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On Optimization of Discontinuous Systems

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Contents

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ABSTRACT.

In this paper stochastic programming techniques are adapted and further developed for applications to discrete event systems. We consider cases when the sample path of the system depend discontinuously on control parameters (e.g. modeling of failures, several competing processes), which could make the computation of estimates of the gradient difficult. Methods which use only samples of the performance criterion are developed, in particular finite differences with reduced variance and concurrent approximation and optimization algorithms. Optimization of the stationary behavior is also considered. Results of numerical experiments and convergence results are reported.

KEYWORDS: Stochastic programming, stochastic quasigradient methods, discrete event systems, simulation, concurrent approximation and optimization.

1. OPTIMIZATION OF DISCRETE EVENT SYSTEMS: INFORMAL DISCUSSION.

The objective of this paper is to address several issues which are important for applications of optimization algorithms to stochastic models of discrete event systems. During last decades considerable efforts were devoted to development of various modeling tools for discrete event systems (DES), in particular Petri nets [1,35], queuing models [21,51], finitely recursive processes [23], and others, for further references see [52]. At the same time the

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development of stochastic programming techniques reached the stage of reasonable theoretical understanding, fairly advanced research software and some sophisticated applications [10]. So far these two fields interacted relatively weakly([17,30,40,46] are among rare exceptions), though discrete event systems seem to be a natural application for stochastic optimization.

We assume that it is possible to identify a set Z of states of DES and the system evolves in time t. The set Z can be finite or infinite, the time can be discrete or continuous. The evolution of the system consists of the sequence of "events" which occur at particular time moments t_i, each event is a change of the state of the system from z_{i-1} to z_i. Thus, the system evolution can be represented as a finite or infinite sequence of pairs

$$U=\left\{(z_0,t_0),(z_1,t_1),\ldots,(z_i,t_i),\ldots\right\} \tag{1}$$

which will be called the <u>path</u> of the system. It is assumed that the system remains in the state z_i at the time interval $[t_i,t_{i+1})$. Optimization will be performed on the <u>simulation model</u> of DES which can reproduce the path U of the system. This model can be built using one of the modeling approaches mentioned above and it would incorporate particular rules which govern the state transitions.

We are interested in the situation when the major structural decisions on the system design are already taken, but the system still depends on the vector of controllable continuous parameters x, and the objective is to select those parameters from an admissible set $X\subseteq R^n$ which would yield the best values of some performance criterion. Examples of such problems can be found in the design of distributed information processing systems [40], manufacturing systems [2], logistics networks. In some DES applications there are ad hoc on-line control strategies which depend on parameters to be adjusted. The objective of optimization here will be to define optimal values of such parameters.

We assume that the system is affected by the presence of uncertainty which can be modelled through uncontrollable stochastic parameters. This stochasticity may be inherently present in the system, for instance it may account for unpredictably changing demand, for the fluctuations in the flow of messages to be processed, for the unpredictable failures of some parts of the system. In other cases it may be a convenient tool to analyze the system.

Thus, both transition times t_i and states z_i which form the path (1) depend on controls and random parameters:

$$U=U(x,\omega)=\left\{(z_0(x,\omega),t_0),(z_1(x,\omega),t_1(x,\omega)),\ldots,(z_i(x,\omega),t_i(x,\omega)),\ldots\right\} \quad (2)$$

where by ω is denoted the possibly infinite sequence of realizations of random parameters:

$$\omega=(\omega(0),\omega(1),\ldots,\omega(i),\ldots)$$

Here each $\omega(i)$ is a random vector with values in $\mathbb{R}^k$ and corresponds to the transition between $z_{i-1}(x,\omega)$ and $z_i(x,\omega)$ in such a way that t_i and z_i depend only on $\omega(s)$, $s=0:i$. For a fixed value x of control parameters and a sample ω of random parameters the simulation run can produce a path $U(x,\omega)$ which will be referred to as a sample path The path $U(x,\omega)$ will be a trajectory of a random process of the special type defined on some probability space $(\Omega,\mathbb{B},P)$ where $\mathbb{B}$ is a Borel field and P is a probability measure. Where it will not cause confusion, we denote an element of this space also by ω. More specifically, this process can be considered to be a generalized semi-Markov process [53]. Precise requirements on the nature of this process will be made later (see Comment 2 to the Theorem 1).

Finally, we assume that some performance criterion $F(x)$ is defined which integrates several desirable features of the system. For instance, in the case of manufacturing system it could be a mixture of a throughput, utilization of important machines, average length of queues, production costs. This performance criterion is expressed as an average over the set of possible sample paths:

$$F(x)=\mathbb{E}_{\omega}f(x,\omega), \quad f(x,\omega)=\varphi(U(x,\omega),x,\omega) \tag{3}$$

Once the sample path is known, the function $f(x,\omega)$ can be either expressed explicitly or by simple recursive formulas. Thus, each simulation run provides the value of $f(x,\omega)$ for some fixed (x,ω). The optimization problem is to minimize the averaged performance criterion (3) on the set $X\subseteq\mathbb{R}^n$ of admissible control parameters:

$$\min_{x\in X} \mathbb{E}_{\omega}f(x,\omega)= \min_{x\in X} \int_{\Omega}f(x,\omega)P(d\omega) \tag{4}$$

This problem is a typical stochastic programming problem, although with the objective function of the special type (3). There have been considerable activities during last two decades concentrated on the development of numerical methods for solving such problems (see [10], where one can find further references). The major difficulty is presented by the expectation operation in (4) since it requires the multidimensional integration which is infeasible for problems of realistic dimension. Therefore the main issue in the algorithmic development was to avoid multidimensional integration and still solve the optimization problem. Two main approaches were used to accomplish this. One is to approximate the probability measure P from (4) by some discrete probability measure P^N. This would reduce the integration in (4) to summation and for important classes of stochastic optimization problems, notably for stochastic programs with recourse, it would lead to a large scale deterministic optimization problem with a special structure [3,5,25,37,42]. Numerical methods were developed which exploit this structure, those methods were particularly efficient for linear programs with recourse. Much work is still needed to adapt these results to the simulation models of discrete event systems.

Another approach makes use of statistical estimates of the values F(x) of the objective function or its gradient $F_x(x)$. It generates a sequence of points $x^0, x^1, \ldots, x^s$ which converges to the optimal solution of the problem (4) and at each step only a small number of observations of the function $f(x,\omega)$ or its gradient is needed, possibly only one observation. One such algorithm is the method of stochastic quasigradients [8,9,13,29,31,39,41,47], among its origins is the stochastic approximation [27]. The method produces a sequence $x^0, x^1, \ldots x^s, \ldots$ according to the rule

$$x^{s+1} = \pi_X(x^s - \rho_s \xi^s) \tag{5}$$

where π_X denotes projection operator on the set X, ρ_s is a stepsize and ξ^s is a stochastic quasigradient with the property

$$\mathbb{E}(\xi^s | x^0, \ldots, x^s) = F_x(x^s) + a_s \tag{6}$$

where a_s vanishes as s tends to infinity. In other words, ξ^s is a statistical estimate of the gradient and in the simplest case one may take $\xi^s = f_x(x^s, \omega^s)$ where ω^s is an independent observation of random parameters.

This paper deals with an application of procedures of the type (5)-(6) to simulation models of discrete event systems. We address some issues which result from the special type of the objective function (3) conditioned by the following specific features of DES.

1. In many cases the performance criterion (3) depends on the stationary behavior of the system which is attained only asymptotically. In such cases, in order to make one observation of the objective function ideally, we should obtain a sample path of infinite length, which is impossible. If we stop a simulation at t=T we would obtain an observation of a function $F^T(x)$ which tends to F(x) with $T \longrightarrow \infty$. Conditions when such convergence occurs for stochastic programming problems were studied in [7,26,28,44,50]. In

this paper in the section 2 we consider algorithmic issues. In particular it is necessary to design a method to minimize F(x) which uses observations of $F^T(x)$ and preferably can work with small values of T on the first iterations, when x^s is far from the solution, and gradually increase T while approaching the solution. It means that method optimizes different functions on different iterations and optimization problem is nonstationary [11]. However we show in the section 2 that under quite general conditions the method on the basis of (5) generates a sequence x^s which converges to the solution of the problem (4).

2. Another important specific feature of DES is that the sample path often depends discontinuously on controlled parameters [18]. This may create difficulties for obtaining statistical estimates ξ^s of the gradient needed in (5)-(6). A straightforward approach for computing such an estimate is to take finite differences, but this would lead to large variance of ξ^s and often prohibitive requirements on the amount of simulation runs even for problems of moderate dimension. Considerable efforts were dedicated recently to the development of differentiation schemes which utilize a knowledge of the structure of DES in order to obtain more precise statistical estimates of the gradient with less simulation effort. Two main approaches are the perturbation analysis [21,51] and the score function (likelihood ratio) method [17,43,45], special notions of derivatives of measures [40] proved to be useful in this respect. However, original versions of these techniques encounter some difficulties. In particular, the perturbation analysis generally gives a biased estimate when a sample path of the system depends discontinuously on control parameters [20]. More rigorous discussion of this issue is contained in the section 3, a simple but illuminating example is contained in the Appendix B. On the other

hand, the score function method deals successfully with discontinuities, but in some cases may yield estimates with large variance [43,46]. Both techniques are now under vigorous development and some of the weak points have been removed [18,33,46].

We consider here the complementary approach intended for the cases when differentiation schemes encounter difficulties. In particular, we deal with discontinuities by developing methods which need only observations of the objective function instead of observations of its gradients, and at the same time represent an improvement compared with ordinary finite differences. Two such methods are presented here.

In the section 3 an enhanced finite difference scheme is presented with reduced variance, it uses the random smoothing and common random numbers. In the section 4 we introduce a new class of algorithms which perform on-line approximation of the objective function on the basis of the current and a number of previous observations. The step direction ξ^s in (6) would be a gradient of the approximation or the direction to the minimum of the approximation. Convergence of one of the algorithms of this type is proved in the Appendix A and a numerical experiment is presented in the Appendix C.

2. OPTIMIZATION OF THE STATIONARY BEHAVIOR

We consider here the case when the system evolves on the infinite time horizon $[t_0,\infty)$. At each t there exists a probability measure $Q(z_0,x,t;dz)$ such that

$$\int_{Z'} Q(z_0,x,t;dz) \tag{7}$$

defines the probability that at the time moment t the state of the system belongs to the set $Z' \subseteq Z$. This measure depends also on the initial state z_0 and control parameters x. Let us assume that there exists the stationary measure $Q(x;dz)$ which defines the stationary state distribution of the system similar to (7), i.e. $Q(z_0,x,t;dz)\longrightarrow Q(x;dz)$ as $t\longrightarrow\infty$ in a sense that will be specified later, and this measure does not depend on the initial state $z_0\in Z$. The performance criterion $F(x)$ is defined in terms of the limiting measure:

$$F(x)=\int_{\Omega}\varphi(U(x,\omega),x,\omega)P(d\omega) = \int_{Z}\psi(x,z)Q(x;dz) \tag{8}$$

and the problem (4) is to be solved with the performance criterion of this type. Many DES optimization problems can be formulated this way, in particular the problems of optimization of Markov systems [40].

The main difficulty of the problem (4),(8) is that neither the measures $Q(z_0,x,t;dz)$ nor especially the measure $Q(x;dz)$ are known explicitly and the solution should be found by observing the values of the function $\varphi(\cdot,x,\omega)$ or related values on finite time intervals. Let us formulate this more precisely.

