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Lyapunov Exponents and Transversality Conditions for an Infinite Horizon Optimal Control Problem

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Foreword

The author presents some necessary conditions for an infinite horizon optimal control problem with restricted controls. The main point is the introduction of transversality conditions which are formulated in terms of Lyapunov exponents.

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Introduction

The aim of this paper is to derive necessary conditions of optimality for an infinite horizon optimal control problem. The Pontryagin maximum principle for this problem without transversality condition at infinity was derived in Pontryagin et al. [1]. Transversality conditions were derived for some dynamical optimization problems arising from mathematical economics [2, 3]. The presence of an exponential factor in the integral functional is a characteristic of these problems and facilitates consideration to a certain extent. In this case the transversality conditions were formulated in terms of the asympotic behavior of solutions to an adjoint equation. The problem without an exponential factor was considered by Gani and Wiese [4] under rather restrictive assumptions, and the transversality conditions were given as initial conditions for the solution to the adjoint equation.

It seems quite natural to use stability theory [5, 6] to analyze infinite time optimal control problems. Following this idea we derive the transversality condition at infinity in terms of a Lyapunov exponent of the solution to the adjoint equation. The problem with the exponential factor is considered as well as the problem without it.

The first section of this paper describes results from stability theory. A property of regular linear differential equations is also established, which is important for further consideration. Necessary conditions of optimality for a general infinite horizon optimal control problem are stated and proved in Section 2. Section 3 investigates of a linear quadratic optimal control problem arising from regulator design theory [7]. The case when the set of controls is a closed convex cone is considered.

1 Regular linear differential equations

In this section results from the stability theory developed by Lyapunov in his famous monograph [5] are recalled.

Let $f : R \longrightarrow R$ be a continuous function. The Lyapunov exponent of the function f is defined by

$$\chi[f(\cdot)] = -\limsup_{t\to\infty} \frac{1}{t} \ln |f(t)|.$$

The Lyapunov exponents possess the following properties

1. $\chi[(f+\phi)(\cdot)] \geq \min\{\chi[f(\cdot)], \chi[\phi(\cdot)]\}$,
2. $\chi[(f\phi)(\cdot)] \geq \chi[f(\cdot)] + \chi[\phi(\cdot)]$,

3. $\chi[(f+1/f)(\cdot)] \leq 0$,

4. $\chi[J \circ f(\cdot)] \geq \chi[f(\cdot)]$, where

$$J \circ f(t) = \begin{cases} \int_t^\infty f(s)ds & \text{if } \chi[f(\cdot)] > 0, \\ \int_0^t f(s)ds & \text{otherwise} \end{cases}$$

5. $\chi[(f\phi)(\cdot)] = \chi[f(\cdot)]$, where $0 < a \leq \phi(t) \leq b < \infty$.

If $f : R \longrightarrow R^n$ is a vector function, then the Lyapunov exponent is defined as the minimal value of the Lyapunov exponents of the components $\chi[f^i(\cdot)]$.

Let us consider the linear differential equation

$$\dot{x}(t) = C(t)x(t), \tag{1}$$

where $n \times n$ matrix $C(t)$ has measurable bounded components. Lyapunov proved that the exponent is finite for any nonzero solution of (1). Moreover, the set of all possible numbers that are Lyapunov exponents of some nonzero solution of (1) is finite, with cardinality less than or equal to n. Lyapunov exponents of nonzero solutions to a linear differential equation with constant matrix C coincide with the real parts of the eigenvalues of C taken with the opposite sign.

A fundamental system of solutions of (1) $x_1(\cdot), \ldots, x_n$ is said to be normal if for all $\alpha_1, \ldots, \alpha_n \in R$

$$\chi[(\sum_{i=1}^n \alpha_i x_i)(\cdot)] = \max\{\chi[x_i(\cdot)] \mid i : \alpha_i \neq 0\}.$$

Lyapunov proved that a normal system of solutions always exists. Lyapunov exponents $\lambda_1, \ldots, \lambda_n$ of a normal system of solutions (there may be equal quantities among them) are called the Lyapunov spectrum of (1).

Let $\lambda_1, \ldots, \lambda_n$ be the Lyapunov spectrum of (1). Then the value $S = \lambda_1 + \cdots + \lambda_n$ does not exceed $\chi[\xi(\cdot)]$ where

$$\xi(t) = \exp \int_0^t \mathrm{tr} C(s) ds.$$

From this fact we obtain the following consequence. If $z_1(\cdot), \ldots, z_n(\cdot)$ is a fundamental system of solutions of (1), $\nu_1, \ldots, \nu_n$ are corresponding Lyapunov exponents, and $\nu_1 + \cdots + \nu_n = \chi[\xi(\cdot)]$, then the system is normal. Equation (1) is called regular if $S = -\chi[(1/\xi)(\cdot)]$. In this case, obviously,

$$S = \chi[\xi(\cdot)] = -\chi[(1/\xi)(\cdot)].$$

As a consequense we derive that the limit

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \mathrm{tr} C(s) dt$$

exists. All linear differential equations with constant or periodic coefficients are regular.

Let us consider along with (1) the adjoint equation

$$\dot{x}^*(t) = -C^*(t)x^*(t), \tag{2}$$

where $C^*(t)$ represents a transposed matrix. An important property of regular equations was established by Perron (see [6], e.g.). If $\lambda_1 \leq \ldots \leq \lambda_n$ is the Lyapunov spectrum of

(1) and $\mu_1 \geq \ldots \geq \mu_n$ is the Lyapunov spectrum of (2), then equation (1) is regular if and only if $\lambda_i + \mu_i = 0, i = 1, \ldots, n$.

