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Hyperbolic Systems of Partial Differential Inclusions

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FOREWORD

This paper is devoted to the study of first-order hyperbolic systems of partial differential inclusions which are in particular motivated by several problems of control theory, such as tracking problems.

The existence of *contingent* single-valued solutions is proved for a certain class of such systems.

Several comparison and localization results (which replace uniqueness results in the case of hyperbolic systems of partial differential equations) allow to derive useful informations on the solutions of these problems.

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Introduction

Let X, Y, Z denote finite dimensional vector-spaces, $f : X \times Y \mapsto X$ be a single-valued map, $G : X \times Y \rightsquigarrow Y$ be a set-valued map and $A \in \mathcal{L}(Y, Y)$ a linear operator. We set throughout this paper $\lambda = \min_{\|x\|=1} \langle Ax, x \rangle$.

We recall that the contingent cone $T_K(x)$ to a subset $K \subset X$ at $x \in K$ is defined by

$$T_K(x) := \left\{ v \in X \mid \liminf_{h \to 0+} \frac{d(x + hv; K)}{h} = 0 \right\}$$

and that the *contingent derivative* $DR(x, y)$ of a set-valued map $R : X \rightsquigarrow Y$ at $(x, y) \in \text{Graph}(R)$ is defined by

$$\text{Graph}(DR(x, y)) := T_{\text{Graph}(R)}(x, y)$$

When $R = r$ is single-valued, we set $Dr(x) := Dr(x, r(x))$. Naturally, $Dr(x)(u) = r'(x)u$ whenever r is differentiable at x.

Usually, a Lipschitz map r is not differentiable, but *contingently differentiable* in the sense that its contingent derivative has nonempty values. In this case, it associates to every direction $u \in X$ the subset

$$Dr(x)(u) := \left\{ v \in Y \mid \liminf_{h \to 0+} \left\| v - \frac{r(x + hu) - r(x)}{h} \right\| = 0 \right\}$$

See [8, Chapter 5] for more details on differential calculus of set-valued maps.

In this paper, we shall look for single-valued and set-valued *contingent solutions* to hyperbolic systems of partial differential inclusions, i.e., single-valued maps $r : X \mapsto Y$ *with closed graph* satisfying

$$\forall\, x \in X, \;\; Ar(x) \in Dr(x)(f(x, r(x))) - G(x, r(x))$$

and set-valued maps $R : X \rightsquigarrow Y$ with *closed graph* satisfying

$$\forall\, x \in X, \;\; \forall\, y \in R(x), \;\; Ay \in DR(x, y)(f(x, y)) - G(x, y)$$

We observe that when r is differentiable, the contingent differential inclusion boils down to a *quasi-linear hyperbolic system of first-order partial differential equations*[1]

$$\forall\, j = 1,\ldots,m, \quad \sum_{k=1}^{m} a_j^k r_k(x) \;=\; \sum_{i=1}^{n} \frac{\partial r_j}{\partial x_i} f_i(x, r(x)) - g_j(x, r(x))$$

Motivations: Tracking Property — Consider the *system of differential inclusions*

$$(1) \qquad \begin{cases} x'(t) \;=\; f(x(t), y(t)) \\ y'(t) \;\in\; Ay(t) + G(x(t), y(t)) \end{cases}$$

The solutions to the inclusion

$$\forall\, x \in X, \;\; Ar(x) \;\in\; Dr(x)(f(x, r(x))) - G(x, r(x))$$

are the maps $r : X \mapsto Y$, regarded as *observation maps*, satisfying what is called the *tracking property*: for every $x_0 \in \mathrm{Dom}(r)$, there exists a solution $(x(\cdot), y(\cdot))$ to this system of differential inclusions (1) starting at $(x_0, y_0 = r(x_0))$ and satisfying

$$\forall\, t \geq 0, \;\; y(t) \;=\; r(x(t))$$

One can also look for set-valued contingent solutions $R : X \rightsquigarrow Y$ to the inclusion

$$(2) \qquad \forall\, (x, y) \in \mathrm{Graph}(R), \;\; Ay \;\in\; DR(x, y)(F(x, y)) - G(x, y)$$

characterizing the *tracking property*: for every $x_0 \in \mathrm{Dom}(R)$ and every $y_0 \in R(x_0)$, there exists a solution $(x(\cdot), y(\cdot))$ to this system of differential inclusions starting at (x_0, y_0) and satisfying

$$\forall\, t \geq 0, \;\; y(t) \;\in\; R(x(t)) \;\; \square$$

Motivations: Inclusions governing feedback controls — The partial differential inclusions governing the feedback controls $r : K \mapsto Y$ regulating solutions of a control system (U, f):

$$(3) \qquad \begin{cases} i) \;\; x'(t) \;=\; f(x(t), u(t)) \text{ for almost all } t \\ ii) \;\; u(t) \;\in\; U(x(t)) \end{cases}$$

belong to the class studied in this paper, as it was mentioned in [9,10,11]. Here, $U : X \rightsquigarrow Y$ is a closed set-valued map, $f : \mathrm{Graph}(U) \mapsto X$ a continuous (single-valued) map with linear growth and $\varphi : \mathrm{Graph}(U) \mapsto \mathbf{R}_+$ a nonnegative continuous function with linear growth (in the sense that $\varphi(x, u) \leq c(\|x\| + \|u\| + 1)$).

[1] For several special types of systems of differential equations, the graph of such a map r (satisfying some additional properties) is called a *center manifold*.

We look for feedback controls r satisfying the following property: *for any* $x_0 \in K$*, there exists a solution to the differential equation*

$$x'(t) \; = \; f(x(t), r(x(t))) \;\; \& \;\; x(0) \; = \; x_0$$

such that $u(t) := r(x(t)) \in U(x(t))$ *is absolutely continuous and fulfils the growth condition*

$$\|u'(t) - Au(t)\| \; \leq \; \varphi(x(t), u(t))$$

for almost all t. Such feedback controls r are *solutions to the following contingent differential inclusion*

$$\forall \, x \in X, \;\; Ar(x) \in Dr(x)(f(x, r(x))) - \varphi(x, r(x))B$$

satisfying the constraints

$$\forall \, x \in X, \;\; r(x) \; \in \; U(x) \;\; \square$$

Outline — We extend in the first section Hadamard's formula of solutions to linear hyperbolic differential equations to the set-valued case. Namely, we shall prove the existence of a set-valued contingent solutions $R_\star$ to the *decomposable system*

$$\forall \, (x,y) \in \text{Graph}(R_\star), \;\; Ay \in DR_\star(x,y)(\Phi(x)) - \Psi(x)$$

where $\Phi : X \leadsto X$ and $\Psi : X \leadsto Y$ are two Peano maps[2] and $A \in \mathcal{L}(Y,Y)$.

If we denote by $\mathcal{S}_\Phi(x,\cdot)$ the set of solutions $x(\cdot)$ to the differential inclusion $x'(t) \in \Phi(x(t))$ starting at x, then the set-valued map $R_\star : X \leadsto Y$ defined by

$$\forall \, x \in X, \;\; R_\star(x) \; := \; -\int_0^\infty e^{-At}\Psi(\mathcal{S}_\Phi(x,t))dt$$

is the *largest contingent solution with linear growth* to this partial differential inclusion when $\lambda := \min_{\|x\|=1}\langle Ax, x\rangle > 0$ is large enough. We also show that it is Lipschitz whenever Φ and Ψ are Lipschitz and compare the solutions associated with maps Φ_i and Ψ_i $(i = 1,2)$.