Let us consider a partition of the time horizon $[t_0,\infty)$ into a sequence of time intervals $\Delta_s=[t_{1s},t_{2s})$, $t_{11}=t_0$, $t_{2s}=t_{1,s+1}$, $t_{1s}=t_{1s}(\omega)$, $t_{2s}=t_{2s}(\omega)$. We would like to define an algorithm which solves the problem (4),(8) during one simulation run, therefore we allow changes in the values of control parameters in the course of simulation. Let us assume that the value x^s of control parameters is set at the beginning of the interval Δ_s and remains unchanged during this interval. Some more notations follow:

$x(s)$ - the sequence $x^1,\ldots,x^s$;

$t(s)$ - the sequence $t_{11},\ldots,t_{1s}$;

$U^s=U^s(x(s),\omega)$ - the section of the sample path which is obtained by discarding all events outside the interval Δ_s;

$U(s)=U(s,x(s),\omega)$ - the section of the sample path from the simulation start at $t=t_0$ to the beginning of the interval Δ_s at $t=t_1$

$\mathbb{B}_s$ - a σ-field defined by $U(s),x(s),t(s)$.

Υ - the set of sequences $\left\{(z_i,t_i),\ i=0,1,\dots\right\}$, finite or infinite, and such that $z_i\in Z$, $t_i\in\mathbb{R}^+$, $t_{i+1}\geq t_i$.

$\varphi(U,x,\omega)$, $\varphi_i(U,x,\omega)$, $i=1:K$ - mappings $\Upsilon\times X\times\Omega\longrightarrow\mathbb{R}$, at this moment we assume only that these functions are such that the following expression is well defined:

$$F(s,x,\omega)=D(\mathbb{E}(\varphi_1(U^s,x,\omega)\,|\,\mathbb{B}_s),\dots,\mathbb{E}(\varphi_K(U^s,x,\omega)\,|\,\mathbb{B}_s))$$

where D is a mapping $\mathbb{R}^K\longrightarrow\mathbb{R}$.

If $F(s,x,\omega)\longrightarrow F(x)$ in some sense then we can use techniques of nonstationary optimization [11] to solve the problem (4),(8). That is, on the step s of the optimization algorithm we make one minimization step of the function $F(s,x,\omega)$, and in this way arrive at the minimum of $F(x)$. This results in the following algorithm which allows to solve (4),(8) in a single simulation run. Other single run simulation optimization algorithms are presented in [30,40,46].

Algorithm 1.

The simulation starts at $t=0$ with some initial value x^0 of control variables and initial state z_0. The algorithm partitions the time horizon $[t_0,\infty)$ into the sequence of intervals $\Delta_1,\dots,\Delta_s,\dots$, and changes the values of control variables x at the end of each time interval as follows.

1. Suppose that the process arrived at the end of the interval Δ_{s-1} and the interval Δ_s starts. The time $t_{2s}=t_{1,s+1}$ of the end of this interval is defined either deterministically or as a stopping time measurable with respect to $\mathbb{B}_{s+1}$.

2. At $t=t_{1,s+1}$ the observation ξ^s is made such that

$$\mathbb{E}(\xi^s | \mathbb{B}_s) = F_x(s, x^s, \omega) + a_{1s} \tag{8}$$

3. At $t = t_{1,s+1}$ the values of control variables are changed as follows:

$$x^{s+1} = \pi_X(x^s - \rho_s \xi^s) \tag{10}$$

where $\rho_s \geq 0$ is the stepsize and π_X is the projection operator on the set X. Let us denote

$$F^* = \min_{x \in X} F(x), \quad X^* = \left\{ x^*: \ x^* \in X, \ F(x^*) = \min_{x \in X} F(x) \right\}$$

Convergence of the Algorithm 1 is established by the following theorem.

Theorem 1. Suppose that the following conditions are satisfied:

1. $X \subset \mathbb{R}^n$ is a convex compact set.

2. $F(x)$ is continuous on X and the set X^* is convex.

3. The function $F(s,x,\omega)$ is a convex function with respect to x with a subdifferential which is bounded on X a.s. uniformly with respect to s, $F(s,x^s,\omega)$ converges to $F(x^s)$ as $s \longrightarrow \infty$ and $\limsup_s F(s,x,\omega) \leq F^*$ a.s. uniformly for $x \in X^*$.

4. $\mathbb{E}(\|\xi^s - F_x(s,x^s,\omega) - a_{1s}\|^2 | x^0, \ldots, x^s) = C_s < \infty$, $a_{1s} \longrightarrow 0$ a.s.,

5. $\rho_s \geq 0$, $\sum_{s=0}^{\infty} \rho_s = \infty$, $\sum_{s=0}^{\infty} C_s^2 \rho_s^2 < \infty$

Then the sequence x^s generated by (9)-(10) has accumulation points and all such points belong to the set X^* of solutions of the problem (4),(8).

Proof of this theorem is given in the Appendix A.

Comments.

1. Similar result holds for differentiable nonconvex functions $F(s,x,\omega)$, but convergence would be to the points where the first order necessary conditions for optimality are fulfilled.

2. We intentionally did not specify precisely the properties of the stochastic process which generates the sample path U and the

properties of the function φ in order to formulate a minimal set of conditions which guarantee applicability of the method (5)-(6) to DES. Now the properties of U and φ are implied by conditions 3 and 4 of the theorem. For example, a convergence part of the condition 3 is obviously satisfied for regenerative case [4] due to representation of the function $F(s,x,\omega)$. Some relevant results for nonregenerative ergodic case are contained in [40], where it was required that the lengths of the intervals Δ_s tend to infinity. More research is needed to translate conditions 3,4 into explicit general requirements on the process in nonregenerative case.

3. Condition 3 is satisfied, for instance, when $F(s,x,\omega)$ converges to $F(x)$ uniformly over (x,ω) as $s\to\infty$.

4. Important issue for implementation of this algorithm is how to select the stepsizes. This can be done similarly to [13,31,39,47].

In the remaining sections of this paper we deal with the problem of determining the step direction ξ^s for the algorithm 1.

3. OBTAINING STATISTICAL ESTIMATES OF THE GRADIENT.

In this section we give a very brief survey of approaches for computing a stochastic quasigradient ξ^s for the method (5) and indicate some of difficulties which result from specific features of DES. We need this to place the methods proposed here in the right context, one in the second part of this section and another in the section 4, and explain why we consider them relevant for DES.

Let us consider properties of the objective function from (3):

$$F(x)=\mathbb{E}_\omega f(x,\omega)=\mathbb{E}_\omega \varphi(U(x,\omega),x,\omega) \tag{11}$$

For the sake of clarity we assume that the sample path $U(x,\omega)$ consists of a finite fixed number N of pairs, which does not depend on ω. Such situation may appear either when the transient behavior of a system is studied or when a section of a sample path is used to make inference on the system behavior, like in the previous section. The case when N depends on ω or is infinite brings nothing conceptually new to the discussion of this section except some technicalities.

One of the important specific features of DES is that the sample path $U(x,\omega)$ often depends discontinuously on (x,ω). This is true for models of systems with several competing concurrent processes, like Petri net models of manufacturing and communication systems, models which include failures and repairs, many queueing models etc. The example in the Appendix B shows that even for very simple problems $f(x,\omega)$ is discontinuous, or more precisely, piecewise continuous with infinite number of continuity sets. The importance of this phenomenon is recognized in the theory of DES (see discussion in [18,51]) where it is known as the event order change.

In such cases also the function $f(x,\omega)$ from (11) depends discontinuously on (x,ω). This creates difficulties for some methods of sensitivity analysis based on differentiation schemes, which can be used for obtaining ξ^s. In particular, event order changes critically affect the infinitesimal perturbation analysis [20,22]. This technique suggests $f_x(x^s,\omega^s)$ for ξ^s with independent ω^s, i.e. simply changes the order of differentiation and expectation in (11). It should be noted that recent developments in perturbation analysis [15,16,18] deal successfully with some of the cases when discontinuities occur.

Another sensitivity analysis techniques called the score function (likelihood ratio) method [17,40,43,45] deals successfully with discontinuities when the objective function has the form

$$F(x)=\mathbb{E}_{\omega}f(\omega)= \int f(\omega)dH(x,\omega) \qquad (12)$$

where $H(x,\omega)$ is a distribution with respect to which expectation is taken (provided $H(x,\omega)$ satisfy additional differentiability conditions). This technique, however, in some cases provide estimate with large variance [43,51]. It is also under vigorous development now and the scope of its applicability has been enlarged recently [33,46]. For further discussion of relative applications domains for these techniques see [43,45,51].

The approach which we pursue here is to design methods of computing stochastic quasigradient ξ^s based not on differentiation schemes, as in the methods mentioned above, but solely on observations $f(x,\omega)$ of the objective function. One such observation can always be made on the basis of one sample path, or its portion, although sometimes it is necessary to make several observations for getting ξ^s. This is not an alternative, but rather a complementary approach to differentiation schemes for cases when such schemes encounter difficulties.

One obvious way to construct statistical estimate of $F_x(x)$ is by using the finite differences:

$$\xi^s = \sum_{i=1}^{n} \frac{f(x^s+\delta_s e_i,\omega^{is}) - f(x^s,\omega^{0s})}{\delta_s} e_i \tag{13}$$

or similar expressions for central finite differences. Here e_i are unit vectors of $\mathbb{R}^n$, ω^{is}, $i=0:n$ are independent observations of ω, each corresponds to the separate run of the model. This approach has two serious shortcomings:

- it requires at least n+1 simulation runs which grows to 2n for central finite differences;

- the variance of the estimate (13) approaches infinity while $\delta_s \longrightarrow 0$ since for independent observations

$$\mathbb{E}(\|\xi^s - \mathbb{E}\xi^s\|^2 \mid x^0,\ldots,x^s) = \frac{1}{\delta_s^2}\sum_{i=1}^{n}(C_{si}+C_{s0}) \tag{14}$$

where

$$C_{si} = \mathbb{E}((f(x^s+\delta_s e_i,\omega^{is}) - F(x^s+\delta_s e_i))^2 \mid x^0,\ldots,x^s),\ i=1:n$$

$$C_{s0}=\mathbb{E}((f(x^s,\omega^{0s})-F(x^s))^2|x^0,\ldots,x^s)$$

On the other hand, taking large values of δ_s would decrease variance, but lead to significant bias.

One might think of using the common random numbers for computing various observations of function values in (13). This would reduce the variance but generally would introduce a bias precisely due to discontinuities in the sample path discussed above.

The number of simulation runs can be reduced by the following device [8]. Suppose that v_i are random vectors uniformly distributed on the unit sphere in $\mathbb{R}^n$ and i=1:M, M≥1. Then one can take

$$\xi^s=\sum_{i=1}^{M}\frac{f(x^s+\delta_s v_i,\omega^{is})-f(x^s,\omega^{0s})}{\delta_s}v_i \qquad (15)$$

if v_i is independent from ω^{is}. This can reduce the simulation effort considerably since M could be equal 1. However the problem with increasing variance would persist. In order to partially alleviate it we propose to use the smoothing.

