Working Paper

Optimal Moving Sensors for Parabolic Systems

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Foreword

This paper continues the investigations in SDS on observability issues motivated by environmental monitoring and related problems. Here the author introduces a specific class of scanning sensors that ensure solvability of the problem and can further lead to numerically robust techniques.

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Optimal Moving Sensors for Parabolic Systems

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1. Introduction, Statement of Problem.

Let A be the infinitesimal generator of a strongly continuous semigroup $S(t)(t > 0)$ in the Hilbert space $L^2(\Omega)$ of square integrable functions that are defined on an open, bounded domain Ω of an n-dimensional Euclidean space R^n with a (sufficiently smooth) boundary $\partial\Omega$, so that

$$A = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}) - a(x),$$

$$\alpha \sum_{i=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij}\ (x)\xi_i\xi_j,\ \alpha = \text{const}\ > 0,\ a(x) \geq 0,\ \text{for a.e. } x \in \Omega;$$

$$a_{ij}(x) = a_{ji}(x);\quad a_{ij}(\cdot), a(\cdot) \in L^\infty(\Omega),\ i,j = 1,2,\dots n.$$

We consider the following initial - boundary value problem:

$$\frac{\partial u(x,t)}{\partial t} = Au(\cdot,t) + f(x,t)\,, \tag{1.1}$$

$$t \in T = (0,\theta)\,,\ x \in \Omega \subset R^n\,,\ Q = \Omega \times T\,,\ \Sigma = \partial\Omega \times T,$$

$$u(x,0) = u_0(x),\ u(\xi,t) = 0, (\xi,t) \in \Sigma,\quad u_0(\cdot) \in L^2(\Omega),\ f(\cdot,\cdot) \in L^2(Q)$$

with unknown initial condition $u_0(x)$ and forcing term $f(x,t)$.

The solution to the problem (1.1) is treated here as a generalized one [18,12,13] from the Banach space $\overset{o}{V}{}_2^{1,0}(Q)$ consisting of all the elements of the Sobolev space $H_0^{1,0}(Q)$ that are continuous in t in the norm of $L^2(\Omega)$,

$$\| u(\cdot,\cdot) \|_{\overset{o}{V}{}_2^{1,0}(Q)} = \max_{0\leq t\leq\theta} \| u(\cdot,t) \|_{L^2(\Omega)} + \| u(\cdot,\cdot) \|_{H_0^{1,0}(Q)}\,.$$

Here and below we use the standard notation for the Sobolev spaces [13].

Denote by $\hat{x}(t)$, $t \in T_\varepsilon = (\varepsilon,\theta)$ a continuous spatial trajectory in the domain $\bar{\Omega}$ so that

$$\hat{x}(t) \in \bar{\Omega}\,,\ t \in\ T_\varepsilon, \tag{1.2}$$

where ε is given , $0\ < \varepsilon\ < \theta$, the symbol "–" stands for the closure of the corresponding set.

We assume next that *measurement data* are taken along the curve (it might be, in general, a fixed point) $\hat{x}(t),\ t \in T_\varepsilon$ and represent at each instant t the spatial average of the quantity $u(x,t)$ over some sensing region :

$$y(t) = \int_\Omega \chi(x, \hat{x}(t)) u(x,t) dx + \zeta(t),\ t \in T_\varepsilon. \qquad (1.3)$$

Here $y(t),\ t \in T_\varepsilon$ stands for observation data,

$$\chi(x, \hat{x}(t)) = \beta(t) \times \begin{cases} 1, & \text{if } x \in S_h(\hat{x}(t)) \cap \Omega, \\ 0, & \text{if } x \bar{\in} S_h(\hat{x}(t)) \cap \Omega, \end{cases}$$

$S_h(\hat{x}(t))$ is the ball in R^n of radius h of point $\hat{x}(t)$, so as

$$S_h(\hat{x}(t)) = \{x \mid \| x - \hat{x}(t) \|_{R^n} \leq h\},$$

$\zeta(\cdot)$ is an unknown *measurement disturbance* and $\beta(\cdot) \in L^\infty(T_\varepsilon)$ is given.

We remark that $\beta^{-1}(t),\ t \in T_\varepsilon$, for example, might be a volume of the set $S_h(\hat{x}(t)) \bigcap \Omega$ or a constant. To simplify calculations we assume, in this paper, that

$$\beta(t) \equiv 1,\ \ t \in T_\varepsilon.$$

The restriction on the *uncertainties* $w(\cdot) = \{u_0(\cdot), f(\cdot,\cdot), \zeta(\cdot)\}$ can, in general, be described as

$$w(\cdot) \in \mathbf{W}, \qquad (1.4)$$

with $\mathbf{W}$ being a given subset of $L^2(\Omega) \times L^2(Q) \times L^2(T_\varepsilon)$.

The deterministic (minimax) state estimation problem is to find a solution $u(x,\theta)$ to the system (1.1) at the terminal instant θ on the basis of measurement data $y(\cdot)$ (given through the equations (1.2)-(1.3)) and the available information (1.4) on uncertainties.

The problem (1.1) - (1.4) has, in general, a set - valued solution [3,17,11] in the Hilbert space $L^2(\Omega)$. Following [8,11] we will denote by $U(\theta, y(\cdot))$ the *informational* set of all those states $u(x,\theta)$ of system (1.1) that are consistent with measurement data $y(\cdot)$ in (1.3) and with restrictions (1.4). In other words, this is the set of all those functions $u(x,\theta)$ for each of which there exists a triplet $\omega^*(\cdot) = \{u_0^*(\cdot), f^*(\cdot,\cdot), \zeta^*(\cdot)\}$ that satisfies (1.4) and generates a pair $\{u^*(\cdot,\theta),\ y^*(\cdot)\}$ (due to (1.1) - (1.3)) that satisfies the equalities $u^*(\cdot,\theta) = u(\cdot,\theta),\ \ y^*(t) = y(t), t \in T_\varepsilon$.

Let us suppose that measurement trajectories are selected from the prescribed set:

$$\hat{x}(\cdot) \in X(\cdot)\ \subset C[\varepsilon, \theta].$$

It is clear that the choice of spatial curve $\hat{x}(\cdot)$ in the equation (1.3) and the observed measurement data $y(\cdot)$ affect the size of the set $U(\theta, y(\cdot))$. To indicate this below we will designate the latter as $U(\theta, y(\cdot), \hat{x}(\cdot))$.

Denote by $Y(\cdot,\hat{x}(\cdot))$ the set of all feasible measurement data $y(t), t \in T_\varepsilon$ that can be obtained due to the system (1.1) - (1.4) under the measurement trajectory $\hat{x}(\cdot)$.

Let

$$J(U(\theta, y(\cdot), \hat{x}(\cdot))) \tag{1.5}$$

stand for a scalar criterion that characterizes the size of the domain $U(\theta, y(\cdot), \hat{x}(\cdot))$.

We define *the problem of optimal choice* of measurement trajectory as follows.

