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Inverse Nonlinear Programming Problem and its Application

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FOREWORD

Inverse nonlinear programming problems for a new class of optimization problems relevant for game theory, system optimization, multicriteria optimization, etc. are considered by the author.

His paper deals with problem definitions, numerical methods and applications of the inverse nonlinear programming problem in multicriteria optimization. Some associated properties of related parametric optimization problems and software implementations are also considered.

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INVERSE NONLINEAR PROGRAMMING PROBLEM AND ITS APPLICATION

Gregory G. Kotkin

Introduction

Inverse nonlinear programming (i.n.p.) problems may by formulated as follows. We have to find out the pair x_* and u_* which satisfy the following system.

$$\begin{cases} a)\ x_* \in \underset{x \in X(u_*)}{\operatorname{Argmin}}\ f(x, u_*); \\ b)\ g(x_*, u_*) \leq 0,\ H(x_*, u_*) = 0, \end{cases} \tag{1}$$

where $X(u) = \{x \in R^n : g(x, u) \leq 0,\ h(x, u) = 0\};\ u \in R^m$;
$f(x, u),\ g(x, u),\ G(x, u),\ H(x, u)$ are continuous functions, $f: R^{n+m} \to R^1,\ g: R^{n+m} \to R^p,\ h: R^{n+m} \to R^s,\ G: R^{n+m} \to R^l,\ H: R^{n+m} \to R^d$.

In other words, we have to find out the solution x_* of the parametric nonlinear programming (p.n.p.) problem (1a) with given parameter u_* and the pair x_*, u_* have to satisfy additional constraints (1b).

Similar problems were studied in game theory, system optimization and multicriteria optimaization (see Kurzhanski, A.B. [1986]).

The multicriteria nonlinear programming (m.n.p.) problem may be formulated as a i.n.p. problem. Let us consider the following m.n.p. problem:

$$\min_{x \in X(0,0)} \bar{f}(x) \tag{2}$$

where $X(y, v) = \{x \in Q \subset R^n : \bar{g}(x) \leq y,\ \bar{h}(x) = v\}$; $Q = \{x \in R^n : a \leq x \leq b\}$ is a rectangular constrained set, $a \in R^n,\ b \in R^n,\ a < b$, are given vectors; $\bar{f}(x),\ \bar{g}(x),\ \bar{h}(x)$ are continuous vector-functions $\bar{f}: Q \to R^{m+1},\ \bar{g}: Q \to R^p,\ \bar{h}: Q \to R^s$.

Usually we have to find out the Pareto optimal solution $x_* \in P_x$ of the problem (2) using some additional information. It is assumed that a decision maker chooses the optimal solution using some additional knowledge about the solution.

In order to write this additional information in the form of the equality and inequality constraints let us consider the following parametrization of the Pareto optimal solution set:

1) $S_1 = \{ \underset{x \in X(0,0)}{\operatorname{Argmin}} < u, \bar{f}(x) >,\ u \in R^{m+1},\ u \geq 0,\ \exists i\,(u^i \neq 0)\}$.

2)

$$S_2 = \{ \underset{\substack{\bar{f}(x) \leq u^i, i=1,\dots,m \\ x \in X(0,0)}}{\mathrm{Argmin}} \bar{f}^0(x),\ u \in R^m \} \tag{3}$$

It is easy to prove that if a function

$$F(u) = \min_{\substack{\bar{f}(x) \leq u^i, i=1,\dots,m \\ x \in X(0,0)}} \bar{f}^0(x)$$

is a continuous function, the set S_2 is a set of weakly efficient solution of the m.n.p. problem (2).

Under the assumption that all functions $\bar{f}(x)$, $\bar{g}(x)$, $\bar{h}(x)$ are strong convex functions any value of parameter u_* associated with unique weakly efficient solution $x(u_*)$ of the m.n.p. problem (2) which is a solution of the following nonlinear programming (n.l.p.) problem:

$$x(u_*) = \underset{\substack{\bar{f}(x) \leq u^i, i=1,\dots,m \\ x \in X(0,0)}}{\mathrm{argmin}} \bar{f}^0(x), \tag{4}$$

and conversely for any weakly efficient solution x_* of the m.n.p. problem (2), we can choose such parameter u_* that x_* would be a solution of the n.l.p. problem (4): $x_4 = x(u_*)$.

Therefore there is point-to-point correspondence between the weakly efficient solution $x_* \in S_x$ and values of parameter $u_* \in U$.

Let us assume that we can write the decision maker's additional information about the solution $x_* \in S_x$ in the form of equality and equality constraints.

For example, if the reference point z_* is given the additional constraints are

$$\bar{f}^0(x_*) - z_*^0 = \bar{f}^1(x_*) - z_*^1 = \ldots = \bar{f}^m(x_*) - z_*^m.$$

If the reservation level z_* is given the additional constraints are $\bar{f}(x_*) \leq z_*$.