We denote by $\Lambda_\tau(\delta)$ the subspace consisting of all points $x_0 \in R^n$ such that a solution of (1) with the initial condition $x(\tau) = x_0$ has a Lyapunov exponent greater than $-\delta$ and by $\Lambda_\tau^+(\delta)$ the subspace consisting of all points $x_0^* \in R^n$ such that a solution of (2) with the initial condition $x(\tau) = x_0$ has a Lyapunov exponent greater than or equal to δ.

Lemma 1.1. If equation (1) is regular, then

$$\Lambda_\tau^\perp(\delta) = \Lambda_\tau^+(\delta).$$

Proof. We first establish the inclusion

$$\Lambda_\tau^+(\delta) \subset \Lambda_\tau^\perp(\delta). \tag{3}$$

Assume that $x_0 \in \Lambda_\tau(\delta), x_0^* \in \Lambda_\tau^+(\delta)$ and that $x(\cdot), x^*(\cdot)$ are solutions of equations (1) and (2), respectively. Then

$$\langle x(t), x^*(t)\rangle = \langle x_0, x_0^*\rangle + \int_\tau^t \frac{d}{ds}\langle x(s), x^*(s)\rangle ds =$$

$$\langle x_0, x_0^*\rangle + \int_\tau^t (\langle C(s)x(s), x^*(s)\rangle + \langle x(s), -C^*(s)x^*(s)\rangle)ds = \langle x_0, x_0^*\rangle.$$

Taking into account properties of the Lyapunov exponents, we obtain $\chi[\langle x, x^*\rangle(\cdot)] > 0$. Thus, $\lim_{t\to\infty}\langle x(t), x^*(t)\rangle = 0$ and $\langle x_0, x_0^*\rangle = 0$. The inclusion (3) is proved.

To prove the equality we consider matrices $\Phi(t,\tau)$ and $\Phi^+(t,\tau)$ of fundamental solutions of equations (1) and (2). Assume that their columns form normal systems of solutions. The subspace $\Lambda_\tau(\delta)$ is spanned by column vectors of the matrix $\Phi(\tau,\tau)$, which correspond to solutions that have Lyapunov exponents greater than $-\delta$, and the subspace $\Lambda_\tau^+(\delta)$ is spanned by column vectors of the matrix $\Phi^+(\tau,\tau)$, which correspond to solutions that have Lyapunov exponents greater than or equal to δ. Let $\dim\Lambda_\tau(\delta) = k$. Since equation (1) is regular,the Perron theorem implies that the Lyapunov spectra $\lambda_1 \leq \ldots \leq \lambda_n$ and $\mu_1 \geq \ldots \geq \mu_n$ of (1) and (2) satisfy the equalities $\lambda_i + \mu_i = 0, i = 1, \ldots, n$. Thus $\dim\Lambda_\tau^+(\delta) = n - k$. If we combine this with (3), we reach $\Lambda_\tau^\perp(\delta) = \Lambda_\tau^+(\delta)$ and the end of the proof.

I conclude this section with two theorems concerning an analogy of solutions of differential equations. These theorems are special cases of more general results which are given in Bylov et al. [6] (Theorems 29.3.1 and 26.1.2).

Let us consider the differential equation

$$\dot{x}(t) = C(t)x(t) + f(t, x(t)), \tag{4}$$

which is obtained from (1) by adding a nonlinear function $f : R \times R^n \longrightarrow R^n$ to the right-hand side.

Assume that the following conditions hold true:

1. $f(t, 0) = 0$ for all t,

2. the function $t \longrightarrow f(t, x)$ is measurable for all x,

3. the function $x \longrightarrow f(t, x)$ is continuously differentiable for almost all t,

4. $|\nabla_x f(t,x)| = O(|x|^\epsilon)$ where $\epsilon > 0$.

Under these assumptions we have the following resulit:

Theorem 1.2. Let $\tau \geq 0, \delta \leq 0$. Suppose that equation (1) is regular and that $\lambda_1 \leq \ldots \leq \lambda_n$ is its Lyapunov spectrum. Then there exists a homeomorphism Φ_τ^δ defined in some neighborhood of the origin U_τ^δ and satisfying the following conditions:

1. $\Phi_\tau^\delta(x) = x + O(|x|^{1+\epsilon})$,
2. if $x_0 \in \Lambda_\tau(\delta) \cap U_\tau^\delta$, $y_0 = \Phi_\tau^\delta(x_0)$, then there exist solutions of the equations (1) and (4) with initial conditions x_0 and y_0, respectively, and with identical Lyapunov exponents.

To give a geometrical interpretation of this theorem we need the definition of a tent introduced by Boltyanski [8].

A convex cone $K \subset R^n$ is called a tent of a set $M \subset R^n$ at a point $x \in M$ if there exist a neighborhood of the origin Ω and a continuous map $\phi : \Omega \longrightarrow R^n$ such that

1. $x + v + \phi(v) \in M$ for all $v \in K \cap \Omega$,
2. $\lim_{v\to 0} |v|^{-1}\phi(v) = 0$.

Theorem 1.2 implies that any solution of equation (4) with the initial condition $x(\tau) \in M_\tau^\delta = \Phi_\tau^\delta(\Lambda_\tau(\delta) \cap U_\tau^\delta)$ has a Lyapunov exponent greater than $-\delta$. Moreover, the subspace $\Lambda_\tau(\delta)$ is a tent of the set M_τ^δ at zero. This statement combined with Lemma 1.1 is of crucial importance for the proof of transversality conditions which is discussed in the next section.

It should be mentiond also that Theorem 1.2 fails to be true when $\delta > 0$.

Example 1.3. Consider the system

$$\dot{x} = x,$$
$$\dot{y} = y.$$

Obviously, $\Lambda_0(2) = R^2$. The system

$$\dot{x} = x + yx,$$
$$\dot{y} = y$$

has, however, a solution that has a Lyapunov exponent $-\infty$ when initial conditions satisfy $x_0 \neq 0$, $y_0 > 0$. Hence, $\Lambda_0(2)$ is not a tent of M_0^2.