We then turn our attention in the second section to partial differential inclusions of the form

$$\forall \, x \in X, \;\; Ar(x) \; \in \; Dh(x)(f(x, h(x))) - G(x, h(x))$$

[2]A Peano map is an upper semicontinuous set-valued map with nonempty compact convex images and with linear growth.

when $\lambda > 0$ is large enough, $f : X \times Y \mapsto X$ is Lipschitz, $G : X \rightsquigarrow Y$ is Lipschitz with nonempty convex compact values and satisfies[3]

$$\forall\, x,\ y,\ \ \|G(x,y)\| \leq c(1+\|y\|)$$

When G is single-valued, we obtain a global *Center Manifold Theorem*, stating the existence and uniqueness of an invariant manifold for systems of differential equations with Lipschitz right-hand sides (existence and uniqueness of a contingent solution r has been proved by viscosity methods in [6,7] when $A = \lambda\mathbf{1}$.)

We end this paper with comparison theorems between single-valued and set-valued solutions to such partial differential inclusions, using both the extension of Hadamard's formula and some kind of maximum principle.

The authors are gratefully indebted to C. Byrnes for stimulating discussions.

Notations — If $r : X \mapsto Y$, we set

$$\|r\|_\infty := \sup_{x\in X} \|r(x)\| \in [0,\infty] \ \ \& \ \ \|r\|_\Lambda := \sup_{x\neq y} \frac{\|r(x)-r(y)\|}{\|x-y\|} \in [0,\infty]$$

When G is Lipschitz with nonempty closed images, we denote by $\|G\|_\Lambda$ its Lipschitz constant, the smallest of the constants l satisfying

$$\forall\, z_1, z_2,\ \ G(z_1) \subset G(z_2) + l\,\|z_1 - z_2\|\, B$$

where B is the unit ball.

When $L \subset X$ and $M \subset X$ are two closed subsets of a metric space, we denote by

$$\Delta(L,M) := \sup_{y\in L} \inf_{z\in M} d(y,z) = \sup_{y\in L} d(y,M)$$

their *semi-Hausdorff distance*[4], and recall that $\Delta(L,M) = 0$ if and only if $L \subset M$. If Φ and Ψ are two set-valued maps, we set

$$\Delta(\Phi,\Psi)_\infty = \sup_{x\in X} \Delta(\Phi(x),\Psi(x)) := \sup_{x\in X} \sup_{y\in\Phi(x)} d(y,\Psi(x))$$

We recall that solutions are always understood as set-valued or single-valued maps *with closed graph*.

[3] We set $\|K\| := \sup_{x\in K} \|x\|$ when $K \subset X$.

[4] The *Hausdorff distance* between L and M is $\max\{\Delta(L,M), \Delta(M,L)\}$, which may be equal to ∞.

1 Contingent Solutions to Decomposable Systems

We need first to establish some properties of contingent set-valued solutions to decomposable systems.

Let $K \subset X$ be a closed subset and $\Phi : K \rightsquigarrow X$ and $\Psi : K \rightsquigarrow Y$ be two Peano maps with nonempty values and $A \in \mathcal{L}(Y,Y)$. We say that K is a ***viability domain*** of Φ if

$$\forall\, x \in K,\ \ \Phi(x) \cap T_K(x)\ \neq\ \emptyset$$

We set $\lambda := \inf_{\|x\|=1}\langle Ax, x\rangle$ and we observe that

$$\forall\, y \in Y,\ \ \left\|e^{-At}y\right\|\ \leq\ e^{-\lambda t}\|y\|$$

We look for a solution $R_\star : K \rightsquigarrow Y$ to the *decomposable system*

$$(4)\qquad \forall\,(x,y) \in \mathrm{Graph}(R_\star),\ \ Ay \in DR_\star(x,y)(\Phi(x)) - \Psi(x)$$

Denote by $\mathcal{S}_\Phi(x,\cdot)$ the set of solutions $x(\cdot)$ to the differential inclusion $x'(t) \in \Phi(x(t))$ starting at x viable in K (in the sense that $x(t) \in K$ for all $t \geq 0$), which exist thanks to the Viability Theorem.

We introduce the set-valued map $R_\star : K \rightsquigarrow Y$ defined[5] by

$$(5)\qquad \forall\, x \in K,\ \ R_\star(x)\ :=\ -\int_0^\infty e^{-At}\Psi(\mathcal{S}_\Phi(x,t))dt$$

(When $A := \lambda 1$, we have proved in [11] that it is a contingent solution to inclusion (4) when $\lambda > 0$ is large enough.)

Theorem 1.1 *Assume that $\Phi : K \rightsquigarrow X$ and $\Psi : K \rightsquigarrow Y$ are Peano maps and that K is a closed viability domain of Φ. If λ is large enough, the contingent solution $R_\star : K \rightsquigarrow Y$ to inclusion (4) defined by (5) is the* largest contingent solution with linear growth *and is bounded whenever Ψ is bounded.*

[5]By definition of the integral of a set-valued map (see [8, Chapter 8] for instance), this means that for every $y \in R_\star(x)$, there exist a solution $x(\cdot) \in \mathcal{S}_\Phi(x,\cdot)$ to the differential inclusion $x'(t) \in \Phi(x(t))$ starting at x and $z(t) \in \Psi(x(t))$ such that

$$y\ :=\ -\int_0^\infty e^{-At}z(t)dt\ \in\ R_\star(x)$$

More precisely, if there exist positive constants α, β and γ such that

$$\forall\, x \in K,\ \ \|\Phi(x)\| \ \leq\ \alpha(\|x\|+1)\ \ \&\ \ \|\Psi(x)\| \ \leq\ \beta + \gamma\|x\|$$

and if $\lambda > \alpha$, then

$$\forall\, x \in K,\ \ \|R_\star(x)\| \ \leq\ \frac{\beta}{\lambda} + \frac{\gamma}{\lambda-\alpha}(\|x\|+1) \tag{6}$$

Moreover, if $K := X$ and Φ, Ψ are Lipschitz, then $R_\star : X \leadsto Y$ is also Lipschitz (with nonempty values) whenever λ is large enough:

$$\text{If } \lambda > \|\Phi\|_\Lambda,\ \ R_\star(x_1) \ \subset\ R_\star(x_2) + \frac{\|\Psi\|_\Lambda}{\lambda - \|\Phi\|_\Lambda}\|x_1 - x_2\|B$$

for every x_1, $x_2 \in X$.

Formula (5) shows also that the graph of $R_\star$ is convex (respectively a convex cone) whenever the graphs of the set-valued maps Φ and Ψ are convex (respectively are convex cones).

Proof

1. — We prove first that the graph of $R_\star$ satisfies contingent inclusion (4).