In common cases these constraints link the solution x_* and the parameter u_*:

$$G(x_*, u_*) \leq 0;\ H(x_*, u_*) = 0$$

where $G(x, u), H(x, u)$ are continuous vector-functions.

In this case the m.n.p. problem (2) is reduced to the following inverse problem:

$$\begin{cases} a) \; x_* \in \underset{\substack{\bar{f}^i(x) \leq u^i, i=1,\ldots,m \\ x \in X(0,0)}}{\operatorname{Argmin}} \bar{f}^0(x) \\ b) \; G(x_*, u_*) \leq 0, H(x_*, u_*) = 0 \end{cases} \tag{5}$$

Using other parametrization of the Pareto set we have the following i.n.p. problem:

$$\begin{cases} a) \; x_* \in \underset{x \in X(0,0)}{\operatorname{Argmin}} < u_*, \bar{f}(x) > ; \\ b) G(x_*, u_*) \leq 0, \; H(x_*, u_*) = 0, \text{ where } u_* \in R^{m+1}. \end{cases}$$

It is easy to know that one cannot include the additional constraints (5b) to the set of the constraints of the problem (5a). Therefore we cannot use the usual optimization technique to solve the inverse problem (5) or (1).

In this paper we will consider a Generalized Newton Method to solve the following inverse nonlinear programming problem:

$$\begin{cases} x_* \in \underset{x \in R^n}{\operatorname{Argmin}} \; f(x, u_*); \\ x_* = A u_* + B, \end{cases}$$

where $f: R^{n+m} \to R^1$ is a sufficiently smooth strong convex function, A and B are given matrixes. It is based on the idea that we can calculate the derivatives $x_u(u)$ of a so-called solution function

$$x(u) = \underset{x \in R^n}{\operatorname{argmin}} \; f(x, u),$$

if we consider the second order derivatives $f_{xx}(x, u)$ and $f_{xu}(x, u)$ of the function $f(x, u)$.

We will present first some results and ideas of the parametric optimization theory and its applications for the i.n.p. problems.

Generalized sensitivity function

Let us denote the weak efficient, the efficient and the proper efficient (Geoffrion optimal) estimations set of problem (2) by $S(0,0)$, $P(0,0)$ and $G(0,0)$ respectively. Clearly, $G(0,0) \subset P(0,0) \subset S(0,0)$. Let us denote for any vector $a = (a^0,\dots,a^m)$ by $\hat{a}$ the following vector $(a^0,\dots,a^{j-1},a^{j+1},\dots,a^m)$ for some j. Without loss of generality we will consider sometimes the case $j=0$ in notation, for example

$$F(\hat{z},y,v) = F^0(\hat{z},y,v).$$

We are concerned with the following objects in the space z_+ of functions $\bar{f},\bar{g},\bar{h}$ values (see Fig. 1):

1) problem image $\check{Z}$:

   $\check{Z} = \check{f}(Q)$, where $\check{f} = (\bar{f},\bar{g},\bar{h})$;

2) generalized sensitivity functions (g.s.f.) $F^j: Y^j \rightarrow R^1$:

   $F^j(\hat{z},y,v) = \min\limits_{x \in X(\hat{z},y,v)} \bar{f}^j(x)$,

   where $X(\hat{z},y,v) = \{x \in Q : \bar{f}^i(x) \leq z^i,\ i=0,\dots,j-1,$

   $j+1,\dots,m,\ \bar{g}(x) \leq y,\ \bar{h}(x) = v\},\ j=0,\dots,m$;

   $Y^j = \{(\hat{z},y,v) : X(\hat{z},y,v) \neq \emptyset\}$;

3) weakly efficient estimations (w.e.e.) set $S(0,0)$:

   $S(0,0) = \{z \in R^{m+1}$: there do not exist $z_0 \in f(X(0,0))$ such that $z_0 < z\}$;

4) convolution functions (c.f.) $M_k(z,y,v)$:

   $M_k : Z_+ \rightarrow R^1$; $M_k(z,y,v)$ are continuous nondecreasing with respect z functions, defined on the domain $\tilde{Z} \subset Z_+$, $k = 1,2,\dots$

5) the function value isolines $V_k(t)$ of c.f.:

   $V_k(t) = \{(z,y,v) \in \tilde{Z} : M_k(z,y,v) = t\}$,

   and the minimal function value isolines $V_k(t_*)$, where

   $t_* = \min\limits_{(z,y,v) \in \check{Z} \cap \tilde{Z}} M_k(z,y,v)$.

The well known sensitivity function (see Elster [1980]) is a g.s.f. in one goal n.l.p. problem (problem (1) in the case $m=0$.

If the arguments y and v are fixed ($y=0$ and $v=0$) the g.s.f. transforms into a function $F^j(\hat{z}) = F^j(\hat{z},0,0)$ which can be used for w.e.e. set parametrization in the following way.

