A New Approach to the Regulator Design Problem

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Foreword

The authors introduce a new method for the analytical regulator design problem. The method is based on a theory of weak asymptotic stability for differential inclusions and yields some new algorithmic techniques.

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The most widespread modern approach to the regulator design problem is based on an analytical solution to the optimal control synthesis problem. Acceptable results in this direction have been obtained only for linear control systems in the following cases. The linear time-optimal control problem was solved by Pontryagin et al. [1] with the help of the Pontryagin maximum principle and the linear infinite horizon control problem with a quadratic functional was investigated by Letov [2] by means of calculus of variations. These methods can be used under very restrictive assumptions and give the appointed transient characteristics only. We propose a new approach to the regulator design problem wich is based on the weak asymptotic stability theory for differential inclusions developed by Smirnov [3]. Our method allows the problem to be solved for many controlled system. It presents a possibility for designing regulators with various transient characteristics and for choosing the most suitable one. Moreover the method admits a very nice numerical realization.

The purpose of our paper is to apply the main ideas and results from [3] to the regulator design problem and emphasize the most important parts of a regulator design algorithm which has been developed by the first author of this paper.

1. The regulator design problem. Let us consider the controlled system

$$\dot{x} = f(x,u), \; u \in U \subset R^k. \tag{1}$$

We assume that $f : R^n \times U \to R^n$ is a continuous function differentiable with respect to x and that $|\nabla_x f(x,u)| \leq L$ for all $(x,u) \in R^n \times U$. Let $f(x,U) \subset R^n$ be a convex set for all $x \in R^n$, and let $u_0 \in U$ be such that $f(0,u_0) = 0$. Our aim is to find a map $u : R^n \to U$ defined for some neighbourhood of the origin satisfying the following conditions:

1. $u(0) = u_0$,
2. equilibrium point $x = 0$ of the differential equation

$$\dot{x} = f(x, u(x)) \tag{2}$$

is asymptotically stable.

The function $u(x)$, in general is not continuous, and therefore we define asymptotic stability following Filippov [4]. Let $\phi : R^n \to R^n$ be a bounded function satisfying $\phi(0) = 0$, and let $\Phi : R^n \to R^n$ be the set-valued map defined by

$$\Phi(x) = \bigcap_{\eta>0} \text{clco}\phi(x + \eta B_n),$$

where B_n is a unit ball in R^n centered at the origin, co is a convex hull and cl means closure. Obviously, the set-valued map Φ has nonempty closed convex images and closed graph. The equilibrium point $x = 0$ of the differential equation

$$\dot{x} = \phi(x) \tag{3}$$

is called asymptotically stable if for any $\epsilon > 0$ there exists $\delta > 0$ such that for all $x_0 \in \delta B_n$ every generalized solution to the differential equation (3), i.e. a solution of the differential inclusion

$$\dot{x} \in \Phi(x), \tag{4}$$

with $x(0) = x_0$ exists for $t \in [0, \infty[$ and satisfies the conditions

$$|x(t)| < \epsilon \ \text{ for all } \ t \in [0, \infty[\ \text{ and } \ \lim x(t) = 0 \ \text{ when } \ t \to \infty.$$

Solutions of the differential inclusion (4) are regarded to be locally absolutly continuous functions satisfying (4) almost everywhere.

2. The scheme of regulator design. The method of solving the regulator design problem is as follows. First of all we investigate the "first approximation" of the system (1), i.e. the linear controlled system

$$\dot{x} = Cx + w, \ w \in K, \tag{5}$$

where $C = \nabla_x f(0, u_0)$ is $(n \times n)$ matrix and K is a closed convex cone spanned by the set $f(0, U)$. For controlled system (5) we derive necessary and sufficient conditions guaranteeing the existence of a norm $V(x)$ in R^n and a number $\theta > 0$ satisfying the following condition:

for all $x \in R^n$ there exists a vector $v \in Cx + K$ such that

$$DV(x)(v) + \theta V(x) \leq 0,$$

where $DV(x)(v)$ is a directional derivative of the convex function V.

Then the map $u(x)$ will be defined to make $V(x)$ a Lyapunov function for the differential equation (2). This will imply asymptotic stability of the equilibrium point $x = 0$. The proof of the existence of the function $V(x)$ is constructive and is the base for the numerical regulator design algorithm.