The second theorem concerning an analogy of solutions of differential equations deals with the linear equations

$$\dot{x}(t) = Cx(t), \tag{5}$$
$$\dot{y}(t) = (C + B(t))y(t), \tag{6}$$

where C is a constant matrix and $B(t)$ is a bounded matrix with measurable coefficients.

Theorem 1.4. Let $\lambda_1 \leq \ldots \leq \lambda_n$ be the Lyapunov spectrum of equation (5). Assume that

$$\int_0^\infty e^{\alpha t}|B(t)|dt < \infty$$

for some $\alpha > 0$. Then there exists a homeomorphism $\Phi : R^n \longrightarrow R^n$ such that solutions $x(\cdot)$ and $y(\cdot)$ of the differential equations (5) and (6), respectively, with corresponding initial conditions $y(0) = \Phi(x(0))$ have identical Lyapunov exponents. Moreover, if $x(\cdot)$ and $y(\cdot)$ have a Lyapunov exponent λ_k, then $\chi[(x-y)(\cdot)] > \lambda_k + \alpha/2$.

2 Necessary conditions of optimality for an infinite horizon optimal control problem

Consider the problem

$$\int_0^\infty \phi(x(t), u(t))dt \longrightarrow \inf \tag{1}$$

$$\dot{x}(t) = f(x(t), u(t)), \tag{2}$$

$$u(t) \in U \subset R^k, \tag{3}$$

$$x(0) = x_0, \ \lim_{t\to\infty} x(t) = 0. \tag{4}$$

Let $f : R^n \times U \longrightarrow R^n$, $\phi : R^n \times U \longrightarrow R$ be continuous functions that are continuously differentiable with respect to x. Arbitrary bounded measurable functions $u : [0, \infty[\longrightarrow U$ are considered admissible controls.

Let $\hat{u}(\cdot)$ be an optimal control for (1)-(4), and let $\hat{x}(\cdot)$ be the corresponding trajectory of equation (2).

Denote

$$C(t) = \nabla_x f(\hat{x}(t), \hat{u}(t)),$$

$$b(t) = \nabla_x \phi(\hat{x}(t), \hat{u}(t)).$$

Suppose that the differential equation

$$\dot{x}(t) = C(t)x(t) \tag{5}$$

is regular and that

$$| \nabla_x f(\hat{x}(t) + x, \hat{u}(t)) - C(t)| + | \nabla_x \phi(\hat{x}(t) + x, \hat{u}(t)) - b(t)| = O(|x|^\epsilon),$$

where $\epsilon > 0$. Under these assumptions we obtain the following result:

Theorem 2.1. There exist a number $\lambda \leq 0$ and a function $p : [0, \infty[\longrightarrow R^n$ such that

1. $\dot{p}(t) = -C^*(t)p(t) - \lambda b(t)$,

2. $$\max_{u\in U}[\langle p(t), f(\hat{x}(t), u)\rangle + \lambda\phi(\hat{x}(t), u)] = \langle p(t), f(\hat{x}(t), \hat{u}(t))\rangle + \lambda\phi(\hat{x}(t), \hat{u}(t)),$$

3. $\chi[p(\cdot)] \geq 0$,

4. $|\lambda| + |p(0)| > 0$.

Proof. Denote

$$\bar{x} = (y, x) \in R \times R^n,$$

$$\bar{f}(\bar{y}, u) = (-\phi(x, u), f(x, u)),$$

$$\hat{\bar{x}}(t) = (\int_t^\infty \phi(\hat{x}(s), \hat{u}(s))ds, \hat{x}(t)),$$

$$\bar{C}(t) = \nabla_{\bar{x}} \bar{f}(\hat{\bar{x}}(t), \hat{u}(t)),$$

$$\bar{K}(t) = \mathrm{cl} \bigcup_{\alpha>0} \alpha \mathrm{co}[\bar{f}(\hat{\bar{x}}(t), U) - \bar{f}(\hat{\bar{x}}(t), \hat{u}(t))].$$

Obviously, $\hat{u}(\cdot)$ solves the problem

$$y(0) \longrightarrow \inf \tag{6}$$

$$\dot{\bar{x}}(t) = \bar{f}(\bar{x}(t), u(t)), \tag{7}$$

$$u(t) \in U, \tag{8}$$

$$\bar{x}(0) = (y(0), x_0),\ \lim_{t\to\infty} \bar{x}(t) = 0, \tag{9}$$

and $\hat{\bar{x}}(\cdot)$ is the corresponding trajectory.

We claim that the differential equation

$$\dot{\bar{x}}(t) = \bar{C}(t)\bar{x}(t) \tag{10}$$

is regular. Indeed, if $\bar{x}(\cdot) = (y(\cdot), x(\cdot))$ is a solution to (10) then $x(\cdot)$ solves (5) and

$$y(t) = y_0 + \int_0^t \langle b(s), x(s)\rangle ds.$$

Let $x_1(\cdot), \ldots, x_n(\cdot)$ be a normal system of solutions of (5) and let $\lambda_1, \ldots, \lambda_n$ be the corresponding Lyapunov exponents and also $\lambda_1 \leq \ldots \leq \lambda_{k-1} \leq 0 < \lambda_k \leq \ldots \leq \lambda_n$. Since the function $b(\cdot)$ is bounded we have $\chi[b(\cdot)] \geq 0$ and, consequently, $\chi[\langle b, x\rangle(\cdot)] \geq \chi[x(\cdot)]$. Consider the collection of $n+1$ functions

$$(\int_0^t \langle b(s), x_1(s)\rangle ds, x_1(t)), \ldots, (\int_0^t \langle b(s), x_{k-1}(s)\rangle ds, x_{k-1}(s)), (1, 0),$$

$$(-\int_t^\infty \langle b(s), x_k(s)\rangle ds, x_k(t)), \ldots, (-\int_t^\infty \langle b(s), x_n(s)\rangle ds, x_n(s)). \tag{11}$$