Let us add vertical lines to the graph T^j of function $F^j(\hat{z})$ through the points where $F^j(\hat{z})$ is discontinuous. We obtain the set Tr^j:

$$Tr^j = \{(z^0,...,z^{j-1}, b, z^{j+1},...,z^m) : \hat{z} \in Y^j, \exists\, \hat{z}_k \to \hat{z}, \hat{z}_k < \hat{z}$$

such that $F^j(\hat{z}) \leq b \leq F^j(\hat{z}_k)\}$.
under the assumption that $F^j(\hat{z}) = +\infty$ $(\forall\, \hat{z} \notin Y^j)$,

Theorem 1 (see Kotkin [1989]).

1) $Tr^0 = Tr^1 = ... = Tr^m$.
2) $S(0,0) = \bar{f}(X(0,0)) \cap Tr^0$.
3) Under the assumption that g.s.f. $F^0(\hat{z}, y\, v)$ or function $F^0(\hat{z})$ is continuous function we have

   $S(0,0) = P(0,0) = T^0 \cap \bar{f}(X(0,0))$.

We will say the n.l.p. problem is stable if and only if the optimal value function $F(y,v)$ is continuous at $(0,0)$. In this case the stability with respect to right hand side perturbations of the constraints is considered.

Let us consider the n.l.p. problem with equaltiy constraints (problem (1) in the case $m=0, s=0$) and denote

$$W(y) = \underset{\bar{g}(x) \leq y,\, x \in Q}{\text{Argmin}} \bar{f}^0(x); \; W_y(y) = \bar{g}(W(y));$$

$$I(y) = \{x \in Q : \bar{g}(x) < y\}.$$

Theorem 2 (see Kotkin [1988]).

Let us suppose that $\bar{f}^0(x)$ and $\bar{g}^0(x)$ are continuous functions, Q is a compact set and $I(y_0) \neq \emptyset$. Then in order that $F(y)$ be a continuous function at y_0 it is necessary and sufficient that

$$cl\, I(y_0) \cap W(y_0) \neq \emptyset.$$

Theorem 2 and some similar results can be used to study that connectedness of various optimal solutions set and to obtain proper efficiency criteria (see Kotkin [1988]).

Theorem 3.

Let us suppose that $F^j(\hat{z})$, $j=0,\ldots,m$, are continuous functions and $X(0,0)$ is a connected (linear-wise connected) set. Then $S(0,0) = P(0,0)$ is a connected (linear-wise connected) set.

Let us denote $\tilde{f}^{-1}(z) = \tilde{f}^{-1}(z) \cap X(0,0)$.

Theorem 4.

Let us suppose that $F^j(\hat{z})$, $j=0,\ldots,m$, are continuous functions, $\tilde{f}^{-1}(z)$ is a connected set for any $z \in f(X(0,0))$ and $S(0,0)$, $P(0,0)$ or $S(0,0) = P(0,0)$ are a connected set. Then $\tilde{f}^{-1}(S)$, $\tilde{f}^{-1}(P)$ or $\tilde{f}^{-1}(G)$ are the associated connected sets, respectively.

Let us consider the stability of the m.n.p. problem (2) with respect to right hand side perturbations of the constraints. We will say problem (2) is stable if and only if the w.e.e. set $S(y,v)$ of the perturbed problem is a continuous point-to-set mapping at $(0,0)$. We use the Hausdorf metric in this definition.

We assume the following regularity condition holds: for any sequence $(y_k, v_k) \to (y_0, v_0)$ and any $\hat{z}_0(\hat{z}_0, y_0, v_0) \in Y^0)$ there exists a $\hat{z}_k \to \hat{z}_0$ such that $(\hat{z}_k, y_k, v_k) \in Y^0$.

Theorem 5 (see Kotkin [1988]).

If for any $\hat{z}_0$ g.s.f. , $F^0(\hat{z}_k, y, v)$ is a continuous function at $(\hat{z}_0, 0, 0)$ then the m.n.p. problem (2) is stable.

A great variety of numerical methods to solve n.l.p. and m.n.p. problems are based on their reduction to a sequence of unconstrained minimization (u.c.m.) problems. We will call these methods sequential unconstrained minimization (s.u.m.) methods. In order to construct each u.c.m. problem, a so-called convolution function (c.f.) is used.

Let us consider, for example, the penalty functions method which reduce the n.l.p. problem to the following problem with rectangular constraints (we will call it the u.c.m. problem):

$$\min_{x \in Q} (\bar{f}^0(x) + t_k \sum_i (\bar{g}^i(x))_+^2,$$

where $(y)_+ = \max(y,0)$, for any $y \in R^1$.