3. Regulator design for the linear controlled system (5). We now consider two linear differential equations

$$\dot{x} = Cx, \tag{6}$$

$$\dot{x} = -C^* x, \tag{7}$$

where C^* is a conjugate matrix. We shall say that $\chi[f]$ is the Lyapunov exponent [5] of a continuous function $f : R \to R^n$ if

$$\chi[f] = -\limsup_{t \to \infty} \frac{1}{t} \ln |f(t)|.$$

We denote by $\mathbf{P}$ the set consisting of all points $x \in R^n$ such that there exists at least one trajectory of (5) starting at x that has a positive Lyapunov exponent. Let Λ be the subspace consisting of all points $x \in R^n$ such that a solution to the equation (6) with the initial condition $x(0) = x$ has positive Lyapunov exponent, let Λ^+ be the subspace consisting of all points $x \in R^n$ such that a solution to (7) with the initial condition $x(0) = x$ has a nonnegative Lyapunov exponent.

It is easy to check that $\mathbf{P}$ is a convex cone and that $\Lambda^\perp = \Lambda^+$.

The next result contains nessessary and sufficient conditions of solvability for the regulator design problem for the linear system (5). We denote by K^* polar cone defined by

$$K^* = \{x^* \in R^n | \langle x, x^* \rangle \geq 0, x \in K\}.$$

Theorem 1. The following conditions are equivalent:

1. $\mathbf{P} = R^n$,
2. the matrix C^* has neither eigenvectors corresponding to nonegative eigenvalues contained in the cone K^* nor proper invariant subspaces contained in the subspace $\Lambda^+ \cap K^* \cap -K^*$,
3. there exist a number $\theta > 0$ and a norm $V(x)$ in R^n such that for all $x \in R^n$ a vector $v \in Cx + K$ satisfying $DV(x)(v) + \theta V(x) \leq 0$ can be found.

We now present the main idea of the proof of implication 2→3 since it is the base of the numerical regulator design algorithm. For the sake of simplicity we shall consider the most important case when the matrix C^* has no proper invariant subspaces contained in the subspace $K^* \cap -K^*$.

For all real λ which are not eigenvalues of the matrix C we define convex cones

$$L_k(\lambda) = -\mathrm{co} \bigcup_{i=0}^{k} (C - \lambda E)^{-i} K, \ k = 0, \ldots, \infty.$$

Let $\sigma^{n+1} \subset R^n$ be a simplex containing the origin as its interior point. We denote by x_i, $i = 0, \ldots, n$ the vertices of the simplex. Using Brauer fixed point theorem one can prove the following result.

Proposition. Assume that the matrix C^* has neither eigenvectors corresponding to nonnegative eigenvalues contained in the cone K^* nor proper invariant subspaces contained in the subspace $K^* \cap -K^*$. Then there exist a number $\lambda < 0$ and vectors

$$y_i \in \bigcup_{k \geq 1} L_k(\lambda), \ b_i \in -K, \ i = 0, \ldots, n$$

satisfying the equality

$$x_i = y_i + b_i.$$

Obviously, for every point y_i there exists a number m_i such that

$$y_i \in L_{m_i}(\lambda), \ i = 0, \ldots, n.$$

From the definition of the cones $L_{m_i}(\lambda_i)$ we derive the existence of a finite set of nonzero vectors $\{y_i^k\}_{k=1}^{m_i}$ which satisfies the following inclusions:

$$
\begin{aligned}
\lambda y_i^1 &\in Cy_i^1 + K, \\
y_i^1 \quad + \lambda y_i^2 &\in Cy_i^2 + K, \\
&\dots \\
y_i^{m_i-1} + \lambda y_i^{m_i} &\in Cy_i^{m_i} + K, \\
y_i^{m_i} &= y_i.
\end{aligned} \tag{8}
$$

Let M be a convex hull of the points

$$b_i,\ i = 0, \dots, n;\ \ z_i^m = (2/|\lambda|)^{m_i - m} y_i^m,\ m = 1, \dots, m_i, i = 0, \dots, n. \tag{9}$$

