Working Paper

Asymptotic Dominance and Confidence for Solutions of Stochastic Programs

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Foreword

The "optimization under uncertainty" project within IIASA has the goal to study structure and solution techniques of stochastic programs. The present paper studies the asymptotic behaviour of solutions of "empirical stochastic programs" that are programs where the unknown expectation functional is replaced by a Monte Carlo estimate. The notion of asymptotic dominance is introduced and its usefulness is indicated.

Asymptotic dominance and confidence for solutions of stochastic programs

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Abstract

For closed–set valued random processes we introduce a stochastic order relation (dominance) and show that the argmins of a sequence of random processes, which are epi–convergent in distribution satisfy this order relation in an asymptotic sense. The result may be used for the construction of confidence regions for the argmin.

1 Introduction.

Consider the stochastic program

$$(\mathcal{P}) \left\| \begin{array}{c} F(x) = \int H(x,\omega)\, d\mu(\omega) = \min! \\ x \in S \subseteq I\!R^m \end{array} \right.$$

where $H(x,\omega)$ is a random lower semicontinuous function, i.e. a real function defined on $S \times \Omega$, where S is a closed subset of $I\!R^m$ and $(\Omega, \mathcal{A}, \mu)$ is a probability space. H satisfies

(i) $x, \omega \mapsto H(x,\omega)$ is jointly measurable
(ii) $x \mapsto H(x,\omega)$ is lower semicontinuous for every $\omega \in \Omega$
(iii) $\int |H(x,\omega)|\, d\mu(\omega) < \infty$ for every $x \in S$.

(see Rockefellar (1976), Castaing and Valadier (1977)). By introducing the characteristic function of a set S

$$\vartheta_S = \begin{cases} 0 & x \in S \\ \infty & x \notin S \end{cases}$$

we may equivalently write

$$(\mathcal{P}) \parallel Z(x) := \int H(x,\omega)\, d\mu(\omega) + \vartheta_S(x) = \min!$$

In cases where the expectation function $x \mapsto \int H(x,\omega)\, d\mu(\omega)$ is not known in its analytic form, a widely used technique approximates $(\mathcal{P})$ by using a sample $(\xi_i)_{i=1,\dots,n}$ of i.i.d random variables with distribution μ. The approximative (random) problem is

$$(\mathcal{P}_n) \parallel Z_n(x) := \frac{1}{n}\sum_{i=1}^{n} H(x,\xi_i) + \vartheta_S(x) = \min!$$

By the law of large numbers $Z_n(x) \to Z(x)$ as $n \to \infty$ for all x.

Motivated by this example, we study the following problem in this paper:
Main problem. If a stochastic process Z_n converges to a limit

$$Z_n(\cdot) \to Z(\cdot)$$

in an appropriate sense, what can be said about

$$\text{argmin } Z_n(\cdot) \text{ in relation to argmin } Z(\cdot)?$$

The main problem arises in many variations: Not only in stochastic optimization but also in the asymptotic theory of statistical estimates, which are defined as minimizers or maximizers of a criterion function such as the least sqares estimate, the maximum likelihood estimate, the M-estimate, the Bayes estimate the minimum distance and the R-estimate.

As the "appropriate sense" mentioned in the statement of the Main Problem, we consider in this paper the notion of epi-convergence in distribution. For a thorough treatment of this kind of convergence for stochastic processes see Salinetti and Wets (1986). We repeat here only some basic facts, the proofs can be found in there.
A sequence of lower semicontinuous functions $(f_n) : \mathbb{R}^m \to \mathbb{R}^1 \cup \{\infty\}$ epi–converges to a l.s.c. function f, if the following properties hold:

(i) for all sequences $x_n \to x$,

$$\liminf_n f_n(x_n) \geq f(x);$$

(ii) there is a sequence $y_n \to x$ such that

$$\limsup_n f_n(y_n) \leq f(x).$$

The following properties are equivalent to (i) resp. (ii):