In this case we have the following goal function in the u.c.m. problem (we will denote it by u.f.):

$$\bar{M}_k(x) = M_k(\bar{f}^0(x), \bar{g}(x)) = \bar{f}^0(x) + t_k \sum_i (\bar{g}^i(x))_+^2,$$

and the following c.f. is used for this method:

$$M_k(z,y) = z + t_k \sum_i (y^i)_+^2, \text{ where } z \in R^1, y \in R^p.$$

We have a sequence of u.c.m. problems and associated u.f. and c.f. when the penalty coefficient tends to infinity the following way: $t_1 = 10, t_{k+1} = 10t_k$.

Therefore, the s.u.m. method would be defined if we define a sequence of c.f. and choose a u.c.m. method. This way we are concerned with external iterative process which is changing the c.f. and the associated u.c.m. problem.

Because of the nature of s.u.m. methods we may consider the problem

$$\min_{(z,y) \in Z'} M_k(z,y), \text{ where } Z' = \check{Z} \cap \tilde{Z},$$

instead of the u.c.m. problem

$$\min_{x \in Q} M_k(\bar{f}^0(x), \bar{g}(x)).$$

Therefore the efficiency of the s.u.m. method is determined by dual properties of the problem image $\check{Z}$ and c.f. $M_k(z,y)$ or minimal function value isolines $V_k(t_*)$.

Let us consider the class of stable n.l.p. problems with finite Lagrange function saddle point which satisfies the Slayter regularity condition.

We will suppose initially that we have the exact solutions of the u.c.m. problem. We will call the c.f. exact if and only if the s.u.m. method converges in a finite numbe of steps. Otherwise, we will call the c.f. smooth.

Let us define conditions that the minimum x of the u.c.m. problem is the exact (a) or approximate (b) solution of the n.l.p. problem with respect to the space Z_+:

a) $\exists k_0 \forall k \geq k_0 \forall (z_k, y_k) \in \underset{(z,y)\in Z'}{\operatorname{Argmin}} M_k(z,y): u_k \leq 0,\ F(0);$

b) $\forall \varepsilon > 0\ \exists k_0 \forall k \geq k_0 \forall (z_k, y_k) \in \underset{(z,y)\in Z'}{\operatorname{Argmin}} M_k(z,y):$

$y_k \leq \varepsilon,\ |\ z_k - F(0)\ | \leq \varepsilon;$

where $Z' = \tilde{Z} \cap \check{Z}$.

There is a stronger condition with respect to the exact solution, namely the sets equality:

a') $\exists k_0 \forall k \geq k_0 \underset{(z,y)\in Z'}{\operatorname{Argmin}} M_k(z,y) = \{F(0)\} \times W_y(0).$

Let us define the following compact set:

$$K(a,b) = \{(z,y): a^0 \leq z \leq b^0 \leq \hat{a} \leq y \leq \hat{b},\ a \in R^{p+1},\ a < 0 < b\}.$$

It can be proved that the characteristic property of the s.u.m. methods, under the assumption that c.f. series does not depend on the n.l.p. problem which is solved, is convergence of so-called "global derivatives surface" to the negative orthant with the origin at $(F(0), 0)$ (see Kotkin [1988], [1989]). The exact c.f. differs from the smooth c.f. on that it has a break-point.

Let us denote

$$D^i_{K_{(\mp)}}(z,y) = -\frac{\partial_{(\mp)} M_k(z,y)}{\partial y^i} \setminus \frac{\partial M_k(z,y)}{\partial z},\quad i = 1, \ldots, p.$$

The function $D^i_{K_{(x)}}(z,y)$ has the following domain:

$$\hat{z}_0 = \{(z,y) \in \tilde{Z}: \frac{\partial M_k z, y)}{\partial z} \neq 0\}.$$

We will use some assumption with respect to the properties of c.f. $M_k(z, y)$ which is satisfied in the case of penalty functions methods, Center methods, Barier methods, etc. (See Kotkin [1989]).

Theorem 6.

1. In order that condition a') holds for any n.l.p. problem it is necessary and sufficient that for any compact set $K(a, b)$

1) for any point $(z,y) \in \hat{Z}_0$ such that $y^i \geq 0$, $y^i \leq 0\,(\forall j \neq i)$:

   $D^i_{k+}(z,y) \rightarrow -\infty$, if $k=\infty$, $i=1,\ldots,p$;

2) for any big enough k and any point

   $(z, y) \in \hat{Z}_0 \cap K(a,b)$ such that $y < 0$:

   $D^i_{k-}(z, y) = 0$, $i=l,\ldots,p$;

2. In order that condition b) holds for any n.l.p. problem it is necessary and sufficient that for any compact set $K(a, b)$

1) for any point $(z, y) \in \hat{Z}_0$ such that $y^i > 0$, $y^j \leq 0\,(\forall j \neq i)$:

   $D^i_{k+}(z, y) \rightarrow -\infty$, if $k \rightarrow \infty$, $i=1,\ldots,p$;

2) for any big enough k and any point

   $(z, y) \in \hat{Z}_0 \cap K(a, b)$ such that $y < 0$:

   $D^i_{k-}(z, y) \leq 0$,

   $D^i_k(z,y) \rightarrow 0$, if $k \rightarrow \infty$, $i=1,\ldots,p$.