It is easy to see that the convex compact set M contains the origin as an interior point. We now prove that the Minkowski function of the set M appears to be a desired norm. For all $i = 0, \dots, n$ the vector $v_i^1 = \lambda z_i^1$ belongs to the set $Cz_i^1 + K$ and satisfies the inclusion

$$z_i^1 + |\lambda|^{-1} v_i^1 = 0 \in \mathrm{int} M,\ i = 0, \dots, n.$$

By $\mathrm{int}M$ we denote an interior of M. For all points z_i^m, $m > 1$ we consider vectors $v_i^m = (1/2)|\lambda| z_i^m + \lambda z_i^m$. Obviously, we have inclusions $v_i^m \in Cz_i^m + K$ and

$$z_i^m + |\lambda|^{-1} v_i^m = (1/2) z_i^{m-1} \in \mathrm{int} M.$$

Since $\gamma B_n \subset \sigma^{n+1} (\gamma > 0)$, we conclude that $(\gamma/2) B_n \subset M$. Let $v_i = Cb_i - 4|Cb_i|b_i$. Then $v_i \in Cb_i + K$ and

$$b_i + 1/(4|Cb_i|) v_i \in (1/4) B_n \subset \mathrm{int} M.$$

We shall denote a boundary of the set M by $\mathrm{bd}M$. From the above consideration we obtain the existence of numbers $\epsilon > 0$ and $\delta \in]0, 1[$ satisfying the following condition: for every point $x \in \mathrm{bd}M$ there exists a vector $v \in Cx + K$ such that

$$x + \epsilon v \in \delta M. \tag{10}$$

Let $V(x)$ be the Minkowski function of the set M . If $x \in \mathrm{bd}M$ and a vector $v \in Cx + K$ is chosen according to the condition (10) then

$$DV(x)(v) \le [V(x + \epsilon v) - V(x)]/\epsilon \le -(1 - \delta)/\epsilon.$$

Since functions $V(\cdot)$ and $DV(x)(\cdot)$ are positively homogeneous, for any $x \in R^n$ there exists a vector $v \in Cx + K$ such that

$$DV(x)(v) + \theta V(x) \le 0, \tag{11}$$

where $\theta = (1 - \delta)/\epsilon$.

Inclusion (10) or inequality (11) actually give the solution to the regulator design problem in the case of the linear controlled system (5) (see the next Section).

4. Regulator design for general controlled system (1). In this case we obtain sufficient conditions of solvability for the regulator design problem.

Theorem 2. Assume that the matrix C^* has neither eigenvectors corresponding to non-negative eigenvalues contained in the cone K^* nor proper invariant subspaces contained in the subspace $\Lambda^+ \cap K^* \cap -K^*$. Then there exist a neighborhood Ω of the origin and a map $u : \Omega \to U$ which satisfies the following conditions: $u(0) = u_0$ and the equilibrium point $x = 0$ of the differential equation

$$\dot{x} = f(x, u(x))$$

is asymptotically stable.

The main idea of the proof is as follows. Suppose that numbers $\epsilon > 0$, $\delta \in]0,1[$ and a norm $V(x)$ are such that for any $x \in \text{bd}M$ ($M = \{x | V(x) \leq 1\}$) there exists a vector $v \in Cx + K$ satisfying $x + \epsilon v \in \delta M$ (see the previous Section). By $d(v, F)$ we shall denote a distance between a point v and a set F. For any $v_0 \in Cx_0 + K$ the equality

$$\lim_{\lambda \downarrow 0,\, x \to x_0} \lambda^{-1} d(\lambda v_0,\ f(\lambda x, U)) = 0,$$

is fulfilled. This implies that if $\epsilon > 0$ is sufficiently small then for all x satisfying inequality $V(x) < \epsilon$ there exists $v \in f(x, U)$ such that $x + \epsilon v \in \frac{1+\delta}{2} V(x) M$.

Let us consider the set-valued maps

$$G(x) = \{v \in R^n \mid x + \epsilon v \in \frac{1+\delta}{2} V(x) M\}, \quad H(x) = G(x) \cap f(x, U),$$

defined on the set $\Omega = \{x | V(x) < \epsilon\}$. We take now any single-valued map $\phi(x) \in H(x)$. Since $\phi(0) = 0$ there exists a map $u : \Omega \to U$ satisfying the conditions: $u(0) = u_0$ and $\phi(x) = f(x, u(x))$. It is easy to prove that

$$DV(x)(\phi(x)) + \theta V(x) \leq 0,$$

where $\theta = \frac{1-\delta}{2}\epsilon$. The last inequality means that the equilibriun point $x = 0$ of differential equation (2) is asymptotically stable.

Further details of this theory can be found in [3].