Indeed, choose an element y in $R_\star(x)$. By definition of the integral of a set-valued map, this means that there exist a solution $x(\cdot) \in \mathcal{S}_\Phi(x,\cdot)$ to the differential inclusion $x'(t) \in \Phi(x(t))$ starting at x and viable in K and $z(t) \in \Psi(x(t))$ such that

$$y \ :=\ -\int_0^\infty e^{-At}z(t)dt \ \in\ R_\star(x)$$

We check that for every $\tau > 0$

$$-\int_0^\infty e^{-At}z(t+\tau)dt \ \in\ R_\star(x(\tau)) \ =\ R_\star\left(x + \tau\left(\frac{1}{\tau}\int_0^\tau x'(t)dt\right)\right)$$

By observing that

$$\begin{cases} \frac{1}{\tau}\int_0^\infty e^{-At}\left(z(t) - z(t+\tau)\right)dt \\ = \ -\frac{e^{A\tau}-1}{\tau}\int_0^\infty e^{-At}z(t)dt + \frac{e^{A\tau}}{\tau}\int_0^\tau e^{-At}z(t)dt \end{cases}$$

we deduce that

$$\begin{cases} y+\tau\left(-\frac{e^{A\tau}-1}{\tau}\int_0^\infty e^{-At}z(t)dt+\frac{e^{A\tau}}{\tau}\int_0^\tau e^{-At}z(t)dt\right) \\ \in\ R_\star\left(x+\tau\left(\frac{1}{\tau}\int_0^\tau x'(t)dt\right)\right) \end{cases}$$

Since Φ is upper semicontinuous, we know that for any $\varepsilon > 0$ and t small enough, $\Phi(x(t)) \subset \Phi(x)+\varepsilon B$, so that $x'(t) \in \Phi(x)+\varepsilon B$ for almost all small t. Therefore, $\Phi(x)$ being closed and convex, we infer that for $\tau > 0$ small enough, $\frac{1}{\tau}\int_0^\tau x'(t)dt \in \Phi(x)+\varepsilon B$ thanks to the Mean-Value Theorem. This latter set being compact, there exists a sequence of $\tau_n > 0$ converging to 0 such that $\frac{1}{\tau_n}\int_0^{\tau_n} x'(t)dt$ converges to some $u \in \Phi(x)$.

In the same way, Ψ being upper semicontinuous, $\Psi(x(t)) \subset \Psi(x)+\varepsilon B$ for any $\varepsilon > 0$ and t small enough, so that $z(t) \in \Psi(x)+\varepsilon B$ for almost all small t. The Mean-Value Theorem implies that

$$\forall\, n > 0,\ \ z_n \ :=\ \frac{1}{\tau_n}\int_0^{\tau_n} z(t)dt\ \in\ \Psi(x)+\varepsilon B$$

since this set is compact and convex. Furthermore, there exists a subsequence of z_n converging to some $z_0 \in \Psi(x)$. Hence, since

$$\frac{1}{\tau_n}\int_0^{\tau_n}\left(e^{-At}-1\right)z(t)dt\ \to\ 0$$

we infer that

$$Ay+z_0\ \in\ DR_\star(x,y)(u)$$

so that $Ay\ \in\ DR_\star(x,y)(\Phi(x))-\Psi(x)$.

2. — Let us prove now that the graph of $R_\star$ is closed when λ is large enough. Consider for that purpose a sequence of elements (x_n, y_n) of the graph of $R_\star$ converging to (x,y). There exist solutions $x_n(\cdot) \in \mathcal{S}_\Phi(x_n,\cdot)$ to the differential inclusion $x' \in \Phi(x)$ starting at x_n and viable in K and measurable selections $z_n(t) \in \Psi(x_n(t))$ such that

$$y_n\ :=\ -\int_0^\infty e^{-At}z_n(t)dt \in R_\star(x_n)$$

The growth of Φ being linear, there exists $\alpha > 0$ such that the solutions $x_n(\cdot)$ obey the estimate

$$\|x_n(t)\| \le (\|x_n\|+1)e^{\alpha t}\ \ \&\ \ \|x_n'(t)\| \le \alpha(\|x_n\|+1)e^{\alpha t}$$

By [8, Theorem 10.1.9], we know that there exists a subsequence (again denoted by) $x_n(\cdot)$ converging uniformly on compact intervals to a solution $x(\cdot) \in \mathcal{S}_\Phi(x, \cdot)$.

The growth of Ψ being also linear, we deduce that, setting $u_n(t) := e^{-At} z_n(t)$,

$$\|z_n(t)\| \leq \beta + \gamma(\|x_n\| + 1)e^{\alpha t} \ \& \ \|u_n(t)\| \leq \beta e^{-\lambda t} + \gamma(\|x_n\| + 1)e^{-(\lambda-\alpha)t}$$

When $\lambda > \alpha$, Dunford-Pettis' Theorem implies that a subsequence (again denoted by) $u_n(\cdot)$ converges weakly to some function $u(\cdot)$ in $L^1(0, \infty; Y)$. This implies that $z_n(\cdot)$ converges weakly to some function $z(\cdot)$ in the space $L^1(0, \infty; Y; e^{-\lambda t} dt)$. The Convergence Theorem [8, Theorem 7.2.2] states that $z(t) \in \Psi(x(t))$ for almost every t. Since the integrals y_n converge to $-\int_0^\infty e^{-At} z(t)dt$, we have proved that

$$y = -\int_0^\infty e^{-At} z(t)dt \in R_\star(x)$$

3. — Estimate (6) is obvious since any solution $x(\cdot) \in \mathcal{S}_\Phi(x, \cdot)$ satisfies

$$\forall\, t \geq 0, \ \|x(t)\| \leq (\|x\| + 1)e^{\alpha t}$$

so that, if $\lambda > \alpha$,

$$\|R_\star(x)\| \leq \int_0^\infty e^{-\lambda t} \left(\beta + \gamma(\|x\| + 1)e^{\alpha t}\right) dt = \frac{\beta}{\lambda} + \frac{\gamma}{\lambda - \alpha}(\|x\| + 1)$$

Assume now that $M : K \rightsquigarrow Y$ is any set-valued contingent solution to inclusion (4) with linear growth: there exists $\delta > 0$ such that for all $x \in X$, $\|M(x)\| \leq \delta(\|x\| + 1)$. Since M enjoys the tracking property, we know that for any $(x, y) \in \text{Graph}(M)$, there exists a solution $(x(\cdot), y(\cdot))$ to the system of differential inclusions

$$(7) \qquad \begin{cases} i) & x'(t) \in \Phi(x(t)) \\ ii) & y'(t) - Ay(t) \in \Psi(x(t)) \end{cases}$$

starting at (x, y) such that $y(t) \in M(x(t))$ for all $t \geq 0$. We also know that $\|x(t)\| \leq (\|x\| + 1)e^{\alpha t}$ so that $\|y(t)\| \leq \delta(1 + (\|x\| + 1)e^{\alpha t})$. The second differential inclusion of the above system implies that $z(t) := y'(t) - Ay(t)$ is a measurable selection of $\Psi(x(t))$ satisfying the growth condition

$$\|z(t)\| \leq \beta + \gamma(1 + (\|x\| + 1)e^{\alpha t})$$

Therefore, if $\lambda > \alpha$, the function $e^{-At}z(t)$ is integrable. On the other hand, integrating by parts $e^{-At}z(t) := e^{-At}y'(t) - e^{-At}Ay(t)$, we obtain

$$e^{-AT}y(T) - y \; = \; \int_0^T e^{-At}z(t)dt$$

which implies that

$$y \; = \; -\int_0^\infty e^{-At}z(t)dt \; \in \; R_\star(x)$$

by letting $T \mapsto \infty$. Hence we have proved that[6] $M(x) \subset R_\star(x)$.