Problem 1.1: *Find a spatial curve* $\hat{x}^*(\cdot)$ *in the set* $X(\cdot)$ *that satisfies the equality*

$$\sup\{\, J(U(\theta, y(\cdot), \hat{x}^*(\cdot))) \mid y(\cdot) \in Y(\cdot, \hat{x}^*(\cdot))\} =$$

$$= \inf_{\hat{x}(\cdot) \in X(\cdot)} \sup\{J(U(\theta, y(\cdot), \hat{x}(\cdot))) \mid y(\cdot) \in Y(\cdot, \hat{x}(\cdot))\}. \tag{1.6}$$

In the present paper we study Problem 1.1 assuming that constraints on disturbances are quadratic and the set $X(\cdot)$ consists of solutions to a system of linear differential equations. Our goal is to derive necessary conditions for optimality.

Remark 1.1. The spatial curve satisfying the equality (1.6) is selected to be the same for any ("worst") possible measurement data. It provides some guaranteed result (the minimum of the estimation error) with respect to the criterion (1.5) for the set-valued solution of the estimation problem (1.1) - (1.4). Problem 1.1 can also be treated as H^∞-optimal control one (see [1]).

Remark 1.2. Instead of the Euclidean neighborhood $S_h(\hat{x})$ one may consider another type of neighborhood, for example:

$$P_h(0) = \{x \mid x = \mathrm{col}[x_1, \ldots, x_n],\ -h \le x_i \le +h,\ i = 1, \ldots, n\}.$$

Remark 1.3. The convex hull of the set $U(\theta, y(\cdot))$ can be described by means of its *support function* [16]:

$$\rho(\varphi(\cdot) \mid U(\theta, y(\cdot))) = \sup\{< \varphi(\cdot), u(\cdot,\theta) > \mid u(\cdot,\theta) \in U(\theta, y(\cdot))\}$$

for any element $\varphi(\cdot)$ of $L^2(\Omega)$.

Here and below the symbol $< (\cdot), (\cdot) >$ stands for a standard scalar product in the respective Hilbert space which will be clearly specified from the context.

The optimal sensor location problem under stationary observations have been studied by many authors in the stochastic setting (mainly with the trace of state covariance operator as a criterion) of the filtering problem (see [9]). The case of moving sensors was considered in [15] for the distributed parameter system identification.

We conclude this section by two examples of the criterion $J(U(\theta, y(\cdot), \hat{x}(\cdot)))$:

1. Diameter of the set $U(\theta, y(\cdot), \hat{x}(\cdot))$:

$$J(U(\theta, y(\cdot), \hat{x}(\cdot))) = \sup_{\|\varphi(\cdot)\|_{L^2(\Omega)}=1} \mid \rho(\varphi(\cdot) \mid U(\theta, y(\cdot), \hat{x}(\cdot))) + \rho(-\varphi(\cdot) \mid U(\theta, y(\cdot), \hat{x}(\cdot))) \mid .$$

As a modification of this criterion one may consider the value of diameter of projection of the set $U(\theta, y(\cdot), \hat{x}(\cdot))$ on the preassigned finite-dimensional subspace of $L^2(\Omega)$.

In the present paper we focus on the following *"weak"* criterion

2. Orthogonal projection on the preassigned direction $l(\cdot) \in L^2(\Omega)$ *:*

$$J(U(\theta, y(\cdot), \hat{x}(\cdot))) = \mid \rho(l(\cdot) \mid U(\theta, y(\cdot), \hat{x}(\cdot))) + \rho(-l(\cdot) \mid U(\theta, y(\cdot), \hat{x}(\cdot))) \mid . \qquad (1.7)$$

Remark 1.4. If

$$l(x) = \gamma \begin{cases} 1, \text{ if } x \in S_\delta(\bar{x}) \cap \Omega, \\ 0, \text{ if } x \bar{\in} S_\delta(\bar{x}) \cap \Omega \end{cases}$$

with $S_\delta(\bar{x})$ being the Euclidean neighborhood (in R^n) of radius δ of point $\bar{x}$ and γ^{-1} being a volume of the set $S_\delta(\bar{x}) \cap \Omega$, the value (1.7) gives us a precise estimate of the averaged value of $u(x, \theta)$ over the spatial region $S_\delta(\bar{x}) \bigcap \Omega$.

2.Preliminary Results , Refined Setting of Problem .

Assume that the set $\mathbf{W}$ is defined by a quadratic inequality, so as

$$\mathbf{W} = \{(u_0(\cdot),\, f(\cdot,\cdot),\, \zeta(\cdot)) \mid \int_\Omega u_0^2(x) m(x) dx +$$

$$+ \int_Q f^2(x,t) k(x,t) dx\, dt + \int_\varepsilon^\theta \zeta^2(t) n(t)\, dt \;\leq\; 1\} \qquad (2.1)$$

with given continuous functions $m(x)$, $k(x,t)$ and $n(t)$ such that

$$\min_{x \in \bar{\Omega}, t \in [0,\theta]} \{m(x), k(x,t)\}, \; \min_{t \in [\varepsilon,\theta]} \{n(t)\} > 0.$$

The set $\mathbf{W}$ is convex and weakly compact in the Hilbert space $L^2(\Omega) \times L^2(Q) \times L^2(T_\varepsilon)$. Therefore, the respective set $U(\theta, y(\cdot), \hat{x}(\cdot))$ will be convex and weakly compact in $L^2(\Omega)$.

It is well-known that the solution to the system (1.1) allows a unique representation as

$$u(\cdot, t) = \mathbf{S}(t) u_0(\cdot) + \int_0^t \mathbf{S}(t-\tau) f(\cdot, \tau) d\tau.$$

Denote by $\{\lambda_i\}_{i=1}^\infty$, $\{\omega_i(x)\}_{i=1}^\infty$ eigenvalues and eigenfunctions for the operator A under the homogeneous boundary condition, so as

$$A\omega_i(\cdot) = -\lambda_i \omega_i(\cdot)\,,\; \omega_i(\cdot) \in H_0^1(\Omega),\; <\omega_i(\cdot), \omega_j(\cdot)> = \delta_{ij},$$

$$\lambda_{i+1} \geq \lambda_i;\; \lambda_i \to +\infty,\;\; i \to +\infty\,;\;\; \delta_{ij} = \begin{cases} 1, \; i = j, \\ 0, \; i \neq j. \end{cases}$$

Then

$$u(x,t) = \sum_{i=1}^\infty e^{-\lambda_i t} < u_0(\cdot), \omega_i(\cdot) > \omega_i(x) + \sum_{i=1}^\infty \int_0^t e^{-\lambda_i (t-\tau)} < f(\cdot, \tau), \omega_i(\cdot) > d\tau \omega_i(x). \quad (2.2)$$

