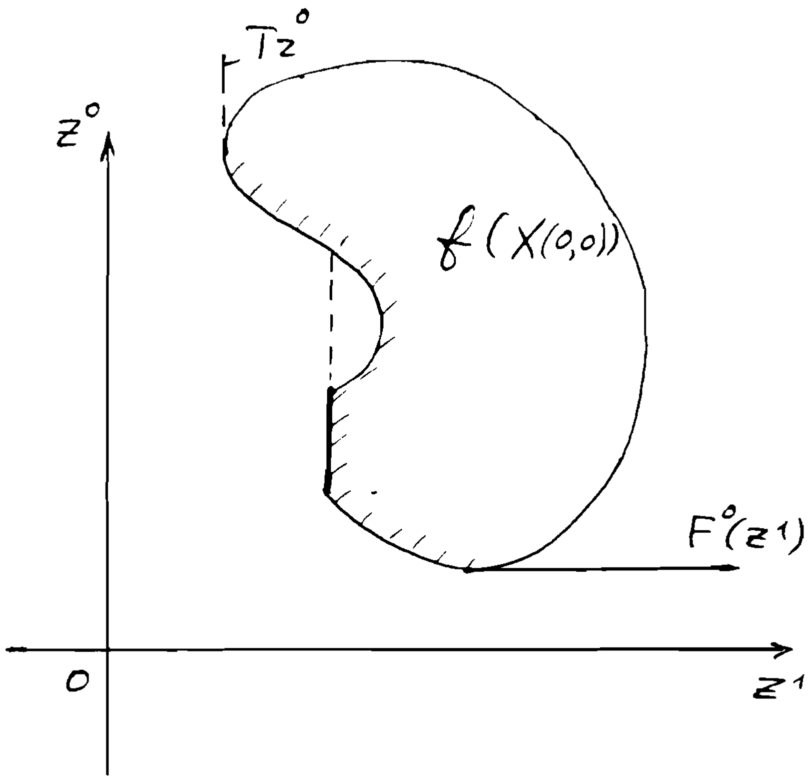


Figure 2: Weakly efficient $S(0,0)$, efficient $P(0,0)$ and proper efficient $G(0,0)$ estimation sets and sensitivity function graphs $F^0(z^1)$ and $F^1(z^0)$.