4. — Assume now that $K = X$ and that Φ and Ψ are Lipschitz, take any pair of elements x_1 and x_2 and choose $y_1 = -\int_0^\infty e^{-At}z_1(t)dt \in R_\star(x_1)$, where

$$x_1(\cdot) \; \in \; \mathcal{S}_\Phi(x_1, \cdot) \;\; \& \;\; z_1(t) \; \in \; \Psi(x_1(t))$$

[6]This proof actually implies that any set-valued contingent solution M with polynomial growth in the sense that for some $\rho \geq 0$,

$$\forall\, x \in X, \;\; \|M(x)\| \; \leq \; \delta(\|x\|^\rho + 1)$$

is contained in $R_\star$ whenever $\lambda > \alpha\rho$, i.e., that there is no contingent solution with polynomial growth other than with linear growth (and bounded when $\gamma = 0$.)

By the Filippov Theorem[7], there exists a solution $x_2(\cdot) \in \mathcal{S}_\Phi(x_2, \cdot)$ such that

$$\forall\, t \geq 0,\ \ \|x_1(t) - x_2(t)\| \ \leq\ e^{\|\Phi\|_\Lambda t}\|x_1 - x_2\|$$

We denote by $z_2(t)$ the projection of $z_1(t)$ onto the closed convex set $\Psi(x_2(t))$, which is measurable thanks to [8, Corollary 8.2.13] and which satisfies

$$\forall t \geq 0,\ \ \|z_1(t) - z_2(t)\| \ \leq\ \|\Psi\|_\Lambda \|x_1(t) - x_2(t)\| \ \leq\ \|\Psi\|_\Lambda e^{\|\Phi\|_\Lambda t}\|x_1 - x_2\|$$

Therefore, if $\lambda > \|\Phi\|_\Lambda$, $y_2 = -\int_0^\infty e^{-At} z_2(t)dt$ belongs to $R_\star(x_2)$ and satisfies

$$\|y_1 - y_2\| \ \leq\ \int_0^\infty \|\Psi\|_\Lambda e^{-t(\lambda - \|\Phi\|_\Lambda)}\|x_1 - x_2\|dt \ \leq\ \frac{\|\Psi\|_\Lambda}{\lambda - \|\Phi\|_\Lambda}\|x_1 - x_2\| \ \ \square$$

Theorem 1.2 *Consider now two pairs* (Φ_1, Ψ_1) *and* (Φ_2, Ψ_2) *of Peano maps defined on* X *and their associated solutions*

$$\forall\, x \in X,\ \ R_{\star i}(x) \ :=\ -\int_0^\infty e^{-At}\Psi_i(\mathcal{S}_{\Phi_i}(x,t))dt \ \ (i = 1,2)$$

---

[7] adapted to the case of solutions defined on $[0, \infty[$. Filippov's Theorem (see [5, Theorem 2.4.1] for instance), yields an estimate on any finite interval $[0, T]$: If Φ is c-Lipschitz with nonempty closed values, and if an absolutely continuous function $y(\cdot)$ and an initial state x_0 are given, then there exists a solution $x(\cdot)$ to the differential inclusion defined on $[0, T]$ starting at x_0 satisfying estimate

$$(8) \qquad \|x(t) - y(t)\| \ \leq\ e^{ct}\left(\|x_0 - y(0)\| \ +\ \int_0^t d(y'(s), \Phi(y(s)))e^{-cs}ds\right)$$

We can extend it to the interval $[0, +\infty[$. Indeed, there exists a solution $x(\cdot)$ to the differential inclusion defined on $[0, T]$ starting at x_0 satisfying estimate (8) and in particular

$$\|x(T) - y(T)\| \ \leq\ e^{cT}\left(\|x_0 - y(0)\| \ +\ \int_0^T d(y'(s), \Phi(y(s)))e^{-cs}ds\right)$$

There also exists a solution $z(\cdot)$ to the differential inclusion starting at $x(T)$ estimating the function $t \mapsto y(t + T)$ and satisfying

$$\|z(t) - y(t + T)\| \ \leq\ e^{ct}\left(\|z(T) - y(T)\| \ +\ \int_0^t d(y'(s + T), \Phi(y(s + T)))e^{-cs}ds\right)$$

Hence we can extend $x(\cdot)$ on the interval $[0, 2T]$ by concatenating it with the function $t \mapsto x(t) := z(t - T)$ on the interval $[T, 2T]$, we check that the above estimates yield (8) for $t \in [0, 2T]$ and we reiterate this process. $\square$

See the forthcoming monograph [22].

If the set-valued maps Φ_2 and Ψ_2 are Lipschitz, and if $\lambda > \|\Phi_2\|_\Lambda$, then

$$\Delta(R_{\star 1}, R_{\star 2})_\infty \leq \frac{1}{\lambda}\Delta(\Psi_1, \Psi_2)_\infty + \frac{\|\Psi_2\|_\Lambda}{\lambda(\lambda - \|\Phi_2\|_\Lambda)}\Delta(\Phi_1, \Phi_2)_\infty$$

Proof — Choose $y_1 = -\int_0^\infty e^{-At} z_1(t)dt \in R_{\star 1}(x)$ where

$$x_1(\cdot) \in \mathcal{S}_{\Phi_1}(x, \cdot) \ \& \ z_1(t) \in \Psi_1(x_1(t))$$

In order to compare $x_1(\cdot)$ with the solution-set $\mathcal{S}_{\Phi_2}(x, \cdot)$ via the Filippov Theorem, we use the estimate

$$d(x_1'(t), \Phi_2(x_1(t))) \leq \sup_{z \in \Phi_1(x_1(t))} d(z, \Phi_2(x_1(t)))) \leq \Delta(\Phi_1, \Phi_2)_\infty$$

Therefore, there exists a solution $x_2(\cdot) \in \mathcal{S}_{\Phi_2}(x, \cdot)$ such that

$$\forall\, t \geq 0, \ \|x_1(t) - x_2(t)\| \leq \Delta(\Phi_1, \Phi_2)_\infty \frac{e^{t\|\Phi_2\|_\Lambda} - 1}{\|\Phi_2\|_\Lambda}$$

by Filippov's Theorem. As before, we denote by $z_2(t)$ the projection of $z_1(t)$ onto the closed convex set $\Psi_2(x_2(t))$, which is measurable and satisfies

$$\begin{cases} \forall\, t \geq 0, \ \|z_1(t) - z_2(t)\| \leq \Delta(\Psi_1, \Psi_2)_\infty + \|\Psi_2\|_\Lambda \|x_1(t) - x_2(t)\| \\ \leq \Delta(\Psi_1, \Psi_2)_\infty + \|\Psi_2\|_\Lambda \Delta(\Phi_1, \Phi_2)_\infty \left(e^{t\|\Phi_2\|_\Lambda} - 1\right) / \|\Phi_2\|_\Lambda \end{cases}$$

Therefore, if $\lambda > \|\Phi_2\|_\Lambda$, $y_2 = -\int_0^\infty e^{-At} z_2(t)dt$ belongs to $R_{\star 2}(x)$ and satisfies

$$\begin{cases} \|y_1 - y_2\| \\ \leq \int_0^\infty e^{-\lambda t}\Delta(\Psi_1, \Psi_2)_\infty dt + \|\Psi_2\|_\Lambda \Delta(\Phi_1, \Phi_2)_\infty \int_0^\infty \frac{e^{t\|\Phi_2\|_\Lambda - 1}}{\|\Phi_2\|_\Lambda} e^{-\lambda t} dt \\ \leq \frac{\Delta(\Psi_1, \Psi_2)_\infty}{\lambda} + \frac{\|\Psi_2\|_\Lambda}{\lambda(\lambda - \|\Phi_2\|_\Lambda)}\Delta(\Phi_1, \Phi_2)_\infty \ \square \end{cases}$$