We derive from the inequalities

$$\chi[J \circ \langle b, x\rangle(\cdot)] \geq \chi[\langle b, x\rangle(\cdot)] \geq \chi[x(\cdot)]$$

(see Section 1) that the functions (11) possess the Lyapunov exponents

$$\mu_1 = \lambda_1, \ldots, \mu_{k-1} = \lambda_{k-1}, \mu_k = 0, \mu_{k+1} = \lambda_k, \ldots, \mu_{n+1} = \lambda_n.$$

The equalities

$$\sum_{i=1}^{n+1} \mu_i = \sum_{i=1}^{n} \lambda_i,\ \mathrm{tr}C(t) = \mathrm{tr}\bar{C}(t)$$

imply that the functions (11) form a normal system of solutions of (10) and that this equation is regular.

Denote by $\bar{\Lambda}_\tau(0) \subset R^{n+1}$ the subspace consisting of all points $\bar{x}_0 \in R^{n+1}$ such that a solution of (10) with the initial condition $\bar{x}(\tau) = \bar{x}_0$ has a positive Lyapunov exponent. Let $P_\tau \subset R^{n+1}$ be the convex cone consisting of all points $\bar{x}_0$ such that there exists a solution to the differential inclusion

$$\dot{\bar{x}} \in \bar{C}(t)\bar{x}(t) + \bar{K}(t),$$

satisfying the boundary conditions

$$\bar{x}(0) = \bar{x}_0,\ \bar{x}(\tau) \in \bar{\Lambda}_\tau(0).$$

Obviously, we have $P_{\tau_1} \subset P_{\tau_2}$ if $\tau_2 > \tau_1$. Denote

$$P = \bigcup_{\tau > 0} P_\tau.$$

Consider the ray $L = \{(\alpha, 0) \in R \times R^n | \alpha < 0\}$. We claim that the ray L and the cone P are separable. Indeed, if $L \cap \mathrm{int}P \neq \emptyset$, then $(-1, 0) \in \mathrm{int}P_\tau$ for some $\tau > 0$. We can apply Theorem 1.2, which states the existence of a set $\bar{M}_\tau$ satisfying the following conditions:

1. $0 \in \bar{M}_\tau$,
2. $\bar{\Lambda}_\tau(0)$ is a tent of $\bar{M}_\tau$ at 0,
3. for all $\bar{y} \in \bar{M}_\tau + \hat{\bar{x}}(t)$ any solution of the Cauchy problem

$$\dot{\bar{x}}(t) = \bar{f}(\bar{x}(t), \hat{u}(t)),$$

$$\bar{x}(\tau) = \bar{y}$$

has a positive Lyapunov exponent.

Define $\bar{\mathbf{M}}_\tau$ as a set of all points $\bar{x}_0$ such that there exist some admissible control $u(\cdot)$ and a corresponding trajectory satisfying

$$\dot{\bar{x}}(t) = \bar{f}(\bar{x}(t), u(t)),$$

$$\bar{x}(0) = \bar{x}_0, \ \bar{x}(\tau) \in \bar{M}_\tau + \hat{\bar{x}}(\tau).$$

Using standard methods developed to prove the maximum principle given in Pontryagin et al. [1] and Boltyanski [8] or applying differential inclusions technique discussed in Smirnov [9], we conclude that P_τ is a tent of $\bar{\mathbf{M}}_\tau$ at $\hat{\bar{x}}(0) = (\int_0^\infty \phi(\hat{x}(s), \hat{u}(s))ds, x_0)$. Since $(-1, 0) \in \mathrm{int}P_\tau$, we obtain

$$\left(\int_0^\infty \phi(\hat{x}(s), \hat{u}(s))ds + \alpha, x_0\right) \in \bar{\mathbf{M}}_\tau$$

for some $\alpha < 0$ (see [1, 8]). This contradicts the optimality of $\hat{u}(\cdot)$. Thus L and P are separable.

Applying the separation theorem, we find a vector $\bar{p}_0 = (\lambda, p_0) \in R \times R^n$ such that $\lambda\alpha \geq \langle \bar{p}_0, \bar{x}_0 \rangle$ for all $\alpha < 0, \bar{x} = (y, x) \in P$. Setting $\bar{x} = 0$, we conclude that $\lambda \leq 0$. Allowing α to tend to zero, we obtain $0 \geq \langle \bar{p}_0, \bar{x} \rangle$ for all $\bar{x} \in P$. The last expression implies that $\langle \bar{p}_0, \bar{x}(0) \rangle \leq 0$ for all $\bar{x}(\cdot)$ satisfying

$$\dot{\bar{x}}(t) \in \bar{C}(t)\bar{x}(t) + \bar{K}(t),$$

$$\bar{x}(\tau) \in \bar{\Lambda}_\tau(0), \ \tau \geq 0.$$

The Lemma 1.1 means that a solution to the Cauchy problem

$$\dot{\bar{p}}(t) = -C^*(t)\bar{p}(t),$$

$$\bar{p}(0) = -\bar{p}_0$$

satisfies

$$\chi[\bar{p}(\cdot)] \geq 0,$$

$$\max_{\bar{w} \in \bar{K}(t)} \langle \bar{w}, \bar{p}(t) \rangle = 0, \ t \in [0, \infty[.$$

Evidently we have $\bar{p}(\cdot) = (-\lambda, p(\cdot))$ where $\lambda = \mathrm{const} \leq 0$ and $p(\cdot)$ is such that

$$\dot{p}(t) = -C^*(t)p(t) - \lambda b(t),$$
$$p(0) = -p_0,$$
$$\chi[p(\cdot)] \geq 0,$$
$$\max_{u \in U}[\langle p(t), f(\hat{x}(t), u)\rangle + \lambda\phi(\hat{x}(t), u)] = [\langle p(t), f(\hat{x}(t), \hat{u}(t))\rangle + \lambda\phi(\hat{x}(t), \hat{u}(t))].$$

The theorem is proved.