5. The main steps of the numerical algorithm. We now describe the most important steps of the numerical regulator design algorithm. The algorithm consists of two parts: the first one is a design of the Lyapunov function $V(x)$ and the second is choosing of control $u(x)$ for any current state x. For the sake of simplicity we suppose that the set U from (1) is a polyhedron, i.e. $U = \text{co}\{u_1, u_2, \ldots, u_r\}$. First of all we have to find the "first approximation" (5) for controlled system (1) at the equilibrium point $x = 0$. Computation of the matrix C is not difficult. The cone K is spanned by the vectors $f(0, u_k)$, $k = 1, \ldots, r$. For further computations we need to have the cone K as an intersection of halfspaces. To obtain such representation we can exclude the variables ν_k, $k = 1, \ldots, r$ from the system

$$\sum_{k=1}^{r} f(0, u_k)\nu_k = x, \quad x \in R^n,$$

$$\nu_k \geq 0, \ k = 1, \ldots, r$$

by first using the Gauss algorithm for equalities and then "rolling up" the system of linear inequalities following Chernikov [6]. As a result we obtain the cone K in the form

$$K = \{x | Wx \leq 0, \ x \in R^n\}, \tag{12}$$

where W is a matrix.

According to the previous consideration we have to find for any simplex vertex x_i a set of vectors $\{y_i^k\}_{k=1}^{m_i}$ satisfying relations (8). Using representation (12) we rewrite the first and the second inclusions from (8) as

$$W(\lambda E - C)y_i^1 \leq 0,$$

$$Wy_i^1 + W(\lambda E - C)y_i^2 \leq 0. \tag{13}$$

If we add to the system (13) the inequality $Wy_i^2 \leq Wx_i$ and find any admissible solution we then obtain a desired set $\{y_i^k\}_{k=1}^{m_i}$ for $m_i = 2$. If this system is incompatible, we consider a similar system but for $m_i = 3$, etc. After a finite number of steps we obtain a compatible system. To find an admissible solution of the linear inequality system we use the first part of the simplex-method. After computation of the sets $\{y_i^k\}_{k=1}^{m_i}$ for all vertices x_i of the simplex σ^{n+1} we obtain the polyhedron M as a convex hull of the points (9). The set (9) can contain another points besides the vertices of M. We exclude such points using the simplex-method and obtain

$$M = \{x|\ x = \mu_1 z_1 + \cdots + \mu_s z_s, \sum_{i=1}^{s} \mu_i = 1, \mu_i \geq 0, i = i, \ldots, s\}$$

where z_i are the vertices of M.

The value $V(x)$ of the Minkowski function of the polyhedron M at the point x is equal to the optimal value of the functional in the following linear programming problem

$$\mu_1 + \cdots + \mu_s \to \min$$

$$\mu_1 z_1 + \cdots + \mu_s z_s = x,$$

$$\mu_i \geq 0,\ i = 1, \ldots, s.$$

Taking the proof of theorem 2 into account we can choose the control $u(x)$ at the point x from the folloving condition

$$V(x + \alpha f(x, u(x))) = \min_{u \in U} V(x + \alpha f(x, u)).$$

The parameter $\alpha > 0$ is defined experimentally for each given problem. This method of $u(x)$ calculation can be realized as follows. Let $G = \mathrm{co}\{x + \alpha f(x, u_i)|\ i = 1, \ldots, r\}$ (u_i are the vertices of M). The solution $\hat{v}$ of the optimization problem

$$\mathrm{minimize}\{V(v)|v \in G\}$$

can be represented as $\hat{v} = \sum_{i=1}^{r} \hat{\xi}_i (x + \alpha f(x, u_i))$, where $\sum_{i=1}^{r} \hat{\xi}_i = 1,\ \hat{\xi}_i \geq 0,\ i = 1, \ldots, r$. To determine values $\hat{\xi}_i,\ i = 1, \ldots, r$ we consider the linear programming problem

$$\mu_1 + \cdots + \mu_s \to \min$$

$$\mu_1 z_1 + \cdots + \mu_s z_s = \xi_1(x + \alpha f(x, u_1)) + \cdots + \xi_r(x + \alpha f(x, u_r)),$$

$$\xi_1 + \cdots + \xi_r = 1,$$

$$\mu_i \geq 0,\ i = 1, \ldots, s,\ \ \hat{\xi}_i \geq 0,\ i = 1, \ldots, r.$$

Then the control $u(x) \in U$ is calculated from the equation $\hat{v} = (x + \alpha f(x, u(x)))$.

The above algorithms have been tested. We present one example below.