(i') for all closed rectangles $\bar{I}$ in $\mathbb{R}^m$

$$\liminf_n \inf_{x\in\bar{I}} f_n(x) \geq \inf_{x\in\bar{I}} f(x)$$

(ii') for all open rectangles I^o in $\mathbb{R}^m$

$$\limsup_n \inf_{x\in I^o} f_n(x) \leq \inf_{x\in I^o} f(x).$$

The epigraph of a function f is

$$\text{epi } f = \{(x,\alpha) : f(x) \le \alpha\} \in \mathbb{R}^{m+1}.$$

epi f is closed if and only if f is l.s.c.
The family $\mathcal{C}^l$ of all closed sets in $\mathbb{R}^l$ (including the empty set $\emptyset$) can be metrizised by the following metric:

$$\delta(C_1, C_2) = \sum_{i=1}^{\infty} \left| \frac{d(x_i, C_1)}{1 + d(x_i, C_1)} - \frac{d(x_i, C_2)}{1 + d(x_i, C_2)} \right|$$

where (x_i) is a countable dense set in $\mathbb{R}^l$ and $d(x, C) = \inf\{\|x - y\| : y \in C\}$. $d(x, \emptyset)$ is defined as ∞ and $\frac{\infty}{1+\infty} = 0$. The topology induced by this metric is called the Painlevé–Kuratowski topology. $(\mathcal{C}^l, \delta)$ is a compact metric space. For a sequence (C_n) of closed sets define the limes superior (ls) and the limes inferior (li) as

$$\text{ls } (C_n) = \{x : \exists x_{n_k} \in C_{n_k} \text{ s.t. } x_{n_k} \to x\}$$

$$\text{li } (C_n) = \{x : \exists x_n \in C_n \text{ s.t. } x_n \to x\}.$$

It is known that $\delta(C_n, C) \to 0$ iff $C =$ ls $(C_n) =$ li (C_n). Moreover, f_n epi-converges to f iff $\delta($ epi f_n, epi $f) \to 0$.
If Z is a m-dimensional l.s.c. random process, then epi Z induces a probability measure on the Borel sets of the compact metric space $(\mathcal{C}^{m+1}, \delta)$. A sequence of random l.s.c. processes (Z_n) epi-converges in distribution to Z (in symbol: $Z_n \overset{\text{epi}-\mathcal{D}}{\longrightarrow} Z$), if the probability distributions induced by epi Z on $(\mathcal{C}^{m+1}, \delta)$ weakly converge to that induced by epi Z.
For an arbitrary set $A \in \mathbb{R}^l$ introduce the "missing family" $\mathcal{M}_A$ and the "hitting family" $\mathcal{H}_A$:

$$\mathcal{M}_A = \{C \in \mathcal{C}^l : C \cap A = \emptyset\}$$
$$\mathcal{H}_A = \{C \in \mathcal{C}^l : C \cap A \ne \emptyset\} = \mathcal{M}_A^c$$

(^{c} denotes the complement). A basis of the Painlevé–Kuratowksi topology is given by the families

$$\mathcal{M}_K \cap \mathcal{H}_{G_1} \cap \ldots \cap \mathcal{H}_{G_k}$$

where K is compact, $G_1, \ldots G_k$ are open and k is arbitrary.

1.1 Definition. Let P and Q be Borel measures on $(\mathcal{C}^l, \delta)$. We say that P dominates Q, if

$$P(\mathcal{M}_{K_1} \cup \ldots \cup \mathcal{M}_{K_k}) \le Q(\mathcal{M}_{K_1} \cup \ldots \cup \mathcal{M}_{K_k})$$

for all k and arbitrary compact sets $K_1, \ldots, K_k$.

1.2 Definition. Let $P, (P_n)_{n \in \mathbb{N}}$ be Borel probability measures on $(\mathcal{C}^l, \delta)$. We say that P asymptotically dominates (P_n), if

$$P(\mathcal{M}_{K_1} \cup \ldots \cup \mathcal{M}_{K_k}) \le \liminf_n P_n(\mathcal{M}_{K_1} \cup \ldots \cup \mathcal{M}_{K_k})$$

for all k and arbitrary compact sets $K_1, \ldots, K_k$.

The reader may easily verify that P asymptotically dominates (P_n) iff P dominates all cluster points of (P_n).

Now, we are ready to state the main theorem.

1.3 Theorem. Let $Z_n(x,\omega)$ be a stochastic process in $\mathbb{R}^m$ which is epi-convergent in distribution to $Z(x,\omega)$. Let P resp. P_n be the distribution induced on $(\mathcal{C}^m, \delta)$ by argmin Z resp. argmin Z_n. Then P asymptotically dominates (P_n).

The proof is contained in section 3.

As an application of this theorem we may determine confidence sets for the argmin as is stated in the following theorem.

