Working Paper

Partial Differential Inclusions Governing Feedback Controls

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FOREWORD

The authors derive partial differential inclusions of hyperbolic type, the solutions of which are *feedbacks governing the viable (controlled invariant) solutions* of a control system.

They show that the ***tracking property***, another important control problem, leads to such hyperbolic systems of partial differential inclusions.

They begin by proving the existence of the largest solution of such a problem, a stability result and provide an explicit solution in the particular case of decomposable systems.

They then state a variational principle and an existence theorem of a (single-valued contingent) solution to such an inclusion, that they apply to assert the existence of a fedback control.

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Partial Differential Inclusions Governing Feedback Controls

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Introduction

Let X, Y, Z denote finite dimensional vector-spaces. We studied in [12] the existence of dynamical closed-loop controls regulating smooth state-control solutions of a control system (U, f):

$$(1) \quad \begin{cases} i) & \text{for almost all } t, \;\; x'(t) \; = \; f(x(t), u(t)) \\ ii) & u(t) \; \in \; U(x(t)) \end{cases}$$

where $U : X \rightsquigarrow Y$ is a closed set-valued map and $f : \text{Graph}(U) \mapsto X$ a continuous (single-valued) map with linear growth.

Consider a nonnegative continuous function $\varphi : \text{Graph}(U) \mapsto \mathbf{R}_+$ with linear growth (in the sense that $\varphi(x, u) \leq c(\|x\| + \|u\| + 1)$) and set $K := \text{Dom}(U)$. We address in this paper the problem of finding feedback controls (or closed-loop) controls $r : K \mapsto Y$ satisfying the constraint

$$\forall \, x \in K, \;\; r(x) \; \in \; U(x)$$

and the regulation property: *for any $x_0 \in K$, there exists a solution to the differential equation*

$$x'(t) \; = \; f(x(t), r(x(t))) \;\; \& \;\; x(0) \; = \; x_0$$

such that $u(t) := r(x(t)) \in U(x(t))$ is absolutely continuous and fulfils the growth condition

$$\|u'(t)\| \; \leq \; \varphi(x(t), u(t))$$

for almost all t.

We observe that *the graphs of such feedback controls are viability domains*[1] *of the system of differential inclusions*

[1] See the appendix for definitions and the main statements of Viability Theory

$$
(2) \quad \begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u'(t) \in \varphi(x(t), u(t))B \end{cases}
$$

contained in the graph of U.

Using the Viability Theorem and the fact that the contingent cone to the graph of a map r at a point $(x, r(x))$ is the graph of the contingent derivative[2] $Dr(x)$ of r at x, we derive that such feedback controls are *solutions to the following contingent differential inclusion*

$$
\forall\, x \in K, \;\; 0 \in Dr(x)(f(x, r(x))) - \varphi(x, r(x))B
$$

satisfying the constraints

$$
\forall\, x \in K, \;\; r(x) \;\in\; U(x) \;\; \square
$$

More generally, we recall that *a closed set-valued map $R : K \rightsquigarrow Y$ is a set-valued feedback regulating φ-smooth viable solutions to the control problem if and only if R is a solution to the* contingent differential inclusion

$$
\forall\, x \in K, \;\; 0 \in DR(x, u)(f(x, u)) - \varphi(x, u)B
$$

satisfying the constraint

$$
\forall\, x \in K, \;\; R(x) \;\subset\; U(x)
$$

and that there exists a largest map with closed graph enjoying this property (See [2,3,12]).

We shall study this partial differential inclusion, provide a variational principle and an existence theorem.

But first, we observe that the existence of a dynamical closed loop is a particular case of the *tracking problem,* which is studied under several

[2] The contingent derivative $DH(x, y)$ of a set-valued map $H : X \rightsquigarrow Y$ at $(x, y) \in \text{Graph}(H)$ is defined by

$$
\text{Graph}(DH(x, y)) \;:=\; T_{\text{Graph}(H)}(x, y)
$$

When $H = h$ is single-valued, we set $Dh(x) := Dh(x, h(x))$. See [11, Chapter 5] for more details on differential calculus of set-valued maps.

names in many fields, and specially, arises in engineering (see for instance [17]).

Indeed, consider two set-valued maps $F : X \times Y \rightsquigarrow X$, $G : X \times Y \rightsquigarrow Y$ and the *system of differential inclusions*

$$\begin{cases} x'(t) \in F(x(t), y(t)) \\ y'(t) \in G(x(t), y(t)) \end{cases}$$

We would like to characterize a set-valued map $H : X \rightsquigarrow Y$, regarded as an *observation map* satisfying what we can call the *tracking property*: for every $x_0 \in \mathrm{Dom}(H)$ and every $y_0 \in H(x_0)$, there exists a solution $(x(\cdot), y(\cdot))$ to this system of differential inclusions starting at (x_0, y_0) and satisfying

$$\forall t \geq 0, \;\; y(t) \in H(x(t))$$

The answer to this question is a solution to a *viability problem*, since we actually look for $(x(\cdot), y(\cdot))$ which remains viable in the graph of the observation map H. So, if the set-valued maps F and G are Peano maps and if the graph of H is closed, the Viability Theorem states that the tracking property is equivalent to the fact that the graph of H is a viability domain of $(x,y) \rightsquigarrow F(x,y) \times G(x,y)$, i.e., that H is a solution to the *contingent differential inclusion*

$$\forall (x,y) \in \mathrm{Graph}(H), \;\; 0 \in DH(x,y)(F(x,y)) - G(x,y)$$

We observe that when F and G are single-valued maps f and g and H is a differentiable single-valued map h, the contingent differential inclusion boils down to the *quasi-linear hyperbolic system of first-order partial differential equations*[3]:

$$\forall j = 1, \ldots, m, \;\; \sum_{i=1}^{n} \frac{\partial h_j}{\partial x_i} f_i(x, h(x)) - g_j(x, h(x)) = 0$$

[3]For several special types of systems of differential equations, the graph of such a map h (satisfying some additional properties) is called a *center manifold*. Theorems providing the existence of local center manifolds have been widely used for the study of stability near an equilibrium and in control theory. See [8,9,19,22] for instance.

It may seem strange to accept set-valued maps as solutions to an hyperbolic system of partial differential inclusions. But this may offer a way to describe shock waves by the set-valued character of the solution (which may happen even for maps with smooth graphs, but whose projection leads to set-valued maps.) Derivatives in the sense of distributions do not offer the unique way to describe weak or generalized solutions. Contingent derivatives offer another way to weaken the required properties of a derivative, loosing the linear character of the differential operator, but allowing a pointwise definition. It provides a convenient way to treat hyperbolic problems. This has been already noticed in [13,14,23,24] for conservation laws.

Knowing F and G, we have to find observation maps H satisfying the tracking property, i.e., we must solve the above contingent differential inclusion.

Furthermore, we can address other questions such as:

a) — Find the largest solution to the contingent differential inclusion contained in a given set-valued map (which then, contains all the other ones if any)

b) — Find single-valued solutions h to the contingent differential inclusion which then becomes

$$\forall\, x \in K, \;\; 0 \in Dh(x)(F(x,h(x))) - G(x,h(x)) \tag{3}$$

In this case, the tracking property states that there exists a solution to the *"reduced" differential inclusion*

$$x'(t) \;\in\; F(x(t),h(x(t)))$$

so that $(x(\cdot), y(\cdot) := h(x(\cdot)))$ is a solution to the initial system of differential inclusions starting at $(x_0, h(x_0))$. Knowing h allows to divide the system by half, so to speak.

This list of problems justifies the study of the contingent inclusion (**3**). Let us mention right now that looking for "weak" solutions to this contingent differential inclusion in Sobolev spaces or other spaces of distributions does not help since we require *solutions h to be defined through their graph, and thus, solutions which are defined everywhere.*

The use of contingent derivatives in some problems (related to the value function of optimal control problems, in particular) is by no means new (see [1], [7, Chapter 6], [27,28]). It has been shown in [27] that "contingent solutions" are related by duality to the "viscosity solutions" introduced in the context of Hamilton-Jacobi equations by Crandall & Lions in [21] (see also [20] and the literature following these papers). In the context of this paper (quasi-linear but set-valued hyperbolic differential inclusions), Proposition 3.4 makes explicit the duality relations between contingent solutions and solutions very closed in spirit to the *viscosity solutions* in the case when $Y = \mathbf{R}$.

The variational principle we prove below (Theorem 3.1) states that for systems of partial differential equations or inclusions, the contingent solutions are adaptations to the vector-valued case of viscosity solutions.

We shall characterize the tracking property in Section 1 and give an explicit formula for a closed solution in the case of *decomposable systems* of differential inclusions. We then devote section 2 to the study of the transpose of contingent derivatives, and in particular, a series of new convergence results.

The *variational principle* is the topic of section 3 and the existence of solutions the object of section 4. These results are applied to characterize and find feedback controls regulating viable solutions in section 5.

1 The Tracking Property

1.1 Characterization of the Tracking Property

Consider two finite dimensional vector-spaces X and Y, two set-valued maps $F : X \times Y \rightsquigarrow X$, $G : X \times Y \rightsquigarrow Y$ and a set-valued map $H : X \rightsquigarrow Y$, regarded as (and often called) the *observation map*:

Definition 1.1 *We shall say that F, G and H satisfy the* tracking property *if for any initial state $(x_0, y_0) \in \mathrm{Graph}(H)$, there exists at least one solution $(x(\cdot), y(\cdot))$ to the* system of differential inclusions

$$(4) \qquad \begin{cases} x'(t) \in F(x(t), y(t)) \\ y'(t) \in G(x(t), y(t)) \end{cases}$$

starting at (x_0, y_0), *defined on* $[0, \infty[$ *and satisfying*

$$\forall\, t \geq 0,\ \ y(t)\ \in\ H(x(t))$$

We now consider the *contingent differential inclusion*

$$(5) \qquad \forall\,(x,y)\ \in\ \mathrm{Graph}(H),\ \ 0 \in DH(x,y)(F(x,y)) - G(x,y)$$

Definition 1.2 *We shall say that a set-valued map* $H : X \rightsquigarrow Y$ *satisfying (5) is a solution to the* contingent differential inclusion *if its graph is a* closed *subset of* $\mathrm{Dom}(F) \cap \mathrm{Dom}(G)$.