Due to [8,11] the set $U(\theta, y(\cdot), \hat{x}(\cdot))$ of the estimation problem (1.1) - (1.3) , (2.1) is an ellipsoid in the Hilbert space $L^2(\Omega)$, such that

$$U(\theta, y(\cdot), \hat{x}(\cdot)) = \{u(\cdot) \mid < u(\cdot) - u^{0*}(\cdot,\theta),\ \hat{\mathbf{P}}^{-1}(\theta)(u(\cdot) - u^{0*}(\cdot,\theta)) > \leq 1 - h^2(\theta)\}, \qquad (2.3)$$

where

$$\hat{\mathbf{P}}(\theta) = \mathbf{P}(\theta) - \mathbf{B}(\theta)\ ,\ \mathbf{P}(\theta) : L^2(\Omega) \to L^2(\Omega),\ \ \mathbf{B}(\theta) : L^2(\Omega) \to L^2(\Omega), \qquad (2.4)$$

$$< \varphi(\cdot), \mathbf{P}(\theta)\varphi(\cdot) > = \sum_{i,j=1}^{\infty} e^{-(\lambda_i+\lambda_j)\theta} \int_\Omega \omega_i(z)\omega_j(z)\, m^{-1}(z)\, dz \ \ < \omega_i(\cdot), \varphi(\cdot) > < \omega_j(\cdot), \varphi(\cdot) >$$

$$+ \sum_{i,j=1}^{\infty} \int_Q e^{-(\lambda_i+\lambda_j)(\theta-t)} \omega_i(z) k^{-1}(z,t) \omega_j(z)\, dz\, dt \ \ < \omega_i(\cdot), \varphi(\cdot) > < \omega_j(\cdot), \varphi(\cdot) > . \qquad (2.5)$$

The operator $\mathbf{B}(\theta)$ is integral:

$$\mathbf{B}(\theta)\varphi(\cdot) = \int_\Omega b(x,y,\theta)\varphi(y) dy.$$

In [8,11] it was shown , that functions $b(x,y,t)$, $u^{0*}(x,t)$ and $h^2(t)$ satisfy on T the joint system of initial-boundary value problems and an ordinary differential equation. We will use below only the respective initial-boundary value problem for $b(x,y,t)$:

$$\frac{\partial b(x,y,t)}{\partial t} = \hat{A} b(\cdot,\cdot,t) + \chi_\varepsilon(t)(q(x,t,t) - \int_\Omega \chi(y,\hat{x}(t)) b(x,y,t) dy) \times \qquad (2.6)$$

$$\times n(t)(q(y,t,t) - \int_\Omega \chi(x,\hat{x}(t)) b(x,y,t) dx),\ x, y \in \Omega,\ t \in (0,\theta),$$

$$b(x,y,0) = 0,\ b(\xi,y,t) = 0\ ,\ b(x,\xi,t) = 0,\quad \xi \in \partial\Omega,$$

where

$$\hat{A} = \sum_{i,j=1}^{n} \{ \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}) + \frac{\partial}{\partial y_i}(a_{ij}(y)\frac{\partial}{\partial y_j})\} - a(x) - a(y),$$

$$\chi_\varepsilon(t) = \begin{cases} 0, & 0 < t < \varepsilon, \\ 1, & \varepsilon \leq t < \theta, \end{cases}$$

$$q(x,t,\theta) = \sum_{i=1}^{\infty} e^{-\lambda_i\theta} \int_\Omega (\sum_{j=1}^{\infty} e^{-\lambda_j t} (\int_\Omega \chi(s,\hat{x}(t))\omega_j(s) ds)\omega_j(z))\, m^{-1}(z)\omega_i(z) dz\ \omega_i(x) +$$

$$+ \sum_{i=1}^{\infty} \int_\varepsilon^t \int_\Omega (\sum_{j=1}^{\infty} e^{-\lambda_j(t-\tau)} (\int_\Omega \chi(s,\hat{x}(t))\omega_j(s) ds)\omega_j(z)) e^{-\lambda_i(\theta-\tau)} k^{-1}(z,\tau)\omega_i(z)\, dz\, d\tau\, \omega_i(x). \quad (2.7)$$

Taking into account (2.3) - (2.4) , one may easily conclude that the functional (1.7) can be written as

$$J(U(\theta, y(\cdot), \hat{x}(\cdot))) = (1 - h^2(\theta))^{1/2} (< \varphi(\cdot), \mathbf{P}(\theta)\varphi(\cdot) > - \int_\Omega \int_\Omega l(x) b(x,y,\theta)) l(y) dy dx)^{1/2}. \ (2.8)$$

We note next that the set $U(\theta, y(\cdot), \hat{x}(\cdot))$ is largest , when $h^2(\theta) = 0$. The latter implies $y(t) \equiv 0, t \in T_\varepsilon$. Thus,

$$\sup\{ J(U(\theta, y(\cdot), \hat{x}(\cdot))) \mid y(\cdot) \in Y(\cdot, \hat{x}(\cdot))\} = J(U(\theta, \{0\}, \hat{x}(\cdot))) =$$

$$= (< \varphi(\cdot), \mathbf{P}(\theta)\varphi(\cdot) > - \int_\Omega \int_\Omega l(x) b(x, y, \theta)) l(y) dy dx)^{1/2}. \tag{2.9}$$

for any admissible trajectory $\hat{x}(\cdot)$.

The formula (2.5) indicates that the operator $\mathbf{P}(\theta)$ does not depend upon the choice of the curve $\hat{x}(t)$ (it is determined only by the system (1.1)). Therefore, we may substitute the criterion (1.7) for the "simplified" one :

$$I(\theta, \hat{x}(\cdot)) = \int_\Omega \int_\Omega l(x) b(x, y, \theta) l(y) dy dx. \tag{2.10}$$

Plainly, the function $\hat{x}(\cdot)$ from $X(\cdot)$ *maximizing* the criterion (2.10) gives us the solution to the problem (1.6) and vice versa.

Let us suppose that $X(\cdot)$ is the set of all those trajectories that are solutions to the following system of ordinary linear differential equations:

$$\frac{d\hat{x}(t)}{dt} = C(t)\hat{x}(t) + D(t)v(t), \quad t \in T_\varepsilon, \tag{2.11}$$

$$\hat{x}(\varepsilon) = x_0 \in \Omega,$$

where $v(t)$ is a (measurable) control,

$$v(t) \in V \subset R^m , t \in T_\varepsilon ; \tag{2.12}$$

the convex compact set V, the (continuous) matrices

$$C(t) = \{c_{ij}(t)\}, \quad D(t) = \{d_{ij}(t)\}$$

and x_0 are given.

It should be noted that behavior of the curve $\hat{x}(t)$ in the vicinity of the boundary $\partial\Omega$ is able to generate a number of serious problems when working with the necessary conditions for optimality. To avoid them we will consider below an "extended" modification of the phase constraints (1.2).

We set

$$\hat{x}(t) \in \bar{\Omega} + S_h(0) , t \in T_\varepsilon. \tag{2.13}$$

Due to (2.13) we will also extend all the functions (that are defined with respect to the spatial variable on Ω) by zero to the "extended" domain $\Omega + S_h(0)$.

Now we may reformulate *the problem of optimal choice* of the measurement trajectory in the precise setting as follows.

Problem 2.1: *Find a solution $\hat{x}^*(t)$, $t \in T_\varepsilon$ to the system (2.11) - (2.13) that satisfies the equality*

$$I(\theta, \hat{x}^*(\cdot)) = \sup\{I(\theta, \hat{x}(\cdot)) \mid \hat{x}(\cdot) \in X(\cdot)\}. \tag{2.14}$$

Problem 2.1 is ***an open loop*** control problem. One has to maximize the functional (2.10) due to solutions to the joint system that includes the infinite-dimensional problem (2.6) of Riccati type and the system of ordinary differential equations (2.11), (2.12) under the phase constraints (2.13).