It can be proved that the conditions of Theorem 6 hold for the m.n.p. problem (2) if we use the appropriate notation (See Kotkin [1988]).

We consider now a technique to solve some class of the i.n.p. problems based on the sensitivity analysis. Considered methods converge under the strong assumption that the behavior of the p.n.p. problem (1a) at the solution similar to the behavior of the associated linear programming problem. The sphere of applications of these methods is restricted but they are interesting from a theoretical point of view.

Consider first the following simplest i.n.p. problem:

$$\begin{cases} a)\ x_* \in \underset{g(x) \le u_*}{\mathrm{Argmin}}\ f(x); \\ b)\ x_* = x_0, \end{cases} \tag{6}$$

where $f: R^n \to R^1$, $g: R^n \to R^m$ are continuous functions. We have to find out the parameter u_* which is associated with the given solution $x_* = x_0$ of the p.n.p. problem (6a).

It is easy to see that the solution is $g(x_*): u_* = g(x_*)$, where $x_* = x_0$. In this case the feasible set of the p.n.p. problem (6a) is bounded by the constraints $g(x) \le g(x_*)$ "just at the point x_*" (see Fig. 1).

This approach can be generalized to solve more interesting classes of the i.n.p. problems:

$$\begin{cases} a)\ x_* \in \underset{g(x) \le \varphi(u_*)}{\mathrm{Argmin}}\ f(x); \\ b)\ x_* = x_0; \end{cases} \tag{7}$$

and

$$\begin{cases} a)\ x_* \in \underset{xDx \le \varphi(u_*)}{\mathrm{Argmin}}\ f(x); \\ b)\ x_* = Au_* + B; \end{cases} \tag{8}$$

where $f: R^n \to R^1$, $g: R^n \to R^p$, $\varphi: R^m \to R^p$, are continuous functions, $u \in R^m$, D, A, C are matrixes.

We will only discuss the main idea of the methods.

Let us consider the i.n.p. problem (7) and assume that gradient $f_x(x)$ is not equal to zero at the solution x_* of the p.n.p. problem (7a). In this case we have the system of active and ot active constraints at the solution (x_*, u_*) of the i.n.p. problem:

$$\begin{cases} g^i(x_*) = \varphi^i(u_*),\ i \in I_\alpha; \\ g^i(x_*) < \varphi^i(u_*),\ i \in M \setminus I_\alpha; \end{cases} \tag{9}$$

where $M = \{1,\dots,p\}$, $I_\alpha \neq \emptyset$, $I_\alpha \subset M$.

It can be shown that under some assumptions the solution of the following nonlinear system with respect to $u \in R^m$ is a solution of the i.n.p. problem (7):

$$\begin{cases} g^i(x_*) = \varphi^i(u), \; i \in I_\alpha; \\ g^i(x_*) \leq \varphi^i(u), i \in M \setminus I_\alpha \end{cases}$$

In order to determine the set of active constraints at the solution (x_*, u_*) we have to consider all systems

$$\begin{cases} < -f_x(x_*),\, w > \; \leq 0; \\ < g_x^i(x_*),\, w > \; \leq 0, \; i \in I_k, \text{ where } k=1,\dots,N \leq 2^p; \end{cases}$$

which have empty solution sets with respect to $w \in R^n$.

Therefore we have to solve the set of N nonlinear systems

$$\begin{cases} g^i(x_*) = \varphi^i(u), \; i \in I_k; \\ g^i(x_*) \leq \varphi^i(u), \; i \in M \setminus I_k, \text{ where } k=1,\dots,N; \end{cases}$$

with respect to the parameter $u \in R^m$. The solution u_* of any of these systems is a solution of the i.n.p. problem (7).

In order to solve the problem (8) we have to insert the dependence (8b) in the constraints of p.n.p. problem (8a).

We have the following system:

$$\begin{cases} < -f_x(x_*),\, w > \; \leq 0; \\ < (Au_* + B)\, D\, (Au_* + B),\, w > \; \leq 0. \end{cases}$$

The idea is to define the region U_k with respect to the parameter u such that the set of active constraints is fixed when the parameter u belong this region.

Therefore we have to solve the set of the following systems

$$\begin{cases} g^i(x_*) = \varphi^i(u),\ i \in I_k; \\ g^i(x_*) \le \varphi^i(u),\ i \in M \setminus I_k; \\ u \in U_k, \end{cases}$$

where $k=1,\dots,N \le 2^p$.

Generalized Newton Method

We will call the dual algorithms the numerical methods to solve the i.n.p. problem (1) which consist of two steps: calculating of the minimum of the p.n.p. problem (1a) with respect to the fixed value of the parameter u, and calculating of the new value of the parameter u using the constraints (1b).