When $H = h : \mathrm{Dom}(h) \mapsto Y$ *is a single-valued map with closed graph contained in* $\mathrm{Dom}(F) \cap \mathrm{Dom}(G)$, *the partial contingent differential inclusion (5) becomes*

$$(6) \qquad \forall\, x \in \mathrm{Dom}(h),\ \ 0 \in Dh(x)(F(x,h(x))) - G(x,h(x))$$

We deduce at once from the viability theorems[4] the following:

Theorem 1.3 *Let us assume that* $F : X \times Y \rightsquigarrow X$, $G : X \times Y \rightsquigarrow Y$ *are Peano maps*[5] *and that the graph of the set-valued map* H *is a closed subset of* $\mathrm{Dom}(F) \cap \mathrm{Dom}(G)$.

1. — The triple (F, G, H) *enjoys the tracking property if and only if* H *is a solution to the* contingent differential inclusion *(5).*

2. — There exists a largest *solution* $H_\star$ *to the contingent differential inclusion (5) contained in* H. *It enjoys the following property: whenever an initial state* $y_0 \in H(x_0)$ *does not belong to* $H_\star(x_0)$, *then all solutions* $(x(\cdot), y(\cdot))$ *to the system of differential inclusions (4) satisfy*

$$(7) \qquad \begin{cases} i) & \forall\, t \geq 0, \quad y(t)\ \notin H_\star(x(t)) \\ ii) & \exists\, T > 0 \quad \textit{such that } y(T)\ \notin\ H(x(T)) \end{cases}$$

We now state a useful Stability Theorem[6]. We recall that the graph of the *graphical upper limit* $H^\sharp$ of a sequence of set-valued maps $H_n : X \rightsquigarrow Y$ is by definition the graph of the upper limit of the graphs of the maps H_n. (See [11, Chapter 7].)

[4]See the Appendix.

[5]See the Appendix.

[6]See the Appendix

Theorem 1.4 (Stability) *Let us consider a sequence of Peano maps* $F_n : X \times Y \rightsquigarrow X$, $G_n : X \times Y \rightsquigarrow Y$ *with uniform linear growth*[7] *and their graphical upper limit* $F^\sharp$ *and* $G^\sharp$.

Consider also a sequence of set-valued map $H_n : X \rightsquigarrow Y$, *solutions to the contingent differential inclusions*

$$(8) \qquad \forall (x,y) \in \mathrm{Graph}(H_n), \ \ 0 \in DH_n(x,y)(F_n(x,y)) - G_n(x,y)$$

Then the graphical upper limit $H^\sharp$ *of the solutions* H_n *is a solution to*

$$(9) \ \ \forall (x,y) \in \mathrm{Graph}(H^\sharp), \ \ 0 \in DH^\sharp(x,y)(\overline{co}F^\sharp(x,y)) - \overline{co}(G^\sharp(x,y))$$

In particular, if the set-valued maps F_n *and* G_n *converge graphically to maps* F *and* G *respectively, then the graphical upper limit* $H^\sharp$ *of the solutions* H_n *is a solution of (5).*

We recall that graphical convergence of single-valued maps is weaker than pointwise convergence. This is why graphical limits of single-valued maps which are converging pointwise may well be set-valued.

Therefore, for single-valued solutions, the stability property implies the following statement: *Let* h_n *be single-valued solutions to the contingent partial differential inclusion (8). Then their graphical upper limit* $h^\sharp$ *is a* (possibly set-valued) solution *to (9).*

Although set-valued solutions to hyperbolic systems make sense to describe shock waves and other phenomena, we may still need sufficient conditions for an upper graphical limit of single-valued maps to be still single-valued. (This is the case when *a sequence of continuous solutions* h_n *to the contingent differential inclusion (8) is equicontinuous and converges pointwise to a function* h. *Then*[8] h *is a single-valued solution to (9).*

[7] In the sense when there exists a constant $c > 0$ such that

$$\sup_{n \geq 0} \max\left(\|F_n(x,y)\|, \|G_n(x,y)\|\right) \ \leq \ c(\|x\| + \|y\| + 1)$$

[8] Indeed, a pointwise limit h of single-valued maps h_n is a selection of the graphical upper limit of the h_n. The latter is equal to h when h_n remain in an equicontinuous subset.

1.2 Decomposable Case

Let $K \subset X$, $\Phi : K \leadsto X$ and $\Psi : K \leadsto Y$ be set-valued maps. Consider the *decomposable* system of differential inclusions

$$(10) \qquad \begin{cases} x'(t) \in \Phi(x(t)) \\ y'(t) \in \lambda y(t) + \Psi(x(t)) \end{cases}$$

which extends to the set-valued case the characteristic system of linear hyperbolic systems.

We denote by $\mathcal{S}_\Phi(x,\cdot)$ the set of solutions $x(\cdot)$ to the differential inclusion $x'(t) \in \Phi(x(t))$ starting at x and viable in K.

Theorem 1.5 *Assume that $\Phi : K \leadsto X$ and $\Psi : K \leadsto Y$ are Peano maps and that K is a viability domain of Φ. The set-valued map $H_\star : K \leadsto Y$ defined by*

$$(11) \qquad \forall\, x \in K,\ \ H_\star(x) \ := \ -\int_0^\infty e^{-\lambda t}\Psi(\mathcal{S}_\Phi(x,t))dt$$

verifies

$$(12) \qquad \forall\,(x,y) \in \mathrm{Graph}(H_\star),\ \ \lambda y \in DH_\star(x,y)(\Phi(x)) - \Psi(x)$$

When λ is large enough (and when $\lambda > 0$ if Ψ is bounded), its graph is closed and $H_\star$ is a solution to the contingent inclusion (5) with $F(x,y) := \Phi(x)$ and $G(x,y) := \lambda y + \Psi(x)$.

Proof

1. — We prove first that the graph of $H_\star$ satisfies contingent inclusion (12).

Indeed, choose an element y in $H_\star(x)$. By definition of the integral of a set-valued map (see [11, Chapter 8] for instance), this means that there exist a solution $x(\cdot) \in \mathcal{S}_\Phi(x,\cdot)$ to the differential inclusion $x'(t) \in \Phi(x(t))$ starting at x and $z(t) \in \Psi(x(t))$ such that

$$y \ := \ -\int_0^\infty e^{-\lambda t}z(t)dt \ \in \ H_\star(x)$$

We check that for every $h > 0$

$$-\int_0^\infty e^{-\lambda t}z(t+h)dt \ \in \ H_\star(x(h)) \ = \ H_\star\left(x + h\left(\frac{1}{h}\int_0^h x'(t)dt\right)\right)$$

By observing that

$$\left\{\begin{array}{l}\frac{1}{h}\int_0^\infty e^{-\lambda t}\left(z(t)-z(t+h)\right)dt \\ \\ = \ -\frac{e^{\lambda h}-1}{h}\int_0^\infty e^{-\lambda t}z(t)dt + \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t}z(t)dt\end{array}\right.$$

we deduce that

$$\left\{\begin{array}{l}y + h\left(-\frac{e^{\lambda h}-1}{h}\int_0^\infty e^{-\lambda t}z(t)dt + \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t}z(t)dt\right) \\ \\ \in \ H_\star\left(x + h\left(\frac{1}{h}\int_0^h x'(t)dt\right)\right)\end{array}\right.$$

Since Φ is upper semicontinuous, we know that for any $\varepsilon > 0$ and t small enough, $\Phi(x(t)) \subset \Phi(x) + \varepsilon B$, so that $x'(t) \in \Phi(x) + \varepsilon B$ for almost all small t. Therefore, $\Phi(x)$ being closed and convex, we infer that for $h > 0$ small enough, $\frac{1}{h}\int_0^h x'(t)dt \in \Phi(x) + \varepsilon B$ thanks to the Mean-Value Theorem. This latter set being compact, there exists a sequence of $h_n > 0$ converging to 0 such that $\frac{1}{h_n}\int_0^{h_n} x'(t)dt$ converges to some $u \in \Phi(x)$.

In the same way, Ψ being upper semicontinuous, $\Psi(x(t)) \subset \Psi(x) + \varepsilon B$ for any $\varepsilon > 0$ and t small enough, so that $z(t) \in \Psi(x) + \varepsilon B$ for almost all small t. The Mean-Value Theorem implies that

$$\forall\, n > 0, \ \ z_n \ := \ \frac{\lambda e^{\lambda h_n}}{e^{\lambda h_n} - 1}\int_0^{h_n} e^{-\lambda t}z(t)dt \ \in \ \Psi(x) + \varepsilon B$$

since this set is compact and convex. Furthermore, there exists a subsequence of z_n converges to some $z_0 \in \Psi(x)$. We thus infer that

$$\lambda y + z_0 \ \in \ DH_\star(x,y)(u)$$

so that $\lambda y \ \in \ DH_\star(x,y)(\Phi(x)) - \Psi(x)$.

2. — Let us prove now that the graph of $H_\star$ is closed when λ is large enough. Consider for that purpose a sequence of elements (x_n, y_n) of the graph of $H_\star$ converging to (x,y). There exist solutions $x_n(\cdot) \in \mathcal{S}_\Phi(x_n,\cdot)$ to the differential inclusion $x' \in \Phi(x)$ starting at x_n and measurable selections $z_n(t) \in \Psi(x_n(t))$ such that

$$y_n \ := \ -\int_0^\infty e^{-\lambda t}z_n(t)dt \in H_\star(x_n)$$

The growth of Φ being linear, there exist $\rho,\ c > 0$ such that the solutions $x_n(\cdot)$ obey the estimate

$$\|x_n(t)\| \leq \rho e^{ct} \ \ \& \ \ \|x_n'(t)\| \leq \rho' e^{ct}$$

By the compactness of the graph of the solution map (which follows from the Convergence Theorem [11, Theorem 7.2.2] and [5]), we know that there exists a subsequence (again denoted by) $x_n(\cdot)$ converging uniformly on compact intervals to a solution $x(\cdot) \in \mathcal{S}_\Phi(x,\cdot)$.

The growth of Ψ being also linear, we deduce that $\|z_n(t)\| \leq \rho_1 e^{ct}$ (with $c = 0$ when Ψ is bounded).