Among the early papers, treating the optimal sensor location problem as one of optimal control on the Riccati equation (describing the evolution of the estimate error covariance operator), was [2].

Remark 2.1. It should be noted that to solve the estimation problem (1.1), (1.3), (2.1) we need to know only the optimal measurement trajectory (from the prescribed set $X(\cdot)$). This circumstance was basically used in the statement of Problem 2.1.

However, to solve the system (2.11) - (2.13) one has to find an associated *optimal control* $v^*(\cdot)$. Accordingly, we will also treat below the solution to Problem 2.1 as a pair

$$\{\hat{x}^*(\cdot), v^*(\cdot)\}.$$

Remark 2.2. In this paper we do not focus our attention on the "ordinary" part (2.11) - (2.12) of the problem. Instead of linear system one may consider a more general case:

$$\frac{d\hat{x}(t)}{dt} = g(\hat{x}(t), t, v(t)), t \in T_\varepsilon, \tag{2.15}$$

$$\hat{x}(\varepsilon) = x_0.$$

Before deriving necessary conditions for optimality of the measurement curve $\hat{x}^*(t)$ let us discuss the problem of existence.

3.Existence of Optimal Measurement Trajectory.

We note first that under the above assumptions the set of solutions to the system (2.11) - (2.12) is a compact subset in the space $C_n[\varepsilon, \theta]$, where the latter stands for n products of $C[\varepsilon, \theta]$.

Remark now, that

$$\| \chi(\cdot, \hat{x}(t)) \|_{L^2(\Omega)} = (\sum_{j=1}^{\infty}(\int_\Omega \chi(x, \hat{x}(t))\omega_j(x)dx)^2)^{1/2} =$$

$$\Big(\int_{S_h(\hat{x}(t))\bigcap\Omega} 1\,dx\Big)^{1/2} = (\mathrm{meas}(S_h(\hat{x}(t))\bigcap\Omega))^{1/2} \tag{3.1}$$

for all those (continuous) curves $\hat{x}(\cdot)$ that satisfy (2.13).

We note next that

$$\int_{\Omega}(\chi(x,\hat{x}_1(t)) - \chi(x,\hat{x}_2(t)))^2 dx = \int_{(S_h(\hat{x}_1(t))\Delta S_h(\hat{x}_2(t)))\bigcap\Omega} 1\,dx \ \leq \tag{3.2}$$

$$\leq M_1 \parallel \hat{x}_1(t)) - \hat{x}_2(t)) \parallel_{R^n}, \ \ \forall t \in [\varepsilon,\theta], \ M_1 = \mathrm{const},$$

where $A\Delta B$ stands for the symmetric difference of the sets A, B.

On the other hand , due to (2.7) , we have

$$\parallel q(\cdot,\cdot,\theta) \parallel_{L_2(Q_\varepsilon)} \leq \ M_2 \ \parallel \chi(\cdot,\hat{x}(\cdot)) \parallel_{L_2(Q_\varepsilon)} , \ M_2 = \mathrm{const}. \tag{3.3}$$

Comparing (3.3) with (3.1) yields

$$\parallel q(\cdot,\cdot,\theta) \parallel_{L_2(Q_\varepsilon)} \leq \ \mathrm{const}. \tag{3.4}$$

From the respective formulae in [8,11] it follows

$$\parallel b_1(\cdot,\cdot,\theta) - b_2(\cdot,\cdot,\theta) \parallel_{L_2(\Omega\times\Omega)} \leq M_3 \parallel q_1(\cdot,\cdot,\theta) - q_2(\cdot,\cdot,\theta) \parallel_{L_2(Q_\varepsilon)} \times \tag{3.5}$$

$$\times \parallel q_1(\cdot,\cdot,\theta) + q_2(\cdot,\cdot,\theta) \parallel_{L_2(Q_\varepsilon)}, \quad M_3 = \mathrm{const},$$

where the subscripts 1,2 indicate that functions $b(\cdot,\cdot,\theta)$ and $q(\cdot,\cdot,\theta)$ are respectively calculated under two different measurement trajectories $\hat{x}_1(\cdot)$ and $\hat{x}_2(\cdot)$.

The latter and (3.2)-(3.4) yield

$$\parallel b_1(\cdot,\cdot,\theta) - b_2(\cdot,\cdot,\theta) \parallel_{L_2(\Omega\times\Omega)} \ \leq \ \ \mathrm{const} \parallel \hat{x}_1(\cdot) - \hat{x}_2(\cdot) \parallel_{C_n[\varepsilon,\theta]} . \tag{3.6}$$

In fact, the estimate (3.6) means the continuity of the criterion (2.10) with respect to measurement curves $\hat{x}(\cdot)$ in the norm of $C_n[\varepsilon,\theta]$.

Therefore , due to compactness of the set of all solutions to the system (2.11) - (2.12) and taking into account that the limit transition remains (2.13) be fulfilled, we come to

Theorem 3.1. Let the set of all those measurement trajectories $\hat{x}(\cdot)$, that satisfy to the system (2.11),(2.12) under the phase constraints (2.13) be non-empty. Then there exists a solution to Problem 2.1.

Indeed, let

$$\hat{x}_i(\cdot), \ \ i = 1,\ldots \tag{3.7}$$

be a sequence that maximizes the functional (2.10) due to the system (2.6), (2.11) - (2.13).

Then, as it was mentioned above, we may select (if necessary) a subsequence of (3.7) and indicate the function $\hat{x}^*(\cdot)$ as a limit of the latter in the norm of $C_n[\varepsilon, \theta]$.

Remark 3.1. From the proof of Theorem 3.1 one can make a conclusion that the latter is also valid in the case of the phase constraints (1.2). However, the phase constraints of type (2.13) allow us, in fact (as it will be shown in the next section), to neglect them.

Remark 3.2. The following chain of estimates

$$| \int_\Omega \chi(x, \hat{x}_1(t))\omega_j(x)dx - \int_\Omega \chi(x, \hat{x}_2(t))\omega_j(x)dx \; |=| \int_{S_h(\hat{x}_1(t))\bigcap\Omega} \omega_j(x)dx -$$

$$- \int_{S_h(\hat{x}_2(t))\bigcap\Omega} \omega_j(x)dx \; |\leq| \int_{(S_h(\hat{x}_1(t))\Delta S_h(\hat{x}_2(t)))\bigcap\Omega} \omega_j(x)dx \; |, \forall t \in [\varepsilon, \theta], \; j = 1, \ldots \qquad (3.8)$$

together with the equalities (3.1) leads to

Lemma 3.1. Let the sequence $\{\hat{x}^i(\cdot), \; i = 1, \ldots\}$ of spatial curves in the domain $\bar{\Omega} + S_h(0)$ converges to the measurement trajectory $\hat{x}(\cdot)$ in the norm of $C_n[\varepsilon, \theta]$. Then

$$\lim_{r\to\infty} \| \chi^*(\cdot, \hat{x}^i(\cdot)) - \chi(\cdot, \hat{x}(\cdot)) \|_{C([\varepsilon,\theta];L_2(\Omega))} = 0, \qquad (3.9)$$

where

$$\chi^*(x, \hat{x}^i(t) = \sum_{j=1}^{i} \int_\Omega \chi(z, \hat{x}^i(t))\omega_j(z)dz \, \omega_j(x).$$

Remark 3.3. The results of this section are also valid for the case of nonlinear system (2.15), if we introduce a number of assumptions that are traditional [7] in the theory of the lumped-parameter system:

Assumption 3.1. All of the solutions to the system (2.15) are uniformly bounded:

$$| \; \hat{x}(t) \; | \leq \text{const}, \forall t \in [\varepsilon, \theta].$$

*Assumption 3.2.*The set

$$L(\hat{x}, t) = \{l \mid l = g(\hat{x}, t, v), v \in V\}$$

is a convex subset of R^n for any pair $\{\hat{x}, t\}$.