We propose here to smooth the function $f(x,\omega)$ and make it differentiable by deliberately introducing some noise into the control variables of the system. Contrary to what might be expected, introduction of the noise would lead to estimates with smaller variance then in (14) because this would make possible the use of common random numbers. Let us consider two independent random vectors $u=(u_1,\ldots,u_n)$ and $v=(v_1,\ldots,v_n)$ with components independently distributed on the interval [-1,1], they are also independent from random parameters ω. Instead of the original system we consider a system whose control variables have the form

$$\bar{x}=x+\delta(u+v),\ \delta\geq 0 \qquad (16)$$

We can simulate a new system by the same model as the original one, it is enough to take $(x+\delta(u^s+v^s),\omega^s)$ instead of the variables (x,ω^s) and run the simulation model. Characteristics of this system are obtained by averaging over such runs, i.e. by averaging over (ω, u, v). In particular, the performance criterion takes the form

$$F(x,\delta)=\mathbb{E}_{\omega uv} f(x+\delta(u+v),\omega) \quad (17)$$

If X is a compact set and F(x) is continuous then $F(x,\delta)\longrightarrow F(x)$ as $\delta\longrightarrow 0$ uniformly over X. Moreover, it is also differentiable, as the following lemma shows:

Lemma 1. Suppose that $\mathbb{E}_{\omega} \sup_{x\in U_{2\Delta\sqrt{n}}(X)} |f(x,\omega)|<\infty$, where

$$U_{2\Delta\sqrt{n}}(X)=\left\{x: \inf_{y\in X} \|y-x\|\leq 2\Delta\sqrt{n}\right\}$$

Then for any δ: $0<\delta<\Delta$ the function (17) is differentiable and

$$\frac{d}{dx}\mathbb{E}_{\omega uv} f(x+\delta(u+v),\omega)=$$

$$\mathbb{E}_{\omega uv} \sum_{i=1}^{n} \frac{f(x+\delta(u+v)+\delta(1-v_i)e_i,\omega)-f(x+\delta(u+v)-\delta(1+v_i)e_i,\omega)}{2\delta} e_i \quad (18)$$

The proof of this lemma is made similarly to general results on smoothing found in [19]. Note, that (18) can be viewed as the special type of the central finite differences. Now it is possible to take independent observations ω^s, u^s, v^s and choose ξ^s as follows:

$$\xi^s= \sum_{i=1}^{n} \frac{f(x^s+\delta_s(u^s+v^s)+\delta_s(1-v_i^s)e_i,\omega^s)-f(x^s+\delta_s(u^s+v^s)-\delta_s(1+v_i^s)e_i,\omega^s)}{2\delta_s} e_i$$

There is one important difference between the last formula and the ordinary finite differences from (13). Here all the observations of the objective function needed to compute the differences are made with the same observation ω^s of random parameters and with slightly different (for small δ_s) control parameters, while in (13) all observations were made with different and independent values for ω. This makes the variance of ξ^s based on (18) considerably smaller, especially for small δ_s. Let us show that for the class of objective functions most commonly found in the models of discrete event systems.

Let us fix $\delta>0$, $x\in X$ and define

$$L(\delta,x,\omega)=\sup_{\substack{x,y\in U_{\delta\sqrt{n}}(x)\\ \|x-y\|=2\delta}} \frac{|f(x,\omega)-f(y,\omega)|}{\|x-y\|}, \quad L(\delta,x)=\mathbb{E}L(\delta,x,\omega)$$

Definition. A function $f(x,\omega)$ is a function with weak Lipchitz property of the order τ if $L(\delta,x)\le L(x)\delta^{-\tau}$ for some $L(x)<\infty$.

This property is closely related to Hoelder continuity. Practically all functions of interest fall within this definition, in particular for $\tau=0$ we obtain Lipchitzian functions and for $\tau=1$ we obtain functions for which $\mathbb{E}\sup_{x\in U_{\delta\sqrt{n}}(x)}|f(x,\omega)|<\infty$. What is more important, for many discontinuous, but piecewise Lipchitzian functions, the value of τ equals 0 or at least $\tau<1$. For such functions ξ^s based on (18) has considerably smaller variance then traditional finite differences due to the following estimate

$$\mathbb{E}(\|\xi^s-\mathbb{E}\xi^s\|^2|x^0,\ldots,x^s)\le\mathbb{E}(\|\xi^s\|^2|x^0,\ldots,x^s)\le nL^2(x)\delta_s^{-2\tau}$$

There will be also a bias here, but in the case if $F(x)$ is differentiable, it will be asymptotically smaller then δ_s. Therefore for such cases introduction of noise in the control variables of the system yields a surprising result: it provides more accurate estimates of the gradient then those obtained without noise.

4. CONCURRENT APPROXIMATION AND OPTIMIZATION

In this section we introduce a general approach for constructing stochastic optimization algorithms which is based on observations of the values of the objective function only. It is not limited to discrete event systems. However, it is particularly useful for optimization of DES when direct application of differentiation schemes is difficult due to discontinuities in the sample paths, see discussion at the beginning of the section 3. It needs considerably less simulation effort compared with other techniques which do not directly involve differentiation. Finally, we specify one new algorithm based on this approach, prove the convergence theorem and present results of numerical experiments.

Informally speaking, the idea behind the proposed approach is the following. Suppose that in the course of optimization the sequence of points $x^0,\dots,x^s$ and the set of observations $\zeta_1,\dots,\zeta_s$ such that $\mathbb{E}(\zeta_i|x^0,\dots,x^i)=F(x^i)$, $i=0:s$ were obtained. These observations are used to approximate the function $F(x)$ by a function $F(s,x)$. Let $\bar{x}^s \in X$ be a point at which $F(s,x)$ attains its minimal value over the set X. Then the next approximation to the optimal solution of the problem (4) is obtained as a linear combination of x^s and $\bar{x}^s$:

$$x^{s+1}=(1-\rho_s)x^s+\rho_s\bar{x}^s$$

or, it is obtained by making a step in the direction opposite to the gradient of the approximating function:

$$x^{s+1}=\pi_X(x^s-\rho_s F_x(s,x^s))$$

After that a new observation is made, the approximation $F(s,x)$ is updated using this observation and the process continues.

Let us compare this approach with two other techniques which does not use derivatives: finite differences and response surface methods. Shortcomings of the finite differences were discussed in the section 3. Here we point out that all observations of the objective function which are made at the point x^s in order to obtain an estimate of the gradient via finite differences (13) are discarded on the next iteration when all observations are made again at the point x^{s+1}. At the same time these observations contain considerable amount of information on the value of $F_x(x^{s+1})$ since the stepsize ρ_s is usually small and $F(x)$ is continuously differentiable. The approach which we propose here use all this information, which result in estimates with smaller variance and/or smaller simulation effort since it can work with only one new observation on each iteration.

The response surface method [24,32,34,36] constructs approximation of the objective function on the basis of observations distributed over some region, then finds the minimum of this approximate function. These steps may be repeated. The novelty of the approach proposed here is that we integrate approximation and optimization into a single on-line procedure. Approximation is updated after each step using new samples made at points (or point) obtained by optimization procedure. In this way excessive sampling in regions far from a vicinity of optimum is avoided. This again results in savings of simulation effort. Of course, an extensive experimentation is needed to further validate these assertions.

In fact, much has to be done to design on its basis a practical algorithm, some of the issues to be clarified are how to choose an appropriate approximation criterion, how to select approximation points properly in order to insure stability of approximations, how to discard old points, etc. Some of those issues are reflected in the following scheme.

<u>Algorithm 2.</u>

1. At the beginning the initial point x^1 is chosen, $\nu_0=0$, $Y^0=\emptyset$, $\Xi^0=\emptyset$ are set.

2. Suppose that prior to making iteration number s the algorithm generated the point x^s, the set of observation points $Y^{s-1}=\left\{y^i, i=1:\nu_{s-1}\right\}$, $Y^{s-1}\subseteq X$, and the set of observations $\Xi^{s-1}=\left\{\zeta_i, i=1:\nu_{s-1}\right\}$ such that $\mathbb{E}(\zeta_i|y^i)=F(y^i)$. The following computations are performed at the iteration number s:

i. The new set of observation points $\bar{Y}^s(x^s)=\left\{y^{1s}\ldots,y^{k_s s}\right\}$ is selected, $\bar{Y}^s\subseteq X$ and observations $\zeta_1^s,\ldots,\zeta_{k_s}^s$ are made such that $\mathbb{E}(\zeta_i^s|y^{is})=F(y^{is})$, the sets Y^s and Ξ^s are obtained:

$$\nu_s=\nu_{s-1}+k_s,\ Y^s=\left\{y^i,\ i=1:\nu_s,\ y^i=y^{i-\nu_{s-1},s},\ i=\nu_{s-1}+1:\nu_s\right\},$$

$$\Xi^s=\left\{\zeta^i,\ i=1:\nu_s,\ \zeta^i=\zeta^{i-\nu_{s-1},s},\ i=\nu_{s-1}+1:\nu_s\right\}$$

ii. The weights $\alpha_s(y)$, $y\in Y^s$ are selected, these weighs are used to define the approximation criterion.

iii. The values of approximation parameters a^s are defined by solving the following approximation problem:

$$\min_{a\in A} \sum_{i=1}^{\nu_s}\alpha_s(y^i)\Phi(s,\zeta^i-F(s,a,y^i)) \tag{19}$$

where $A\subseteq\mathbb{R}^k$, $F(s,a,x)$ is some predefined class of functions, which is used to approximate $F(x)$ and the function $\Phi(s,w)$ measures the closeness of fit of the approximation $F(s,a,y)$ at the point y.

iv. The next approximation x^{s+1} to the optimal solution is obtained either by

$$x^{s+1}=(1-\rho_s)x^s+\rho_s\bar{x}^s,\ F(s,a^s,\bar{x}^s)=\min_{x\in X} F(s,a^s,x),\ \bar{x}^s\in X \tag{20}$$

or by

$$x^{s+1}=\pi_X(x^s-\rho_s F_x(s,a^s,x^s)) \tag{21}$$

In order to specify implementable algorithm on the basis of this scheme it is necessary to choose the approximating function $F(s,a,x)$, the approximation criterion $\Phi(s,w)$, the set of observation points Y^s and weights $\alpha_s(y)$. Some of the issues concerning convergence of this method to the optimal solution of the problem (4) for particular choices of $F(s,a,x)$, $\Phi(s,w)$, Y^s, $\alpha_s(y)$, were clarified in [12]. In the remainder of this section we shall present one algorithm not covered there.

Let us take

$$a=(b,d),\ b\in\mathbb{R}^1,\ d\in\mathbb{R}^n,\ A=\mathbb{R}^{n+1},\ F(a,x)=b+d^T(x-x^s),\ \Phi(s,w)=w^2 \tag{22}$$

Then the problem (19) has the explicit solution

$$d^s=Q^s u^s \tag{23}$$

where

$$u^s=\sum_{i=1}^{\nu_s}\alpha_s(y^i)\left(\zeta_i-\frac{1}{\sigma_s}\sum_{j=1}^{\nu_s}\alpha_s(y^j)\zeta_j\right)(y^i-x^s),\ \sigma_s=\sum_{i=1}^{\nu_s}\alpha_s(y^i)$$

$$Q^s=\left(\sum_{i=1}^{\nu_s}\alpha_s(y^i)(y^i-x^s)\left((y^i-x^s)^T-\frac{1}{\sigma_s}\sum_{j=1}^{\nu_s}\alpha_s(y^j)(y^j-x^s)^T\right)\right)^{-1}$$

Let us specify now the rule for selection of observation points. Here we consider the case when only one observation point is added on each iteration, in order to minimize simulation requirements:

$$\bar{Y}^s=\{y^{1s}\},\quad Y^s=\left\{y^1,\dots,y^s\right\},\quad y^s=x^s+r_s v^s \tag{24}$$

where v^s are independent random vectors with zero mean. Introduction of the term $r_s v^s$ is necessary in order to stabilize the approximation process.

Finally, let us specify the rule for choosing approximation weights:

$$\alpha_s(y^i)=\alpha_{is}=\begin{cases}(1-\beta_s)\alpha_{i,s-1} & \text{if } i<s\\ \beta_s & \text{if } i=s\end{cases} \tag{25}$$

where $\beta_s\le 1$, $\beta_1=1$. Now it is possible to represent (23)-(25) in recursive form in order to avoid the matrix inversion on each iteration. Using the identity

$$(I+ab^T)^{-1}=I-\frac{ab^T}{1+b^Ta}$$

we obtain

$$d^s=\left(I-\beta_s\frac{Q^{s-1}\delta_{sx}\delta_{sx}^T}{1+\beta_s\delta_{sx}^TQ^{s-1}\delta_{sx}}\right)\left(d^{s-1}+\beta_s(\zeta^{s-1}-\zeta(y^s))Q^{s-1}\delta_{sx}\right) \tag{26}$$

$$Q^s=\frac{1}{1-\beta_s}\left(Q^{s-1}-\beta_s\frac{Q^{s-1}\delta_{sx}\delta_{sx}^TQ^{s-1}}{1+\beta_s\delta_{sx}^TQ^{s-1}\delta_{sx}}\right),\quad \delta_{sx}=\chi^{s-1}-(y^s-x^{s-1})$$

$$\zeta^s=(1-\beta_s)\zeta^{s-1}+\beta_s\zeta(y^s),\quad \chi^s=(1-\beta_s)\chi^{s-1}+\beta_s(y^s-x^{s-1})-\Delta^s,\quad \Delta^s=x^s-x^{s-1},$$

The iterations of the algorithm proceed as follows:

$$x^{s+1}=\pi_X(x^s-\rho_s\gamma_s d^s),\quad \gamma_s=\begin{cases}C_0/\|d^s\| & \text{if } \|d^s\|\ge C_0\\ 1 & \text{otherwise}\end{cases} \tag{27}$$

The following theorem confirms convergence of the algorithm (22)-(27). By $\mathbb{B}_s$ will be denoted the σ-field defined by $x^0,\dots,x^s$.