When $\lambda > c$, setting $u_n(t) := e^{-\lambda t} z_n(t)$, Dunford-Pettis' Theorem implies that a subsequence (again denoted by) $u_n(\cdot)$ converges weakly to some function $u(\cdot)$ in $L^1(0,\infty;Y)$. This means that $z_n(\cdot)$ converges weakly to some function $z(\cdot)$ in $L^1(0,\infty;Y;e^{-\lambda t}dt)$. The Convergence Theorem states that $z(t) \in \Psi(x(t))$ for almost every t. Since the integrals y_n converge to $-\int_0^\infty e^{-\lambda t} z(t)dt$, we have proved that

$$y \ = \ -\int_0^\infty e^{-\lambda t} z(t)dt \ \in \ H_\star(x) \ \ \square$$

Remark — When $\Phi = \varphi$ and $\Psi = \psi$ are smooth single-valued maps, this formula yields the classical formula

$$h(x) \ := \ -\int_0^\infty e^{-\lambda t} \psi(\mathcal{S}_\varphi(x,t))dt \tag{13}$$

of the solution to the linear system of partial differential equations

$$\lambda h(x) \ = \ h'(x)\varphi(x) - \psi(x)$$

It was also proved in [8,9] that the map h defined by (13) is a solution when φ and ψ are Lipschitz and ψ is bounded. $\square$

1.3 Energy Maps (or Zero Dynamics)

The simplest dynamics are obtained when $G \equiv 0$. Therefore, when F is a Peano map, H enjoys the tracking property if and only if it is a solution to

$$\forall\,(x,y) \in \mathrm{Graph}(H), \ \ 0 \in DH(x,y)(F(x,y)) \tag{14}$$

Since the tracking property of H amounts to saying that each subset $H^{-1}(y)$ enjoys the viability property for $F(\cdot, y)$, we observe that this condition is also equivalent to condition

$$\forall\, y \in \mathrm{Im}(H),\ \forall\, x \in H^{-1}(y),\ \ F(x,y) \cap T_{H^{-1}(y)}(x) \neq \emptyset$$

We may say that such a set-valued map H is an *energy map* of F. □

In the general case, the evolution with respect to a parameter y of the viability kernels of the closed subsets $H^{-1}(y)$ under the set-valued map $F(\cdot, y)$ is described in terms of $H_\star$:

Proposition 1.6 *Let $F : X \times Y \rightsquigarrow X$ be a Peano map and $H : X \rightsquigarrow Y$ be a closed set-valued map. Then there exists a largest solution $H_\star : X \rightsquigarrow Y$ contained in H to (14).*

The inverse images $H_\star^{-1}(y)$ are the viability kernels of the subsets $H^{-1}(y)$ under the maps $F(\cdot, y)$:

$$\mathrm{Viab}_{F(\cdot,y)}(H^{-1}(y)) \;=\; H_\star^{-1}(y)$$

The graphical upper limit of energy maps is still an energy map.

Then *the graph of the map $y \rightsquigarrow \mathrm{Viab}_{F(\cdot,y)}(H^{-1}(y))$ is closed, and thus upper semicontinuous whenever the domain of H is bounded.*

When the observation map H is a single-valued map h, the contingent differential inclusion becomes[9]

$$\forall\, x,\ \exists\, u \in F(x,h(x)) \ \text{ such that }\ 0 \in Dh(x)(u)$$

The largest closed energy map $h_\star$ contained in h is necessarily the restriction of h to a closed subset $K_\star$ of the domain of h. Therefore, for all $y \in \mathrm{Im}(h)$, $K_\star \cap h^{-1}(y)$ *is the viability kernel of* $h^{-1}(y)$. The restriction of the differential inclusion $x'(t) \in F(x(t), y)$ to the viability kernel of $h^{-1}(y)$ is (almost) what Byrnes and Isidori call *zero dynamics of F* (in the framework of smooth nonlinear control systems). See [9,16,17,18] for instance.

[9]When $F(x) := f(x, U(x))$ is derived from a control problem, it is the "contingent version" of the Hamilton-Jacobi equation. See [26,27,28] the forthcoming monograph [29] for its exhaustive study and the connections with the *viscosity solutions*.

2 Codifferentials

A set-valued map whose graph is a closed cone is called a *closed process*. It is a *closed convex process* if its graph is furthermore convex. Closed convex processes enjoy most of the properties of continuous linear operators, as it is shown in [11, Chapter 2]. The transpose of a closed process $A : X \rightsquigarrow Y$ is the closed convex process $A^\star : Y^\star \rightsquigarrow X^\star$ defined by

$$p \in A^\star(q) \text{ if and only if } \forall\,(x,y) \in \text{Graph}(A),\ \langle p,x\rangle \leq \langle q,y\rangle$$

We define in a symmetric way the *bitranspose* $A^{\star\star} : X \rightsquigarrow Y$ of A, the graph of which is the closed convex cone spanned by the graph of A:

$$\text{Graph}(A) = (\text{Graph}(A))^{--}$$

Definition 2.1 *Let $H : X \rightsquigarrow Y$ be a set-valued map and (x,y) belong to its graph. We shall say that the transpose $DH(x,y)^\star : Y^\star \rightsquigarrow X^\star$ of the contingent derivative $DH(x,y)$ is the* codifferential of H at (x,y). *When $H := h$ is single-valued, we set $Dh(x)^\star := Dh(x,h(x))^\star$.*

Before proceeding further, we need more informations about *transposes of the contingent derivatives* of set-valued and single-valued maps which are involved in the formulation of the variational principle and the proof of the existence theorem.

We recall that whenever h is Lipschitz around x, $Dh(x)(u) \neq \emptyset$ for every $u \in X$ (See [11, Proposition 5.1.4]).

Lemma 2.2 *Let X and Z be finite dimensional vector-spaces, $K \subset X$ and $h : K \mapsto Z$ be a single-valued map Lipschitz around $x \in K$. Then $p \in Dh(x)^\star(q)$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$(15)\forall\, y \in B(x,\delta) \cap K,\ < p, y-x > - < q, h(y)-h(x) > \leq \varepsilon\|x-y\|$$

Proof — The sufficient condition being straightforward, let us prove the necessary one. Assume the contrary: there exists $\varepsilon > 0$ and a sequence of elements $x_n \in K$ converging to x such that

$$< p, x_n - x > - < q, h(x_n) - h(x) > > \varepsilon\|x - x_n\|$$

We set $\varepsilon_n := \|x_n - x\|$ which converges to 0 and $u_n := (x_n - x)/\varepsilon_n$, a subsequence of which converges to some u of the unit sphere. Since h is Lipschitz around x, there exists a cluster point $v \in Dh(x)(u)$ of the sequence

$$(h(x + \varepsilon_n u_n) - h(x))/\varepsilon_n$$

We thus deduce that both

$$< p, u > - < q, v > \leq 0 \ \ \& \ \ < p, u > - < q, v > \geq \varepsilon$$

hold true, i.e., a contradiction. □

We recall the the *contingent epiderivative* of an extended function $V : X \mapsto \mathbf{R} \cup \{+\infty\}$ at a point x of its domain is defined by

$$D_\uparrow V(x)(u) := \liminf_{h \to 0+,\, u' \to u} \frac{V(x + hu') - V(x)}{h}$$

so that its epigraph coincides with the contingent cone to the epigraph of V at $(x, V(x))$. (See [11, Chapter 6] for more details on this topic).

The following result characterizes the *transpose of the contingent derivative* of a map H in terms of the contingent epiderivatives of its support function:

Proposition 2.3 *Assume that $H : X \rightsquigarrow Y$ has compact convex values. We associate with any $q \in Y^\star$ the functions $H_q^\flat : X \mapsto \mathbf{R}_+$ and $H_q^\sharp : X \mapsto \mathbf{R}_+$ defined by*

$$\forall\, x \in X,\ \ H_q^\flat(x) := \inf_{y \in H(x)} \langle q, y\rangle \ \ \& \ \ H_q^\sharp(x) := \sup_{y \in H(x)} \langle q, y\rangle$$

Let $y_q^\flat \in H(x)$ satisfy $\langle q, y\rangle = H_q^\flat(x)$ and $y_q^\sharp \in H(x)$ satisfy $\langle q, y\rangle = H_q^\sharp(x)$. Then

$$\left\{p \in X^\star \mid \forall\, u \in X,\ \langle p, u\rangle \leq D_\uparrow H_q^\flat(x)(u) \right\} \subset DH(x, y_q^\flat)^\star(q)$$

If H is Lipschitz at x (in the sense that there exists $l > 0$ such that $H(x) \subset H(y) + l\|x - y\|B$ for every y in a neighborhood of x), then

$$DH(x, y_q^\sharp)^\star(q) \subset \left\{p \in X^\star \mid \forall\, u \in X,\ \langle p, u\rangle \leq D_\uparrow H_q^\sharp(x)(u) \right\}$$

Consequently, when $H = h$ is single-valued and Lipschitz at x, we set $h_q^\star(x) := \langle q, h(x)\rangle = h_q^\sharp(x) = h_q^\flat(x)$ and we obtain the equality

$$Dh(x)^\star(q) = \left\{p \in X^\star \mid \forall\, u \in X,\ \langle p, u\rangle \leq D_\uparrow h_q^\star(x)(u) \right\}$$

Proof — Assume first that $p \in X^\star$ satisfies

$$\forall\, u \in X,\ \langle p, u\rangle \ \le\ D_\uparrow H_q^\flat(x)(u)$$

We prove that for every $v \in DH\left(x, y_q^\flat\right)(u)$,

$$D_\uparrow H_q^\flat(x)(u) \ \le\ \langle q, v\rangle$$

Indeed, by definition of the contingent derivative, there exist sequences $t_n > 0$, $u_n \in X$ and $v_n \in Y$ converging to 0, u and v respectively such that

$$\forall\, n \ge 0,\ \ y + t_n v_n \ \in\ H(x + t_n u_n)$$

Therefore,

$$\left\{\begin{array}{l} D_\uparrow H_q^\flat(x)(u) \\ \le\ \liminf_{n\to\infty} \frac{H_q^\flat(x+t_n u_n) - H_q^\flat(x)}{t_n} \\ \le\ \liminf_{n\to\infty} \langle q, v_n\rangle \ =\ \langle q, v\rangle \end{array}\right.$$

Consequently, $\langle p, u\rangle \le \langle q, v\rangle$ for every $(u, v) \in \mathrm{Graph}(DH(x, y_q^\flat))$, so that $p \in DH(x, y_q^\flat)^\star(q)$.