4.Suboptimal Solutions.

Let us consider the impact of the phase constraints (2.13) in more details.

Denote by e^* the set of all those instants of time when the optimal trajectory $\hat{x}^*(\cdot)$ lies on the boundary of the domain $\Omega + S_h(0)$, so that

$$e^* = \{t \mid t \in T_\varepsilon, \; \hat{x}^*(t) \in \partial(\Omega + S_h(0))\}.$$

Lemma 4.1. Let $\hat{x}^{**}(\cdot)$ be an arbitrary spatial curve in the domain $\bar{\Omega} + S_h(0)$ that coincides with $\hat{x}^*(\cdot)$ on the set $T_\epsilon \backslash e^*$. Then

$$U(\theta, \{0\}, \hat{x}^{**}(\cdot)) \subseteq U(\theta, \{0\}, \hat{x}^*(\cdot)).$$

Proof. Indeed, let $u^*(\cdot, \theta)$ be an element of the set $U(\theta, \{0\}, \hat{x}^{**}(\cdot))$ generated due to the system (1.1),(1.3),(2.1) by the triplet

$$\omega^*(\cdot) = \{u_0^*(\cdot), f^*(\cdot, \cdot), \zeta^*(\cdot)\}.$$

We note next that

$$\chi(x, \hat{x}^*(t)) = 0, \ \forall t \in e^*.$$

The latter indicates that the triplet

$$\omega^{**}(\cdot) = \{u_0^*(\cdot), f^*(\cdot, \cdot), \zeta^{**}(\cdot)\},$$

with

$$\zeta^{**} = \begin{cases} \zeta^*, & \text{if } t \in T_\epsilon \backslash e^*, \\ 0, & \text{if } t \in e^*, \end{cases}$$

satisfies all the relations (1.1), (1.3), (2.1) with $\hat{x}^*(\cdot)$. This yields the desired conclusion of Lemma 4.1.

From Lemma 4.1 it immediately follows

Lemma 4.2. The optimal measurement curve $\hat{x}^*(\cdot)$ is completely described by its interior part that entirely lies in the domain $\Omega + S_h(0)$.

Let $L^{2(r)}(\Omega)$ be an r-dimensional subspace of $L^2(\Omega)$ spanned by the functions

$$\omega_i(\cdot), \ i = 1, \ldots, r.$$

Using the Fourier-series expansion, we consider the sequence of finite - dimensional optimal control problems that are defined as orthogonal projections of Problem 2.1 on the series of $L^{2(r)}(\Omega)$, $r = 1, \ldots$.

Problem 4.1(r): *Find a solution $\hat{x}^{*(r)}(t)$, $t \in T_\epsilon$ to the system (2.11) - (2.13) that maximizes the functional*

$$\sum_{i,j=1}^{r} l_i b_{ij}^r(\theta) l_j, \tag{4.1}$$

where

$$\dot{b}_{ij}^r(t) = -(\lambda_i + \lambda_j) b_{ij}^r(t) + \chi_\epsilon(t)(q_i(t) - \sum_{s=1}^{r} b_{is}^r(t) d_s(t)) n(t) (q_j(t) - \sum_{s=1}^{r} b_{sj}^r(t) d_s(t)), \ t \in T_\epsilon, \tag{4.2}$$

$$b_{ij}^r(0) = 0, \ \ i, j = 1, \ldots, r; \ \ q_i(t) = \int_\Omega q(x, t, t) \omega_i(x) dx,$$

$$l_i = \int_\Omega l(x)\omega_i(x)dx\ ,\ d_i(t) = \int_\Omega \chi(x, \hat{x}(t))\omega_i(x)dx.$$

We remark that the scheme introduced in the previous section provides existence of solutions to Problems 4.1(r), $r = 1, \ldots$.

Due to the formulae derived in [8,11](see also (2.7)), we can write

$$\lim_{r\to\infty} \sum_{i,j=1}^{r} l_i b_{ij}^r(\theta) l_j = \int_\Omega \int_\Omega l(x) b(x,y,\theta) l(y) dx dy,\ \ \forall \hat{x}(\cdot) \in X(\cdot), \tag{4.3}$$

$$\lim_{r\to\infty} \| b^r(\cdot,\cdot,\cdot) - b(\cdot,\cdot,\cdot) \|_{C([\epsilon,\theta];L^2(\Omega\times\Omega))} = 0,\ \ \forall \hat{x}(\cdot) \in X(\cdot), \tag{4.4}$$

$$\lim_{r\to\infty} \| \int_\Omega \chi(y, \hat{x}(\cdot)) b^r(\cdot, y, \cdot) dy - \int_\Omega \chi(y, \hat{x}(\cdot)) b(\cdot, y, \cdot) dy \|_{L^2(Q_\epsilon)} = 0,\ \ \forall \hat{x}(\cdot) \in X(\cdot), \tag{4.5}$$

where

$$b^r(x,y,t) = \sum_{i,j=1}^{r} b_{ij}^r(t)\omega_i(x)\omega_j(y).$$

We note next, that assertions of Theorem 3.1 and of Lemma 4.2 are also valid for the series of Problems 4.1(r).

Set

$$\psi_j(\hat{x}) = \int_{S_h(\hat{x})\bigcap\Omega} \omega_j(x)dx,\ \ \hat{x} = \mathrm{col}\,[\hat{x}_1, \ldots, \hat{x}_n] \in \bar{\Omega} + S_h(0).$$

Assumption 4.1. Assume that the following condition

$$\frac{d\psi_j(\hat{x})}{d\hat{x}_k} \in C(\bar{\Omega} + S_h(0)),\ \ k = 1, \ldots, n;\ j = 1, \ldots .$$

will be fulfilled below.

Let $\{\hat{x}^{(r)}(\cdot), v^{(r)}(\cdot)\}$ be a pair of optimal trajectory and control that solves Problem 4.1(r).