Under the assumption that the functions $f(x,u)$, $g(x,u)$, $h(x,u)$, are strong convex with respect to x functions we have the unique solution of the p.n.p. problem (1a):

$$x(u) = \operatorname*{argmin}_{x \in X(u)} f(x,u). \tag{10}$$

We will call the function $x(u): R^m \to R^n$ a solution function.

Let us use the solution function $x(u)$ in the constraints (1b):

$$\tilde{G}(u) = G(x(u),u);\ \tilde{H}(u) = H(x(u),u).$$

We have to solve the following system with respect to u:

$$\tilde{G}(u) \le 0;\ \tilde{H}(u) = 0. \tag{11}$$

We can calculate the values of the vector-functions $\tilde{G}(u)$ and $\tilde{H}(u)$ at any point $u \in R^m$ because we can calculate the values of the solution functions $x(u)$. Therefore we may try to use usual optimization technique to solve the system (11) but we do not know anything about the properties of the functions $\tilde{G}(u)$ and $\tilde{H}(u)$ because we do not know anything about the properties of the solution function $x(u)$.

We will present the so-called Generalized Newton Dual Method to solve the following i.n.p. problem:

$$\begin{cases} a)\ x_* \in \underset{x \in R^n}{\text{Argmin}}\ f(x, u_*); \\ b)\ x_* = Au^* + B. \end{cases} \tag{12}$$

It is a generalization of the Newton Method to solve the system.

$$x(u) = Au + B.$$

Let us consider the solution function $x(u)$ of the i.n.p. problem (12) under the assumption that $f(x,u)$ is a sufficiently smooth and strong convex with respect to x function. Characteristic property of the solution function $x(u)$ is the gradient $f_x(x,u)$ equal to zero on the "surface" $x(u)$:

$$f_x(x(u), u) = 0. \tag{13}$$

We will use the equation (13) to calculate the derivatives of the solution function $x(u)$.

Let us consider the function $f_x(x,u)$ at the neighborhood of the point $(x_0,u_0) = (x(u_0), u_0)$:

$$f_x(x, u) = f_x(x_0, u_0) + < [f_{xx}, f_{xu}], [x - x_0, u - u_0] > + 0([x - x_0, u - u_0]).$$

We have

$$f_x(x(u), u) = f_x(x_0, u_0) + < [f_{xx}, f_{xu}], [x(u) - x_0, u - u_0]> + 0([x(u) - x_0, u - u_0]).$$

Let us assume that the solution function $x(u)$ is a sufficiently smooth function and

$$x(u) = x(u_0) + < x_u(u_0), u - u_0 > + 0(u - u_0).$$

We have

$$f_x(x(u), u) = f_x(x_0, u_0) + < [f_{xx}, f_{xu}], [< x_u(u_0), u - u_0 >, u - u_0] > + 0(u - u_0).$$

Using (13) we conclude

$$< [f_{xx}(x(u_0), u_0),\ f_{xu}\, x(u_0), u_0)],\ \ [<x_u(u_0),\ u-u_0>,\ u-u_0] > = 0. \qquad (14)$$

We have the following system of n linear equations

$$\sum_{k=1}^{n} \frac{\partial^2 f}{\partial x^i \partial x^k} w_j^k + \frac{\partial^2 f}{\partial x^i \partial u^j} = 0,\ i=1,\ldots,n;$$

which have an unique solution $\frac{\partial x}{\partial u^i}(u_0) = (w_j^1, \ldots, w_j^n)$, where $j=1,\ldots,m$.

Therefore we can calculate derivatives $x_u(u_0)$ using the system (14).

Let us prove that the solution function $x(u)$ is a locally sufficiently smooth function in the case $u \in R^1$ under the assumption that function $f(x,u)$ is a sufficiently smooth and strong convex function with respect to x function.

Let us consider the system (14) with respect to unknown function $\tilde{x}(u)$:

$$\begin{cases} < [f_{xx}(\tilde{x}(u), u),\ f_{xu}(\tilde{x}(u), u)],\ \ [< \tilde{x}_u(u), u-u_0>,\ u-u_0] > = 0; \\ \tilde{x}(u_0) = x(u_0). \end{cases} \qquad (15)$$

We have the system of differential equations (15) with initial condition that unknown function $\tilde{x}(u)$ equal to the solution function $x(u)$ at some point u_0 . System (15) has a unique and sufficiently smooth solution $x(u)$ which equal to solution function at some neighborhood of $u_0 : \tilde{x}(u) = x(u)$.

In order to find out the solution of i.n.p. problem (12) we have to find out the root of the following vector-function

$$q(u) = x(u) - (Au + B). \qquad (16)$$

We can solve the system

$$q(u) = 0,\ u \in R^m, \qquad (17)$$

using the usual Newton Method (see Fig. 2).