Conversely, assume that H is Lipschitz at x, $p \in DH\left(x, y_q^\sharp\right)^\star(q)$ and fix $u \in X$. By definition of the contingent epiderivative, there exist sequences $t_n > 0$ and u_n converging to 0 and u such that

$$D_\uparrow H_q^\sharp(x)(u) \ =\ \lim_{n\to\infty} \frac{H_q^\sharp(x + t_n u_n) - H_q^\sharp(x)}{t_n}$$

Since H is Lipschitz at x, there exists $l > 0$ such that, for n large enough, $y_q^\sharp$ belongs to $H(x + t_n u_n) + l t_n \|u_n\| B$, so that it can be written $y_q^\sharp = y_n - t_n v_n$ where $y_n \in H(x + t_n u_n)$ and $\|v_n\| \le l\|u_n\|$. Therefore a subsequence (again denoted by) v_n converges to some v, which belongs to $DH(x, y_q^\sharp)(u)$. Since $\langle q, y_n\rangle \le H_q^\sharp(x + t_n u_n)$ and $\left\langle q, y_q^\sharp\right\rangle = H_q^\sharp(x)$, we infer that

$$D_\uparrow H_q^\sharp(x)(u) \ \ge\ \langle q, v\rangle \ \ge\ \langle p, u\rangle$$

because $v \in DH(x, y_q^\sharp)(u)$ and $p \in DH(x, y_q^\sharp)^\star(q)$.

Remark — Furthermore, when h is real-valued, we need only to know the values of $Dh(x)^\star$ at the points 0, +1 and −1 to reconstruct the whole set-valued map $Dh(x)^\star$.

We observe that for $q = +1$, $D_\uparrow h_q^\star(x)(u) = D_\uparrow h(x)(u)$ and that for $q = -1$, $D_\uparrow h_q^\star(x)(u) = D_\uparrow(-h)(x)(u)$ and that for $q = 0$, $Dh(x)^\star(0) = (\mathrm{Dom}(Dh(x)))^-$.

We recall (see [11, Definition 6.4.7] and [11, Proposition 6.4.8]) that:

$$\begin{cases} \{ p \in X^\star \mid \forall\, u \in X,\ \langle p,u\rangle \leq D_\uparrow h(x)(u) \} = \partial_- h(x) \\ \quad \text{is the } \textit{local subdifferential} \\ \text{and} \\ \{ p \in X^\star \mid \forall\, u \in X,\ \langle p,u\rangle \leq D_\uparrow(-h)(x)(u) \} = -\partial_+ h(x) \\ \quad \text{is the } \textit{local superdifferential} \end{cases}$$

The above characterization then becomes

Proposition 2.4 *Let $h : X \mapsto \mathbf{R}$ be a function continuous at x. Then*

$$Dh(x)^\star(+1) = \partial_- h(x) \ \&\ Dh(x)^\star(-1) = -\partial_+ h(x)$$

Proof — We already know that

$$\partial_- h(x) \subset Dh(x)^\star(+1)$$

Assume now that $p \in Dh(x)^\star(+1)$. We have to show that for every $u \in X$, $\langle p,u\rangle \leq D_\uparrow h(x)(u)$.

There is nothing to prove if $D_\uparrow h(x)(u) = +\infty$.

If $D_\uparrow h(x)(u)$ is finite, then $v := D_\uparrow h(x)(u)$ belongs to $Dh(x)(u)$ by [11, Proposition 6.1.5], so that $\langle p,u\rangle \leq D_\uparrow h(x)(u)$.

We finally claim that the continuity of h at x implies that $D_\uparrow h(x)(u) > -\infty$ for any $u \in X$, which is equivalent to $D_\uparrow h(x)(0) = 0$ thanks to [11, Propositions 6.1.3].

If not, by [11, Propositions 6.1.4 and Lemma 6.1.1], the pair $(0,-1)$ belongs to the contingent cone to the epigraph of h at $(x, h(x))$. Then there exist sequences $t_n > 0$ converging to 0, u_n converging to 0 and a sequence of $v_n > 0$ going to 1 such that

$$\forall\, n \geq 0,\ \ h(x + t_n u_n) \leq h(x) - t_n v_n$$

On the other hand, h being continuous at x, the continuous function φ defined by $\varphi(t) := h(x + tu_n)$ satisfies

$$\varphi(t_n) \leq h(x) - t_n v_n \leq \varphi(0)$$

and therefore, there exists $s_n \in [0, t_n]$ such that $\varphi(s_n) = h(x) - t_n v_n$. Setting $\tilde{u}_n := \frac{s_n}{t_n} u_n$, which also converges to 0, we observe that $h(x + t_n \tilde{u}_n) = h(x) - t_n v_n$. This means that

$$(0, -1) \in Dh(x)(0)$$

But $p \in Dh(x)^\star(1)$, and thus, we then obtain the contradiction

$$0 = \langle p, 0 \rangle \leq \langle 1, -1 \rangle = -1 \ \Box$$

Remark — The above proposition allows to reformulate the notion of viscosity solution of a scalar Hamilton-Jacobi equation $\Psi(x, h'(x)) = 0$ in the following way: h is a viscosity solution if and only if

$$(16) \qquad \begin{cases} i) & \forall\, p \in Dh(x)^\star(+1), \ \ \Psi(x, p) \geq 0 \\ ii) & \forall\, p \in Dh(x)^\star(-1), \ \ \Psi(x, -p) \leq 0 \ \Box \end{cases}$$

The variational principle of section 3 is based on the following convergence result:

Proposition 2.5 *Let X, Y be finite dimensional vector-spaces and $K \subset X$ be a closed subset. Assume that h is the pointwise limit of an equicontinuous family of maps $h_n : K \mapsto Y$. Let $x \in K$ and $p \in Dh(x)^\star(q)$ be fixed. Then there exist subsequences of elements $x_{n_k} \in K$ converging to x, q_{n_k} converging to q and $p_{n_k} \in Dh_{n_k}(x_{n_k})^\star(q_{n_k})$ converging to p.*

If the functions h_n are differentiable, we deduce that there exist subsequences of elements $x_{n_k} \in K$ converging to x and q_{n_k} converging to q such that $h'_{n_k}(x_{n_k})^\star(q_{n_k})$ converges to p.

Proof — We can reformulate the statement in the following way: we observe that $p \in Dh(x)^\star(q)$ if and only if

$$(p, -q) \in \left(T_{\mathrm{Graph}(h)}(x, h(x))\right)^-$$

so that we have to prove that there exist subsequences $x_{n_k} \in K$ and

$$(p_{n_k}, -q_{n_k}) \in \left(T_{\mathrm{Graph}(h_{n_k})}(x_{n_k}, h_{n_k}(x_{n_k}))\right)^-$$

converging to x and $(p, -q)$ respectively. Therefore the proposition follows from the

Theorem 2.6 *Let us consider a sequence of closed subsets K_n and an element $x \in \mathrm{Liminf}_{n\to\infty} K_n$ (assumed to be nonempty). Set $K^\sharp := \mathrm{Limsup}_{n\to\infty} K_n$.*

Then, for any $p \in (T_{K^\sharp}(x))^-$, there exist subsequences of elements $x_{n_k} \in K_{n_k}$ and $p_{n_k} \in \left(T_{K_{n_k}}(x_{n_k})\right)^-$ converging to p and x respectively:

$$(T_{K^\sharp}(x))^- \subset \mathrm{Limsup}_{n\to\infty,\, x_n \to_{K_n} x} (T_{K_n}(x_n))^-$$

Proof — First, it is sufficient to consider the case when x belongs to the intersection $\bigcap_{n=1}^{\infty} K_n$ of the subsets K_n. If not, we set $\widehat{K_n} := K_n + x - u_n$ where $u_n \in K_n$ converges to x. We observe that $x \in \bigcap_{n=1}^{\infty} \widehat{K_n}$ and that $T_{\widehat{K}_n}(x_n) = T_{K_n}(x_n - x + u_n)$.

Let $p \in (T_{K^\sharp}(x))^-$ be given with norm 1. We associate with any positive λ the projection x_n^λ of $x + \lambda p$ onto K_n:

$$\left\|x + \lambda p - x_n^\lambda\right\| = \min_{x_n \in K_n} \|x + \lambda p - x_n\| \tag{17}$$

and set

$$v_n^\lambda := \frac{x_n^\lambda - x}{\lambda} \quad \& \quad p_n^\lambda := p - v_n^\lambda \in \left(T_{K_n}(x_n^\lambda)\right)^-$$

because $x + \lambda p - x_n^\lambda = \lambda(p - v_n^\lambda)$ belongs to the polar cone $\left(T_{K_n}(x_n^\lambda)\right)^-$ to the contingent cone $T_{K_n}(x_n^\lambda)$.

Let us fix for the time $\lambda > 0$. By taking $x_n = x \in K_n$ in (17), we infer that $\|v_n^\lambda\| \leq 2$. Therefore, the sequences x_n^λ and v_n^λ being bounded, some subsequences $x_{n'}^\lambda$ and $v_{n'}^\lambda$ converge to elements $x^\lambda \in K^\sharp$ and $v^\lambda = \frac{x^\lambda - x}{\lambda}$ respectively.

Furthermore, there exists a sequence $\lambda_k \to 0+$ such that v^{λ_k} converge to some $v \in T_{K^\sharp}(x)$ because $\|v^\lambda\| \leq 2$ and because for every λ,

$$x^\lambda = x + \lambda v^\lambda \in K^\sharp$$

Therefore $\langle p, v\rangle \leq 0$ since $p \in (T_{K^\mathfrak{l}}(x))^-$.

On the other hand, we deduce from (17) the inequalities

$$\left\|p - v_n^\lambda\right\|^2 \;=\; \|p\|^2 + \|v_n^\lambda\|^2 - 2\langle p, v_n^\lambda\rangle \;\leq\; \|p\|^2$$

which imply, by passing to the limit, that $\|v\|^2 \leq 2\langle p, v\rangle \leq 0$.

