Working Paper

Ellipsoidal Techniques: Guaranteed State Estimation

A.B. Kurzhanski and I. Vályi

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IIASA International Institute for Applied Systems Analysis □ A-2361 Laxenburg □ Austria
Telephone: +43 2236 715210 □ Telex: 079 137 iiasa a □ Telefax: +43 2236 71313

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Foreword

This is the third of a series of papers giving an early account of the application of ellipsoidal techniques to various problems in dynamical systems. It deals with guaranteed state estimation – also to be interpreted as a tracking problem – again under unknown but bounded disturbances.

Contents

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Introduction

This paper gives a concise description of effective solutions to the "guaranteed" state estimation problems for dynamic systems with unknown but bounded uncertainty. It indicates a rather unconventional, rigorous theory for these problems based on the notion of evolution equations of the "funnel" type which could be further transformed – through *exact* ellipsoidal approximations – into algorithmic procedures that allow effective simulation particularly with computer graphics. The estimation problem is also interpreted as a problem of tracking a partially known system under incomplete measurements.

Mathematically, the technique described in this paper is based on a theory of set-valued evolution equations with the approximation of solutions formulated in terms of set-valued calculus by ellipsoidal-valued functions.

1 Uncertain Systems

An *uncertain system* is said to be one of type

$$\dot{x}(t) \in A(t)x(t) + u(t), \quad t_0 \leq t \leq t_1, \qquad x(t_0) = x_0, \tag{1}$$

where $u = u(t)$ is the *unknown but bounded input* (disturbance). It is presumed that the *initial state* x_0 is also unknown but bounded, so that

$$u(t) \in \mathcal{P}(t), \quad t_0 \leq t \leq t_1, \qquad x_0 \in \mathcal{X}_0, \tag{2}$$

where the set $\mathcal{X}_0 \subset$ conv $\mathcal{R}^n$ and the continuous set-valued function $\mathcal{P}(t) \in$ conv $\mathcal{R}^n$ are given (conv $\mathcal{R}^n$ stands for the variety of all convex compact subsets of $\mathcal{R}^n$).

Equation (1) of the plant may be complemented by a *state constraint*

$$G(t)x(t) \in \mathcal{Y}(t), \qquad t_0 \leq t \leq t_1 \tag{3}$$

where $\mathcal{Y}(t) \in$ conv $\mathcal{R}^m$, $m < n$. The constraint (2) may be particularly generated by a *measurement equation*

$$y(t) = G(t)x(t) + v(t), \qquad t_0 \leq t \leq t_1, \tag{4}$$

with an *unknown but bounded error*

$$v(t) \in \mathcal{Q}(t), \qquad t_0 \le t \le t_1 \tag{5}$$

where $\mathcal{Q}(t) \in$ conv $\mathcal{R}^m$ is a Lipschitz-continuous set-valued map, [1]. With the realization $y = y[t]$ being known, restriction (4), (5) turns into

$$G(t)x(t) \in y[t] - \mathcal{Q}(t), \qquad t_0 \le t \le t_1, \tag{6}$$

so that $y[t] - \mathcal{Q}(t)$ now substitutes for $\mathcal{Y}(t)$ (the function $y[t]$ may however not be known in advance, arriving *on-line*).

Our objective will be to *estimate the system output*

$$z(t) = Hx(t), \qquad z \in \mathcal{R}^k, \quad k \le n, \quad t_0 \le t \le t_1 \tag{7}$$

at a prescribed instant of time θ, — either for the system (1), (2), (3) (the *attainability problem* under state constraints) or for the system (1), (2), (7) (the problem of *guaranteed state estimation*).

The solutions to both problems are well known (see e.g. [14], [5], [7]). Our aim however is not to repeat this information but to rewrite the theoretical results focusing on the main objective – a constructive algorithmic procedure based on ellipsoidal techniques that allows a simulation with graphical representations. We will now specify the problems considered here.