Let us rewrite the system (18) using the notation

$$v_x = x_u(u_k),\ v_u = u - u_k:$$

$$< [f_{xx}, f_{xu}],\ [< v_x, v_u >,\ v_u] > = 0$$

We have the dual methods which consist of two steps:

1) solving of the p.n.p. problem (12a) with a fixed value of the parameter u ; 2) usual Newton step to solve the system (16):

1) $x_{k+1} \in \underset{f \in R^n}{\text{Argmin}}\ f(x, u_k)$

2) $(v_x, v_u) \in \underset{v_x, v_u}{\text{Argmin}} \{ | < [f_{xx}, f_{xu}],\ [< v_x, v_u >,\ v_u] > |^2 +$

$+ | x_{k+1} - (A u_k + B) - < v_x + A,\ v_u > |^2 +$

$+ \alpha | [< v_x, v_u >, v_u] |^2 \}$, where $\alpha > 0$;

$u_{k+1} = u_k + v_u$.

This method converges to the solution $u_* : q(u_*) = 0$, under the assumption that $f(x,u)$ is sufficiently smooth and strong convex function ($x(u)$ is sufficiently smooth in the case $u \in R^m$, $m > 1$) and function $q(u)$ satisfy the usual assumption of the Newton Method. So we have the local convergence of this method.

Parametric programming system DISO / PC-MCNLP

The DISO / PC-MCNLP system is developed for IBM-PC/XT compatible computers in the Computer Center of the U.S.S.R. Academy of Sciences for multicriteria nonlinear programming solving.

DISO / PC-MCNLP is based on ideas of multicriteria and nonlinear optimization associated with parametric optimization and sensitivity analysis. The great variety of numerical algorithms, interactive procedures, parametric study possibilities and flexible control are the main features of the DISO / PC-MCNLP system.

The flexible control system Field Manager and analytical differentiation language DIFALG are the user level. One can use the "C" language to make a problem definition if one prefers "C" to DIFALG.

The unconstrained optimization algorithm, nonlinear programming algorithm, interactive procedure of multicriteria search, any parameters of numerical method and some parameters of the applied problem can be changed assynchronically with respect to the calculation process. Field Manager allows one to adjust the interface to one's own applied problem. The beginning of the possible adjustments is choosing the numerical algorithm and its parameters, followed by preparing (if one needs) windows, the form of presentation of the system and problem objects (numeric, histograms, graphs), defining the applied problem objects, names, etc. One can change the values of parameters which lie in the basis of one's own applied problem, write and read these parameters and other information from a floppy disk assynchronically to the calculation process. All these features of DISO / PC-MCNLP allow the easy construction of the interactive system for applied optimization problems.

Such interactive systems have been constructed for water resources distribution problems (see Kotkin and Mironov [1989]), metalworking production and other applications.

A great variety of numerical algorithms and interactive procedures are available in the DISO / PC-MCNLP system. They are needed to choose the appropriate algorithm for solving the problem. The DISO / PC-MCNLP has a multi level structure with respect to numerical methods. Several unconstrained minimization methods are at the base level. They are the result of the long time experience of a group of scientists from the Computer Center of the U.S.S.R. Academy of Sciences (see Evtushenko [1985]).

The next level consists of a number of nonlinear programming techniques because any multicriteria programming problem is usually reduced to one goal programming problem. The last level consists of a series of decision making procedures.

The DISO / PC-MCNLP includes the following methods.

Unconstrained minimization methods:

1u) coordinate descent;

2u) direct search (two modifications);

3u) random search method;

4u) conjugate gradient;

5u) Newton method.

Nonlinear programming methods:

1n) center method modifications;

2n) penalty functions method modifications;

3n) barrier methods;

4n) exact penalty function method modifications.

Decision making methods:

1d) gradient method (Geoffrion);

2d) parametric programming method (Guddat);

3d) reference point method modifications;

4d) scalarization method modifications;

5d) nonlinear parametric programming method.

In order to construct nonlinear programming method and decision making procedure we used a convolution which satisfying conditions of Theorem 9 and in special cases Theorem 10.

$$\sum_{i=0}^{m+p+s+1} (\check{f}^i(x) - v^i)^{\begin{Bmatrix} p^i, \text{ if } \check{f}^i(x) \le v^i \\ p^i_+, \text{ if } \check{f}^i(x) > v^i \end{Bmatrix}} \ x \ \begin{Bmatrix} u^i, \text{ if } \check{f}^i(x) \le v^i \\ u^i, \text{ if } \check{f}^i(x) > v^i \end{Bmatrix}.$$

We can construct the methods 1n) - 4n) and 1d) - 5d) if we choose the parameters $v^i, u^i_-, u^i_+ \in R^1$; $p^i_-, p^i_+ \in \{-1,1,2,4\}, i=0,\dots, m+1+p+s$, using Table 1.

DISO / PC-MCNLP can be extended to solve i.n.p. problems. First of all it is easy to consider the i.n.p. problem which has the parameters of the convolution (19) $v, u_{-,+} \in R^{m+1+p+s}$ as parameters of the i.n.p. problem. In this case the parameters of i.n.p. problem are used to construct goal or constraint function from the defined set of the "initial" functions. For example we can consider the i.n.p. problem (5) in order to solve m.n.p. problem (2).