We have proved that a subsequence v^{λ_k} converges to 0, and thus, that a subsequence $v_{n_k}^{\lambda_k} = p - p_{n_k}^{\lambda_k}$ converges also to 0. The lemma ensues. □

We shall need stronger convergence results, where in the conclusion of Proposition 2.5 we require that q_n and/or x_n remain constant. We have to pay some price for that: stronger convergence assumptions and the use of graphical derivatives $D_\flat h(x)$ contained in the graph of $Dh(x)$ *which are closed convex processes*. For instance, the *circatangent derivative* $Ch(x)$, defined in the following way from the Clarke tangent cone:

$$\text{Graph}(Ch(x)) \;:=\; C_{\text{Graph}(h)}(x, h(x))$$

is a closed convex process contained in the contingent derivative $Dh(x)$. They coincide whenever h is sleek at x. We can also use the *asymptotic derivative* $D_\infty h(x)$, whose graph is the *asymptotic cone* to the graph of h at $(x, h(x))$. (See [11, Chapters 4,5] for further details.)

We prove for instance the following

Proposition 2.7 *Let X be a finite dimensional vector-space and $K \subset X$ be a closed subset. Assume that h is Lipschitz around x on K and consider a sequence of continuous maps h_n converging to h uniformly on compact subsets of K. Let $x \in K$ and $p \in Dh(x)^\star(q)$ be fixed. Then there exist a sequence of elements $x_n \in K$ converging to x and a sequence of elements $p_n \in D_\flat h_n(x_n)^\star(q)$ converging to p.*

If the functions h_n are differentiable, we infer that there exists a sequence of elements $x_n \in K$ converging to x such that $h_n'(x_n)^\star(q)$ converges to p.

Proof — Let $\mu > 0$, $L := K \cap B(x, \mu)$ be a compact neighbourhood of x on which the maps h_n converge uniformly to h. We apply Ekeland's Theorem to the functions $y \mapsto < q, h_n(y) > - < p, y >$ defined on this subset. Fix $\varepsilon \in]0, \mu[$. Then there exists $x_n \in L$ satisfying

$$\begin{cases} i) & < q, h_n(x_n) > - < p, x_n > + \varepsilon\|x_n - x\| \ \leq < q, h_n(x) > - < p, x > \\ ii) & \forall\, y \in L, \ < q, h_n(x_n) > - < p, x_n > \\ & \leq < q, h_n(y) > - < p, y > + \varepsilon\|y - x_n\| \end{cases}$$

The first inequality implies that

$$\begin{cases} \varepsilon\|x - x_n\| \ \leq < q, h_n(x) - h(x) > + < q, h(x_n) - h_n(x_n) > \\ + < p, x_n - x > - < q, h(x_n) - h(x) > \\ \leq 2\|q\|\|h_n - h\| + < p, x_n - x > - < q, h(x_n) - h(x) > \end{cases}$$

By Lemma 2.2, there exists $0 < \delta \leq \mu$ such that

$$\forall\, y \in B_K(x, \delta), \ < p, x_n - x > - < q, h(x_n) - h(x) > \leq \ \varepsilon\|x_n - x\|/2$$

Hence, $\|x_n - x\| \leq 4\|q\|\|h_n - h\|/\varepsilon < \mu$ for n large enough.

On the other hand, consider any $v \in Dh_n(x_n)(u)$: There exist $\varepsilon_p > 0$ converging to 0, u_p converging to u and v_p converging to v such that $h_n(x_n + \varepsilon_p u_p) = h_n(x_n) + \varepsilon_p v_p$ for all p. Taking $y = x_n + \varepsilon_p u_p \in K \cap B(x, \mu)$ for p large enough in the second inequality, we infer that

$$0 \ \leq \ \langle q, v_p\rangle - \langle p, u_p\rangle + \varepsilon\|u_p\|$$

and thus, by letting u_p and v_p converge to u and v,

$$\forall\, (u, v) \in \mathrm{Graph}(Dh_n(x)), \ 0 \ \leq \ \langle q, v\rangle - \langle p, u\rangle + \varepsilon\|u\|$$

In particular, taking the restriction to $\mathrm{Graph}(D_\delta h_n(x_n))$ and noticing that $\|u\| = \sup_{e \in B_\star}\langle u, e\rangle)$, this inequality can be written in the form:

$$0 \leq \inf_{(u,v) \in \mathrm{Graph}(D_\delta h_n(x))} \sup_{e \in B_\star}(\langle q, v\rangle - \langle p, u\rangle + \varepsilon\langle e, u\rangle)$$

Since $B_\star$ is convex compact and since the graph of $D_\delta h_n(x)$ is convex, the lop-sided minimax theorem (see for instance [10]) implies the existence of $e_0 \in B_\star$ such that

$$0 \leq \inf_{(u,v) \in \mathrm{Graph}(D_\delta h_n(x))} (\langle q, v\rangle - \langle p, u\rangle + \varepsilon\langle e_0, u\rangle)$$

Consequently, $(p - \varepsilon e_0, -q)$ belongs to the polar cone to Graph$(D_\delta h_n(x_n))$, so that $p_n := p - \varepsilon e_0 \in D_\delta h_n(x_n)^\star(q)$. Summarizing, for any $\varepsilon > 0$ and for any n such that $\|h_n - h\| \leq \varepsilon^2/4\|q\|$, we have proved the existence of $x_n \in K$ and $p_n \in D_\delta h_n(x_n)^\star(q)$ such that

$$\|x_n - x\| \leq \varepsilon \ \ \& \ \ \|p_n - p\| \leq \varepsilon \quad \square$$

Let $K \subset X$ be a closed subset and $\mathcal{C}_\Lambda(K, Z)$ denote the space of Lipschitz (single-valued) bounded maps from K to a finite dimensional vector-space Z,

$$\|h\|_\Lambda := \sup_{x \neq y} \frac{\|h(x) - h(y)\|}{\|x - y\|} \ \ \& \ \ \|h\|_\infty := \sup_{x \in K} \|h(x)\|$$

denote the Lipschitz semi-norm and the sup-norm. Set

$$\|h\|_1 \ = \ \|h\|_\Lambda + \|h\|_\infty$$

It denotes the norm of the Banach space $\mathcal{C}_\Lambda(K, Z)$.

We observe the following continuous properties of the contingent derivative:

Lemma 2.8 *Let $x \in K$ be fixed. Then the map*

$$(h, u) \in \mathcal{C}_\Lambda(K, Y) \times X \ \rightsquigarrow \ Dh(x)(u)$$

is Lipschitz:

$$\forall\, h,\ g\ \in \mathcal{C}_\Lambda(K, Y),\ \ Dh(x)(u) \subset Dg(x)(v) + \|h - g\|_\Lambda \|u\| + \|g\|_\Lambda \|u - v\|$$

Proof — The proof is straightforward from the inequality

$$\left\| \frac{h(x + tu) - h(x)}{t} - \frac{g(x + tv) - g(x)}{t} \right\| \ \leq \ \|h - g\|_\Lambda \|u\| + \|g\|_\Lambda \|u - v\| \quad \square$$

We shall need the following stronger statement than Proposition 2.7:

Proposition 2.9 *Let X be a finite dimensional vector-space and $K \subset X$ be a closed subset. Assume that h is Lipschitz and consider a sequence of Lipschitz maps h_n converging to h in $\mathcal{C}_\Lambda(K, Y)$. Let $x \in K$ and $p \in Dh(x)^\star(q)$ be fixed. Then there exists a sequence of elements $p_n \in D_\delta h_n(x)^\star(q)$ converging to p.*

In particular, if the maps h_n are differentiable, we infer that $h_n'(x)^\star q$ converges to p.

Proof — Set $\varepsilon_n := 2\|q\|\|h_n - h\|_\Lambda$. By Lemma 2.2, there exist $\mu > 0$ such that

$$< p, y - x > - < q, h(y) - h(x) > \leq \varepsilon_n \|y - x\|/2$$

whenever $y \in K \cap B(x, \mu)$. Therefore

$$\begin{cases} < p, y - x > - < q, h_n(y) - h_n(x) > \\ \leq \ \varepsilon_n \|y - x\|/2 + \|q\|\|(h_n - h)(y) - (h_n - h)(x)\| \\ \leq (\varepsilon_n/2 + \|q\|\|h_n - h\|_\Lambda)\|y - x\| \ \leq \ \varepsilon_n \|y - x\| \end{cases}$$

On the other hand, consider any $v \in Dh_n(x)(u)$: There exist $t_p > 0$ converging to 0, u_p converging to u and v_p converging to v such that $h_n(x + t_p u_p) = h_n(x) + t_p v_p$ for all p. Taking $y = x + t_p u_p \in K \cap B(x, \mu)$ for n large enough and observing that $h_n(y) = h_n(x) + t_p v_p$ in the second inequality, we infer that

$$0 \ \leq < q, v_p > - < p, u_p > + \varepsilon_n \|u_p\|$$

and thus, by letting u_p and v_p converge to u and v,

$$\forall\, (u, v) \in \text{Graph}(Dh_n(x)), \ \ 0 \ \leq < q, v > - < p, u > + \varepsilon_n \|u\|$$

In particular, taking the restriction to Graph$(D_\delta h_n(x))$ which is convex, the lop-sided minimax theorem implies that inequality

$$0 \leq \inf_{(u,v) \in \text{Graph}(D_\delta h_n(x))} \sup_{e \in B_\star} (< q, v > - < p, u > + \varepsilon_n < e, u >)$$

provides the existence of $e_n \in B_\star$ such that $(p - \varepsilon_n e_n, -q)$ belongs to the negative polar cone to Graph$(D_\delta h_n(x))$, i.e., such that

$$p_n := p - \varepsilon_n e_n \ \in \ D_\delta h_n(x)^\star(q) \quad \square$$

3 The Variational Principle

We characterize in this section solutions to the contingent differential inclusion (6) through a *variational principle*. For that purpose, we denote by

$$\sigma(M, p) \ := \ \sup_{z \in M} < p, z > \ \ \& \ \ \sigma^\flat(M, p) \ := \ \inf_{z \in M} < p, z >$$

the support functions of $M \subset X$ and by $B_\star$ the unit ball of $Y^\star$.