Applying the Pontryagin's maximum principle[14] yields for each of Problems 4.1(r):

$$\max_{v\in V} H^r(t, \hat{x}^{(r)}(t), v) = H^r(t, \hat{x}^{(r)}(t), v^{(r)}(t)) \ \text{ for a.e. } t \in T_\epsilon \backslash e^{(r)}, \tag{4.6}$$

where

$$H^r(t, \hat{x}^{(r)}(t), v) = p^{(r)\prime}(t) D(t) v, \tag{4.7}$$

$$e^{(r)} = \{t \mid t \in T_\epsilon,\ \ \hat{x}^{(r)}(t) \in \partial(\Omega + S_h(0))\}. \tag{4.8}$$

The vector-function $p^{(r)}(t) = \mathrm{col}[p_1^{(r)}(t), \ldots, p_n^{(r)}(t)]$ is a solution to the following *adjoint system*

$$\dot{p}_{ij}^{(r)}(t) = (\lambda_i + \lambda_j) p_{ij}^{(r)}(t) - \sum_{s=1}^{r} p_{is}^{(r)}(t) d_j^{*r}(t) n(t) \times$$

$$\times (q_s^{*r}(t) - \sum_{k=1}^{r} b_{ks}^{*r}(t) d_k^{*r}(t)) - \sum_{s=1}^{r} p_{sj}^{(r)}(t) d_i^{*r}(t) n(t) (q_s^{*r}(t) - \sum_{k=1}^{r} b_{sk}^{*r}(t) d_k^{*r}(t)),\ \ t \in T_\epsilon, \tag{4.9}$$

$$p_{ij}^{(r)}(\theta) = l_i l_j,\ \ i,j = 1, \ldots, r,$$

$$\dot{p}_k^{(r)}(t) = -\sum_{i,j=1}^{r} p_{ij}^{(r)}(t)\frac{\partial}{\partial \hat{x}_k}\{(q_i^{*r}(t) - \sum_{s=1}^{r} b_{is}^{*r}(t)d_s^{*r}(t))n(t)(q_j^{*r}(t) - \sum_{s=1}^{r} b_{sj}^{*r}(t)d_s^{*r}(t))\} -$$

$$-\sum_{i=1}^{n} c_{ik}(t)p_i^{(r)}(t), \quad k = 1, \ldots, n, \tag{4.10}$$

$$p_k^{(r)}(\theta) = 0, \quad g(\hat{x}^{(r)}(t), t, v^{(r)}(t)) = \text{ col } [g_1(\hat{x}^{(r)}(t), t, v^{(r)}(t)), \ \ldots \ , g_n(\hat{x}^{(r)}(t), t, v^{(r)}(t))],$$

the combination of the symbols "$*, r$" indicates that an appropriate value is calculated under

$$\hat{x}(\cdot) = \hat{x}^{(r)}(\cdot), \quad v(\cdot) = v^{(r)}(\cdot).$$

From the limit relation (4.3) it immediately follows

Lemma 4.3. The pairs $\{\hat{x}^{(r)}(\cdot), v^{(r)}(\cdot), \ r = 1, \ldots\}$ that satisfy the sequence of finite-dimensional maximum principles (4.2),(4.6) -(4.10) form the sequence of suboptimal solutions to Problem 2.1.

Theorem 3.1 allows us to select a subsequence of measurement trajectories $\{\hat{x}^{(r_i)}(\cdot), \ i = 1, \ldots\}$ that converges in the norm of $C_n[\varepsilon, \theta]$ to the optimal trajectory $\hat{x}^*(\cdot)$ solving Problem 2.1, so that

$$\lim_{i \to \infty} \| \hat{x}^*(\cdot) - \hat{x}^{(r_i)}(\cdot) \|_{C_n[\varepsilon,\theta]} = 0. \tag{4.11}$$

Furthermore, due to weak compactness of the set of all the admissible controls $v(\cdot)$ in $L_m^2(T_\varepsilon)$, we can select the above sequence of trajectories in such a way that

$$v^{(r_i)}(\cdot) \to v^*(\cdot) \text{ weakly in } L_m^2(T_\varepsilon), \tag{4.12}$$

where

$$L_m^2(T_\varepsilon) = \underbrace{L^2(T_\varepsilon) \times \ldots \times L^2(T_\varepsilon)}_{m}.$$

Thus, we have obtain

Theorem 4.1. There exists a sequence of pairs $\{\hat{x}^{(r_i)}(\cdot), v^{(r_i)}(\cdot), \ i = 1, \ldots\}$, satisfying the respective sequence of finite-dimensional maximum principles (4.2),(4.6) - (4.10), such that its limit in the sense of the relations (4.11), (4.12) is a solution to Problem 2.1.

5.Necessary Conditions for Optimality.

The scheme for deriving of optimality conditions is based on a limit transition along the sequence of finite-dimensional maximum principles described in the previous section.

Denote

$$P_r(x, y, t) = \sum_{i,j=1}^{r} p_{ij}^{(r)}(t)\omega_i(x)\omega_j(y).$$

Multiplying both sides of the equation (4.9) by $\omega_i(x)\omega_j(y)$ and summing them up over indices $i, j = 1, \ldots, r$ yield the initial-boundary value problem for the function $P_r(x, y, t)$:

$$\frac{\partial P_r(x,y,t)}{\partial t} = -\hat{A}P_r(\cdot,\cdot,t) - \chi^*(x,\hat{x}^r(t))n(t)\times \tag{5.1}$$

$$\times \int_\Omega P_r(z,y,t)(q^{*r}(z,t,t) - \int_\Omega \chi^*(s,\hat{x}^{(r)}(t))b^{*r}(s,z,t)ds)dz-$$

$$-\chi^*(y,\hat{x}^{(r)}(t))n(t)\int_\Omega P_r(x,z,t)(q^{*r}(z,t,t) - \int_\Omega \chi^*(s,\hat{x}^{(r)}(t))b^{*r}(z,s,t)ds)dz, \ x,y\in\Omega, \ t\in T_\varepsilon,$$

$$P_r(x,y,t)\mid_{\Omega\times\Omega} = 0, \ \ P_r(x,y,\theta) = l^r(x)l^r(y),$$

where

$$q^{*r}(x,t,t) = \sum_{i=1}^r q_i^{*r}(t)\omega_i(x), \ \ l^r(x) = \sum_{i=1}^r l_i\omega_i(x),$$

$$b^{*r}(x,y,t) = \sum_{i,j=1}^r b_{ij}^{*r}(t)\omega_i(x)\omega_j(y), \ \ \chi^*(x,\hat{x}^{(r)}(t) = \sum_{i=1}^r d_i^{*r}(t)\omega_i(x).$$

Let us denote now the sum of integral terms in the right-hand side of the equation (5.1) by

$$\rho(x,y,t,P_r(\cdot,\cdot,t)) =$$

$$= \chi^*(x,\hat{x}^{(r)}(t))n(t)\times\int_\Omega P_r(z,y,t)(q^{*r}(z,t,t) - \int_\Omega \chi^*(s,\hat{x}^{(r)}(t))b^{*r}(s,z,t)ds)dz+$$

$$+\chi^*(y,\hat{x}^{(r)}(t))n(t)\times\int_\Omega P_r(x,z,t)(q^{*r}(z,t,t) - \int_\Omega \chi^*(s,\hat{x}^{(r)}(t))b^{*r}(z,s,t)ds)dz. \tag{5.2}$$

We note next that eigenfunctions of the operator $\hat{A}$ are as follows:

$$\omega_i(x)\omega_j(y), \ \ i,j = 1,\ldots$$

Hence, using the Fourier-series expansion (along the system of eigenfunctions) for representation of solutions to the linear stationary parabolic system , we obtain

$$P_r(x,y,t) = \sum_{i,j=1}^r \int_t^\theta e^{(\lambda_i+\lambda_j)(t-\tau)} < \rho(\cdot,\cdot,\tau,P_r(\cdot,\cdot,\tau)),\omega_i(\cdot)\omega_j(\cdot) > d\tau\omega_i(x)\omega_j(y)+ \tag{5.3}$$

$$+\sum_{i,j=1}^r e^{(\lambda_i+\lambda_j)(t-\theta)} < l^r(\cdot),\omega_i(\cdot) >< l^r(\cdot),\omega_j(\cdot) > \omega_i(x)\omega_j(y), \ \ t\in T_\varepsilon.$$

In other words $P_r(x, y, t)$ is a mild solution to the system (5.1).