The most restrictive assumption of Theorem 2.1 is regularity of the differential equation (2). However, this assumption is essential as we can see from the following example:

Example 2.2. Consider the problem

$$\int_0^\infty x dt \longrightarrow \inf$$
$$\dot{x}(t) = a(t)x(t) + u(t),$$
$$u(t) \geq 0,$$
$$x(0) = 1, \ \lim_{t\to\infty} x(t) = 0,$$

where $a(t) = \cos\ln t - 1$. Obviously, $\hat{u}(t) \equiv 0$ is an optimal control and $\hat{x}(t) = \exp\int_0^t a(s)ds$ is the corresponding trajectory. We claim that the statement of Theorem 2.1 fails to be true. Assume that there exist a number $\lambda \leq 0$ and a function $p(\cdot)$ satisfying

$$\dot{p} = -a(t)p(t) - \lambda,$$
$$\chi[p(\cdot)] \geq 0,$$
$$p(t) \leq 0.$$

Since $\chi[p(\cdot)] \geq 0$, we conclude that $\lambda \neq 0$. Let $\lambda = -1$. Then

$$p(t) = (\hat{x}(t))^{-1}(p_0 + \int_0^t \hat{x}(s)ds).$$

Observe that $\chi[(1/\hat{x})(\cdot)] < 0$. Consequently, $\lim_{t\to\infty}\int_0^t \hat{x}(s)ds = -p_0$. Thus we have $p(t) = (\hat{x}(t))^{-1}(-\int_t^\infty \hat{x}(s)ds)$. The inequality

$$\chi[p(\cdot)] \leq \chi[(1/\hat{x})(\cdot)] + \chi[J \circ \hat{x}(\cdot)] \leq \chi[(1/\hat{x})(\cdot)] + \chi[\hat{x}(\cdot)] = -\sqrt{2}$$

leads to a contradiction.

Now we consider an optimal control problem with the functional

$$\int_0^\infty e^{-\delta t}\phi(x, u)dt,$$

where $\delta > 0$. Since we can consider trajectories with negative Lyapunov exponents as possible variations, it is natural to expect that, for an optimal control problem without restrictions at infinity under suitable growth assumptions on the function ϕ, a transversality condition will be $\chi[p(\cdot)] \geq \alpha > 0$. According to Example 1.3 we know that such variations cannot be used for nonlinear controlled system. Therefore we shall investigate the linear case only.

Consider the following problem

$$\int_0^\infty e^{-\delta t}\phi(x(t),u(t))dt \longrightarrow \inf \tag{12}$$

$$\dot{x}(t) = C(t)x(t) + u(t), \tag{13}$$
$$u(t) \in U \subset R^n, \tag{14}$$
$$x(0) = x_0, \tag{15}$$

where $\delta > 0$. Let $\phi : R^n \times U \longrightarrow R$ be a continuous function that is continuously differentiable with respect to x and satisfies the growth conditions

$$\liminf_{x\to\infty} |x|^{-\alpha} \inf_{u\in U} \phi(x,u) > 0,\ \alpha \geq 1 \tag{16}$$

$$|\nabla_x \phi(x,u)| \leq c(1+|x|^\beta),\ \beta \geq 0. \tag{17}$$

Suppose that $C(t)$ is a matrix with bounded measurable components and that the differentional equation

$$\dot{x}(t) = C(t)x(t) \tag{18}$$

is regular. Arbitrary bounded measurable functions $u : [0,\infty[\longrightarrow U$ are considered as admissible controls.

Denote

$$\gamma = \begin{cases} \delta/(1+\beta) & \text{if } \alpha > \beta + 1 \\ \delta(1-\beta/\alpha) & \text{otherwise.} \end{cases}$$

Theorem 2.3. Let $\hat{u}(\cdot)$ be an optimal control for the problem (12)-(15) and let $\hat{x}(\cdot)$ be the corresponding trajectory. Then there exist a number $\lambda \leq 0$ and a function $p : [0,\infty[\longrightarrow R^n$ such that

1. $\dot{p}(t) = -C^*(t)p(t) - \lambda e^{-\delta t} \nabla_x \phi(\hat{x}(t),\hat{u}(t))$,
2. $\max_{u\in U}[\langle p(t),u\rangle + \lambda e^{-\delta t}\phi(\hat{x}(t),u)] = \langle p(t),\hat{u}(t)\rangle + \lambda e^{-\delta t}\phi(\hat{x}(t),\hat{u}(t))$,
3. $\chi[p(\cdot)] \geq \gamma$,
4. $|\lambda| + |p(0)| > 0$.

Proof. This proof is similar to the proof of Theorem 2.1 except for a variation of the optimal trajectory at infinity. The presence of an exponential factor permits us to extend the set of variations. To this end consider the subspace $\Lambda_\tau(\gamma) \subset R^n$ consisting of all points $x_0 \in R^n$ such that a solution of (18) with the initial condition $x(\tau) = x_0$ has a Lyapunov exponent greater than $-\gamma$. We prove first that the integral

$$\int_\tau^\infty e^{-\delta t}\phi(\hat{x}(t)+x(t),\hat{u}(t))dt$$

exists when $x(\cdot)$ is a solution to (18) satisfying $x(\tau) \in \Lambda_\tau(\gamma)$. Observe that existence of the integral

$$\int_0^\infty e^{-\delta t}\phi(\hat{x}(t),\hat{u}(t))dt$$

together with (16) imply that

$$\int_0^\infty e^{-\delta t}|\hat{x}(t)|^\alpha dt < \infty.$$

Since $|C(t)| \leq b, |\hat{u}(t)| \leq b$ for all $t \geq 0$, we obtain

$$|\frac{d}{dt}(e^{-\delta t}|\hat{x}(t)|^\alpha)| \leq e^{-\delta t}(\delta|\hat{x}(t)|^\alpha + \alpha b|\hat{x}(t)|^\alpha + \alpha b|\hat{x}(t)|^{\alpha-1}) \leq$$

$$e^{-\delta t}((\delta + \alpha b)|\hat{x}(t)|^\alpha + \alpha b \max\{1, |\hat{x}(t)|^\alpha\}).$$