Theorem 2. Suppose that the following conditions are satisfied:

1. The set $X\subset\mathbb{R}^n$ is convex and compact.

2. The function $F(x)$ is convex continuously differentiable and $F_x(x)$ satisfy the Lipchitz condition on X.

3. $\mathbb{E}(v^s v^{s^T} | \mathbb{B}_s) = \mathbb{E} v^s v^{s^T} = V > 0$, $\mathbb{E}(v^s | \mathbb{B}_s) = 0$, $\|v^s\| < C < \infty$, $\mathbb{E}(\zeta^s - F(y^s) | \mathbb{B}_s, v^s) = 0$, $\mathbb{E}((\zeta^s - F(y^s))^2 | \mathbb{B}_s, v^s) < C < \infty$.

4. $\beta_s \geq 0$, $\sum_{i=1}^{\infty} \beta_i = \infty$, $(1-\beta_s) \dfrac{r_{s-1}^2}{r_s^2} = 1 - \beta_{1s}$, $\dfrac{\beta_s}{\beta_{1s}} \longrightarrow 1$, $r_s \longrightarrow 0$, $r_{s-1} \geq r_s$,

$$\frac{r_{s-1}}{r_s} \longrightarrow 1, \quad \frac{\rho_{s-1}}{\beta_s r_s^2} \longrightarrow 0, \quad \sum_{i=1}^{\infty} \frac{\beta_i^2}{r_i^2} < \infty, \quad \sum_{i=1}^{\infty} \frac{\rho_{i-1}^2}{r_i^4} < \infty, \quad \frac{1}{\beta_s} \left| \frac{\rho_{s-2}\beta_s}{\rho_{s-1}\beta_{s-1}} - 1 \right| \longrightarrow 0,$$

$$\rho_s \geq 0, \quad \sum_{i=1}^{\infty} \rho_i = \infty$$

Then the sequence x^s has accumulation points and all such points belong to the set X^* of solutions of the problem (4).

The proof of this theorem is contained in the Appendix A, numerical experiments are contained in the Appendix C.

Comments.

1. With minor changes in the theorem conditions similar result holds for nonconvex $F(x)$ with gradient which satisfies the Lipchitz condition. In this case convergence would occur to points which satisfy the first order optimality conditions.

2. Although the stepsize condition 4 of the theorem looks complicated, it is satisfied for a reasonable range of possible sequences r_s, β_s and ρ_s. For example if those sequences behave asymptotically like s^{-r}, $s^{-\beta}$ and $s^{-\rho}$ then the condition 4 is satisfied for

$$\rho \leq 1, \ \beta < 1, \ \rho - \beta - 2r > 0, \ 2\beta - 2r > 1, \ 2\rho - 4r > 1$$

for instance for $\rho = 1$, $\beta = 0.7$, $r = 0.14$. Those conditions have only an asymptotic value and for practical implementation β_s and ρ_s would be taken constant and ρ_s would be selected according to one of the adaptive rules [12,39,47].

3. The algorithm (23)-(27) is one of many possible variants of the general scheme described in the Algorithm 2. Due to explicit

formulas for the step direction, it is easier to prove convergence for (23)-(27), but other variants could be more advantageous from practical point of view. We tried, for instance, a similar algorithm based on L_1 approximation and found it to be more stable. M-estimates, trimming and other techniques of robust statistics [22] can be applied here. In order to select the measure for generating identification step v^s the methods of optimal experiment design can be used [6,49].

APPENDIX A. PROOF OF THEOREMS 1,2.

In what follows we denote by C, C_1, C_2 some finite constants, to simplify notations different such constants are denoted by the same letter. The same convention holds for a_s by which we denote an arbitrary sequence which tends to zero.

At the beginning we need several lemmas.

Lemma 2. Suppose that for a nonnegative sequence a_s the following inequality is satisfied:

$$a_{s+1} \le a_s - \beta_s(a_s(1-\varepsilon_s)-C),\ C\ge 0,\ \beta_s \ge 0,\ \beta_s \longrightarrow 0,\ \sum_{i=1}^{\infty} \beta_i = \infty,\ \varepsilon_s \longrightarrow 0 \qquad (A.1)$$

Then $\limsup_i a_i \le C$

Proof.

Let us fix some δ: $0<\delta<1$ and take such k that $\varepsilon_s<\delta$, $\beta_s<\delta/C$ for $s\ge k$. Then

$$a_{s+1} - a_s \le \delta, \quad s \ge k. \qquad (A.2)$$

Suppose that $a_s(1-\delta)-C>\delta$ for $s>k$. Then (A.1) yields for $s>k$:

$$a_{s+1} \le a_s - \delta\beta_s,\ a_s \le a_k - \delta \sum_{i=k}^{s} \beta_i$$

which contradicts with nonnegativity of a_s due to $\sum_{i=1}^{\infty} \beta_i = \infty$. Therefore there exists $l \ge k$ such that $a_l(1-\delta)-C\le\delta$. Now for any $s>l$ there are the following two possibilities:

i. $a_{s-1}(1-\delta)-C\le\delta$, then due to (A.2)

$$a_s(1-\delta)-C\le a_{s-1}(1-\delta)-C+(1-\delta)(a_s-a_{s-1})\le\delta+(1-\delta)\delta<2\delta$$

ii. $a_{s-1}(1-\delta)-C>\delta$, then $a_s\le a_{s-1}$ and

$$a_s(1-\delta)-C\le a_{s-1}(1-\delta)-C$$

Therefore $a_s(1-\delta)-C<2\delta$ for $s\ge 1$ and

$$\limsup_i a_i < (C+2\delta)/(1-\delta)$$

which yields the required assertion since δ can be taken arbitrary small. ∎

In what follows we deal with the convergence with probability 1 (a.s.) of random sequences defined on some probability space $(\Omega,\mathbb{B},P)$ where $\mathbb{B}$ is a Borel field and P is a probability measure. An element of this space is denoted by ω.

Lemma 3. Suppose that

$$a_{s+1}=(1-\beta_s)a_s+\beta_s\varepsilon_s,\ \varepsilon_s\longrightarrow 0 \text{ a.s. } \sum_{i=1}^{\infty}\beta_i=\infty,\ \beta_s\le 1 \qquad (A.3)$$

Then $a_s\longrightarrow 0$ a.s.

Proof.

From (A.3) we obtain

$$\|a_{s+1}\|\le(1-\beta_s)\|a_s\|+\beta_s\|\varepsilon_s\|,\ \|a_k\|\le\|a_l\|-\sum_{i=l+1}^{k}\beta_i(\|a_i\|-\|\varepsilon_i\|)$$

now if for some ω there exist l and $\delta>0$ such that $\|a_i\|-\|\varepsilon_i\|>\delta$ for $k>l$ then

$$\|a_k\|\le\|a_l\|-\delta\sum_{i=l+1}^{k}\beta_i$$

which contradicts nonnegativity of $\|a_k\|$ for sufficiently large k. Therefore for any ω, δ and l there exists $k=k(\omega,\delta,l)\ge l$ such that $\|a_k\|-\|\varepsilon_k\|<\delta$. Then (A.3) implies

$$\|a_{s+1}\|\le\max\{\|a_s\|,\|\varepsilon_s\|\}$$

which yields

$$\|a_s\|\le\max\{\|a_k\|,\max_{i\ge k}\|\varepsilon_i\|\}\le\max_{i\ge k}\|\varepsilon_i\|+\delta$$

Since $\varepsilon_s\longrightarrow 0$ a.s. the last inequality implies $\|a_s\|\longrightarrow 0$ a.s. ∎

The assertion of this lemma can be alternatively obtained from results contained in [48].

Lemma 4. Suppose that

$$a_{s+1}=(1-\beta_{1s})a_s+\beta_{2s}\xi^s,\ \mathbb{E}(\xi^s|a_1,\dots,a_s)=\varepsilon_s,\ \varepsilon_s\beta_{2s}/\beta_{1s}\to 0 \text{ a.s.}$$

$$\beta_{1s}\le 1,\ \sum_{i=1}^{\infty}\beta_{1i}=\infty,\ \sum_{i=1}^{\infty}\beta_{2i}^2<\infty,\ \mathbb{E}(\|\xi^s-\varepsilon_s\|^2|a_1,\dots a_s)<C<\infty$$

Then $a_s\to 0$ a.s.

Proof.

Let us denote

$$a_{1,1}=0,\quad a_{1,s+1}=(1-\beta_{1s})a_{1,s}+\beta_{2s}(\xi^s-\varepsilon_s),$$

$$a_{2,1}=a_1,\ a_{2,s+1}=(1-\beta_{1s})a_{2,s}+\beta_{2s}\varepsilon_s$$

Then $a_s=a_{1,s}+a_{2,s}$ and $\|a_{2,s}\|\to 0$ a.s. due to the Lemma 3.

$$\|a_{1,s+1}\|^2=(1-\beta_{1s})^2\|a_{1,s}\|^2+2\beta_{2s}(1-\beta_{1s})(\xi^s-\varepsilon_s,a_{1,s})+\beta_{2s}^2\|\xi^s-\varepsilon_s\|^2 \quad (A.4)$$

which implies that

$$\|a_{1,s}\|^2+\sum_{i=s}^{\infty}\beta_{2i}^2\mathbb{E}(\|\xi^s-\varepsilon_s\|^2|a_1,\dots a_s)$$

is a nonnegative supermartingale. Therefore $\|a_{1,s}\|^2$ converges with probability 1 [38]. From (A.4) follows that

$$\mathbb{E}\|a_{1,s+1}\|^2\le(1-\beta_{1s})\mathbb{E}\|a_{1,s}\|^2+C\beta_{2s}^2$$

which yields

$$\mathbb{E}\|a_{1,s}\|^2\le\prod_{i=k}^{s-1}(1-\beta_{1i})\mathbb{E}\|a_{1,k}\|^2+C\sum_{i=k}^{s-1}\beta_{2i}^2$$

for any $k\ge 1$, $s>k$. Due to $\sum_{i=1}^{\infty}\beta_{1i}=\infty$ we obtain now:

$$\limsup_s \mathbb{E}\|a_{1,s}\|^2\le C\sum_{i=k}^{\infty}\beta_{2i}^2$$

which is true for an arbitrary $k\ge 1$. Therefore $\mathbb{E}\|a_{1,s}\|^2\to 0$ because $\sum_{i=1}^{\infty}\beta_{2i}^2<\infty$. This together with the convergence of $a_{1,s}$ gives $a_{1,s}\to 0$ a.s.∎

We shall use these lemmas to derive the asymptotic expression for the matrix Q^s from (23),(26).