In a traditional way this gives the estimate

$$\| P_r(\cdot,\cdot,t) \|_{L_2(\Omega\times\Omega)} \leq c_1\{\| l^r(\cdot) \|^2_{L_2(\Omega)} + (\int_t^\theta \| \rho(\cdot,\cdot,t,P_r(\cdot,\cdot,t) \|^2_{L_2(\Omega\times\Omega)} dt)^{1/2}\}, \tag{5.4}$$

$$\forall t\in[\varepsilon,\theta], \ \ c_1 \ = \ \text{const}.$$

Due to (5.2) and (3.1)-(3.6) we obtain

$$\| \rho(\cdot,\cdot,t,P_r(\cdot,\cdot,t)) \|^2_{L_2(\Omega\times\Omega)} \leq c_2 \| \chi^*(\cdot,\hat{x}^{(r)}(t)) \|^2_{L_2(\Omega)} \| P_r(\cdot,\cdot,t) \|^2_{L_2(\Omega\times\Omega)} \times$$

$$\times \| q^{*r}(\cdot,t,t) - \int_\Omega \chi^*(s,\hat{x}^{(r)}(t)) b^{*r}(s,\cdot,t) ds \|^2_{L_2(\Omega)} \leq c_3^2 \| P_r(\cdot,\cdot,t) \|^2_{L_2(\Omega\times\Omega)}, \quad (5.5)$$

where c_2, c_3 are positive constants independent of r.

Now the estimate (5.4) can be presented in the form:

$$\max_{[t^*,\theta]} \| P_r(\cdot,\cdot,t) \|_{L_2(\Omega\times\Omega)} \leq c_1 \| l^r(\cdot) \|^2_{L_2(\Omega)} +$$

$$+ c_1 c_3 (\theta - t^*)^{1/2} \max_{[t^*,\theta]} \| P_r(\cdot,\cdot,t) \|_{L_2(\Omega\times\Omega)} dt, \ \forall t^* \in [\varepsilon,\theta]. \quad (5.6)$$

Finally, using the Gronwall's inequality, we have the estimate

$$\max_{[\varepsilon,\theta]} \| P_r(\cdot,\cdot,t) \|_{L_2(\Omega\times\Omega)} \leq c_4 \| l(\cdot) \|^2_{L_2(\Omega)}, \quad c_4 = \text{const}. \quad (5.7)$$

The latter is a technical issue for a limit transition in the mixed problem (5.1).

Multiplying the equation (5.1) by an arbitrary function $\phi(x,y,t)$,

$$\phi(\cdot,\cdot,\cdot) \in H^{2,2,1}(\Omega\times\Omega\times T_\varepsilon), \ \ \phi(\cdot,\cdot,t) |_{\partial(\Omega\times\Omega)} = 0, \ ; \ \phi(x,y,\varepsilon) = 0$$

and applying Green's theorems yield the integral identity :

$$\int_\varepsilon^\theta \int_\Omega \int_\Omega \{ P_r(x,y,t) [- \frac{\partial \phi(x,y,t)}{\partial t} + \hat{A}\phi(\cdot,\cdot,t)] + [\chi^*(x,\hat{x}^r(t)) n(t) \times \quad (5.8)$$

$$\times \int_\Omega P_r(z,y,t)(q^{*r}(z,t,t) - \int_\Omega \chi^*(s,\hat{x}^{(r)}(t)) b^{*r}(s,z,t) ds) dz +$$

$$+ \chi^*(y,\hat{x}^{(r)}(t)) n(t) \times \int_\Omega P_r(x,z,t)(q^{*r}(z,t,t) - \int_\Omega \chi^*(s,\hat{x}^{(r)}(t)) b^{*r}(z,s,t) ds) dz] \phi(x,y,t) \} dx dy dt =$$

$$= - \int_\Omega \int_\Omega l^r(x) l^r(y) \phi(x,y,\theta) dx dy,$$

where the Sobolev space $H^{2,2,1}(\Omega\times\Omega\times T_\varepsilon)$ is defined similarly to $H^{2,1}(Q)$.

On the other hand the estimate (5.7) provides us by a subsequence of functions $P_r(\cdot,\cdot,\cdot)$ such that

$$P_{r_i}(\cdot,\cdot,\cdot) \to P(\cdot,\cdot,\cdot) \ \text{weakly in} \ L_2(\Omega\times\Omega\times T_\varepsilon). \quad (5.9)$$

Without loss of generality , we can assume that for an associated subsequence of measurement trajectories the equality (4.11) is fulfilled.

Combining of (5.8) and (5.9) gives the identity (5.8) with the limit function $P(x,y,t)$ substituted for $P_r(x,y,t)$. The latter may be also represented as an initial-boundary value problem

$$\frac{\partial P(x,y,t)}{\partial t} = -\hat{A}P(\cdot,\cdot,t) - \chi(x,\hat{x}^*(t)) n(t) \times \quad (5.10)$$

$$\times \int_\Omega P(z,y,t)(q^*(z,t,t) - \int_\Omega \chi(s,\hat{x}^*(t))b^*(s,z,t)ds)dz -$$

$$-\chi(y,\hat{x}^*(t))n(t) \times \int_\Omega P(x,z,t)(q^*(z,t,t) - \int_\Omega \chi(s,\hat{x}^*(t))b^*(z,s,t)ds)dz, \;\; x,y \in \Omega, \; t \in T_\varepsilon,$$

$$P(x,y,t)\mid_{\partial\Omega\times\Omega} = 0, \;\; P(x,y,\theta) = l(x)l(y).$$

Here the symbol "$*$" indicates that respective values are calculated under $\hat{x}(t) \equiv \hat{x}^*(t), t \in T_\varepsilon$.