Therefore the integral

$$\int_0^\infty \frac{d}{dt}(e^{-\delta t}|\hat{x}(t)|^\alpha)dt$$

exists. This implies that the function $e^{-\delta t}|\hat{x}(t)|^\alpha$ tends to zero as $t \to \infty$. Thus, $\chi[\hat{x}(\cdot)] \geq -\delta/\alpha$.

By condition (17)

$$|\phi(\hat{x}(t) + x(t), \hat{u}(t))| \leq |\phi(\hat{x}(t), \hat{u}(t))| + c(1 + (|\hat{x}(t)| + |x(t)|)^\beta)|x(t)|.$$

Since $\chi[x(\cdot)] > -\gamma$, the Lyapunov exponent of the second term is greater than $-\delta$. Thus, the integral

$$\int_\tau^\infty e^{-\delta t}\phi(\hat{x}(t) + x(t), \hat{u}(t))dt$$

does exist.

Consider the set $\bar{M}_\tau \subset R^n \times R$ consisting of all points

$$(\hat{x}(\tau) + x_\tau, \int_\tau^\infty e^{-\delta t}\phi(\hat{x}(t) + x(t), \hat{u}(t))dt),$$

where $x_\tau \in \Lambda_\tau(\gamma)$ and $x(\cdot)$ is a solution to (18) with the initial condition $x(\tau) = x_\tau$. Using estimates similar to those obtained above, it is not difficult to show that the subspace $\bar{\Lambda}_\tau$ consisting of points

$$(x_\tau, \int_\tau^\infty e^{-\delta t}\langle \nabla_x \phi(\hat{x}(t), \hat{u}(t)), x(t)\rangle dt)$$

is a tent of the set $\bar{M}_\tau$ at the point

$$(\hat{x}(\tau), \int_\tau^\infty e^{-\delta t}\phi(\hat{x}(t), \hat{u}(t))dt).$$

Then following the proof of Theorem 2.1 we achieve the result.

There is no end-point constraint at infinity in the problem (12)-(15), but, as we can see from the example below, the Lagrange multiplier λ can be equal to zero. This is a payment for the transversality condition.

Example 2.4. Consider the problem

$$\int_0^\infty e^{-t/2}(x_1(t) + x_1^2(t) + x_2^2(t))dt \longrightarrow \inf$$

$$\dot{x}_1(t) = x_2(t),$$

$$\dot{x}_2(t) = x_1(t) + u(t),$$

$$u(t) \geq 0,$$

$$x_1(0) = x_2(0) = 0.$$

Here all assumptions of Theorem 2.3 are satisfied. Obviously, $\hat{u}(t) \equiv 0$ is an optimal control and $\hat{x}_1(t) \equiv \hat{x}_2(t) \equiv 0$ is the corresponding trajectory. Growth conditions (16) and (17) are satisfied when $\alpha = 2$ and $\beta = 1$, e.g. By Theorem 2.3 there exist a number $\lambda \leq 0$ and a function $(p_1(\cdot), p_2(\cdot))$ such that

$$\dot{p}_1(t) = -p_2(t) - \lambda e^{-t/2},$$

$$\dot{p}_2(t) = -p_1(t),$$

$$p_2(t) \leq 0,$$

$$\chi[(p_1, p_2)(\cdot)] \geq 1/4,$$

$$|\lambda| + |(p_1(0), p_2(0))| > 0.$$

Suppose that $\lambda = -1$. Then $\ddot{p}_2(t) = p_2(t) - e^{-t/2}$. Therefore $p_2(t) = ae^t + be^{-t} + \frac{4}{3}e^{-t/2}$. Since $\chi[p_2(\cdot)] \geq 1/4$, we obtain $a = 0$. Then for t sufficiently large $p_2(t) > 0$. This contradicts to the inequality $p_2(t) \leq 0$. Thus, $\lambda = 0$.

3 The linear quadratic problem

In this section we deal with the linear quadratic optimal control problem

$$\int_0^\infty (x^2(t) + u^2(t))dt \longrightarrow \inf \tag{1}$$

$$\dot{x}(t) = Cx(t) + u(t), \tag{2}$$

$$u(t) \in K, \tag{3}$$

$$x(0) = x_0, \tag{4}$$

where C is a constant $(n \times n)$ matrix and K is a closed convex cone. We shall derive necessary and sufficient conditions of optimality for the problem under the following hypothesis

(H) for any $x_0 \in R^n$ there exists a solution to the controlled system (2) and (3), with the initial condition (4) satisfying

$$\lim_{t\to\infty} x(t) = 0.$$

To test this hypothesis one can use the following result [10].

We denote by K^* the polar cone of K, the closed convex cone defined by

$$K^* = \{x^* | \forall x \in K, \langle x^*, x\rangle \geq 0\}.$$

Theorem 3.1. The following conditions are equivalent:

1. the hypothesis **(H)** holds true,

2. the matrix C^* has neither eigenvectors corresponding to nonnegative eigenvalues contained in K^* nor proper invariant subspaces corresponding to eigenvalues with nonnegative real parts contained in K^*,

3. the differential equation $\dot{p}(t) = -C^*p(t)$ does not have nontrivial solutions that have nonnegative Lyapunov exponents contained in K^*.