1.4 Theorem. Suppose that the assumptions of the theorem 1.3 are fulfilled. Assume further that there is a compact set K such that

$$\liminf_n P(\text{ argmin } Z_n \subseteq K) \geq 1 - \alpha/2$$

and an open D with $P(\text{ argmin } Z \subseteq D) \geq 1 - \alpha/2$. Then D is an asymptotic confidence set for argmin Z_n in the sense that

$$\liminf_n P(\text{ argmin } Z_n \subseteq D) \geq 1 - \alpha.$$

The proof is also contained in section 3.

2 Weak convergence in $\mathcal{C}^l$.

Let $\mathcal{V}$ be a collection of compact sets in $\mathbb{R}^l$ such that

(a) $\mathcal{V}$ is closed under finite union and intersection

(b) each compact set K is representable as the intersection of a decreasing sequence in $\mathcal{V}$, i.e.

$$K = \bigcap_{i=1}^{\infty} V_i; \qquad V_{i+1} \subseteq V_i; \qquad V_i \in \mathcal{V}$$

(in symbol $V_i \downarrow K$)

(c) each open set G is representable as the union of an increasing sequence in $\mathcal{V}$, i.e.

$$G = \bigcup_{i=1}^{\infty} V_i; \qquad V_{i+1} \supseteq V_i; \qquad V_i \in \mathcal{V}$$

(in symbol $V_i \uparrow G$)

Typical examples for $\mathcal{V}$ are the family of compact rectangles or the family of compact rectangles with rational endpoints.

2.1 Lemma. The Borel σ–algebra on $(\mathcal{C}^l, \delta)$ is generated by the sets

$$\{\mathcal{M}_V : V \in \mathcal{V}\}.$$

Proof. It is sufficient to show that each basic open set

$$\mathcal{M}_K \cap \mathcal{H}_{G_1} \cap \ldots \cap \mathcal{H}_{G_k}$$

is contained in the σ–algebra generated by $\{\mathcal{M}_V : V \in \mathcal{V}\}$. Let $V_i^{(0)} \in \mathcal{V}$ such that $V_i^{(0)} \downarrow K$ and $V_i^{(j)} \in \mathcal{V}$ such that $V_i^{(j)} \uparrow G_j$; $1 \leq j \leq k$. We claim that $\mathcal{M}_K = \bigcup_{i=1}^{\infty} \mathcal{M}_{V_i^{(0)}}$. Obviously $\mathcal{M}_K \supseteq \bigcup_{i=1}^{\infty} \mathcal{M}_{V_i^{(0)}}$. Suppose that $C \in \mathcal{M}_K \setminus \bigcup_{i=1}^{\infty} \mathcal{M}_{V_i^{(0)}}$. Then $C \cap K = \emptyset$, but $C \cap V_i^{(0)} \neq \emptyset$. Let $x_i \in C \cap V_i^{(0)}$. The sequence (x_i) has a cluster point x^*, which is in $\bigcap_{i=1}^{\infty} V_i^{(0)}$, a contradiction.
Since $V_i^{(j)} \uparrow G_j$, $\mathcal{H}_{V_i^{(j)}} \uparrow \mathcal{H}_{G_j}$ for $1 \leq j \leq k$. Consequently, by $\mathcal{H}_{V_i^{(j)}} = \mathcal{M}^c_{V_i^{(j)}}$,

$$\mathcal{M}_{V_i^{(0)}} \cap \mathcal{M}^c_{V_i^{(1)}} \cap \ldots \cap \mathcal{M}^c_{V_i^{(k)}} \uparrow \mathcal{M}_K \cap \mathcal{H}_{G-1} \cap \ldots \cap \mathcal{H}_{G_k}$$

as $i \to \infty$. □

2.2 Lemma. Let P be a probability measure on $(\mathcal{C}^l, \delta)$. Then P is determined by its values on $\mathcal{M}_V; V \in \mathcal{V}$, i.e. by

$$\{P(\mathcal{M}_V) : V \in \mathcal{V}\}.$$

Proof. Let $\mathcal{S}_{\mathcal{V}}$ be the family

$$\mathcal{S}_{\mathcal{V}} = \{\mathcal{M}_{V_0} \cap \mathcal{M}^c_{V_1} \cap \ldots \cap \mathcal{M}^c_{V_k} : V_0, V_1, \ldots V_k \in \mathcal{V}\}.$$