Let us consider now the system of ordinary differential equations (4.10). Similarly to (5.8) we may represent it in the integral form:

$$p_k^{(r)}(t) = +\int_t^\theta \int_\Omega \int_\Omega P_r(x,y,\tau)\frac{\partial}{\partial \hat{x}_k}[(q^{*r}(x,\tau,\tau) - \int_\Omega \chi^*(z,\hat{x}^r(\tau))b^{*r}(x,z,\tau)dz)\times \tag{5.11}$$

$$\times n(\tau)(q^{*r}(y,\tau,\tau) - \int_\Omega \chi^*(z,\hat{x}^r(\tau))b^{*r}(z,y,\tau)dz)]dxdyd\tau +$$

$$+\int_t^\theta \sum_{i=1}^n c_{ik}(\tau)p_i^{(r)}(\tau)d\tau, \;\; k = 1,\dots n.$$

The crucial point to ensure the limit transition in (5.11) is the derivative in the first integral term:

$$Q_k(x,y,t,\hat{x}^{(r)}(\cdot)) = \frac{\partial}{\partial \hat{x}_k}[(q^{*r}(x,t,t) -$$

$$-\int_\Omega \chi^*(z,\hat{x}^r(t))b^{*r}(z,y,t)dz)n(t)(q^{*r}(y,t,t) - \int_\Omega \chi^*(z,\hat{x}^r(t))b^{*r}(z,y,t)dz)].$$

Introduce the following

Assumption 5.1. There exists a subsequence of integers $\{r_i\}_{i=1}^\infty$ such that

$$\lim_{i\to\infty} \| Q_k(\cdot,\cdot,\cdot,\hat{x}^{(r_i)}(\cdot)) - Q_k(\cdot,\cdot,\cdot,\hat{x}^*(\cdot)) \|_{L_2(\Omega\times\Omega\times T_\varepsilon)} = 0, \; k = 1,\dots n,$$

whereas the equality (4.11) is fulfilled.

Remark 5.1. Let Assumption 4.1 be complemented by the following uniform estimate

$$\mid \frac{d\psi_j(\hat{x})}{d\hat{x}_k} \mid \le \text{const}, \;\; \forall \hat{x} \in \bar{\Omega} + S_h(0), \;\; k = 1,\dots,n; \; j = 1,\dots .$$

Consider two cases when Assumption 5.1 is fulfilled.

1. If the dimensionality of the spatial variable in the system (1.1) is equal to 1, then Assumption 5.1 is valid due to the asymptotics [4] of eigenvalues of the operator A.

2. If the dimensionality of the spatial variable in the system (1.1) is more than 1, then Assumption 5.1 is fulfilled when the forcing term in (1.1) is absent.

Indeed, in this case all the above formulae can be simplified. For example, the formula (2.7) will be as follows

$$q(x,t,\theta) = \sum_{i=1}^\infty e^{-\lambda_i\theta} \int_\Omega (\sum_{j=1}^\infty e^{-\lambda_j t}(\int_\Omega \chi(s,\hat{x}(t))\omega_j(s)ds)\omega_i(z))\, m^{-1}(z)\omega_i(z)dz\, \omega_i(x).$$

The latter allows us to provide Assumption 5.1, using asymptotic behavior of eigenvalues.

We note next that applying Gronwall's inequality to the equation (5.11) yields the boundedness of the values $p_k^{(r)}(\cdot)$:

$$\| p_k^{(r)}(\cdot) \|_{L_2(T_\epsilon)} \leq \text{ const}, \quad k = 1, \ldots, n; \; r = 1, \ldots . \tag{5.12}$$

Selecting (if necessary) subsequence of functions $\{p^{(r_i)}(\cdot)\}_{i=1}^{\infty}$ and taking $i \to \infty$, we obtain

$$\frac{d\,p_k(t)}{dt} = -\int_\Omega \int_\Omega P(x,y,t) \frac{\partial}{\partial \hat{x}_k} [(q^*(x,t,t) - \int\limits_\Omega \chi(z, \hat{x}^*(t)) b^*(x,z,t) dy) n(t) \times \tag{5.13}$$

$$\times (q^*(y,t,t) - \int\limits_\Omega \chi(z, \hat{x}^r(t)) b^*(z,y,t) dz)] dx dy dt - \sum_{i=1}^{n} c_{ik}(t) p_i(t), \quad p_k(\theta) = 0, \; k = 1, \ldots n,$$

where

$$p_k^{(r_i)}(\cdot) \;\; \to p_k(\cdot) \text{ in } L^2(T_\epsilon). \tag{5.14}$$

Finally, we proceed to the limit transition in the relation of maximum (4.6).

First, we rewrite it again in the integral form:

$$\frac{1}{\nu} \int\limits_{(t,t+\nu) \bigcap (T_\epsilon \backslash e^{(r)})} H^r(\tau, \hat{x}^{(r)}(\tau), v) d\tau \leq \frac{1}{\nu} \int\limits_{(t,t+\nu) \bigcap (T_\epsilon \backslash e^{(r)})} H^r(\tau, \hat{x}^{(r)}(\tau), v^{(r)}(\tau)) d\tau \tag{5.15}$$

$$\text{for a.e. } t \in T_\epsilon \backslash e^{(r)}, \;\; \forall \nu > 0, \;\; r = 1, \ldots$$

Now, taking into account (5.14), we can realize the limit transition in (5.15) with $r \to \infty$ (selecting, if necessary a subsequence of r_i).

After that, taking $\nu \to \infty$, we come to

Theorem 5.1. Let the pair $\{\hat{x}^*(\cdot), v^*(\cdot)\}$ be a solution to Problem 2.1. Then it satisfies the maximum principle:

$$\max_{v \in V} H(t, \hat{x}^*(t), v) = H(t, \hat{x}^*(t), v^*(t)) \;\; \text{for a.e. } t \in T_\epsilon \backslash e^*, \tag{5.16}$$

where

$$H(t, \hat{x}^*(t), v) = p'(t) D(t) v.$$

Remark 5.2. The typical criterion, characterizing the quality of sensors in the stochastic setting of the optimal sensor location problem, is the trace of the error covariance operator[9]. Due to the formula (2.4) in the above minimax setting the latter corresponds to the functional

$$I_1(\theta, \hat{x}(\cdot)) = \int_\Omega b(x,x,\theta) dx \;\; \to \;\; \sup . \tag{5.17}$$

For Problem 2.1 with the criterion (5.17) substituted for (2.14) necessary conditions for optimality may be presented in the form (5.13),(5.16) and (5.10) under the terminal condition

$$P(x, y, \theta) = \delta(x - y).$$

Remark 5.3. Instead of the spatially averaged observations of type (1.3) we may also consider a dynamic pointwise one:

$$y(t) = u(\hat{x}(t), t) + \zeta(t), \; t \in T_\varepsilon. \tag{5.18}$$

It is clear that this type of sensors requires a corresponding smoothness of solutions to the problem (1.1) (for example: $u(\cdot, \cdot) \in H^{2,1}(\Omega \times T_\varepsilon)$ under $n \leq 3$, see [12,13]).

Deriving of necessary conditions for optimality of measurement trajectories in the case of pointwise observations (5.18) might be (due to arising technical problems) a subject for a separated paper (the one dimensional example of such a problem was considered in [11]). Here we remark only that Problem 2.1 with the spatial observations of type (1.3) might be useful for constructing suboptimal solutions to the latter one, when taking

$$\beta(t) = V^{-1}(S_h(\hat{x}(t)) \bigcap \Omega).$$

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