When $\Phi := \varphi$, $\Psi := \psi$ are single-valued, we obtain:

Proposition 1.3 *Assume that φ and ψ are Lipschitz and that ψ is bounded. Then when $\lambda > 0$, the map $r := \Gamma(\varphi, \psi)$ defined by*

$$r(x) = -\int_0^\infty e^{-At}\psi(S_\varphi(x, t))dt$$

is the unique bounded single-valued solution to the contingent inclusion

$$(9) \qquad Ar(x) \in Dr(x)(\varphi(x)) - \psi(x)$$

and satisfies

$$(10) \qquad \|r\|_\infty \leq \frac{\|\psi\|_\infty}{\lambda} \ \& \ \forall \lambda > \|\varphi\|_\Lambda, \ \|r\|_\Lambda \leq \frac{\|\psi\|_\Lambda}{\lambda - \|\varphi\|_\Lambda}$$

and

$$(11) \quad \|\Gamma(\varphi_1, \psi_1) - \Gamma(\varphi_2, \psi_2)\|_\infty \leq \frac{\|\psi_1 - \psi_2\|_\infty}{\lambda} + \frac{\|\psi_2\|_\Lambda}{\lambda(\lambda - \|\varphi_2\|_\Lambda)}\|\varphi_1 - \varphi_2\|_\infty$$

The proof can be derived from Theorems 1.1 and 1.2 or directly from the properties of linear systems of hyperbolic equations established in [7].

2 Existence of a Lipschitz contingent solution

We shall now prove the existence of a contingent single-valued solution to inclusion

$$(12) \qquad \forall x \in X, \ Ar(x) \in Dr(x)(f(x, r(x))) - G(x, r(x))$$

Theorem 2.1 *Assume that the map $f : X \times Y \mapsto X$ is Lipschitz, that $G : X \rightsquigarrow Y$ is Lipschitz with nonempty convex compact values and that*

$$\forall x, \ y, \ \|G(x, y)\| \leq c(1 + \|y\|)$$

for some $c > 0$.

Then if $\lambda > \max(c, 4\nu\|f\|_\Lambda\|G\|_\Lambda)$ (where ν is the dimension of X), there exists a bounded Lipschitz solution to the contingent inclusion (12).

Proof— Since for every Lipschitz single-valued map $s(\cdot)$, $x \rightsquigarrow G(x, s(x))$ is Lipschitz (with constant $\|G\|_\Lambda (1 + \|s\|)_\Lambda)$ and has convex compact values, [8, Theorem 9.4.1] implies that the subset G_s of Lipschitz selections ψ of the set-valued map $x \rightsquigarrow G(x, s(x))$ with Lipschitz constant less than $\nu \|G\|_\Lambda (1 + \|s\|_\Lambda)$ is not empty (where ν denotes the dimension of X). We denote by φ_s the Lipschitz map defined by $\varphi_s(x) := f(x, s(x))$, with Lipschitz constant equal to $\|f\|_\Lambda(1 + \|s\|_\Lambda)$.

The solutions r to inclusion (12) are the fixed points to the set-valued map $\mathcal{H} : \mathcal{C}(X,Y) \rightsquigarrow \mathcal{C}(X,Y)$ defined by

$$\text{(13)} \qquad \mathcal{H}(s) \; := \; \{\Gamma(\varphi_s, \psi)\}_{\psi \in G_s}$$

Indeed, if $r \in \mathcal{H}(r)$, there exists a selection $\psi \in G_r$ such that

$$Ar(x) \; \in \; Dr(x)(f(x,r(x))) - \psi(x) \; \subset \; Dr(x)(f(x,r(x))) - G(x,r(x))$$

Since $\|G(x,y)\| \leq c(1 + \|y\|)$, we deduce that any selection $\psi \in G_s$ satisfies

$$\|\psi\|_\infty \; \leq \; c(1 + \|s\|_\infty)$$

Therefore, Proposition 1.3 implies that if λ is large enough,

$$\forall\, r \in \mathcal{H}(s), \;\; \|r\|_\infty \; \leq \; \frac{c}{\lambda}(1 + \|s\|_\infty) \;\; \& \;\; \|r\|_\Lambda \; \leq \; \frac{\nu\|G\|_\Lambda(1 + \|s\|_\Lambda)}{\lambda - \|f\|_\Lambda(1 + \|s\|_\Lambda)}$$

We first observe that when $\lambda > c$,

$$\forall\, s \in \mathcal{C}(X,Y) \;\; \text{such that} \;\; \|s\|_\infty \; \leq \; \frac{c}{\lambda - c}, \;\; \forall\, r \in \mathcal{H}(s), \;\; \|r\|_\infty \; \leq \frac{c}{\lambda - c}$$

When $\lambda > 4\nu\|f\|_\Lambda \, \|G\|_\Lambda$, we denote by

$$\rho(\lambda) \; := \; \frac{\lambda - \|f\|_\Lambda - \nu\|G\|_\Lambda - \sqrt{\lambda^2 - 2\lambda(\|f\|_\Lambda + \nu\|G\|_\Lambda) + (\|f\|_\Lambda - \nu\|G\|_\Lambda)^2}}{2\|f\|_\Lambda}$$

the smallest root of the equation

$$\lambda\rho \; = \; \|f\|_\Lambda \rho^2 + (\|f\|_\Lambda + \nu\|G\|_\Lambda)\rho + \nu\|G\|_\Lambda$$

which is positive. We observe that

$$\lim_{\lambda \to +\infty} \lambda\rho(\lambda) \; = \; \nu\|G\|_\Lambda$$

and infer that

$$\forall\, s \in \mathcal{C}(X,Y) \;\; \text{such that} \;\; \|s\|_\Lambda \leq \rho(\lambda), \;\; \forall\, r \in \mathcal{H}(s), \;\; \|r\|_\Lambda \; \leq \; \rho(\lambda)$$

because r being of the form $\Gamma(\varphi_s, \psi_s)$, satisfies by Proposition 1.3:

$$\|r\|_\Lambda \leq \frac{\|\psi_s\|_\Lambda}{\lambda - \|\varphi_s\|_\Lambda} \leq \frac{\nu\,\|G\|_\Lambda\,(1 + \|s\|_\Lambda)}{\lambda - \|f\|_\Lambda(1 + \|s\|_\Lambda)} \leq \frac{\nu\,\|G\|_\Lambda\,(1 + \rho(\lambda))}{\lambda - \|f\|_\Lambda(1 + \rho(\lambda))} = \rho(\lambda)$$

Let us denote by $B^1_\infty(\lambda)$ the subset defined by

$$B^1_\infty(\lambda) := \left\{ r \in \mathcal{C}(X,Y) \mid \|r\|_\infty \leq \frac{c}{\lambda - c} \ \& \ \|r\|_\Lambda \leq \rho(\lambda) \right\}$$

which is compact (for the compact convergence topology) thanks to Ascoli's Theorem.

We have therefore proved that if $\lambda > \max(c, 4\nu\|f\|_\Lambda\|G\|_\Lambda)$, the set-valued map $\mathcal{H}$ sends the compact subset $B^1_\infty(\lambda)$ to itself.