Lemma 5. Suppose that

$$\beta_s \longrightarrow 0,\ \sum_{i=1}^{\infty}\beta_i=\infty,\quad \sum_{i=1}^{\infty}\beta_i^2<\infty,\quad \frac{1}{\beta_s}\left|\frac{\rho_{s-2}\beta_s}{\rho_{s-1}\beta_{s-1}}-1\right|\longrightarrow 0,\quad (1-\beta_s)\frac{r_{s-1}^2}{r_s^2}=1-\beta_{1s},$$

$$\frac{\beta_s-\beta_{1s}}{\beta_{1s}}\longrightarrow 0,\ \sum_{i=1}^{\infty}\beta_{1i}=\infty,\ \sum_{i=1}^{\infty}\beta_{1i}^2<\infty,\ r_{s-1}\ge r_s,\quad \frac{r_{s-1}}{r_s}\longrightarrow 1,\quad \frac{\rho_{s-1}}{\beta_s r_s}\longrightarrow 0,$$

$\mathbb{E}v^s v^{s^T}=V,\ 0<V<\infty,\ \mathbb{E}(v^s|\mathbb{B}_s)=0,\ \mathbb{E}((v_i^s v_j^s-\mathbb{E}v_i^s v_j^s)^2|\mathbb{B}_s)<C<\infty,$

$\mathbb{E}(v_i^s v_j^s|\mathbb{B}_s)=\mathbb{E}v_i^s v_j^s$

Then

$$Q^s=\frac{1}{r_s^2}(V+a_s)^{-1},\ a_s\longrightarrow 0 \text{ a.s.}$$

Proof.

From (23),(26) follows that

$$Q^s=\left(\sum_{i=1}^{s}\alpha_{is}(y^i-x^s)(y^i-x^s)^T-\chi^s\chi^{s^T}\right)^{-1},\ \chi^s=\sum_{i=1}^{s}\alpha_{is}(y^i-x^s),\ \sum_{i=1}^{s}\alpha_{is}=1 \quad (A.5)$$

Let us consider various terms in (A.5).

1. Let us estimate $w_s=\sum_{i=1}^{s}\alpha_{is}(x^i-x^s)$. We obtain:

$$w_s=(1-\beta_s)(w_{s-1}-(x^s-x^{s-1})),\ \|w_s\|\le(1-\beta_s)(\|w_{s-1}\|+C\rho_s) \quad (A.6)$$

since

$$x^s-x^{s-1}=\tau_{0s}\rho_{s-1},\ \|\tau_0\|\le C_0 \quad (A.7)$$

due to (27). Let us substitute $\|w_s\|=a_s\rho_{s-1}/\beta_s$. Then (A.6) yields:

$$a_s\le a_{s-1}-\beta_s\left(a_{s-1}\left(1-\frac{1}{\beta_s}\left|\frac{\rho_{s-2}\beta_s}{\rho_{s-1}\beta_{s-1}}-1\right|\right)-C_0\right)$$

Applying the Lemma 2 to this inequality we obtain

$$\limsup_i a_i \le C_0$$

and finally for sufficiently large s we have:

$$\sum_{i=1}^{s}\alpha_{is}(x^i-x^s) = \frac{\rho_{s-1}}{\beta_s}\tau_{1s},\ \|\tau_{1s}\|\le 2C \quad (A.8)$$

2. Let us estimate $w_s=\sum_{i=1}^{s}\alpha_{is}(y^i-x^i)$. Due to (24) we obtain:

$$w_s=\sum_{i=1}^{s}\alpha_{is}r_i v^i=(1-\beta_s)w_{s-1}+\beta_s r_s v^s$$

Taking $a_s=w_s/r_s$ we obtain from this inequality:

$$a_s=(1-\beta_{2s})a_{s-1}+\beta_s v^s, \quad (1-\beta_s)\frac{r_{s-1}}{r_s}=1-\beta_{2s}$$

therefore $\beta_{2s}\geq\beta_{1s}$ since $r_{s-1}\geq r_s$. Thus, the Lemma 4 can be applied here, which yields $a_s\longrightarrow 0$ a.s. and finally

$$\sum_{i=1}^{s}\alpha_{is}(y^i-x^i) = r_s\tau_{2s}, \ \tau_{2s}\longrightarrow 0 \text{ a.s.} \tag{A.9}$$

3. Let us take $R^s=Q^{s^{-1}}$. From (A.5),(A.8),(A.9) we obtain:

$$R^s=(1-\beta_s)\left(R^{s-1}+\beta_s\left[\chi^{s-1}-(y^s-x^{s-1})\right]\left[\chi^{s-1}-(y^s-x^{s-1})\right]^T\right)=$$

$$(1-\beta_s)(R^{s-1}+\beta_s r_s^2(v^s+\tau_{3s})(v^s+\tau_{3s})^T), \ \tau_{3s}\longrightarrow 0 \text{ a.s.} \tag{A.10}$$

where

$$\tau^{3s}=-\tau_{1,s-1}\frac{\rho_{s-2}}{\beta_{s-1}r_s}-\tau_{2,s-1}\frac{r_{s-1}}{r_s}+\tau_{0s}\frac{\rho_{s-1}}{r_s}$$

and τ^{3s} is $\mathbb{B}_{s-1}$-measurable. This gives the following inequality for the element R_{ij}^s of the matrix R^s:

$$R_{ij}^s=(1-\beta_s)R_{ij}^{s-1}+\beta_s r_s^2(v_i^s v_j^s+\tau_{ij}^{4s}), \ \tau_{ij}^{4s}=\tau_i^{3s}\tau_j^{3s}+v_i^s\tau_j^{3s}+v_j^s\tau_i^{3s}, \ \tau_{ij}^{4s}\longrightarrow 0 \text{ a.s.}$$

Let us substitute $R_{ij}^s=r_s^2(\mathbb{E}v_i^s v_j^s+a_s)$ in this inequality, then

$$a_s=(1-\beta_{1s})a_{s-1}+\beta_{1s}\left(\frac{\beta_s}{\beta_{1s}}v_i^s v_j^s-\mathbb{E}v_i^s v_j^s+\frac{\beta_s}{\beta_{1s}}\tau_{ij}^{4s}\right)$$

and Lemma 4 yields $a_s\longrightarrow 0$ a.s. and $R_{ij}^s=r_s^2(V_{ij}+a_s)$. ∎

The following lemma establishes the fundamental property of the step direction d^s.

<u>Lemma 6.</u> Suppose that the following conditions are satisfied:

1. The set $X\subset\mathbb{R}^n$ is a compact set.

2. The function $F(x)$ is continuously differentiable and $F_x(x)$ satisfies a Lipchitz condition.

3. There exists a.s. $k=k(\omega)$ such that $\|Q^s\|\leq\frac{C}{r_s^2}$, $C<\infty$ for $s\geq k$

4. $\beta_s\geq 0$, $\sum_{i=1}^{\infty}\beta_i=\infty$, $\sum_{i=1}^{\infty}\beta_i^2<\infty$, $(1-\beta_s)\frac{r_{s-1}^2}{r_s^2}=1-\beta_{1s}$, $\sum_{i=1}^{\infty}\beta_{1i}=\infty$, $r_s\longrightarrow 0$,

$$\frac{\rho_{s-1}}{\beta_{1s} r_s^2} \longrightarrow 0, \quad \sum_{i=1}^{\infty} \frac{\beta_i^2}{r_i^2} < \infty, \quad \sum_{i=1}^{\infty} \frac{\rho_{i-1}^2}{r_i^4} < \infty, \quad \frac{r_{s-1}}{r_s} < C < \infty, \quad \frac{\beta_s}{\beta_{1s}} < C < \infty$$

$\mathbb{E}(\zeta^s - F(y^s) \mid \mathbb{B}_s, v^s) = 0$, $\mathbb{E}((\zeta^s - F(y^s))^2 \mid \mathbb{B}_s, v^s) < C < \infty$, $\mathbb{E}(v^s \mid \mathbb{B}_s) = 0$, $\|v^s\| < C < \infty$

Then $d^s = F(x^s) + a_s$ where $a_s \longrightarrow 0$ a.s.

Proof.

1. Let us derive an expression for $d^s - F(x^s)$. Denoting

$$\varepsilon^i = \zeta^i - F(y^i), \quad \Delta_{is} = F(y^i) - F(x^s) - (y^i - x^s)^T F_x(x^s)$$

we obtain from (23):

$$u^s = Q^{s^{-1}} F_x(x^s) + \sum_{i=1}^{s} \alpha_{is}(y^i - x^s)\left(\Delta_{is} + \varepsilon^i - \sum_{j=1}^{s} \alpha_{js}\Delta_{js} - \sum_{j=1}^{s} \alpha_{js}\varepsilon^j\right)$$

Combining this with (23) we obtain:

$$d^s - F_x(x^s) = Q^s \sum_{i=1}^{s} \alpha_{is}(y^i - x^s)\left(\Delta_{is} + \varepsilon^i - \sum_{j=1}^{s} \alpha_{js}\Delta_{js} - \sum_{j=1}^{s} \alpha_{js}\varepsilon^j\right) \tag{A.11}$$

Let us consider different terms in the right hand side of (A.11)

2. Let us estimate $w_s = \sum_{i=1}^{s} \alpha_{is}(x^i - x^s)\varepsilon^i$. We obtain

$$w_s = (1-\beta_s) w_{s-1} - (1-\beta_s)(x^s - x^{s-1}) \sum_{i=1}^{s-1} \alpha_{i,s-1}\varepsilon^i$$

which gives

$$\|w^s\| \le (1-\beta_s)\|w^{s-1}\| + C_0 \rho_{s-1} \tau_{1s} \tag{A.12}$$

where

$$\tau_{1s} = \left\| \sum_{i=1}^{s-1} \alpha_{i,s-1}\varepsilon^i \right\| \longrightarrow 0 \text{ a.s.} \tag{A.13}$$

due to the Lemma 4 and

$$\sum_{i=1}^{\infty} \beta_i = \infty, \quad \sum_{i=1}^{\infty} \beta_i^2 < \infty, \quad \mathbb{E}(\varepsilon^i \mid \mathbb{B}_i) = 0, \quad \mathbb{E}((\varepsilon^i)^2 \mid \mathbb{B}_i) < C < \infty$$

Taking $a_s = \|w^s\| / r_s^2$ we obtain from (A.12):

$$a_s \le (1-\beta_{1s}) a_{s-1} + C_0 \beta_{1s} \frac{\rho_{s-1}}{\beta_{1s} r_s^2} \tau_{1s}$$

This yields $a_s \longrightarrow 0$ a.s. due to the Lemma 3 and finally

$$\frac{1}{r_s^2} \sum_{i=1}^{s} \alpha_{is}(x^i - x^s)\varepsilon^i \longrightarrow 0 \text{ a.s.} \tag{A.14}$$

3. Let us estimate $w_s=\sum_{i=1}^{s}\alpha_{is}r_iv^i\varepsilon^i$. We obtain

$$w_s=(1-\beta_s)w_{s-1}+\beta_sr_sv^s\varepsilon^s$$

Taking $a_s=\|w^s\|/r_s^2$ we obtain:

$$a_s=(1-\beta_{1s})a_{s-1}+\frac{\beta_s}{r_s}v^s\varepsilon^s$$

The Lemma 4 yields now $a_s\longrightarrow 0$ a.s. and finally

$$\frac{1}{r_s^2}\sum_{i=1}^{s}\alpha_{is}r_iv^i\varepsilon^i\longrightarrow 0 \text{ a.s.} \tag{A.15}$$

4. Let us estimate $w_s=\sum_{i=1}^{s}\alpha_{is}(x^i-x^s)\sum_{j=1}^{s}\alpha_{js}\varepsilon^j$. We obtain:

$$w_s=(1-\beta_s)^2w_{s-1}+\beta_s\varepsilon^s\sum_{i=1}^{s}\alpha_{is}(x^i-x^s)-(1-\beta_s)^2(x^s-x^{s-1})\sum_{j=1}^{s-1}\alpha_{j,s-1}\varepsilon^j$$

Due to (A.8),(A.13) we have the following estimates:

$$(1-\beta_s)^2(x^s-x^{s-1})\sum_{j=1}^{s-1}\alpha_{j,s-1}\varepsilon^j=\rho_{s-1}\tau_{1s},\ \tau_{1s}\longrightarrow 0 \text{ a.s.}$$

$$\beta_s\varepsilon^s\sum_{i=1}^{s}\alpha_{is}(x^i-x^s)=\varepsilon^s\rho_{s-1}\tau_{2s},\ \|\tau_{2s}\|<C<\infty \text{ a.s.}$$

where τ_{2s} is measurable with respect to $\mathbb{B}_s$. This yields the following equality:

$$w_s=(1-\beta_s)^2w_{s-1}+\rho_{s-1}(\varepsilon^s\tau_{2s}-\tau_{1s})$$

Taking $a_s=\|w_s\|/r_s^2$ we obtain:

$$a_s=(1-\beta_s)(1-\beta_{1s})a_{s-1}+\frac{\rho_{s-1}}{r_s^2}(\varepsilon^s\tau_{2s}-\tau_{1s})$$