Consider a closed subset $K \subset X$. We introduce the nonnegative functional Φ defined on the space $\mathcal{C}(K,Y)$ of continuous maps by

$$\Phi(h) := \sup_{q\in B_\star} \sup_{x\in K} \sup_{p\in Dh(x)^\star(q)} \left(\sigma^\flat(F(x,h(x)),p) - \sigma(G(x,h(x)),q)\right)$$

Theorem 3.1 (Variational Principle) *Assume that the set-valued maps F and G are upper semicontinuous with convex and compact values. Let $c > 0$. Then a single-valued map $h : K \mapsto Y$ is a solution to the contingent differential inclusion*

$$\forall\, x \in K,\ \ 0 \in Dh(x)(F(x,h(x))) - G(x,h(x)) + cB$$

if and only if $\Phi(h) \leq c$.

Consequently, h is a solution to the contingent differential inclusion (6) if and only if $\Phi(h) = 0$.

Proof— The first inclusion is easy: let $u \in F(x,h(x))$, $v \in G(x,h(x))$ and $e \in cB$ be such that $v - e \in Dh(x)(u)$. Then, for any $q \in B_\star$ and $p \in Dh(x)^\star(q)$, we know that

$$< p,u > - < q, v - e > \leq\ 0$$

so that

$$\sigma^\flat(F(x,h(x)),p) - \sigma(G(x,h(x)),q)\ \leq < p,u > - < q,v > \leq < q,e > \leq\ c$$

By taking the supremum with respect $x \in K$, $q \in B_\star$ and $p \in Dh(x)^\star(q)$, we infer that $\Phi(h) \leq c$.

Conversely, we can write inequality $\Phi(h) \leq c$ in the form of the minimax inequality: for any $x \in K$, $q \in Y^\star$,

$$\sup_{p\in Dh(x)^\star(q)} \inf_{u\in F(x,h(x))} \inf_{v\in G(x,h(x))} (< p,u > - < q,v >) \leq c\|q\|$$

Noticing that $c\|q\| = \sigma(cB,q)$ and setting

$$\beta(p,q;u,v,e) := < p,u > - < q, v - e >$$

this inequality can be written in the form: for every $x \in K$,

$$\sup_{(p,-q)\in \text{Graph}(Dh(x))^-} \inf_{(u,v,e)\in F(x,h(x))\times G(x,h(x))\times cB} \beta(p,q;u,v,e) \leq 0$$

Since the set $F(x,h(x)) \times G(x,h(x)) \times cB$ is convex compact and since the negative polar cone to the graph of $Dh(x)$ is convex, the lop-sided minimax theorem (see for instance [10]) implies the existence of $u_0 \in F(x,h(x))$, $v_0 \in G(x,h(x))$ and $e_0 \in cB$ such that

$$\sup_{(p,-q)\in \text{Graph}(Dh(x))^-}(< p, u_0 > - < q, v_0 - e_0 >) =$$

$$\sup_{(p,-q)\in \text{Graph}(Dh(x))^-} \inf_{(u,v,e)\in F(x,h(x))\times G(x,h(x))\times cB} \beta(p,q;u,v,e) \leq 0$$

This means that $(u_0, v_0 - e_0)$ belongs to the bipolar of the graph of $Dh(x)$, i.e., its closed convex hull $\overline{co}(\text{Graph}(Dh(x)))$. In other words, we have proved that

$$(F(x,h(x)) \times (G(x,h(x)) + cB)) \cap \overline{co}\left(T_{\text{Graph}(h)}(x,h(x))\right) \neq \emptyset$$

But by Proposition 6.1 of the Appendix, this is equivalent to the condition

$$(F(x,h(x)) \times (G(x,h(x)) + cB)) \cap T_{\text{Graph}(h)}(x,h(x)) \neq \emptyset$$

i.e., h is a solution to the contingent differential inclusion. □

Remark — Since

$$\text{Graph}(Dh(x))^{--} = \text{Graph}(Dh(x)^{\star\star})$$

the graph of the bipolar cone of Graph($Dh(x)$) is the graph of the bitranspose $Dh(x)^{\star\star}$, we have actually proved that h is a solution to the contingent differential inclusion if and only if it is a solution to the "relaxed" contingent differential inclusion

$$0 \in Dh(x)^{\star\star}(F(x,h(x))) - G(x,h(x)) + cB \quad \square$$

Theorem 3.2 *Assume that the set-valued maps F and G are upper semicontinuous with nonempty convex compact images. Let $\mathcal{H} \subset \mathcal{C}(K,Y)$ be a compact subset for the compact convergence topology.*

Assume that $c := \inf_{h\in\mathcal{H}} \Phi(h) < +\infty$. Then there exists a solution $h \in \mathcal{H}$ to the contingent differential inclusion

$$0 \in Dh(x)(F(x,h(x))) - G(x,h(x)) + cB$$

Since $\mathcal{H}$ is a compact subset for the compact convergence topology, it is sufficient to prove that the fonctional Φ is lower semicontinuous on the space $\mathcal{C}(K,Y)$ for this topology: If it is proper (i.e., different from the constant $+\infty$), it achieves its minimum at some $h \in \mathcal{H}$, which is a solution to the above contingent differential inclusion thanks to Theorem 3.1.

Proposition 3.3 *Assume that the set-valued maps F and G are upper semicontinuous with nonempty convex compact images. Then the functional Φ is lower semicontinuous on equicontinuous subsets of the space $\mathcal{C}(K,Y)$ for the compact convergence topology.*

Proof — Assume that Φ is proper. Let h_n be a sequence of Φ satisfying for any n, $\Phi(h_n) \leq c$ and converging to some map h. We have to check that $\Phi(h) \leq c$. Indeed, fix $x \in K$, $q \in B_\star$ and $p \in Dh(x)^\star(q)$. By Proposition 2.5, there exist subsequences (again denoted by) $x_n \in K$ converging to x, q_n converging to q and $p_n \in Dh_n(x_n)^\star(q_n)$ converging to p such that $h_n(x_n)$ converges to $h(x)$.

We can always assume that $\|q_n\| \leq 1$. If not, we replace q_n by $\widehat{q}_n := \frac{\|q\|}{\|q_n\|} q_n$ and p_n by

$$\widehat{p}_n \; := \; \frac{\|q\|}{\|q_n\|} p_n \; \in \; Dh_n(x_n)^\star(\widehat{q}_n)$$

Since F and G are upper semicontinuous with compact values, we know that for any (p,q) and $\varepsilon > 0$, we have

$$\begin{cases} \sigma^\flat(F(x,h(x)),p) - \sigma(G(x,h(x)),q) \\ \leq \sigma^\flat(F(x_n,h_n(x_n)),p_n) - \sigma(G(x_n,h_n(x_n)),q) + \varepsilon \leq \Phi(h_n) + \varepsilon \end{cases}$$

for n large enough. Hence, by letting n go to ∞, we infer that for any $\varepsilon > 0$,

$$\sigma^\flat(F(x,h(x)),p) - \sigma(G(x,h(x)),q) \leq c + \varepsilon$$

Letting ε converge to 0 and taking the supremum on $q \in B_\star$, $x \in K$ and $p \in Dh(x)^\star(q)$, we infer that $\Phi(h) \le c$. □

Remark — In the case when $Y = \mathbf{R}$, the contingent solutions are very closed in spirit to the *viscosity solutions*:

Proposition 3.4 *Assume that $Y = \mathbf{R}$ and that the values of the set-valued maps F and G are convex and compact. Then a continuous function $h : K \mapsto R$ is a solution if and only if for every $x \in K$,*

$$\begin{cases} i) & \sup_{p\in\partial_- h(x)} \left(\sigma^\flat(F(x,h(x)),p) - \sup(G(x,h(x)))\right) \le 0 \\ ii) & \inf_{p\in\partial_+ h(x)} (\sigma(F(x,h(x)),p) - \inf(G(x,h(x)))) \ge 0 \\ iii) & F(x,h(x)) \cap (\mathrm{Dom}(Dh(x)))^{-\,-} \neq \emptyset \end{cases}$$

Remark — When h is locally Lipschitz, then the domain fo the contingent derivative $Dh(x)$ is the whole space anf the third condition is automatically satisfied. □

Proof — Indeed, in the case when $Y = \mathbf{R}$, the functional Φ can be written in the form

$$\Phi(h) = \sup_{x\in K} \max(\Phi_0(h,x), \Phi_+(h,x), \Phi_-(h,x))$$

where

$$\begin{cases} \Phi_+(h,x) &= \sup_{p\in Dh(x)^\star(+1)} \left(\sigma^\flat(F(x,h(x)),p) - \sigma(G(x,h(x)),+1)\right) \\ \Phi_-(h,x) &= \sup_{p\in Dh(x)^\star(-1)} \left(\sigma^\flat(F(x,h(x)),p) - \sigma(G(x,h(x)),-1)\right) \\ \Phi_0(h,x) &= \sup_{p\in Dh(x)^\star(0)} \left(\sigma^\flat(F(x,h(x)),p)\right) \end{cases}$$

The two first properties follow from Proposition 2.4 and Theorem 3.1 with $c = 0$. The last one can be derived from

$$\begin{cases} \sup_{p\in Dh(x)^\star(0)} \left(\sigma^\flat(F(x,h(x)),p)\right) = \sup_{p\in(\mathrm{Dom}(Dh(x)))^-} \inf_{v\in F(x,h(x))} \langle p, v\rangle \\ = \sup_{p\in(\mathrm{Dom}(Dh(x)))^-} \langle p, v_0\rangle \le 0 \end{cases}$$

for some $v_0 \in F(x, h(x))$ thanks to the lop-sided Minimax Theorem. □

Remark — We can relate solutions to the contingent differential inclusion (6) to viscosity solutions when the set-valued map $F : X \rightsquigarrow X$ does not depend on y and when G is equal to 0. The above proposition implies that both h and $-h$ are viscosity subsolutions to the Hamilton-Jacobi equation

$$-\sigma(F(x), h'(x)) = 0$$

The apparent discrepency comes from the fact that solutions h of the contingent partial differential inclusion are energy functions and not the value function of an optimal control problem. □

4 Single-Valued Solutions to Contingent Differential Inclusions

We shall look for solutions in a compact convex subset $\mathcal{H}$ of the space $\mathcal{C}_\Lambda(K, Y)$ of Lipschitz maps from K to Y.