It is obvious that the values of $\mathcal{H}$ are convex. Kakutani's Fixed-Point Theorem implies the existence of a fixed point $r \in \mathcal{H}(r)$ if we prove that the graph of $\mathcal{H}$ is closed.

Actually, the graph of $\mathcal{H}$ is compact. Indeed, let us consider any sequence $(s_n, r_n) \in \text{Graph}(\mathcal{H})$. Since $B^1_\infty(\lambda)$ is compact, a subsequence (again denoted by) (s_n, r_n) converges to some function

$$(s, r) \in B^1_\infty(\lambda) \times B^1_\infty(\lambda)$$

But there exist bounded Lipschitz selections $\psi_n \in G_{s_n}$ with Lipschitz constant $\nu\|G\|_\Lambda(1 + \rho(\lambda))$ such that

$$\forall\, n \geq 0, \ r_n = \Gamma(\varphi_{s_n}, \psi_n)$$

Therefore a subsequence (again denoted by) ψ_n converges to some function $\psi \in G_s$. Since φ_{s_n} converges obviously to φ_s, we infer that r_n converges to $\Gamma(\varphi_s, \psi)$ where $\psi \in G_s$, i.e., that $r \in \mathcal{H}(s)$, since Γ is continuous by formula (11) of Proposition 1.3. □

3 Comparison Results

The point of this section is to compare two solutions to inclusion (12), or even, a single-valued solution and a contingent set-valued solution $M : X \rightsquigarrow Y$.

We first deduce from Theorem 1.2 the following "localization property":

Theorem 3.1 *We posit the assumptions of Theorem 2.1 with $A \in \mathcal{L}(Y,Y)$ such that $\lambda > \max(c, 4\nu\|f\|_\Lambda\|G\|_\Lambda)$ (where ν is the dimension of X). Let $\Phi : X \rightsquigarrow X$ and $\Psi : X \rightsquigarrow Y$ be two Lipschitz and Peano maps with which we associate the set-valued map $R_\star$ defined by*

$$\forall\, x \in X, \ R_\star(x) := -\int_0^\infty e^{-At}\Psi(\mathcal{S}_\Phi(x,t))dt$$

Then any bounded single-valued contingent solution $r(\cdot)$ to inclusion (12) satisfies the following estimate

$$\begin{cases} \forall\, x \in X,\;\; d(r(x), R_\star(x)) \;\le \\ \frac{1}{\lambda}\sup_{x\in X} \Delta(G(x,r(x)),\Psi(x)) + \frac{\|\Psi\|_\Lambda}{\lambda(\lambda-\|\Phi\|_\Lambda)}\sup_{x\in X} d(f(x,r(x)),\Phi(x)) \end{cases}$$

In particular, if we assume that

$$\forall\, y \in Y,\;\; f(x,y) \;\in\; \Phi(x) \;\;\&\;\; G(x,y) \;\subset\; \Psi(x)$$

then all bounded single-valued contingent solutions $r(\cdot)$ to inclusion (12) are selections of $R_\star$.

Proof — Let r be any bounded single-valued contingent solution to inclusion (12). One can show that r can be written in the form

$$r(x) \;=\; -\int_0^\infty e^{-At} z(t)dt \;\text{ where }\; z(t) \;\in\; G(x(t), r(x(t)))$$

by using the same arguments as in the third part of the proof of Theorem 1.1.

We also adapt the proof of Theorem 1.2 with $\Phi_1 := f(x,r(x))$, $z_1(t) := z(t)$, $\Phi_2 := \Phi$ and $\Psi_2 := \Psi$, to show that the estimates stated in the theorem hold true. □

The next comparison results are consequences of the following kind of *maximum principle*.

We recall that when M is Lipschitz, its *adjacent derivative* $D^\flat M(x,y) \subset DM(x,y)$ is defined by

$$v \in D^\flat M(x,y)(u) \;\text{ if and only if }\; \lim_{h\to 0+} d\left(v, \frac{M(x+hu)-y}{h}\right) \;=\; 0$$

A set-valued map M is said to be *derivable* at (x,y) if the contingent and adjacent derivatives coincide at (x,y) and derivable if it is derivable at every point of its graph. See [8] for more details.

Lemma 3.2 (Maximum Principle) *We posit the assumptions of Theorem 2.1 with $A \in \mathcal{L}(Y,Y)$ such that $\lambda > \max(c, 4\nu\|f\|_\Lambda\|G\|_\Lambda)$. Let M be a Lipschitz set-valued map such that $D^\flat M(x,y)(f(x,y))$ is nonempty for every*

$(x,y) \in \text{Graph}(M)$. *Let r be any Lipschitz bounded single-valued solution to (12). If the supremum*

$$\delta := \sup_{(x,y)\in\text{Graph}(M)} \|r(x) - y\|$$

is finite, then

$$\delta \le \frac{1}{\lambda} \sup_{(x,y)\in\text{Graph}(M)} \Delta\left(Ay + G(x, r(x)), \overline{co}(D^\flat M(x,y)(f(x,r(x))))\right)$$

The same conclusion holds true if we assume that the solution r is derivable and when we replace the adjacent derivative of M by its contingent derivative.

Proof — It is sufficient to consider the case when the supremum

$$\delta := \sup_{(x,y)\in\text{Graph}(M)} \|r(x) - y\| = \|r(\bar{x}) - \bar{y}\|$$

is achieved[8] at some $(\bar{x}, \bar{y})$ of the graph of M and when $\delta > 0$.

We know that there exist $v \in Dr(\bar{x})(f(\bar{x}, r(\bar{x})))$ and $\psi \in G(\bar{x}, r(\bar{x}))$ such that $Ar(\bar{x}) = v - \psi$. Set $u := f(\bar{x}, r(\bar{x}))$. Since r is Lipschitz, there exists a sequence $h_n > 0$ converging to 0 such that

$$\frac{r(\bar{x} + h_n u) - r(\bar{x})}{h_n} \text{ converges to } v$$

Since M is Lipschitz, we deduce that for any $w \in D^\flat M(\bar{x}, \bar{y})(u)$, there exists a sequence w_n converging to w such that $\bar{y} + h_n w_n \in M(\bar{x} + h_n u)$. Thus

$$\|r(\bar{x}) - \bar{y}\| \ge \left\| r(\bar{x}) - \bar{y} + h_n \left(\frac{r(\bar{x} + h_n u) - r(\bar{x})}{h_n} - w_n \right) \right\|$$

Therefore,

$$\forall\, w \in D^\flat M(\bar{x}, \bar{y})(u),\ \langle r(\bar{x}) - \bar{y}, v - w\rangle \le 0$$

and we infer that

$$\forall\, w \in \overline{co}(D^\flat M(\bar{x}, \bar{y})(f(\bar{x}, r(\bar{x})))),\ \langle r(\bar{x}) - \bar{y}, A(r(\bar{x}) - \bar{y}) + A\bar{y} + \psi - w\rangle \le 0$$

[8] **If the nonnegative bounded function $\chi(x,y) := \|r(x) - y\|$ does not achieve its maximum, we use a standard argument which can be found in [16,25] for instance. One can find approximate maxima (x_n, y_n) such that $\chi(x_n, y_n)$ converges to $\sup_{x\in\text{Graph}(M)} \chi(x,y)$ and $\chi'(x_n, y_n)$ converges to 0.**

from which we obtain the estimate

$$\lambda\|r(\bar{x}) - \bar{y}\| \leq \inf_{w \in \overline{co}(D^\flat M(\bar{x},\bar{y})(f(\bar{x},r(\bar{x}))))} \|A\bar{y} + \psi - w\| \quad \square$$

We use this Lemma to compare two solutions to inclusion (12):

Theorem 3.3 *We posit the assumptions of Theorem 2.1. Let r_1 and r_2 be two Lipschitz contingent solutions to (12). If r_2 is differentiable and if $\lambda > \|r_2\|_\Lambda \|f\|_\Lambda$, then*

$$\|r_1 - r_2\|_\infty \leq \sup_{x \in X} \frac{\|G(x, r_1(x)) - G(x, r_2(x))\|}{\lambda - \|r_2\|_\Lambda \|f\|_\Lambda}$$

When f does not depend on y, we can take $\|f\|_\Lambda = 0$ in the above estimate.