$\mathcal{S}_{\mathcal{V}}$ is a semi-ring: It is closed under intersection, since

$$\begin{aligned} &\mathcal{M}_{V_0} \cap \mathcal{M}^c_{V_1} \cap \ldots \cap \mathcal{M}^c_{V_k} \cap \mathcal{M}_{V_0'} \cap \mathcal{M}^c_{V_1'} \cap \ldots \cap \mathcal{M}^c_{V_k'} \\ &= \mathcal{M}_{V_0 \cup V_0'} \cap \mathcal{M}^c_{V_1} \cap \ldots \cap \mathcal{M}^c_{V_k} \cap \mathcal{M}^c_{V_1'} \cap \ldots \cap \mathcal{M}^c_{V_k'} \end{aligned}$$

and the complement of each set from $\mathcal{S}_{\mathcal{V}}$ is representable as a finite union of sets from $\mathcal{S}_{\mathcal{V}}$. P is uniquely determined on $\mathcal{S}_{\mathcal{V}}$ by

$$P(\mathcal{M}^c_{V_1}) = 1 - P(\mathcal{M}_{V_1})$$

$$P(\mathcal{M}_{V_0} \cap \mathcal{M}^c_{V_1}) = P(\mathcal{M}_{V_0}) - P(\mathcal{M}_{V_0 \cup V_1})$$

and – by induction –

$$\begin{aligned} &P(\mathcal{M}_{V_0} \cap \mathcal{M}^c_{V_1} \cap \ldots \cap \mathcal{M}^c_{V_{k+1}}) \\ = \ &P(\mathcal{M}_{V_0} \cap \mathcal{M}^c_{V_1} \cap \ldots \cap \mathcal{M}^c_{V_k}) - P(\mathcal{M}_{V_0 \cup V_{k+1}} \cap \mathcal{M}^c_{V_1} \cap \ldots \cap \mathcal{M}^c_{V_k}). \end{aligned}$$

Consequently, P is uniquely determined on the generated algebra (which consists of finite unions of sets from $\mathcal{S}_\mathcal{V}$) and hence also on the generated σ–algebra. □

2.3 Theorem. A sequence of Borel probability measures (P_n) on $(\mathcal{C}^l, \delta)$ weakly converges to a limit P, if and only if for all $V \in \mathcal{V}$

$$P(\mathcal{M}_V) \le \liminf_n P_n(\mathcal{M}_V) \le \limsup_n P_n(\mathcal{M}_{V^o}) \le P(\mathcal{M}_{V^o}), \qquad (1)$$

where V^o is the open interior of V.

Proof. Suppose that $P_n \Rightarrow P$ weakly. It is not difficult to show that the closure of $\mathcal{M}_V$ is $\mathcal{M}_{V^o}$. Thus $\mathcal{M}_V$ is open and $\mathcal{M}_{V^o}$ is closed and (??) follows from the well known Portmanteau theorem ([2]).

Suppose conversely that the condition (??) is fulfilled. If $V = \bigcup_{i=1}^k [\alpha_1^{(i)}, \beta_1^{(i)}] \times \cdots \times [\alpha_l^{(i)}, \beta_l^{(i)}]$ then V^ϵ is defined as

$$V^\epsilon = \bigcup_{i=1}^k [\alpha_1^{(i)} - \epsilon, \beta_1^{(i)} + \epsilon] \times \cdots \times [\alpha_l^{(i)} - \epsilon, \beta_l^{(i)} + \epsilon]$$

for ϵ near 0. The function $\epsilon \mapsto P(\mathcal{M}_{V^\epsilon})$ is monotonically decreasing and right–continuous. There are only denumerably many jumps and the set of all ϵ, for which

$$P(\mathcal{M}_{V^\epsilon}) = \lim_{\eta \uparrow \epsilon} P(\mathcal{M}_{V^\eta})$$

is dense. Let $\mathcal{V}_o$ be the family of subsets of $\mathcal{V}$ which are P–continuous, i.e. for which

$$P(\mathcal{M}_V) = \lim_{\epsilon \uparrow 0} P(\mathcal{M}_{V^\epsilon}).$$

$\mathcal{V}_o$ has the same properties (a) – (c) as $\mathcal{V}$ and the assertion of Lemma 2.1 and 2.2 are valid also for $\mathcal{V}_o$.