For a_s all assumptions of the Lemma 4 are satisfied, which implies

$$\frac{1}{r_s^2}\sum_{i=1}^{s}\alpha_{is}(x^i-x^s)\sum_{j=1}^{s}\alpha_{js}\varepsilon^j\longrightarrow 0 \text{ a.s.} \tag{A.16}$$

5. Let us estimate $w_s=\sum_{i=1}^{s}\alpha_{is}r_iv^i\sum_{j=1}^{s}\alpha_{js}\varepsilon^j$. We obtain:

$$w_s=(1-\beta_s)^2w_{s-1}+\beta_s(1-\beta_s)r_sv^s\sum_{j=1}^{s-1}\alpha_{j,s-1}\varepsilon^j+$$

$$\beta_s(1-\beta_s)\varepsilon^s\sum_{j=1}^{s-1}\alpha_{j,s-1}r_jv^j+\beta_s^2r_sv^s\varepsilon^s \tag{A.17}$$

We need now to estimate $b_s=\sum_{j=1}^{s} \alpha_{j,s} r_j v^j$

$$b_s=(1-\beta_s)b_{s-1}+\beta_s r_s v^s$$

Making the substitution $c_s=b_s/r_s^2$ we obtain:

$$c_s=(1-\beta_{1s})c_{s-1}+\frac{\beta_s}{r_s}v^s$$

with all conditions of the Lemma 4 being satisfied for $a_s=c_s$, therefore $c_s\longrightarrow 0$ a.s. Substituting this and (A.13) in (A.17) we obtain:

$$w_s=(1-\beta_s)^2 w_{s-1}+\beta_s(1-\beta_s)r_s v^s\tau_{1s}+\beta_s(1-\beta_s)\varepsilon^s r_{s-1}^2 c_{s-1}+\beta_s^2 r_s v^s\varepsilon^s$$

after another substitution $a_s=w_s/r_s^2$ we obtain:

$$a_s=(1-\beta_s)(1-\beta_{1s})a_{s-1}+\frac{\beta_s}{r_s}(1-\beta_s)v^s\tau_{1s}+\beta_s(1-\beta_s)\varepsilon^s\frac{r_{s-1}^2}{r_s^2}c_{s-1}+\frac{\beta_s^2}{r_s}v^s\varepsilon^s$$

All conditions of the Lemma 4 are satisfied and $a_s\longrightarrow 0$ a.s., which yields

$$\frac{1}{r_s^2}\sum_{i=1}^{s}\alpha_{is}r_i v^i\sum_{j=1}^{s}\alpha_{js}\varepsilon^j\longrightarrow 0 \text{ a.s.} \qquad (A.18)$$

6. Let us estimate $w_s=\sum_{i=1}^{s}\alpha_{is}(x^i-x^s)\Delta_{is}$. We obtain:

$$w_s=(1-\beta_s)w_{s-1}-(1-\beta_s)(x^s-x^{s-1})\sum_{i=1}^{s-1}\alpha_{i,s-1}\Delta_{is}$$

$$+(1-\beta_s)\sum_{i=1}^{s-1}\alpha_{i,s-1}(x^i-x^{s-1})(\Delta_{is}-\Delta_{i,s-1}) \qquad (A.19)$$

where

$$\Delta_{is}-\Delta_{i,s-1}=F(x^{s-1})-F(x^s)-(\bar{x}^i-x^s)^\tau(F_x(x^s)-F_x(x^{s-1}))+(x^s-x^{s-1})^\tau F_x(x^{s-1})$$

We obtain the following estimates for the first and the second term in (A.19):

$$\|(x^s-x^{s-1})\sum_{i=1}^{s-1}\alpha_{i,s-1}\Delta_{is}\|\le C\rho_{s-1}\|\sum_{i=1}^{s-1}\alpha_{i,s-1}\Delta_{is}\|\le C\rho_{s-1} \qquad (A.20)$$

since Δ_{is} is bounded due to the compactness of the set X and the differentiability of F(x). The Lipchitz property of $F_x(x)$ yields:

$$\|\sum_{i=1}^{s-1}\alpha_{i,s-1}(x^i-x^{s-1})(\Delta_{is}-\Delta_{i,s-1})\|\le C_1\rho_{s-1} \qquad (A.21)$$

Here we assumed in addition that $F_x(x)$ has the Lipchitz property on X. Combining (A.19)-(A.21) we obtain:

$$\|w_s\| \le (1-\beta_s)\|w_{s-1}\| + C_2\rho_{s-1}$$

After the substitution $\|w_s\| = a_s/r_s^2$ this yields:

$$a_s \le (1-\beta_{1s})a_{s-1} + C_2\frac{\rho_s}{r_s^2}$$

We now obtain from the Lemma 3 that $a_s \longrightarrow 0$ a.s. and finally

$$\frac{1}{r_s^2}\sum_{i=1}^{s}\alpha_{is}(x^i - x^s)\Delta_{is} \longrightarrow 0 \text{ a.s.} \tag{A.22}$$

7. Let us estimate $w_s = \sum_{i=1}^{s}\alpha_{is}r_i v^i\Delta_{is}$. We obtain:

$$w_s = (1-\beta_s)w_{s-1} + \beta_s r_s v^s\Delta_{ss} + (1-\beta_s)\sum_{i=1}^{s-1}\alpha_{i,s-1}r_i v^i(\Delta_{is} - \Delta_{i,s-1}) \tag{A.23}$$

Since $\|v^s\|$ is bounded and due to conditions 1,2 we get

$$(1-\beta_s)\left\|\sum_{i=1}^{s-1}\alpha_{i,s-1}r_i v^i(\Delta_{is} - \Delta_{i,s-1})\right\| \le C\rho_{s-1} \tag{A.24}$$

Δ_{ss} can be estimated as follows:

$$\Delta_{ss} = F(y^s) - F(x^s) - r_s(v^s, F_x(x^s)) = r_s(v^s, F_x(x^s + \theta r_s v^s) - F_x(x^s)),\quad 0 \le \theta \le 1$$

therefore

$$\|\Delta_{ss}\| \le Cr_s^2 \tag{A.25}$$

Combining (A.23)-(A.25) we obtain

$$\|w_s\| \le (1-\beta_s)\|w_{s-1}\| + C\beta_s r_s^3 + C\rho_{s-1}$$

which yields after the substitution $a_s = \|w_s\|/r_s^2$:

$$a_s \le (1-\beta_{1s})a_{s-1} + C\beta_s r_s + C\frac{\rho_{s-1}}{r_s^2}$$

and all conditions of the Lemma 3 are satisfied which yields $a_s \longrightarrow 0$ a.s. and finally

$$\frac{1}{r_s^2}\sum_{i=1}^{s}\alpha_{is}r_i v^i\Delta_{is} \longrightarrow 0 \text{ a.s.} \tag{A.26}$$

8. Let us estimate $w_s = \sum_{j=1}^{s}\alpha_{js}\Delta_{js}$. We obtain:

$$w_s = (1-\beta_s)w_{s-1} + \beta_s\Delta_{ss} + (1-\beta_s)\sum_{j=1}^{s-1}\alpha_{j,s-1}(\Delta_{j,s} - \Delta_{j,s-1}) \tag{A.27}$$

Similarly to (A.21) we obtain:

$$\|\sum_{j=1}^{s-1}\alpha_{j,s-1}(\Delta_{j,s}-\Delta_{j,s-1})\|\le C\rho_{s-1} \tag{A.28}$$

Expressions (A.25), (A.27) and (A.28) yield:

$$\|w_s\|\le(1-\beta_s)\|w_{s-1}\|+C_1\beta_s r_s^2+C\rho_{s-1}$$

after making the substitution $\|w_s\|=a_s/r_s^2$ we obtain:

$$a_s\le(1-\beta_{1s})a_{s-1}+C_1\beta_s+C\frac{\rho_{s-1}}{r_s^2}=a_{s-1}-\beta_{1s}(a_{s-1}-C_1\frac{\beta_s}{\beta_{1s}}-C\frac{\rho_{s-1}}{r_s^2\beta_{1s}})$$

Under assumption 4 a_s satisfies conditions of the Lemma 2 which yields $\limsup_k a_k \le C$ and finally

$$\|\sum_{j=1}^{s}\alpha_{js}\Delta_{js}\| \le Cr_s^2,\ C<\infty \tag{A.29}$$

9. Let us estimate $w_s=\sum_{i=1}^{s}\alpha_{is}(y^i-x^s)\sum_{j=1}^{s}\alpha_{js}\Delta_{js}$. Similarly to (A.8),(A.9) we obtain

$$\|\sum_{i=1}^{s}\alpha_{is}(y^i-x^s)\|\longrightarrow 0 \text{ a.s.}$$

Combining this with (A.29) we get the following estimate:

$$\frac{1}{r_s^2}\|w_s\|\le\frac{1}{r_s^2}\|\sum_{i=1}^{s}\alpha_{is}(\bar{x}^i-x^s)\|\|\sum_{j=1}^{s}\alpha_{js}\Delta_{js}\|\le C\|\sum_{i=1}^{s}\alpha_{is}(\bar{x}^i-x^s)\| \longrightarrow 0 \text{ a.s.}$$

Thus

$$\frac{1}{r_s^2}\sum_{i=1}^{s}\alpha_{is}(\bar{x}^i-x^s)\sum_{j=1}^{s}\alpha_{js}\Delta_{js} \longrightarrow 0 \text{ a.s.} \tag{A.30}$$

10. Combining (A.11),(A.14),(A.15),(A.16),(A.18),(A.22),(A.26),(A.30) we obtain:

$$d^s-F_x(x^s)=r_s^2Q^s a_s,\ a_s\longrightarrow 0 \text{ a.s.}$$

which due to the condition 3 completes the proof. ■

Lemma 7. Suppose that for a nonnegative sequence a_s the following conditions are satisfied

$$a_{s+1}\le a_s-C\rho_s\varphi(a_s)+C_1\rho_s\tau_s,\ \tau_s\longrightarrow 0 \text{ a.s.},\ \rho_s\ge 0,\ \sum_{i=1}^{\infty}\rho_i=\infty,\ C>0,$$

$$\inf_{b\ge c}\varphi(b)>0 \text{ for } c>0$$

Then $a_s\longrightarrow 0$ a.s.

Proof.

We may assume without loss of generality that $\varphi(b)\ge\varphi(c)>0$ for $b\ge c>0$. Suppose that for some $\omega\in\Omega$ exists k and $\delta>0$ such that $a_s>\delta$ for $s\ge k$. We may assume without loss of generality that $\tau_s\le\varphi(\delta)/2$ for $s\ge k$. Then

$$a_s\le a_k-\frac{1}{2}C\varphi(\delta)\sum_{i=k}^{s-1}\rho_i$$

which contradicts nonnegativity of a_s for sufficiently large s. Therefore for any k and $\delta>0$ a.s. there exists $m=m(k,\delta)$ such that $a_m<\delta$. Suppose that there exists a number $l=l(m,\delta)$ such that $l>m$ and $a_l>3\delta$. We may assume without loss of generality that there exists r: $m<r<l$, $\delta<a_r\le2\delta$, $2\delta\le a_s\le3\delta$ for $r<s\le l$, since $\max\left\{0,a_{s+1}-a_s\right\}\longrightarrow0$. We assumed already that $\tau_s\le\varphi(\delta)/2$ for $s\ge k$, thus $a_s\ge a_{s+1}$ for $r<s\le l$. Therefore $a_l\le3\delta$ which contradicts assumption $a_l>3\delta$. This contradiction completes the proof ∎

Lemma 8. Suppose that for a nonnegative sequence a_s the following conditions are satisfied:

$$a_{s+1}\le a_s-C\rho_s\varphi(a_s)+C_1\rho_s\kappa^s, \tag{A.31}$$
$$\mathbb{E}(\kappa^s|a_0,\ldots,a_s)=\tau_s,\ \tau_s\longrightarrow0\text{ a.s.},\quad \mathbb{E}(\|\kappa^s-\tau_s\|^2|a_0,\ldots,a_s)=C_s^2,$$
$$\rho_s\ge0,\ \sum_{i=1}^{\infty}\rho_i=\infty,\ \sum_{i=1}^{\infty}\rho_i^2C_i^2<\infty,\ C>0,\ 0\le a_s\le C_2\text{ for some }C_2<\infty,$$

Then $a_s\longrightarrow0$ a.s.