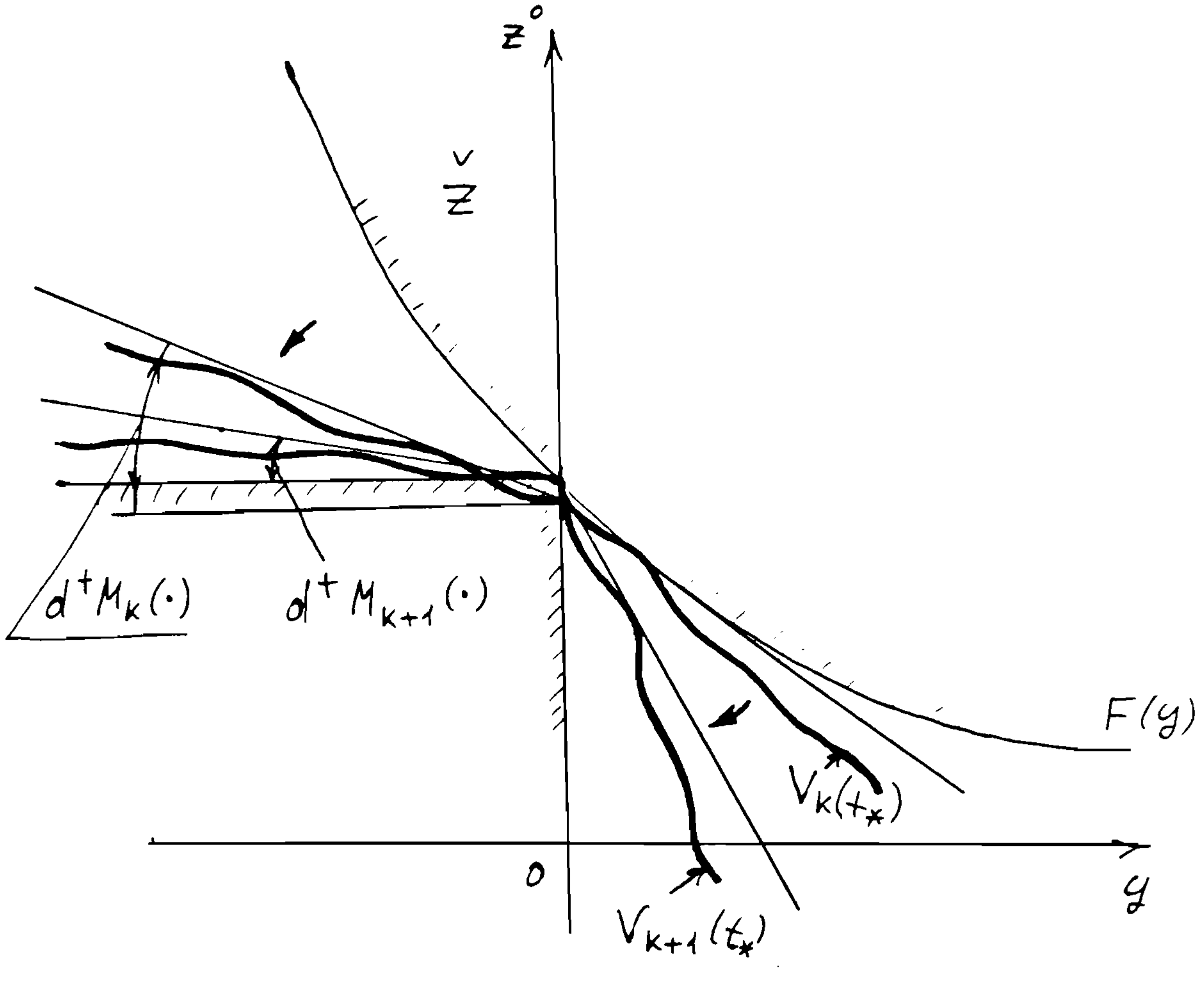


Figure 3: Convolution function value isolines $v_k(t_*)$, $v_{k+1}(t_*)$ and "global derivatives" d^+M_k, d^+M_{k+1}.