Theorem 4.1 *Let X and Y be two finite dimensional vector-spaces, $F : X \times Y \rightsquigarrow X$, $G : X \times Y \rightsquigarrow Y$ be two upper semicontinuous set-valued maps with compact convex values and K be a closed subset of X.*

Let $\mathcal{H}$ be a compact convex subset of $\mathcal{C}_\Lambda(K, Y)$. When $h \in \mathcal{H}$, we denote by $T_{\mathcal{H}}(h(\cdot)) \subset \mathcal{C}(K, Y)$ the tangent cone to $\mathcal{H}$ at h for the pointwise convergence topology.

Assume that for every $h \in \mathcal{H}$, there exist v, $w \in \mathcal{C}(K, Y)$ such that $\forall x \in K$,

$$w(x) \in D_\delta h(x)(F(x, h(x))),\ v(x) \in G(x, h(x))\ \&\ w(\cdot) - v(\cdot) \in T_{\mathcal{H}}(h(\cdot))$$

Then there exists a solution $h \in \mathcal{H}$ to the contingent differential inclusion:

$$\forall x \in K,\ 0 \in Dh(x)(F(x, h(x))) - G(x, h(x))$$

Proof — We assume that there is no solution to the contingent differential inclusion and we shall derive a contradiction.

Indeed, thanks to Proposition 6.1, this amounts to assume that for any $h \in \mathcal{H}$, there exists $x \in K$ such that

$$0 \notin \overline{co}\left(T_{\mathrm{Graph}(h)}(x, h(x))\right) - F(x, h(x)) \times G(x, h(x))$$

Since the images of F and G are compact and convex, the Separation Theorem implies that there exists also $(p, -q) \in X^\star \times Y^\star$ such that

$$0 \ < \ \sigma^\flat(F(x, h(x)), p) - \sigma(G(x, h(x)), q) \ \ \& \ \ p \in Dh(x)^\star(q)$$

Set

$$a(h; x, q) := \sup_{p \in D_\delta h(x)^\star(q)} \sigma^\flat(F(x, h(x)), p) - \sigma(G(x, h(x)), q)$$

Since $Dh(x)^\star(q) \subset D_\delta h(x)^\star(q)$, we observe that

$$0 \ < \ a(h; x, q)$$

On the other hand, the function $(y, p) \mapsto \sigma^\flat(F(x, y), p) - \sigma(G(x, y), q)$ being lower semicontinuous (because F and G are upper semicontinuous with compact values), there exist neighborhoods $N_1(h(x))$ and $N_2(p)$ such that

$$\forall\, y \in N_1(h(x)), \ \forall\, p' \in N_2(p), \ \ 0 \ < \ \sigma^\flat(F(x, y), p') - \sigma(G(x, y), q)$$

By Proposition 2.9, there exists $\eta(x) > 0$ such that whenever $\|l - h\|_\Lambda \leq \eta(x)$, there exists $p' \in D_\delta l(x)^\star(q)$ satisfying $l(x) \in N_1(h(x))$ and $p' \in N_2(p)$. Hence

$$0 \ < \ \sigma^\flat(F(x, l(x)), p') - \sigma(G(x, l(x)), q) \ \leq \ a(l; x, q)$$

Consequently, h belongs to the subset $N(x, q)$ defined by

$$N(x, q) \ := \ \{l \in \mathcal{C}_\Lambda(K, Y) \mid 0 \ < \ a(l; x, q)\}$$

which is *open in* $\mathcal{C}_\Lambda(K, Y)$ by Proposition 2.9.

Summing up, we just have proved that if there is no solution to the contingent differential inclusion, then $\mathcal{H}$ can be covered by the open subsets $N(x, q)$. Being compact, it can be covered by a finite number m of such neighborhoods $N(x_i, q_i)$. Let $\alpha_i(\cdot)$ be a continuous partition of unity associated with this covering.

We introduce now the function $\varphi : \mathcal{H} \times \mathcal{H} \mapsto \mathbf{R}$ defined by

$$\varphi(h,l) \;:=\; \sum_{i=1}^{m} \alpha_i(h)\langle q_i, l(x_i) - h(x_i)\rangle$$

It is continuous with respect to h on $\mathcal{C}_\Lambda(K,Y)$ (because the α_i's are so and $h \mapsto < q_i, h(x_i) >$ are continuous for the pointwise topology), affine with respect to l and satisfies $\varphi(l,l) = 0$. Hence, $\mathcal{H}$ being convex and compact, Ky Fan Inequality (see [11, Theorem 3.1.1]) implies the existence of $\bar{h} \in \mathcal{H}$ such that for every $l \in \mathcal{H}$, $\varphi(\bar{h},l) \leq 0$. This means that the discrete measure $\sum_{i=1}^{m} \alpha_i(\bar{h}) q_i \otimes \delta(x_i)$ belongs to the normal cone to $\mathcal{H}$ at $\bar{h}$ since for every $l \in \mathcal{H}$,

$$\langle \sum_{i=1}^{m} \alpha_i(\bar{h}) q_i \otimes \delta(x_i), l - \bar{h}\rangle \;=\; \sum_{i=1}^{m} \alpha_i(\bar{h})\langle q_i, l(x_i) - \bar{h}(x_i)\rangle \;\leq\; 0$$

We then deduce from the assumption that

$$\sum_{i=1}^{m} \alpha_i(\bar{h}) a(\bar{h}; x_i, q_i) \;\leq\; 0$$

Indeed, there exist continuous functions $v(\cdot)$ and $w(\cdot)$ such that

$$\forall\, x \in K,\;\; v(x) \in G(x, \bar{h}(x)) \;\;\&\;\; w(x) \in D_\delta \bar{h}(x)(F(x, \bar{h}(x)))$$

and $w(\cdot) - v(\cdot) \in T_{\mathcal{H}}(\bar{h}(\cdot))$. Consequently, whenever $\alpha_i(\bar{h}) > 0$, then $\bar{h}$ belongs to $N(x_i, q_i)$. Therefore, for any $p_i \in D_\delta \bar{h}(x_i)^\star(q_i)$, there exists $u_i \in F(x_i, \bar{h}(x_i))$ such that

$$\begin{cases} \sigma^\flat(F(x_i, \bar{h}(x_i)), p_i) - \sigma(G(x_i, \bar{h}(x_i)), q_i) \\ \leq\; \langle p_i, u_i > - < q_i, v(x_i)\rangle \;\leq\; \langle q_i, w(x_i) - v(x_i)\rangle \end{cases}$$

so that, by taking the supremum on $p_i \in D_\delta \bar{h}(x_i)^\star(q_i)$, we obtain

$$a(\bar{h}; x_i, q_i) \;\leq < q_i, w(x_i) - v(x_i) >$$

Multiplying by $\alpha_i(\bar{h}) \geq 0$ and summing from $i = 1$ to m, we obtain

$$\sum_{i=1}^{m} \alpha_i(\bar{h}) a(\bar{h}; x_i, q_i) \;\leq\; \left\langle \sum_{i=1}^{m} \alpha_i(\bar{h}) q_i \otimes \delta(x_i), w(\cdot) - v(\cdot)\right\rangle \;\leq\; 0$$

because $w(\cdot) - v(\cdot)$ belongs to the tangent cone to $\mathcal{H}$ at $\bar{h}$ and

$$\sum_{i=1}^{m} \alpha_i(\bar{h}) q_i \otimes \delta(x_i)$$

belongs to the normal cone to $\mathcal{H}$ at $\bar{h}$.

Now, we claim that

$$0 \; < \; \sum_{i=1}^{m} \alpha_i(\bar{h}) a(\bar{h}; x_i, q_i)$$

Indeed, whenever $\alpha_i(\bar{h}) > 0$, then $\bar{h}$ belongs to $N(x_i, q_i)$, which implies that $0 < a(\bar{h}, x_i, q_i)$. We have therefore obtained a contradiction. □

Lemma 4.2 *Let $H : K \rightsquigarrow Y$ be a set-valued map and let $\mathcal{H}$ be a subset of continuous selections of H. Then*

$$T_{\mathcal{H}}(h(\cdot)) \; \subset \; \{v \in \mathcal{C}(K,Y) \; \mid \; \forall\, x \in K, \;\; v(x) \in T_{H(x)}(h(x))\}$$

Assume that for any finite sequence $(x_i, y_i) \in \mathrm{Graph}(H)$ $(i = 1, \ldots, m)$ such that $x_i \neq x_j$ whenever $i \neq j$, there exists a selection $s \in \mathcal{H}$ interpolating it:

$$\forall\, i = 1, \ldots, m, \;\; s(x_i) = y_i$$

Then equality holds true:

$$T_{\mathcal{H}}(h(\cdot)) \; = \; \{v \in \mathcal{C}(K,Y) \; \mid \; \forall\, x \in K, \;\; v(x) \in T_{H(x)}(h(x))\}$$

Proof — Indeed, let $v \in \mathcal{C}(K,Y)$ such that $v(x) \in T_{H(x)}(h(x))$ for all $x \in K$. Then there exists $\varepsilon_\lambda(x)$ converging to 0 with λ for the pointwise topology such that $h(x) + \lambda v(x) + \lambda \varepsilon_\lambda(x) \in H(x)$. Let us consider any neighbourhood of 0 for the pointwise topology

$$\mathcal{V} := \{l \in \mathcal{C}(K,Y) \; \mid \; \sup_{i=1,\ldots,n} \|l(x_i)\| \leq \varepsilon\}$$

associated with a finite subset $\{x_1, \ldots, x_n\}$ and λ small enough for $\varepsilon_\lambda(\cdot)$ to belong to it. Since by the interpolation assumption there exists $l_\lambda \in \mathcal{H}$ such that

$$\forall\, x_i, \;\; l_\lambda(x_i) = h(x_i) + \lambda v(x_i) + \lambda \varepsilon_\lambda(x_i) \in H(x_i)$$

then the continuous function $u_\lambda := (l_\lambda - h)/\lambda$ is such that $h + \lambda u_\lambda \in \mathcal{H}$ and belongs to the neighbourhood $v + \mathcal{V}$ of v for the topology of the pointwise convergence. In other words, v belongs to the tangent cone to $\mathcal{H}$ at h for the pointwise topology. □

5 Feedback Controls Regulating Smooth Evolutions

Consider a control system (U, f):

$$(18) \quad \begin{cases} i) & \text{for almost all } t, \ x'(t) = f(x(t), u(t)) \\ ii) & \text{where } u(t) \in U(x(t)) \end{cases}$$

Let $(x, u) \to \varphi(x, u)$ be a non negative continuous function with linear growth.