Proof.

Let us note that conditions of the lemma imply that

$$\sum_{i=1}^{\infty}\rho_i(\kappa^i-\tau_i)\quad\text{converges a.s.} \tag{A.32}$$

(see [38]). Denoting

$$\bar\varphi(\bar a)=\inf_{\mathbb{E}a=\bar a,\,0\le a\le C_2}\mathbb{E}\varphi(a)$$

we obtain $\inf_{b\ge c}\bar\varphi(b)>0$ for $c>0$. Taking expectation from both sides of (A.31) we get:

$$\mathbb{E}a_{s+1} \le \mathbb{E}a_s - C\rho_s \bar{\varphi}(\mathbb{E}a_s) + C_1 \rho_s \mathbb{E}\tau_s$$

and for $\mathbb{E}a_s$ all conditions of Lemma 7 are satisfied, which yields $\mathbb{E}a_s \longrightarrow 0$. Therefore for any k and $\delta>0$ a.s. there exists $m=m(k,\delta)$ (which depend on an element of probability space Ω) such that $a_m<\delta$. Let us suppose that $a_n>3\delta$ for some $n>m$. Due to (A.31),(A.32) we have $\max\left\{a_{s+1}-a_s, 0\right\} \longrightarrow 0$ a.s., therefore for sufficiently large k there exists l: $m<l<n$ such that $\delta \le a_l \le 2\delta$, $\delta \le a_i \le 3\delta$ and $C\varphi(a_i) > C_1 \tau_i$ for $l \le i \le n$. Thus,

$$a_n \le a_l + C_1 \sum_{i=l}^{n-1} \rho_i (\kappa^i - \tau_i) \tag{A.33}$$

Due to (A.32) we have

$$\sum_{i=l}^{n-1} \rho_i (\kappa^i - \tau_i) \longrightarrow 0 \text{ a.s. for } l,n \longrightarrow \infty.$$

Thus, (A.33) contradicts assertion $a_n>3\delta$ for sufficiently large k ∎

Proof of the Theorem 1.

Let us denote

$$F^* = \min_{x \in X} F(x),\quad X^* = \left\{x:\ x \in X,\ F(x) = F^*\right\},\quad W^s = \min_{x \in X^*} \|x^s - x\|^2,\quad \|x^s - x(s)\|^2 = W^s,$$

$$x(s) \in X^*,\quad \varphi(w) = \inf\left\{F(x) - F^* : x \in X, \min_{z \in X^*} \|x - z\|^2 \ge w\right\},\quad \Delta(s,x,\omega) = F(s,x,\omega) - F(x)$$

Note, that $\varphi(w)>0$ for $w>0$ due to compactness of X. Taking into account convexity of the set X and the function $F(s,x,\omega)$ we obtain the following inequality for W^{s+1}:

$$\begin{gathered}
W^{s+1} = \|x^{s+1} - x(s+1)\|^2 \le \|x^{s+1} - x(s)\|^2 \le \|x^s - \rho_s \xi^s - x(s)\|^2 = \\
W^s - 2\rho_s(\xi^s, x^s - x(s)) + \rho_s^2 \|\xi^s\|^2 = \\
W^s - 2\rho_s(F_x(s,x^s,\omega), x^s - x(s)) - 2\rho_s(\xi^s - F(s,x^s,\omega), x^s - x(s)) + \rho_s^2 \|\xi^s\|^2 \le \\
W^s - 2\rho_s(F(s,x^s,\omega) - F(s,x(s),\omega)) - 2\rho_s(\xi^s - F(s,x^s,\omega), x^s - x(s)) + \rho_s^2 \|\xi^s\|^2 \le \\
W^s - 2\rho_s \varphi(W^s) + \rho_s \kappa^s
\end{gathered} \tag{A.34}$$

where

$$\kappa^s = -2(\Delta(s,x^s,\omega) - \Delta(s,x(s),\omega)) - 2(\xi^s - F(s,x^s,\omega), x^s - x(s)) + \rho_s \|\xi^s\|^2$$

all conditions of the Lemma 8 are satisfied for $a_s = W^s$ and (A.34), therefore $W^s \longrightarrow 0$ a.s. ∎

Proof of the Theorem 2.

We are using here notations introduced in the proof of the Theorem 1. Similar to (A.34) we obtain:

$$W^{s+1}= W^s-2\rho_s\gamma_s(d^s,x^s-x(s))+\rho_s^2\gamma_s^2\|d^s\|^2 =$$

$$W^s-2\rho_s\gamma_s(F_x(x^s),x^s-x(s))-2\rho_s\gamma_s(d^s-F(x^s),x^s-x(s))+\rho_s^2\gamma_s^2\|d^s\|^2 \le$$

$$W^s-2\rho_s\gamma_s(F(x^s)-F^*)-2\rho_s\gamma_s(d^s-F(x^s),x^s-x(s))+\rho_s^2\gamma_s^2\|d^s\|^2 \qquad (A.35)$$

Under assumptions of the theorem all conditions of lemmas 5,6 are satisfied, therefore

$$d^s=F_x(x^s)+a_s,\ a_s\longrightarrow 0 \text{ a.s.} \qquad (A.36)$$

This together with the boundedness of $F_x(x^s)$ on the set X implies the existence a.s. of the number $k=k(\omega)$ and $C_1>0$ such that

$$\gamma_s\ge C_1>0\ ,\ s\ge k \qquad (A.37)$$

(A.36) and the compactness of the set X yield:

$$2\rho_s\gamma_s(d^s-F(x^s),x^s-x(s))+\rho_s^2\gamma_s^2\|d^s\|^2\le C_1\rho_s a_s,\ a_s\longrightarrow 0 \text{ a.s.} \qquad (A.38)$$

After the substitution of (A.37),(A.38) in (A.35) we get

$$W^{s+1}\le W^s-C\rho_s\varphi(W^s)+C_1\rho_s a_s,\ a_s\longrightarrow 0 \text{ a.s.}$$

which together with the Lemma 7 yields $W^s\longrightarrow 0$ a.s. This completes the proof due to the compactness of the set X. ■

APPENDIX B. AN EXAMPLE OF DISCRETE EVENT SYSTEM WITH DISCONTINUITIES

Suppose that the manufacturing system contains two machines M_1, M_2 and the buffer B. The buffer contains items which should be processed consecutively by both machines (Figure 1).

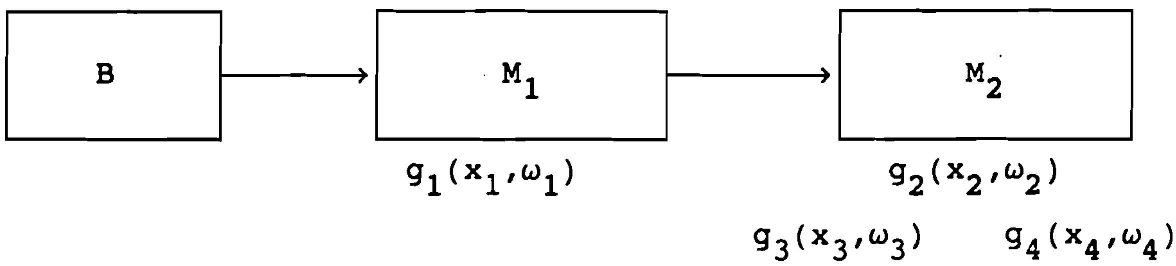

Figure 1.

The processing time of each machine is $g_i(x_i,\omega_i)$, $i=1,2$, $x_i\in\mathbb{R}^1$, ω_i

is distributed uniformly on [0,1]. If, for example, the processing time is distributed exponentially and x_i is the processing rate then

$$g_i(x_i,\omega_i)=-\frac{1}{x_i}\ln(1-\omega_i)$$

The performance capability of the second machine can deteriorate and is monitored by a separate process. If it is detected that the second machine has deteriorated below certain level and the machine is idle then the maintenance is started. If it is busy then the maintenance is started immediately after finishing the job. If an item arrives at the input of the second machine during a maintenance period then it waits till the end of maintenance, and immediately after that the processing is started. The time elapsed between the end of one maintenance period and the detection of necessity for another maintenance is $g_3(x_3,\omega_3)$, the length of maintenance is $g_4(x_4,\omega_4)$. Suppose for simplicity that the buffer contains only one item. Then the system can be in one of the following states:

z^1 - M_1 is busy, M_2 is idle and ready for a job

z^2 - M_1 is busy, M_2 is under maintenance

z^3 - M_1 is idle, M_2 is busy

z^4 - M_1 is idle, M_2 is under maintenance and the item waits at the input of M_2

z^5 - an item is at the output of the M_2.

At the initial moment t=0 the item arrives at the input of M_1 and M_2 is considered to be just after maintenance. Suppose that the probability of coincidence of the item arrival at the input of the second machine and the detection of the need for maintenance is zero. Then the following sample paths are possible in this system:

$$U^{1k}(x,\omega)=\left\{\Big[(z^1(i),t^1(i)),(z^2(i),t^2(i))\Big]_{i=1}^{k},(z^1(k+1),t^1(k+1)),\right.$$
$$\left.(z^3(1),t^3(1)),\ (z^5(1),t^5(1))\right\},\ k=0,2,\ldots$$

$$U^{2k}(x,\omega)=\left\{\Big[(z^1(i),t^1(i)),(z^2(i),t^2(i))\Big]_{i=1}^{k},(z^4(1),t^4(1)),\right.$$

$$(z^3(1),t^3(1)),\ (z^5(1),t^5(1))\Big\},\ k=1,2,\ldots$$

where $(z^j(i),t^j(i))$ denotes event which consists of the i-th transition to the state j from the beginning of simulation, in order to simplify notations we omitted dependence on (x,ω). Here

$$t^1(1)=0,\ t^1(k)=G(k-1,x,\omega),\ k\geq 2,\ G(k,x,\omega)=\sum_{i=1}^{k}\left[g_3(x_3,\omega_3^i)+g_4(x_4,\omega_4^i)\right]$$

$$t^2(k)=t^1(k)+g_3(x_3,\omega_3^k),\ t^3(1)=\begin{cases} g_1(x_1,\omega_1^1) & \text{for path } Z^{1k}(x,\omega) \\ G(k,x,\omega) & \text{for path } Z^{2k}(x,\omega)\end{cases}$$

$$t^4(1)=g_1(x_1,\omega_1^1),\ t^5(1)=t^3(1)+g_2(x_2,\omega_2^1), \tag{B.1}$$

The path $U^{1k}(x,\omega)$ is taken if $(x,\omega)\in\Theta_{1k}$ and the path $U^{2k}(x,\omega)$ is taken in the case $(x,\omega)\in\Theta_{2k}$, where

$$\Theta_{1k}=\left\{(x,\omega):\ G(k,x,\omega)\leq g_1(x_1,\omega_1^1)\leq G(k,x,\omega)+g_3(x_3,\omega_3^{k+1})\right\},k=0,1,\ldots \tag{B.2}$$

$$\Theta_{2k}=\left\{(x,\omega):\ G(k-1,x,\omega)+g_3(x_3,\omega_3^k)<g_1(x_1,\omega_1^1)<G(k,x,\omega)\right\},k=1,2,\ldots \tag{B.3}$$

Suppose that the objective function is the weighted sum of average processing time and cost terms. The average processing time in this case is the time of arrival for the first time at the state z^5, since only one item is in the buffer, i.e. it equals $t^5(1)$. Summarizing (B.1)-(B.3) we obtain:

$$F(x)=F^1(x)+F^2(x),\ F^1(x)=E_\omega f(x,\omega),$$

$$f(x,\omega)=\begin{cases} g_1(x_1,\omega_1^1)+g_2(x_2,\omega_2^1) & \text{if} \quad (x,\omega)\in\Theta_{1k} \\ G(k+1,x,\omega)+g_2(x_2,\omega_2^1) & \text{if} \quad (x,\omega)\in\Theta_{2(k+1)}\end{cases} \tag{B.4}$$

where $k=0,1,\ldots$ and $G(0,x,\omega)=0$. Therefore the function $f(x,\omega)$ is discontinuous with respect to (x,ω), but it is differentiable on each set Θ_{1k},Θ_{2k} if $g_i(x_i,\omega_i)$ are differentiable. Note that the function $F(x)$ may be differentiable too, depending on the properties of $g_i(x_i,\omega_i)$. In particular, it would be differentiable in the case when $g_i(x_i,\omega_i)$ are distributed exponentially.