When G does not depend on y, we deduce that

$$\|r_1 - r_2\|_\infty \leq \sup_{x \in X} \frac{\mathrm{Diam}(G(x))}{\lambda - \|r_2\|_\Lambda \|f\|_\Lambda}$$

More generally, let us consider a set-valued contingent solution $M : X \rightsquigarrow Y$ to the inclusion

(14) $\quad \forall\, (x, y) \in \mathrm{Graph}(M), \quad Ay \in DM(x, y)(f(x, y)) - G(x, y)$

Theorem 3.4 *We posit the assumptions of Theorem 2.1. Let r be a Lipschitz contingent solution to (12) and M be a Lipschitz set-valued contingent solution to inclusion (14) in the stronger sense that for every $(x, y) \in \mathrm{Graph}(M)$, there exists a Lipschitz closed convex process $E(x, y) \subset \overline{co}(D^\flat M(x, y))$ satisfying*

$$\forall\, (x, y) \in \mathrm{Graph}(M), \quad Ay \in E(x, y)(f(x, y)) - G(x, y)$$

and

$$\|E\|_\Lambda := \sup_{(x,y) \in \mathrm{Graph}(M)} \|E(x, y)\|_\Lambda < +\infty$$

Assume also that the supremum

$$\delta := \sup_{(x,y) \in \mathrm{Graph}(M)} \|r(x) - y\|$$

is finite and that $\lambda > \|E\|_\Lambda \|f\|_\Lambda$. *Then*

$$\sup_{(x,y)\in \mathrm{Graph}(M)} \|r(x) - y\| \leq \sup_{(x,y)\in \mathrm{Graph}(M)} \frac{\|G(x,r(x)) - G(x,y)\|}{\lambda - \|E\|_\Lambda \|f\|_\Lambda}$$

or, equivalently,

$$\forall\, (x,y) \in \mathrm{Graph}(M),\;\; M(x) \subset r(x) + \sup_{(x,y)\in \mathrm{Graph}(M)} \frac{\|G(x,r(x)) - G(x,y)\|}{\lambda - \|E\|_\Lambda \|f\|_\Lambda} B$$

When f *does not depend on* y, *we can take* $\|f\|_\Lambda = 0$ *in the above estimates.*
When G *does not depend on* y, *we deduce that*

$$\forall\, (x,y) \in \mathrm{Graph}(M),\;\; M(x) \;\subset\; r(x) + \sup_{(x,y)\in \mathrm{Graph}(M)} \frac{\mathrm{Diam}(G(x))}{\lambda - \|E\|_\Lambda \|f\|_\Lambda} B$$

Proof — It is sufficient to consider the case when the supremum

$$\delta \;:=\; \sup_{(x,y)\in \mathrm{Graph}(M)} \|r(x) - y\| \;=\; \|r(\bar{x}) - \bar{y}\|$$

is achieved at some $(\bar{x}, \bar{y})$ of the graph of M.

By assumption, we know that the norms of the closed convex processes $E(x,y)$ are bounded by $\|E\|_\Lambda$ and that

$$\begin{cases} A\bar{y} \;\in\; E(\bar{x},\bar{y})(f(\bar{x},\bar{y})) - G(\bar{x},\bar{y}) \\ \;\subset\; E(\bar{x},\bar{y})(f(\bar{x},r(\bar{x}))) + E(\bar{x},\bar{y})(f(\bar{x},\bar{y}) - f(\bar{x},r(\bar{x}))) - G(\bar{x},\bar{y}) \end{cases}$$

Then there exist $w \in E(\bar{x},\bar{y})(f(\bar{x},r(\bar{x}))) \subset \overline{co}(D^\flat M(\bar{x},\bar{y})(f(\bar{x},r(\bar{x}))))$ and $\psi' \in G(\bar{x},\bar{y})$ satisfying

$$\|A\bar{y} - w + \psi'\| \;\leq\; \|E\|_\Lambda \|f\|_\Lambda \|r(\bar{x}) - \bar{y}\| \;=\; \|E\|_\Lambda \|f\|_\Lambda \delta$$

Let $\psi \in G(\bar{x}, r(\bar{x}))$ such that $Ar(\bar{x}) \in Dr(\bar{x})(f(\bar{x},r(\bar{x}))) - \psi$. We thus deduce from Lemma 3.2 that

$$\lambda\delta \leq \|\psi - \psi'\| + \|E\| \|f\|_\Lambda \delta \leq \sup_{(x,y)\in \mathrm{Graph}(M)} \|G(x,r(x)) - G(x,y)\| + \|E\|_\Lambda \|f\|_\Lambda \delta$$

from which the conclusion of Theorem 3.4 follows. □

Uniqueness follows when λ is large enough and when we assume the existence of a set-valued map M the graph of which is an *invariance domain* of the set-valued map $(x,y) \rightsquigarrow f(x,y) \times (Ay + G(x,y))$, in the sense that[9]

$$\forall (x,y) \in \text{Graph}(M), \;\; G(x,y) + Ay \subset DM(x,y)(f(x,y))$$

We need to use the *circatangent derivative* $CM(x,y)$ of M at (x,y) defined by

$$v \in CM(x,y)(u) \text{ if and only if } \lim_{\substack{(x',y') \to_G (x,y) \\ h \to 0+}} d\left(v, \frac{M(x'+hu) - y'}{h}\right) = 0$$

See [8, Chapter 4] for more details.

Theorem 3.5 *We posit the assumptions of Theorem 2.1. Assume that the graph of the Lipschitz set-valued map M is an invariance domain of $(x,y) \rightsquigarrow f(x,y) \times (Ay + G(x,y))$ and that there exists Lipschitz closed convex process E satisfying*

$$\forall (x,y) \in \text{Graph}(M), \;\; CM(x,y) \subset E(x,y) \subset \overline{co}(D^b M(x,y))$$

and

$$\|E\|_\Lambda := \sup_{(x,y) \in \text{Graph}(M)} \|E(x,y)\|_\Lambda < +\infty$$

If λ is large enough, then $M(x) = \{r(x)\}$ for any single-valued contingent solution r to inclusion (12) such that the supremum

$$\delta := \sup_{(x,y) \in \text{Graph}(M)} \|r(x) - y\|$$

is finite.