We have proved in [12] that there exists a closed regulation map $R^{\varphi} \subset U$ larger than any closed regulation map $R : K \rightsquigarrow Z$ contained in U and enjoying the following viability property: *For any initial state* $x_0 \in \mathrm{Dom}(R)$ *and any initial control* $u_0 \in R(x_0)$*, there exists a solution* $(x(\cdot), u(\cdot))$ *to the control system (18) starting at* (x_0, u_0) *such that*

$$\forall\, t \geq 0, \ \ u(t) \in R(x(t))$$

and

$$\text{for almost all } \ t \geq 0, \ \|u'(t)\| \leq \varphi(x(t), u(t))$$

Let $K \subset \mathrm{Dom}(U)$ be a closed subset. We also recall that *a closed set-valued map* $R : K \rightsquigarrow Z$ *is a feedback control regulating viable solutions to the control problem satisfying the above growth condition if and only if* R *is a solution to the* contingent differential inclusion

$$\forall\, x \in K, \ \ 0 \in DR(x, u)(f(x, u)) - \varphi(x, u)B$$

satisfying the constraint

$$\forall\, x \in K, \ \ R(x) \subset U(x)$$

In particular, a *closed graph single-valued regulation maps* $r : K \mapsto Z$ is a solution to the *contingent differential inclusion*

$$(19) \quad \forall\, x \in K, \ \ 0 \in Dr(x)(f(x, r(x))) - \varphi(x, r(x))B$$

satisfying the constraint

$$\forall\, x \in K, \ \ r(x) \in U(x)$$

Such solution can be obtained by a variational principle. We introduce the functional Φ defined by

$$\Phi(r) := \sup_{q\in B_\star}\sup_{x\in K}\sup_{p\in Dr(x)^\star(q)}(<p, f(x,r(x))> -\varphi(x,r(x))\|q\|)$$

Theorem 5.1 *Let $\mathcal{R}\subset\mathcal{C}(K,Y)$ be a nonempty compact subset of selections of the set-valued map U (for the compact convergence topology).*

Suppose that the functions f and φ are continuous and that

$$c := \inf_{r\in\mathcal{R}}\Phi(r) < +\infty$$

Then there exists a solution $r(\cdot)$ to the contingent differential inclusion

$$\forall x\in K,\ \ 0\in Dr(x)(f(x,r(x))) - (\varphi(x,r(x))+c)B$$

As for the existence of such a feedback, we deduce from Theorem 4.1 the following consequence:

Theorem 5.2 *Consider a nonempty convex subset $\mathcal{R}\subset\mathcal{C}_\Lambda(K,Z)$ of selections of the set-valued map U which is compact in $\mathcal{C}_\Lambda(K,Z)$.*

Suppose that the functions f and φ are continuous.

Assume that for every $r\in\mathcal{R}$, there exist v, $w\in\mathcal{C}(K,Y)$ such that $\forall x\in K$,

$$w(x)\in D_\delta r(x)(f(x,r(x))),\ v(x)\in\varphi(x,r(x))B\ \&\ w(\cdot)-v(\cdot)\in T_{\mathcal{R}}(r(\cdot))$$

Then there exists a solution to the contingent differential inclusion (19).

6 Appendix: Viability Theorems

We recall here some definitions and the statement of the Viability Theorem. Let $F : X \rightsquigarrow X$ be a set-valued map and $K\subset \mathrm{Dom}(F)$ be a nonempty subset.

The subset K enjoys the *viability property* for the differential inclusion $x'\in F(x)$ if for any initial state $x_0\in K$, there exists a solution starting at x_0 which is viable in K (in the sense that $x(t)\in K$ for all $t\geq 0$.) The viability property is said to be *local* if for any initial state $x_0\in K$, there exist $T_{x_0} > 0$ and a solution starting at x_0 which is viable in K on the interval $[0, T_{x_0}]$ in the sense that for every $t \in [0, T_{x_0}]$, $x(t) \in K$.

We denote by

$$T_K(x) := \left\{ v \in X \mid \liminf_{h \to 0+} \frac{d(x + hv; K)}{h} = 0 \right\}$$

the contingent cone to K at $x \in K$. We say that K is a *viability domain* of F if

$$\forall\, x \in K,\ R(x) := F(x) \cap T_K(x) \neq \emptyset$$

The Viability Theorem states that if F is upper semicontinuous with nonempty compact convex images, then a locally compact set K enjoys the local viability property if and only if it is a viability domain. In this case, if the growth $R := F \cap T_K$ is linear in the sense that for some $c > 0$,

$$\forall\, x \in K,\ \ \|R(x)\| := \inf_{u \in R(x)} \|u\| \leq c(\|x\| + 1)$$

and if K is closed, then K enjoys the viability property.

For simplicity, we say that a set-valued map G is a *Peano map* if it is upper semicontinuous with nonempty compact convex images and with linear growth.

The global Viability Theorem states that when F is a Peano map, the upper limit of closed viability domains K_n is still a viability domain, and that, for any nonempty closed subset $K \subset \mathrm{Dom}(F)$, there exists a largest closed viability domain $\mathrm{Viab}_F(K)$ contained in K, possibly empty, called the *viability kernel* of K. If F_n is a sequence of Peano maps enjoying a uniform linear growth and if K_n is a sequence of closed viability domains of F_n, then the upper limit $K^\sharp$ of the K_n's is a viability domain of $\overline{co}(F^\sharp)$, where $F^\sharp$ denotes the graphical upper limit of the F_n's. (See for instance [4,5] and [11, Chapter 10].)

The following result provides a very useful *duality* characterization of viability domains:

Proposition 6.1 *Assume that the set-valued map $F : K \rightsquigarrow X$ is upper semicontinuous with convex compact values. Then the three following prop-*

erties are equivalent:

$$
(20) \quad \begin{cases} i) & \forall\, x \in K,\ \ F(x) \cap T_K(x) \neq \emptyset \\ ii) & \forall\, x \in K,\ \ F(x) \cap \overline{co}(T_K(x)) \neq \emptyset \\ iii) & \forall\, x \in K,\ \ \forall\, p \in (T_K(x))^-,\ \ \sigma(F(x), -p) \geq 0 \end{cases}
$$

Proof — The equivalence between ii) and iii) follows obviously from the Separation Theorem. The equivalence between i) and ii) has been proved in a different context in [25]. We provide here a simpler proof.

Asume that ii) holds true and fix $x \in K$. Let $u \in F(x)$ and $v \in T_K(x)$ achieve the distance between $F(x)$ and $T_K(x)$:

$$\|u - v\| = \inf_{y \in F(x),\, z \in T_K(x)} \|y - z\|$$

and set $w := \frac{u+v}{2}$. We have to prove that $u = v$. Assume the contrary.

Since v is contingent to K at x, there exist sequences $h_n > 0$ converging to 0 and v_n converging to v such that $x + h_n v_n$ belongs to K for every $n \geq 0$. We also introduce a projection of best approximation

$$x_n \in \Pi_K(x + h_n w) \text{ of } x + h_n w \text{ onto } K \text{ and we set } z_n := \frac{x_n - x}{h_n}$$

so that, by [11, Proposition 4.1.2], we know that

$$w - z_n \in (T_K(x_n))^- = (\overline{co}(T_K(x_n)))^-$$

By assumption ii), there exists an element $y_n \in F(x_n) \cap \overline{co}(T_K(x_n))$. Consequently,

$$(21) \qquad \langle w - z_n, y_n \rangle \leq 0$$

Since x_n converges to x, the upper semicontinuity of F at x implies that for any $\varepsilon > 0$, there exists N_ε such that for $n \geq N_\varepsilon$, y_n belongs to the neighborhood $F(x) + \varepsilon B$, which is compact. Thus a subsequence (again denoted by) y_n converges to some element $y \in F(x)$.

We shall now prove that z_n converges to v. Indeed, inequality

$$\|w - z_n\| = \frac{1}{h_n}\|x + h_n w - x_n\| \leq \frac{1}{h_n}\|x + h_n w - x - h_n v_n\| = \|w - v_n\|$$

implies that the sequence z_n has a cluster point and that every cluster point z of the sequence z_n belongs to $T_K(x)$, because $x + h_n z_n = x_n \in K$ for every $n \geq 0$. Furthermore, every such z satisfies $\|w - z\| \leq \|w - v\|$.

We now observe that v *is the unique best approximation of* w *by elements of* $T_K(x)$. If not, there would exist $p \in T_K(x)$ satisfying either $\|w - p\| < \|w - v\|$ or $p \neq v$ and $\|w - p\| = \|w - v\| = \|w - u\|$. In the latter case, we have $\langle u - w, w - p\rangle < \|u - w\|\|w - p\|$, since the equality holds true only for $p = v$. Each of these conditions together with the estimates

$$\begin{cases} \|u-p\|^2 \;=\; \|u-w\|^2 + \|w-p\|^2 + 2\langle u-w, w-p\rangle \\ \leq\; (\|u-w\| + \|w-p\|)^2 \;\leq\; \|u-v\|^2 \end{cases}$$

imply the strict inequality $\|u - p\| < \|u - v\|$, which is impossible since v is the projection of u onto $T_K(x)$. Hence $z = v$.

Consequently, all the cluster points being equal to v, we conclude that z_n converges to v.

Therefore, we can pass to the limit in inequalities (21) and obtain, observing that $w - v = (u - v)/2$,

$$(22) \qquad \langle u - v, y\rangle \;=\; 2\langle w - v, y\rangle \;\leq\; 0 \text{ where } y \in F(x)$$

Since $F(x)$ is closed and convex and since $u \in F(x)$ is the projection of v onto $F(x)$, we infer that

$$(23) \qquad \langle u - v, u - y\rangle \;\leq\; 0$$

Finally, $T_K(x)$ being a cone and $v \in T_K(x)$ being the projection of u onto this cone, and in particular, onto the half-line $v\mathbf{R}_+$, we deduce that

$$(24) \qquad \langle u - v, v\rangle \;=\; 0$$

Therefore, properties (22, 23, 24) imply that

$$\|u - v\|^2 \;=\langle u - v, -v\rangle + \langle u - v, u - y\rangle + \langle u - v, y\rangle \;\leq\; 0$$

and thus, that $u = v$. □

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