Thus, even in such simple example as this, there are infinite number of sets in the continuity partition defined by (B.2)-(B.3).

Some of differentiation schemes can experience difficulties in this situation. For example, a sample derivative $f_x(x,\omega)$ gives in this case a biased estimate of the gradient $F^1(x)$. In order to see this let us compute the partial derivative of $F_1(x)$ with respect to x_1. Let us denote

$$a(k,x,\omega)=a(k):\ g_1(x_1,a(k,x,\omega))=G(k,x,\omega)+g_3(x_3,\omega_3^{k+1}),\ k\geq 0$$

$$b(k,x,\omega)=b(k):\ g_1(x_1,b(k,x,\omega))=G(k,x,\omega),\quad k\geq 1,\ b(0,x)=0$$

Then

$$F^1_{x_1}(x)=\sum_{k=0}^{\infty}\int_0^1\int_{b(k)}^{a(k)} g_{1x_1}(x_1,\omega_1^1)d\omega_1^1 d\omega_3^1\cdots d\omega_3^{k+1}d\omega_4^1\cdots d\omega_4^k +$$

$$\sum_{k=0}^{\infty}\int_0^1 (a_{x_1}(k)(g_1(x_1,a(k))-G(k+1,x,\omega))d\omega_3^1\cdots d\omega_3^{k+1}d\omega_4^1\cdots d\omega_4^{k+1} \quad (B.5)$$

Now let us try to compute the same derivative using only one sample path, which amounts to the differentiation of $f(x,\omega)$. We obtain

$$f_{x_1}(x,\omega)=\begin{cases} g_{1x_1}(x_1,\omega_1^1) & \text{if} \quad (x,\omega)\in\Theta_{1k} \\ 0 & \text{if} \quad (x,\omega)\in\Theta_{2(k+1)} \end{cases} \quad (B.6)$$

Note, that under general assumptions this derivative exists almost everywhere. Taking the expectation in (B.6) we obtain only the first term in (B.5) and lose the second term, which appears due to discontinuities.

APPENDIX C. NUMERICAL EXPERIMENT

The example from the Appendix B was used for the numerical experiment reported here. The objective function of the problem (4) was

$$F(x)=F^1(x)+F^2(x),\ F^1(x)=\mathbb{E}_\omega f(x,\omega), \quad (C.1)$$

where $f(x,\omega)$ is described in (B.4) and the cost term $F^2(x)$ was the following:

$$F^2(x) = 1.32x_1 + 0.25x_2 - 1.28x_3 + 0.4x_3^2 + 1.92x_4 + 0.4$$

The operation times $g_i(x_i,\omega_i)$ were taken to be exponential in order to allow exact computation of the objective function values, this was necessary for the verification of algorithm results:

$$g_i(x_i,\omega_i)=-\frac{1}{x_i}\ln(1-\omega_i),\ i=1:4$$

where ω_i are uniformly distributed on [0,1). In this case it is possible to obtain the explicit formula for $F^1(x)$:

$$F^1(x)=\frac{1}{x_2}+\sum_{k=0}^{\infty} x_1\frac{1}{\left(1+\frac{x_1}{x_3}\right)^{k+1}\left(1+\frac{x_1}{x_4}\right)^{k}}\times$$

$$\left(\frac{1}{x_3}\left(\frac{k+1}{x_1+x_3}+\frac{k}{x_1+x_4}\right)+\frac{1}{x_1+x_4}\left(\frac{k+1}{x_1+x_3}+\frac{k+1}{x_1+x_4}+\frac{1}{x_4}\right)\right) \tag{C.2}$$

The admissible set X was the following:

$$X=\left\{x:\ x\in\mathbb{R}^4,\ \underline{x}\le x\le\bar{x},\ \underline{x}=(0.5,0.5,0.5,0.2),\ \bar{x}=(4,4,4,4)\right\}$$

This problem has the optimal solution $x^*=(1,2,1,0.5)$ with the optimal value $F^*=4.6$

The simulation model which provided the values of $f(x,\omega)+F^2(x)$ was a general simulation program intended for simulation of one of the modifications of the Petri Nets. This program supplied the observations of the objective function to the interactive program SQG-PC which is an advanced implementation of stochastic quasigradient methods [14]. This program was supplemented by the implementation of the algorithm (23)-(27).

The objective of the numerical experiment was to compare stochastic quasigradients (10) with finite differences (13) with the concurrent approximation (23)-(27). Therefore the algorithm parameters in both cases were taken as similar as possible. We used forward finite differences (13) with the fixed value of difference step which was equal 0.2. Thus, five simulation runs were needed on each step in order to obtain a step direction. The finite difference direction was normalized in order to be comparable with direction generated by (23)-(27). In the concurrent approximation algorithm the expression (25) was modified as follows:

$$\alpha_{is} = \begin{cases} 0.95\alpha_{i,s-1} & \text{if} \quad s-50<i<s \\ 0.05 & \text{if} \quad i=s \\ 0 & \text{otherwise} \end{cases}$$

Instead of probabilistic selection of v^s in (24), the deterministic scheme was adopted here:

$$r_s=0.1,\ v^s=\theta_s e_{i(s)},\ i(s)=s-4[(s-1)/4],\ \theta_s = \begin{cases} -\theta_{s-1} & \text{if } i(s)=1,\ s>1 \\ \theta_{s-1} & \text{if } i(s)=2,3,4 \\ 1 & \text{if } s=1 \end{cases}$$

where e_i is the i-th unit vector of the basis in $\mathbb{R}^n$. Thus, the step in finite differences equals the size of vicinity of x^s in which observations are made for concurrent approximation. The value of γ_s from (27) always was equal $1/\|d^s\|$. The step size ρ_s in both methods was selected according to one of adaptive rules implemented in SQG-PC.

The initial point x^1 for both algorithms was

$$x^1=(3,3,3,3),\ F(x^1)=11.4078$$

Starting from this point two sequences of points were generated: x^{1s} by (10),(13) and x^{2s} by (27), $x^{11}=x^{21}=x^1$, with the same sequence of random numbers used to generate function observations. Each algorithm performed the number of iterations for which 2500 independent observations of the objective function were needed, 500 iterations in the case of finite differences and 2500 iterations in the case of concurrent approximation. After that, exact function values for both sequences of points were computed using expression (C.2). The results are displayed on the Figure 2. The number of observations of the objective function is depicted on the horisontal axis and corresponding exact values of F(x) are depicted on the vertical axis. The straight dashed line is the optimal value F^*, the solid line corresponds to the concurrent approximation and the dotted line corresponds to the finite differences.

Both algorithms exhibit behavior typical of the stochastic optimization procedures: comparatively fast convergence to a certain vicinity of the optimal solution and slow convergence with oscillations in this vicinity. However, the concurrent approximation method shows more regular behavior, converges faster and to smaller vicinity of the solution.

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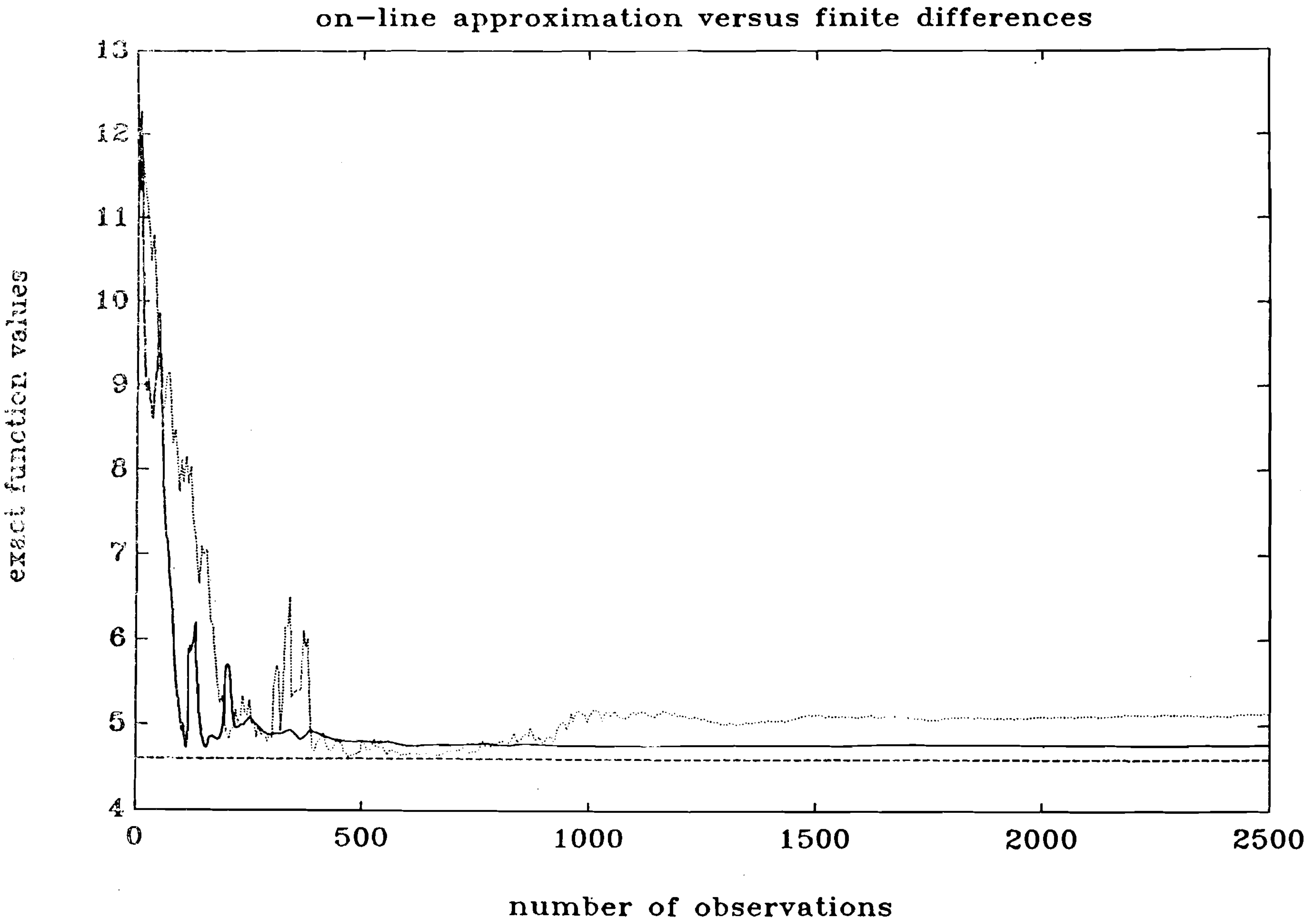
on−line approximation versus finite differences
exact function values
number of observations
13
12
11
10
9
8
7
6
5
4
0
500
1000
1500
2000
2500