[9] One can prove that when F is Lipschitz with closed values, Graph(M) is an *invariance domain* if and only if it is invariant in the sense that for any $(x_0, y_0) \in$ Graph(M), every solution to the system of differential inclusions

$$\begin{cases} x'(t) = f(x(t), y(t)) \\ y'(t) \in Ay(t) + G(x(t), y(t)) \end{cases}$$

starting at (x_0, y_0) satisfies

$$\forall t \geq 0, \;\; y(t) \in M(x(t))$$

Proof — Since f and G are lower semicontinuous, we know from [8, Theorem 4.1.9] that inclusion

$$\forall\,(x,y)\in \mathrm{Graph}(M),\;\; G(x,y)+Ay\;\subset\; DM(x,y)(f(x,y))$$

holds true with the circatangent derivative $CM(x,y)$ (which is a closed convex process), so that

$$\forall\,(x,y)\in \mathrm{Graph}(M),\;\; G(x,y)+Ay\;\subset\; CM(x,y)(f(x,y))\;\subset\; E(x,y)(f(x,y))$$

Let $(\bar{x},\bar{y})$ in the graph of M achieve the supremum

$$\delta \;:=\; \sup_{(x,y)\in \mathrm{Graph}(M)} \|r(x)-y\| \;=\; \|r(\bar{x})-\bar{y}\|$$

Take $\psi\in G(\bar{x},r(\bar{x}))$ such that $Ar(\bar{x})\in Dr(\bar{x})(f(\bar{x},r(\bar{x})))-\psi$. Since G is Lipschitz, we infer that

$$\psi\;\in\; G(\bar{x},r(\bar{x}))\;\subset\; G(\bar{x},\bar{y})+\|G\|_\Lambda\|r(\bar{x})-\bar{y}\|B \;=\; G(\bar{x},\bar{y})+\|G\|_\Lambda\delta B$$

Therefore,

$$A\bar{y}+\psi\;\in\; A\bar{y}+G(\bar{x},\bar{y})+\|G\|_\Lambda\delta B\;\subset\; CM(\bar{x},\bar{y})(f(\bar{x},\bar{y}))+\|G\|_\Lambda\delta B$$

and, $E(\bar{x},\bar{y})$ being a closed convex process with a norm less than or equal to $\|E\|_\Lambda$,

$$\left\{\begin{array}{l} E(\bar{x},\bar{y})(f(\bar{x},\bar{y}))\;\subset\; E(\bar{x},\bar{y})(f(\bar{x},r(\bar{x})))+E(\bar{x},\bar{y})(f(\bar{x},\bar{y})-f(\bar{x},r(\bar{x})))\\ \subset\; E(\bar{x},\bar{y})(f(\bar{x},r(\bar{x})))+\|E\|_\Lambda\|f\|_\Lambda\delta \end{array}\right.$$

We thus deduce from Lemma 3.2 that

$$\lambda\delta\;\leq\;\|G\|_\Lambda\delta+\|E\|_\Lambda\|f\|_\Lambda\delta$$

which implies that $\delta=0$ whenever $\lambda>\|G\|_\Lambda+\|E\|_\Lambda\|f\|_\Lambda$. □

References

[1] AUBIN J.-P. (1981) *Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions*, Advances in Mathematics, Supplementary studies, Ed. Nachbin L., 160-232

[2] AUBIN J.-P. (1990) *A Survey of Viability Theory*, SIAM J. on Control and Optimization, 28, 749-788

[3] AUBIN J.-P. (to appear) VIABILITY THEORY

[4] AUBIN J.-P., BYRNES C. & ISIDORI A. (1990) *Viability Kernels, Controlled Invariance and Zero Dynamics for Nonlinear Systems*, Proceedings of the 9th International Conference on Analysis and Optimization of Systems, Nice, June 1990, Lecture Notes in Control and Information Sciences, Springer-Verlag

[5] AUBIN J.-P. & CELLINA A. (1984) DIFFERENTIAL INCLUSIONS, Springer-Verlag, Grundlehren der Math. Wiss.

[6] AUBIN J.-P. & DA PRATO G. (1990) *Solutions contingentes de l'équation de la variété centrale*, Comptes Rendus de l'Académie des Sciences, PARIS, Série 1,

[7] AUBIN J.-P. & DA PRATO G. (to appear) *Contingent Solutions to the Center Manifold Equation*, Annales de l'Institut Henri Poincaré, Analyse Non Linéaire

[8] AUBIN J.-P. & FRANKOWSKA H. (1990) SET-VALUED ANALYSIS, Systems and Control: Foundations and Applications, Birkhäuser, Boston, Basel, Berlin

[9] AUBIN J.-P. & FRANKOWSKA H. (1990) *Inclusions aux dérivées partielles gouvernant des contrôles de rétroaction*, Comptes Rendus de l'Académie des Sciences, PARIS, Série 1,

[10] AUBIN J.-P. & FRANKOWSKA H. (to appear) *Viability Kernels of Control Systems*, in NONLINEAR SYNTHESIS, Eds. Byrnes & Kurzhanski, Birkhäuser

[11] AUBIN J.-P. & FRANKOWSKA H. (to appear) *Partial Differential Inclusions Governing Feedback Controls*,

[12] BRENIER Y. (1984) *Averaged Multivalued solutions for scalar conservation laws*, SIAM J. Num. Anal., 21, 1013-1037

[13] BYRNES C.I. & ISIDORI A. (1989) *Feedback Design From the Zero Dynamics Point of View*, in COMPUTATION AND CONTROL, Bowers K. & Lund J. Eds., Birkhäuser, 23-52

[14] BYRNES C.I. & ISIDORI A. (1990) *Output Regulation of Nonlinear Systems*, IEEE Trans. Autom. Control, 35, 131-140

[15] BYRNES C.I. & ISIDORI A. (to appear) *Asymptotic Stabilization of Minimum Phase Nonlinear Systems*, Preprint

[16] CANNARSA P. & DA PRATO G. (to appear) *Direct Solutions of a Second-Order Hamilton-Jacobi Equation in Hilbert Spaces*, Preprint

[17] CARR J. (1981) APPLICATIONS OF CENTRE MANIFOLD THEORY, Springer Verlag

[18] DA PRATO G. & LUNARDI A. (1988) *Stability, Instability and Center Manifold Theorem for Fully Nonlinear Autonomous Parabolic Equations in Banach Spaces*, Arch. Rat. Mech. Anal., 101, 115-141

[19] FRANKOWSKA H. (1987) *L'équation d'Hamilton-Jacobi contingente*, Comptes Rendus de l'Académie des Sciences, PARIS, Série 1, 304, 295-298

[20] FRANKOWSKA H. (1989) *Optimal trajectories associated to a solution of contingent Hamilton-Jacobi Equations*, Applied Mathematics and Optimization, 19, 291-311

[21] FRANKOWSKA H. (1989) *Nonsmooth solutions of Hamilton-Jacobi-Bellman Equations*, Proceedings of the International Conference Bellman Continuum, Antibes, France, June 13-14, 1988, Lecture Notes in Control and Information Sciences, Springer Verlag

[22] FRANKOWSKA H. (to appear) SET-VALUED ANALYSIS AND CONTROL THEORY, Systems and Control: Foundations and Applications, Birkhäuser, Boston, Basel, Berlin

[23] LIONS P.-L. (1982) GENERALIZED SOLUTIONS OF HAMILTON-JACOBI EQUATIONS, Pitman

[24] LIONS P.-L. & SOUGANIDIS P. (to appear)

[25] LUNARDI A. (1988) *Existence in the small and in the large in fully nonlinear parabolic equations*, in DIFFERENTIAL EQUATIONS AND APPLICATIONS, Ed. Aftabizadeh, Ohio University Press