Now, we derive from **(H)** the solvability of the problem (1)-(4) for any initial condition.

Theorem 3.2. Assume that hypothesis **(H)** holds true. Then the problem (1)-(4) has a unique solution $\hat{u}(\cdot), \hat{x}(\cdot)$. Moreover, $\hat{x}(t) \to 0$ as $t \to \infty$.

Proof. We prove first that there exist numbers $\gamma > 0, a > 0, b > 0$ such that for any $x_0 \in R^n$ one can find an admissible control $u(\cdot)$ and a corresponding solution $x(\cdot)$ of the Cauchy problem (2) and (4) satisfying

$$|x(t)| \leq a|x_0|e^{-\gamma t}, \ |u(t)| \leq b|x_0|e^{-\gamma t}, \ t \geq 0. \tag{5}$$

Consider a simplex $\sigma^{n+1} \subset R^n$ a containing unit ball centered at zero. Let $z_k, k = 0, \dots, n$ be its vertices. By hypothesis **(H)** there exist admissible controls $u_k(\cdot)$ such that solutions of the Cauchy problems

$$\dot{x}_k(t) = Cx_k(t) + u_k(t),$$
$$x_k(0) = z_k$$

tend to zero as t becomes infinite. There exists $\tau \geq 0$ satisfying $|x_k(\tau)| \leq 1/e, k = 0, \dots, n$. Without loss of generality $|u_k(t)| \leq \eta$ for all $t \in [0, \tau], k = 0, \dots, n$. Let $y \in R^n, |y| = 1$. Then $y = \sum_{k=0}^n \lambda_k z_k$ for some $\lambda_k \geq 0, k = 0, \dots, n$ satisfying $\sum_{k=0}^n \lambda_k = 1$. Obviously, the trajectory $x(\cdot, y)$ of the controlled system (2) with $x(0, y) = y$ corresponding to the control $u(\cdot, y) = \sum_{k=0}^n \lambda_k u_k(\cdot)$ satisfies $|x(\tau, y)| \leq 1/e, \ |u(t, y)| \leq \eta, \ t \in [0, \tau]$. We define for $x_0 \in R^n$

$$u_{x_0}(t) = |x_0|u(t, x_0/|x_0|), t \in [0, \tau].$$

Let $x_{x_0}(t)$ be the corresponding trajectory with $x_{x_0}(0) = x_0$. Then $|x_{x_0}(\tau)| \leq |x_0|/e$, $|u_{x_0}(t)| \leq \eta|x_0|$ when $t \in [0, \tau]$. For $t \geq 0$, we set

$$u(t) = \begin{cases} u_{x_0}(t) & t \in [0, \tau], \\ u_{x(m\tau)}(t - m\tau) & t \in]m\tau, (m+1)\tau]. \end{cases}$$

This control and the corresponding trajectory $x(\cdot)$ satisfy (5) with $\gamma = 1/\tau, \ b = e\eta$, and

$$a = e \max\{|x_k(t)| \ |t \in [0, \tau], \ k = 0, \dots, n\}.$$

By (5) the functional (1) in the problem (1)-(4) is finite for any x_0. Using standard reasoning based upon weak compactness of a unit ball in Hilbert space and Mazur lemma, we obtain existence of an optimal control.

To prove uniqueness suppose the opposite. Let $u(\cdot)$ and $w(\cdot)$ be optimal controls, and let $x(\cdot)$ and $y(\cdot)$ be the corresponding trajectories. Then

$$I = \int_0^\infty (x^2(t) + u^2(t))dt = \int_0^\infty (y^2(t) + w^2(t))dt.$$

The inequality

$$\int_0^\infty \left(\left(\frac{x(t) + y(t)}{2} \right)^2 + \left(\frac{u(t) + w(t)}{2} \right)^2 \right) dt = \frac{1}{4} \int_0^\infty (2(x^2(t) + y^2(t)) -$$

$$(x(t)-y(t))^2+2(u^2(t)+w^2(t))-(u(t)-w(t))^2)dt < I$$

contradicts optimality of $u(\cdot)$ and $w(\cdot)$.

We prove now that the optimal trajectory $\hat{x}(t)$ tends to zero as t becomes infinite. Since $\hat{x}(\cdot), \dot{\hat{x}}(\cdot) \in L_2([0,\infty[, R^n)$, the function $\frac{d}{dt}|\hat{x}(t)|^2 = 2\langle \hat{x}(t), \dot{\hat{x}}(t)\rangle$ belongs to $L_1([0,\infty[, R)$. Therefore, $\lim_{t\to\infty} |\hat{x}(t)|^2$ exists and is equal to zero. This ends the proof.

We denote by $\pi_K(p)$ the point $x \in K$ such that $|x-p| = \inf\{|y-p| \; |y \in K\}$.

Theorem 3.3. Suppose that hypothesis **(H)** holds true. Then the control $\hat{u}(\cdot)$ and the corresponding trajectory $\hat{x}(\cdot)$ are optimal for the problem (1)-(4) if and only if $\chi[\hat{x}(\cdot)] > 0$ and there exists a function $p : [0,\infty[\longrightarrow R^n$ satisfying

$$\dot{p}(t) = -C^*p(t) + \hat{x}(t),$$

$$\hat{u}(t) = \pi_K(p(t)),$$

$$\chi[p(\cdot)] > 0.$$

Proof. To begin with, note that by Theorem 3.2 optimal control exists and belongs to L_2. Since Theorem 2.1 was proved under the assumption of boundness of optimal control, it is not applied directly. But because this problem has a linear quadratic form, following the proof of Theorem 2.1, it is possible to derive that there exist a number $\lambda \le 0$ and a function $p : [0,\infty[\longrightarrow R^n$ such that