Suppose that $\mathcal{A}$ is an open set in $(\mathcal{C}^l, \delta)$. We know that $\mathcal{A}$ is of the form

$$\mathcal{A} = \bigcup_{i=1}^\infty \mathcal{M}_{K_i} \cap \mathcal{M}^c_{G_i^{(1)}} \cap \ldots \cap \mathcal{M}^c_{G_i^{(k_i)}}.$$

for compact K_i and open $G_i^{(j)}$. Since every set $\mathcal{M}_{K_i} \cap \mathcal{M}^c_{G_i^{(1)}} \cap \ldots \cap \mathcal{M}^c_{G_i^{(k_i)}}$ may be approximated from below by sets from $\mathcal{S}_\mathcal{V}$ and even by sets from $\mathcal{S}_{\mathcal{V}_o}$ we may write

$$\mathcal{A} = \bigcup_{j=1}^\infty \mathcal{M}_{V_j^{(0)}} \cap \mathcal{M}^c_{V_j^{(1)}} \cap \ldots \cap \mathcal{M}^c_{V_j^{(n_j)}},$$

where $V_j^{(\cdot)} \in \mathcal{V}_o$. Let ϵ be arbitrary small and J such large that

$$P(\bigcup_{j=1}^J \mathcal{M}_{V_j^{(0)}} \cap \mathcal{M}^c_{V_j^{(1)}} \cap \ldots \cap \mathcal{M}^c_{V_j^{(n_j)}} \ge P(\mathcal{A}) - \epsilon.$$

Since, by assumption (cf. Lemma 2.2),

$$P_n(\mathcal{M}_V) \to P(\mathcal{M}_V) \qquad \text{for all } V \in \mathcal{V}_o$$

ans $\mathcal{V}_o$ is closed w.r.t. intersection, we have

$$P_n(\bigcup_{j=1}^{J} \mathcal{M}_{V_j^{(0)}} \cap \mathcal{M}^c_{V_j^{(1)}} \cap \ldots \cap \mathcal{M}^c_{V_j^{(n_j)}}) \to$$
$$P(\bigcup_{j=1}^{J} \mathcal{M}_{V_j^{(0)}} \cap \mathcal{M}^c_{V_j^{(1)}} \cap \ldots \cap \mathcal{M}^c_{V_j^{(n_j)}})$$

and therefore

$$\liminf_n P_n(\mathcal{A}) \geq \lim_n P_n(\bigcup_{j=1}^{J} \mathcal{M}_{V_j^{(0)}} \cap \mathcal{M}^c_{V_j^{(1)}} \cap \ldots \cap \mathcal{M}^c_{V_j^{(n_j)}}) = P(\mathcal{A}) - \epsilon.$$

Since ϵ was arbitrary, $\liminf_n P_n(\mathcal{A}) \geq P(\mathcal{A})$ which is equivalent to $P_n \Rightarrow P$ weakly.

□

2.4 Corollary. Let $Z_n(x,\cdot)$ be a l.s.c. process. Then Z_n epi–converges in distribution ($Z_n \stackrel{\text{epi}-\mathcal{D}}{\longrightarrow} Z$) if and only if for all k, all collections of closed rectangles $R_1, \ldots, R_k$ and all reals $\alpha_1, \ldots, \alpha_k$

$$\begin{aligned} & P(\inf_{x\in R_1} Z(x,\cdot) > \alpha_1, \cdots, \inf_{x\in R_k} Z(x,\cdot) > \alpha_k) \\ \leq\ & \liminf_n P(\inf_{x\in R_1} Z_n(x,\cdot) > \alpha_1, \cdots, \inf_{x\in R_k} Z_n(x,\cdot) > \alpha_k) \\ \leq\ & \limsup_n P(\inf_{x\in R_1^o} Z_n(x,\cdot) \geq \alpha_1, \cdots, \inf_{x\in R_k^o} Z_n(x,\cdot) \geq \alpha_k) \\ \leq\ & P(\inf_{x\in R_1^o} Z(x,\cdot) \geq \alpha_1, \cdots, \inf_{x\in R_k^o} Z(x,\cdot) \geq \alpha_k). \end{aligned}$$

Proof. The corollary is a direct consequence of Theorem 2.3, since

$$\begin{aligned} & \{\inf_{x\in R_1} Z(x,\cdot) > \alpha_1, \cdots, \inf_{x\in R_k} Z(x,\cdot) > \alpha_k\} \\ =\ & \{\text{ epi } Z(\cdot,\cdot) \in \bigcap_{i=1}^{k} \mathcal{M}_{R_i \times [\alpha_i, \alpha_i - 1]}\}. \end{aligned}$$