6. Example. The controlled system which describes a motion of an oscillator subjected to a unilateral force

$$\dot{x}_1 = -x_2 + u,$$
$$\dot{x}_2 = x_1, \tag{14}$$
$$0 \leq u \leq 1.$$

is considered. The polyhedron with the vertices (1,0), (0,1),(-1,-1) is taken as σ^3. The parameter λ is chosen -2.0 and -0.5. In the first (second) case the set $\{y_i^k\}_{k=1}^{m_i}$ for the first vertex of the simplex σ^3 is found when $m_1 = 7$ ($m_1 = 4$), for the second vertex when $m_2 = 4$ ($m_2 = 2$) and for the third vertex when $m_3 = 7$ ($m_3 = 5$). As the result 21 (14) points of the set (9) are obtained. Only 8 (5) of them turn out to be the vertices of the polyhedron M. Examples of trajectories for the case $\lambda = -2$ are in Fig. 1 and for $\lambda = -0.5$ in Fig. 2. Numerical integration of the differential equation

$$\dot{x}_1 = -x_2 + u(x_1, x_2),$$
$$\dot{x}_2 = x_1$$

is fulfilled following the Euler scheme with the step $\Delta t = 0.1$. The markers are put on the trajectories after every ten steps of integration.

The parameter λ influences the duration of the transient. If we allow $|\lambda|$ to drop, the duration of the transient grows. With the help of markers we can see that when $\lambda = -2$ (Fig. 1) the regulator is "fast" and when $\lambda = -0.5$ (Fig. 2) the regulator is "slow" but it spends control resources more carefully and gives nonoscillating trajectories.

7. Concluding remarks. In this paper we have discussed a new approach to the regulator design problem. This is a computational approach which uses only linear programming techniques. The algorithm described above contains many parameters. We can dispose of these parameters as needed through some optimization methods or heuristically to obtain desired result. These possibilities allow us to expect that our method will be a useful tool for regulator design.

References

1. L.S. Pontryagin, V.G. Boltyanski, R.V. Gamkrelidze, E.F. Mischenko, *The Mathematical Theory of Optimal Processes*, Wiley Interscience, New-York, 1962.
2. A.M. Letov, Analytical design of regulators, I-III, *Avtomatika i telemekchanika*, **21**(1960) N4-6 (in Russian).
3. G.V. Smirnov, Weak asymptotic stability for differential inclusions, I-II, *Avtomatika i telemekchanika*, **51**(1990), N7, N8 (in Russian).
4. A.F. Filippov, *Differential Equations with Discontinuous Right-hand Side*, Nauka, Moscow, 1985 (in Russian).
5. A.M. Lyapunov, Probléme general de la stabilité du mouvement, 1892. Reprinted in *Annual of Mathematical Studies*, **17**, Princeton University Press, Princeton, N.J. 1949.
6. S.N. Chernikov, *Linear Inequalities*, Nauka, Moscow, 1968 (in Russian).