$$\dot{p}(t) = -C^*p(t) - 2\lambda\hat{x}(t), \tag{6}$$

$$\max_{u\in K}[\lambda u^2 + \langle p(t), u\rangle] = \lambda\hat{u}^2(t) + \langle p(t), \hat{u}(t)\rangle, \tag{7}$$

$$\chi[p(\cdot)] \ge 0, \tag{8}$$

$$|\lambda| + |p(0)| > 0. \tag{9}$$

To show that $\lambda \neq 0$ suppose the opposite. Then by (6)-(9) we have

$$\dot{p}(t) = -C^*p(t),$$

$$p(t) \in -K^*,$$

$$\chi[p(\cdot)] \ge 0,$$

$$|p(0)| > 0.$$

This contradicts hypothesis **(H)** because of Theorem 3.1. Thus, we can set $\lambda = -1/2$. Using subdifferential calculus (see [11], e.g.) we obtain that (7) implies $0 \in \hat{u}(t) - p(t) - K^* \cap \{\hat{u}(t)\}^\perp$ or $\hat{u}(t) \in K \cap (K^* \cap \{\hat{u}(t)\}^\perp + p(t))$. Therefore $\hat{u}(t) = \pi_K(p(t))$. Thus, if $\hat{u}(\cdot)$ is an optimal control and $\hat{x}(\cdot)$ is a corresponding trajectory, then there exists a function $p : [0,\infty[\longrightarrow R^n$ satisfying

$$\dot{\hat{x}}(t) = C\hat{x}(t) + \pi_K(p(t)), \tag{10}$$

$$\dot{p}(t) = -C^*p(t) + \hat{x}(t), \tag{11}$$

$$\chi[p(\cdot)] \ge 0. \tag{12}$$

We now prove that $\chi[\hat{x}(\cdot)] > 0$. For this purpose consider the value function

$$V(x) = \min \int_0^\infty (x^2(t) + u^2(t))dt,$$

where minimum is taken over all admissible controls and over all corresponding trajectories with $x(0) = x$. By Theorem 3.2 there exists an optimal control $\tilde{u}(\cdot)$. Let $\tilde{x}(\cdot)$ be a corresponding trajectory with $\tilde{x}(0) = x$. We observe that

$$V(\tilde{x}(t)) = \int_t^\infty (\tilde{x}^2(s) + \tilde{u}^2(s))ds.$$

Furthermore,

$$\tau^{-1}(V(\tilde{x}(t+\tau)) - V(\tilde{x}(t))) = -\tau^{-1}\int_t^{t+\tau} (\tilde{x}^2(s) + \tilde{u}^2(s))ds.$$

Hence, $\frac{d}{dt}V(\tilde{x}(t)) \leq -\tilde{x}^2(t)$. Since the function $[V(x)]^{1/2}$ is a norm in R^n, we obtain $\frac{d}{dt}V(\tilde{x}(t)) \leq -cV(\tilde{x}(t))$, $c > 0$. This implies that $[V(\hat{x}(t))]^{1/2} \leq [V(\hat{x}(0))]^{1/2}e^{-ct/2}$. Thus, $\chi[\hat{x}(\cdot)] > 0$.

We claim that $p(t)$ tends to zero as t becames infinite. If this is not the case there exist a sequence $t_k \to 0$ as $k \to \infty$ and a number $\alpha > 0$ such that $|\hat{x}(t)| < 1/k$ for all $t \geq t_k$ and $|p(t_k)| \geq \alpha$. We denote by $z(\cdot)$ a pair of functions $(\hat{x}(\cdot), p(\cdot))$ and consider a sequence of solutions to differential equations (10) and (11), $z_k(t) = z(t + t_k)/|p(t_k)|$, $t \geq 0$. Since the sequence $z_k(0)$ is bounded, we conclude without loss of generality that $z_k(0)$ tends to some z_0. Obviously, $z_0 = (0, p_0)$, $|p_0| = 1$. There exists a solution $z_0(\cdot)$ to (10) and (11) with $z_0(0) = z_0$, which is a uniform limit of the sequence $z_k(\cdot)$ on finite intervals. Taking the limit in (10) and (11) we obtain that $z_0(\cdot) = (0, p_0(\cdot))$ where

$$\dot{p}_0(t) = -C^* p_0(t),$$

$$\pi_K(p_0(t)) = 0.$$

The latter equality implies that $p_0(t) \in -K^*$, $t \geq 0$. By Theorem 3.1 $\chi[p_0(\cdot)] < 0$. To obtain the contradiction we shall prove that $p_0(\cdot)$ is bounded.

Set $\bar{p} = (p^0, p) \in R \times R^n$ and consider the differential equations

$$\dot{\bar{p}}(t) = \mathbf{C}\bar{p}(t), \tag{13}$$

$$\dot{\bar{q}}(t) = (\mathbf{C} + \mathbf{B}(t))\bar{q}(t), \tag{14}$$

where $(n+1) \times (n+1)$ matrices $\mathbf{C}$ and $\mathbf{B}(t)$ are equal to

$$\mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & -C^* \end{pmatrix}, \ \mathbf{B}(t) = \begin{pmatrix} 0 & 0 \\ \hat{x}(t) & 0 \end{pmatrix}.$$

The solution of the equation (14) $\bar{q}(t) = (1, p(t))$ has zero Lyapunov exponent. By Theorem 1.4 the difference between $\bar{q}(\cdot)$ and some solution of (13) is a function with a positive Lyapunov exponent. The solution of (13) with zero Lyapunov exponent is a polynomial with bounded functions as coefficients. Hence, the function $\lim_{k\to\infty} |p(t + t_k)|/|p(t_k)|$ is bounded. Thus, $\lim_{t\to\infty} p(t) = 0$.

Applying Theorem 1.4 to the equations (13) and (14) once more we obtain $\chi[p(\cdot)] > 0$.

The necessity of the theorem statement is proved. The sufficiency is a consequence of the standard argument given in Lee and Markus [12].

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