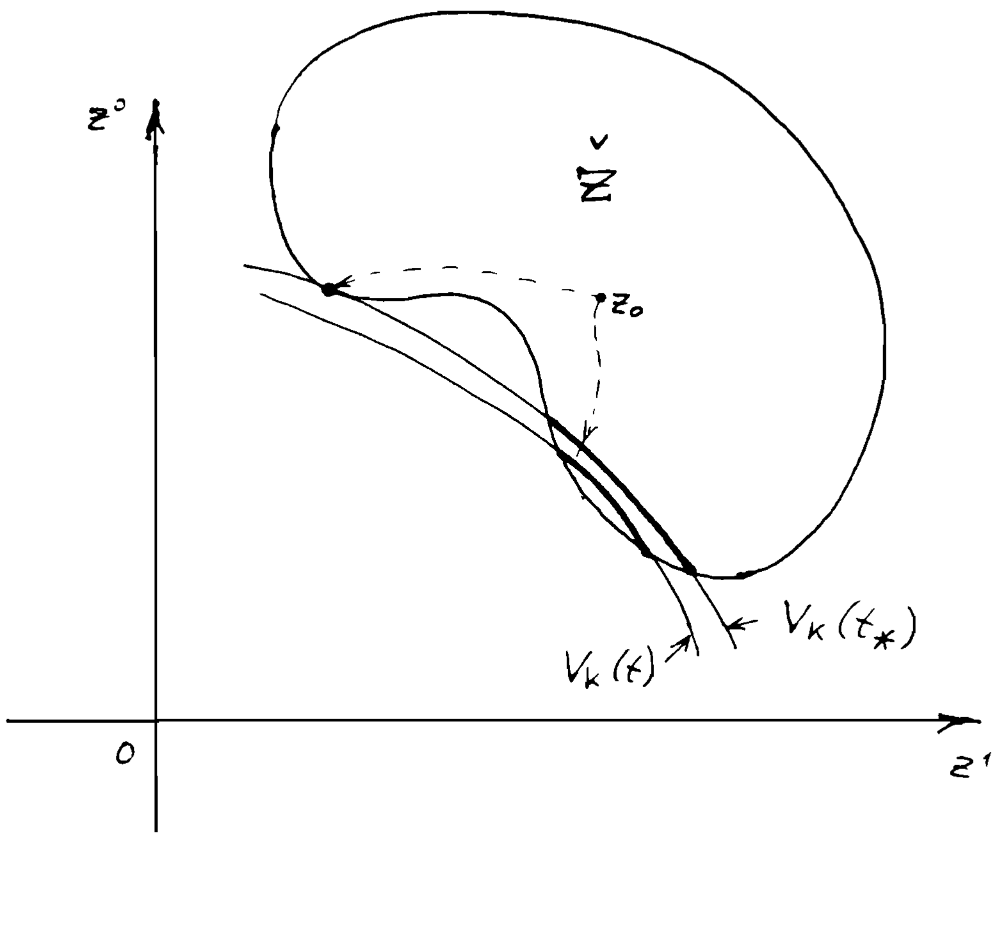


Figure 4: Quasiconvex problem study.