□

3 Proofs of the main theorems

3.1 Proof of Theorem 1.3. Suppose that $Z_n(\cdot,\cdot) \stackrel{\text{epi}-\mathcal{D}}{\longrightarrow} Z(\cdot,\cdot)$. By the well-known Skorohod–Dudley–Wichura theorem one may construct a probability space $(\Omega', \mathcal{A}', P')$ and random l.s.c. functions Z'_n resp. Z' on Ω' such that

(i) Z_n and Z'_n resp. Z and Z' coincide in distribution,

(ii) $\delta(\text{ epi } Z'_n, \text{ epi } Z') \to 0 \qquad P' - a.s.$

Let $A_n =$ argmin Z_n, $A'_n =$ argmin Z'_n, $A =$ argmin Z, $A' =$ argmin Z'. Clearly A_n and A'_n, resp. A and A' coincide in distribution. By (ii)

$$\text{ls } A'_n \subseteq A' \qquad\qquad P' - a.s.$$

since $f_n \overset{\text{epi}}{\longrightarrow} f$ implies that ls argmin $f_n \subseteq$ argmin f. Let K be a compact set and suppose that $A'(\omega) \cap K = \emptyset$. Then $A'_n(\omega) \cap K \neq \emptyset$ only for finitely many n. Otherwise this would be a contradiction to ls $A'_n(\omega) \subseteq A'(\omega)$. Consequently

$$\mathbb{1}_{\{A' \in \mathcal{M}_K\}} \leq \liminf_n \mathbb{1}_{\{A'_n \in \mathcal{M}_K\}} \qquad P' - a.s.$$

where $\mathbb{1}$ denotes the indicator function. Let $K_1, \ldots, K_k$ be a collection of compact sets. By Fatou's Lemma

$$\begin{aligned}
& P(A \in \mathcal{M}_{K_1} \cup \ldots \cup \mathcal{M}_{K_k}) = P'(A' \in \mathcal{M}_{K_1} \cup \ldots \cup \mathcal{M}_{K_k}) \\
= \;& \mathbb{E}_{P'}(\max_i \mathbb{1}_{\{A' \in \mathcal{M}_{K_i}\}}) \leq \mathbb{E}_{P'}(\max_i \liminf_n \mathbb{1}_{\{A'_n \in \mathcal{M}_{K_i}\}}) \\
\leq \;& \mathbb{E}_{P'}(\liminf_i \max_i \mathbb{1}_{\{A' \in \mathcal{M}_{K_i}\}}) \leq \liminf_n \mathbb{E}_{P'}(\max_i \mathbb{1}_{\{A' \in \mathcal{M}_{K_i}\}}) \\
= \;& \liminf_n P'(A'_n \in \mathcal{M}_{K_1} \cup \ldots \cup \mathcal{M}_{K_k}) \\
= \;& \liminf_n P(A_n \in \mathcal{M}_{K_1} \cup \ldots \cup \mathcal{M}_{K_k}).
\end{aligned}$$

□

3.2 Proof of Theorem 1.4.

We continue with the notation of the previous proof. Since $D \subseteq K$ it follows that

$$\mathcal{M}_{D^c} = \mathcal{M}_{K \cap D^c} \backslash \mathcal{M}^c_{K^c}.$$

Therefore, by Theorem 1.3,

$$\begin{aligned}
& \liminf_n P(\text{ argmin } Z_n \subseteq D) = \liminf_n P(A_n \in \mathcal{M}_{D^c}) \\
\geq \;& \liminf_n [P(A_n \in \mathcal{M}_{K \cap D^c}) - P(A_n \in \mathcal{M}^c_{K^c})] \\
\geq \;& \liminf_n P(A_n \in \mathcal{M}_{K \cap D^c}) + \liminf_n P(A_n \in \mathcal{M}_{K^c}) - 1 \\
\geq \;& P(A \in \mathcal{M}_{K \cap D^c}) + \liminf_n P(\text{ argmin } Z_n \subseteq K) - 1 \\
\geq \;& P(A \in \mathcal{M}_{D^c}) + \alpha/2 \\
= \;& 1 - P(\text{ argmin } Z \subseteq D) + \alpha/2 = 1 - \alpha.
\end{aligned}$$

□

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