Table 1.

Parameters of c.f.															
Method	With respect to					With respect to					With respect to				
—	v^i	u^i	u^i_+	p^i_-	p^i_+	v^i	u^i_-	u^i_+	p^i_-	p^i_+	v^i	u^i_-	u^i_+	p^i_-	p^i_+
2n+ 4d	0	1	1	1	1	0	0	t1 →∞	2	2	0	t2 →∞	t2 →∞	2	2
3n+ 4d	0	1	1	1	1	0	-t1 →0	t1 →∞	-1	2	0	t2 →	t2 →∞	2	2
1n+ 3d	z var	0	1	2	2	0	0	t1 →∞	2	2	0	t2 →∞	t2 →∞	2	2
3n+ 3d	z var	0	1	2	2	0	0 →0	t1 →∞	1	1	0	t2 →∞	t2 →∞	2	2
4n+ 3d	z var	0	1	2	2	0	0	t1 →∞	1	1	0	t2 →∞	t2 →∞	1	1
1n+ 5d	z var	0	1	2	2	$v1$ var	0	t1 →∞	2	2	$v2$ var	t2 →∞	t2 →∞	2	2

Other definitions of the inverse nonlinear programming problem

Let us consider other problem definitions for i.n.p. problems. We have to find out a pair (x_*, u_*) such that

$$\begin{cases} x_* \in \operatorname*{Argmin}_{x \in X^1(u_*)} f^1(x, u_*); \\ x_* \in \operatorname*{Argmin}_{x \in X^2(u_*)} f^2(x, u_*) \end{cases} \tag{18}$$

The i.n.p. problem (1) is a special case of this problem under the assumption that at any fixed value of parameter $u \in R^m$ exist a solution of the system of the constraints (1b). In this case the problem (1) is reduced to the following problem

$$\begin{cases} x_* \in \operatorname*{Argmin}_{x \in X u_*)} f(x, u_*); \\ x_* \in \operatorname*{Argmin}_{x \in R^n} (G(x, u_*)^2_+ + H(x, u_*)). \end{cases}$$

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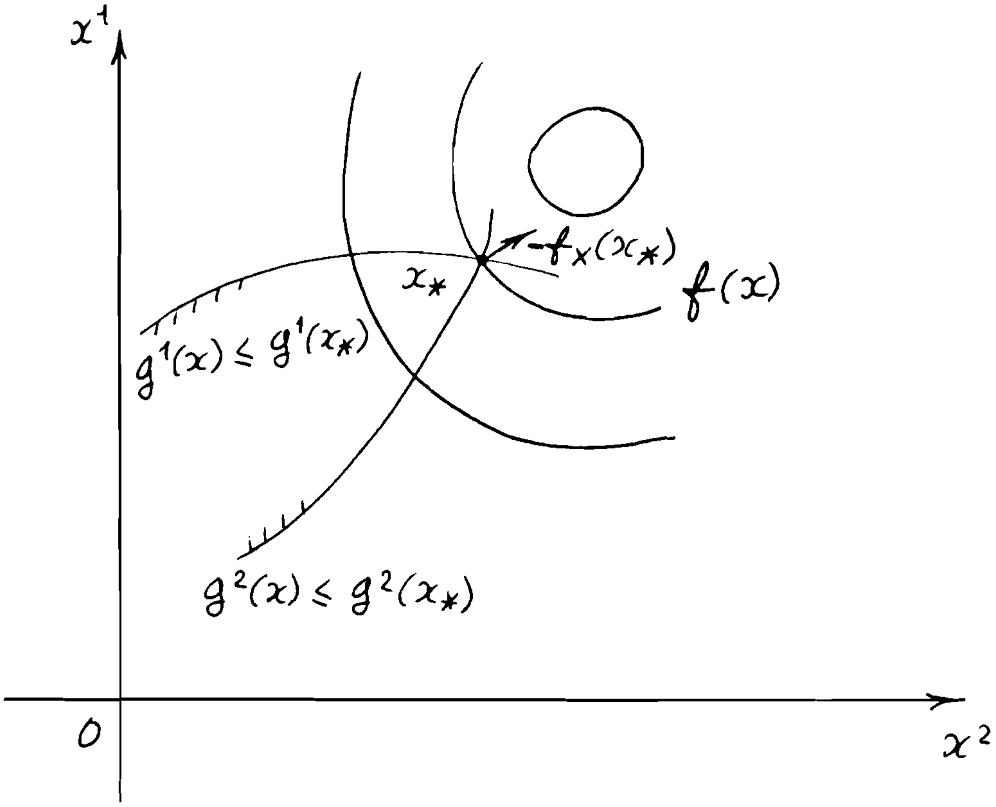

Figure 1.

Figure 2.