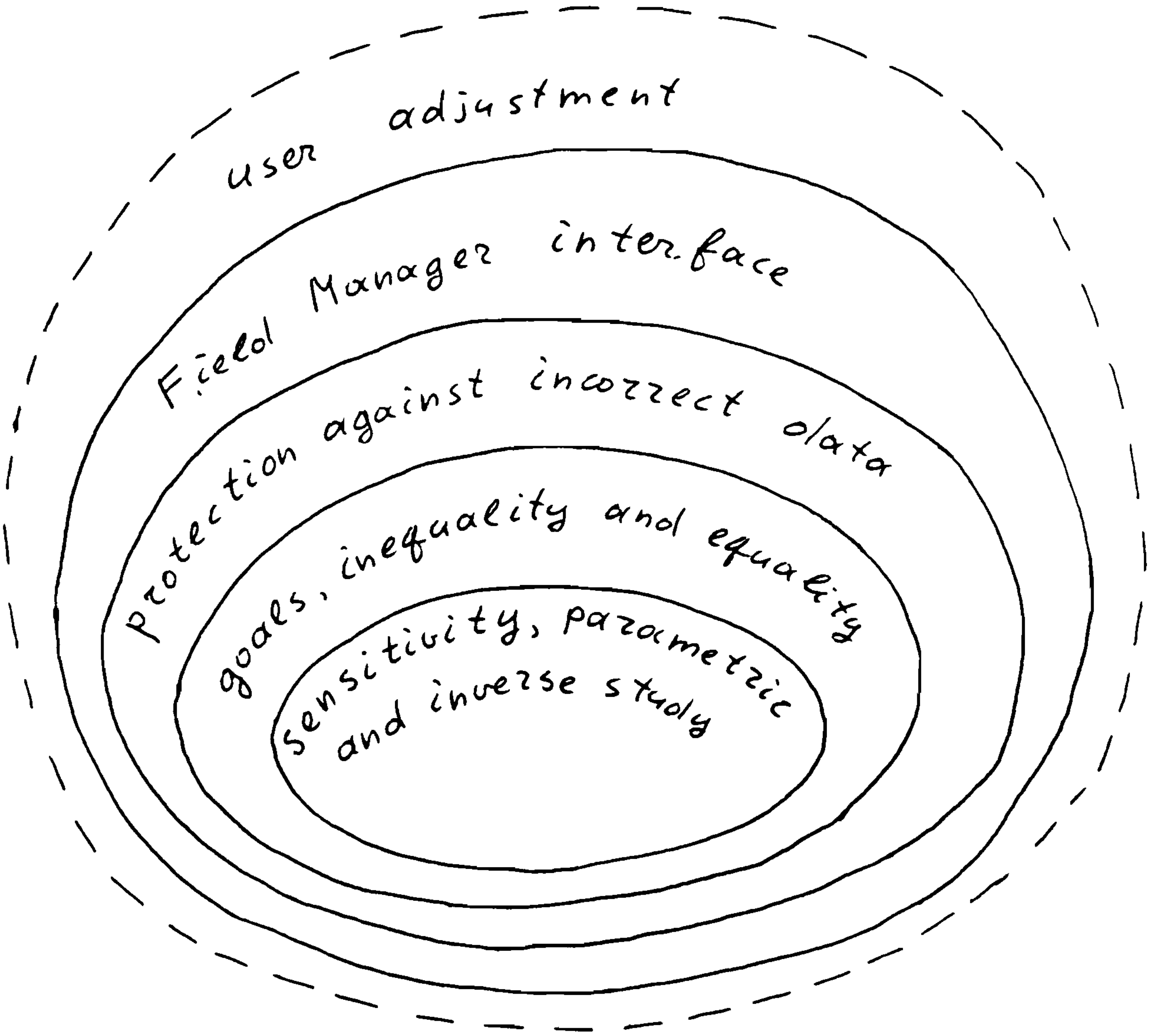


Figure 5: DISO/PC-MCNLP structure with respect to user.

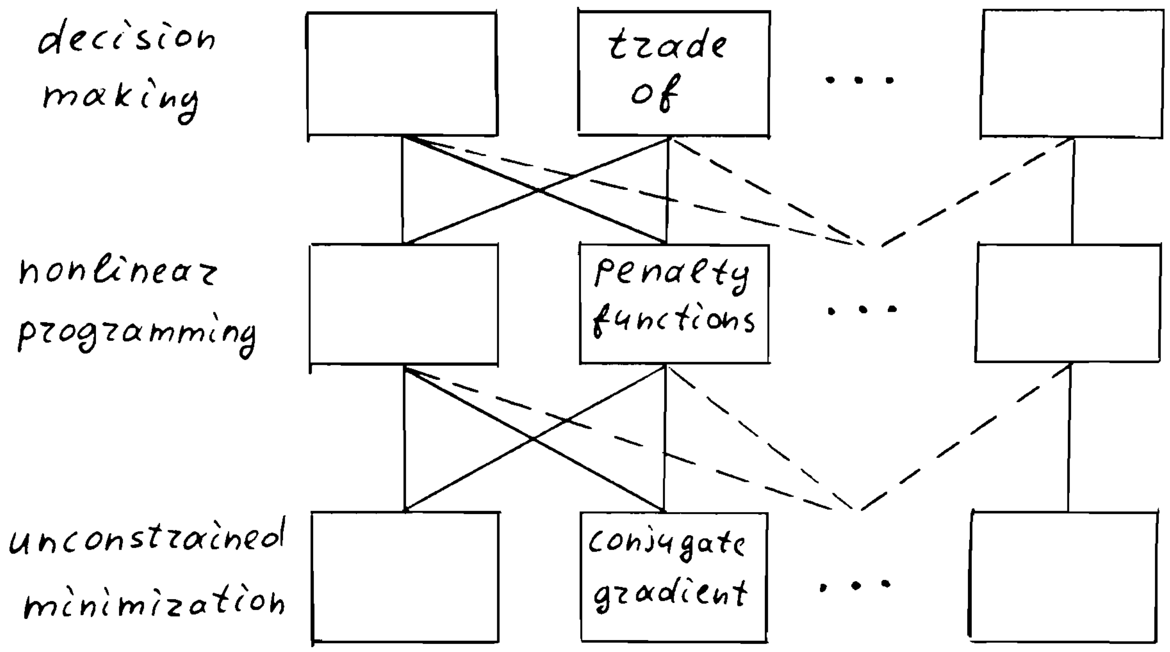


Figure 6: DISO/PC-MCNLP structure with respect to numerical methods.