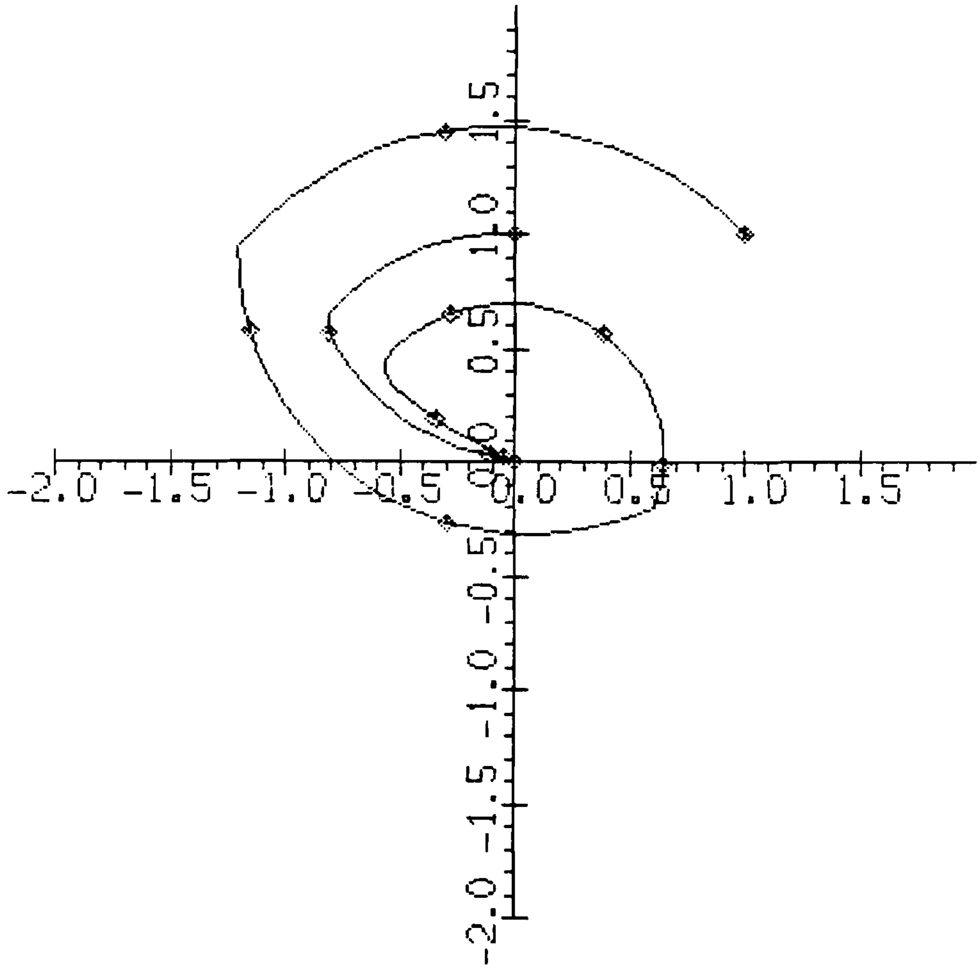

FIG 1

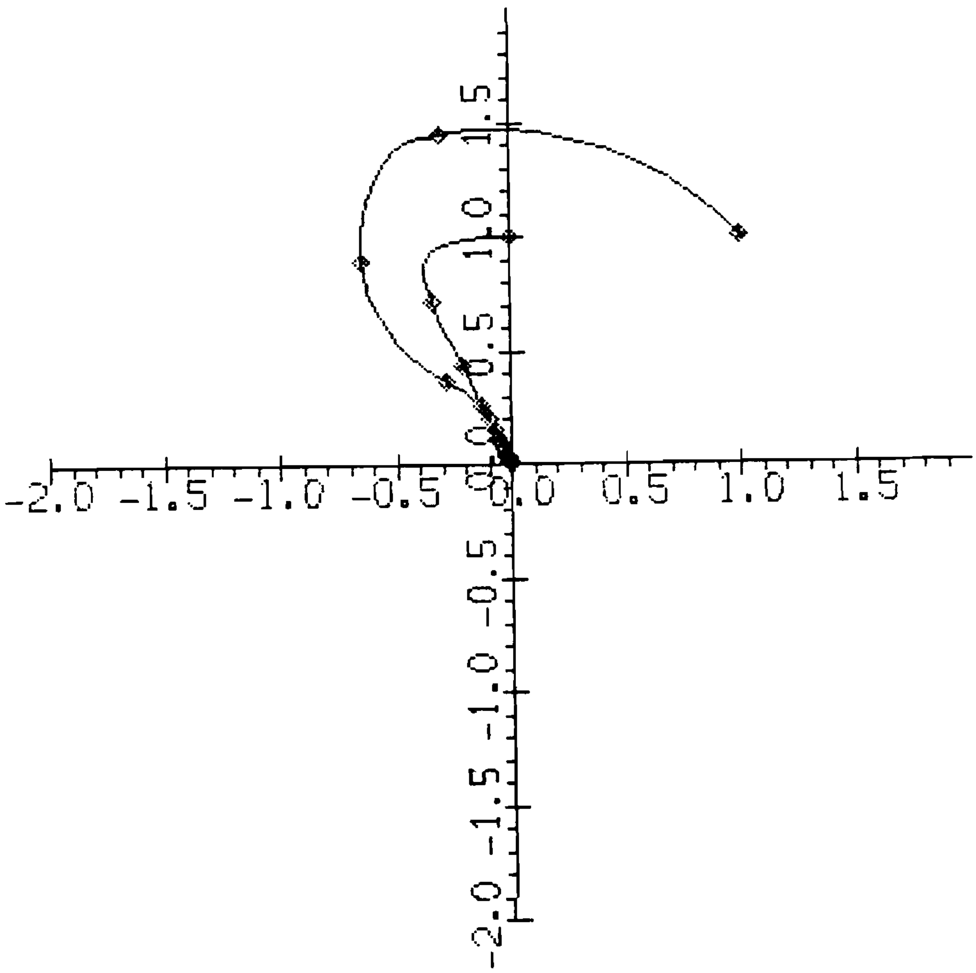

FIG 2