2 The Estimation Problems

We start with the *attainability problem*. Let $x[t] = x(t, t_0, x^0)$ stand for an isolated solution of system (1.1) that starts at point $x^0 = x(t_0)$. As is well known the *attainability domain* for (1), (2), (3) at time θ from point x^0 is the cross-section at $t = \theta$ of the tube $\mathcal{X}(t, t_0, x^0)$ of all trajectories $x[t] = x(t, t_0, x^0)$ that satisfy (1), (2), (3). The union

$$\bigcup\{x(\theta, t_0, x^0) \mid x^0 \in \mathcal{X}_0\} = \mathcal{X}(\theta, t_0, \mathcal{X}_0) = \mathcal{X}[\theta]$$

is the *attainability domain* at time θ *from set* $\mathcal{X}_0$.

The multivalued map $\mathcal{X}[t]$ generates a *generalized dynamic system*. Namely the mapping

$$\mathcal{X}(t', t, \cdot) \quad : \quad \text{conv } \mathcal{R}^n \rightarrow \text{ conv } \mathcal{R}^n$$

satisfies a *semigroup property:* whatever are the values $t \le \tau \le \theta$, $(t_0 \le t, \quad \theta \le t_1)$ we have

$$\mathcal{X}(\theta, t, \mathcal{X}[t]) = \mathcal{X}(\theta, \tau, \mathcal{X}(\tau, t, \mathcal{X}[t])) \tag{8}$$

The set-valued map, or in other words, the tube $\mathcal{X}[t]$, $(\mathcal{X}[t_0] = \mathcal{X}_0)$ satisfies an evolution equation – a "funnel" equation, ([14], [7]) – which is

$$\lim_{\sigma \to +0} \sigma^{-1} h(\mathcal{X}[t+\sigma], [(I + A(t)\sigma)\mathcal{X}[t] + \sigma\mathcal{P}(t)] \cap \mathcal{Y}(t+\sigma)) = 0, \qquad t_0 \leq t \leq t_1, \tag{9}$$

$$\mathcal{X}[t_0] = \mathcal{X}_0,$$

Here $h(\mathcal{X}'\mathcal{X}'')$ stands for the *Hausdorff distance* between $\mathcal{X}', \mathcal{X}'' \in \text{conv } \mathcal{R}^n$, namely

$$h(\mathcal{X}'\mathcal{X}'') = \max\{h_+(\mathcal{X}', \mathcal{X}''), h_-(\mathcal{X}', \mathcal{X}'')\},$$

where

$$h_+(\mathcal{X}', \mathcal{X}'') = \min\{r : \mathcal{X}'' \subset \mathcal{X}' + r\mathcal{S}\}$$

and $h_-(\mathcal{X}'\mathcal{X}'') = h_+(\mathcal{X}'', \mathcal{X}')$ are the *Hausdorff semi-distances*, $\mathcal{S}$ is a unit ball in $\mathcal{R}^n$.

Equation (9) is correctly posed and has a unique solution that defines the tube $\mathcal{X}[t] = \mathcal{X}(t, t_0, \mathcal{X}_0)$ for system (1), (2), (3) if the map $\mathcal{Y}(t)$ is such that the support function

$$f(t, \ell) = \max\{(\ell, p) \mid p \in \mathcal{Y}(t)\} = \rho(\ell \mid \mathcal{Y}(t))$$

is Lipschitz continuous in t, [7].

Using only one of the Hausdorff semi-distances in (9) leads to the loss of *uniqueness* of the solutions, but complemented with an extremality condition we obtain alternative descriptions the multivalued map $\mathcal{X}[t]$.

On one hand, consider

$$\lim_{\sigma \to +0} \sigma^{-1} h_+(\mathcal{Z}[t+\sigma], [(I + A(t)\sigma)\mathcal{Z}[t] + \sigma\mathcal{P}](t) \cap \mathcal{Y}(t+\sigma)) = 0, \qquad t_0 \leq t \leq t_1, \tag{10}$$

$$\mathcal{Z}[t_0] = \mathcal{X}_0,$$

A set-valued map $\mathcal{Z}_*[t]$ will be defined as a *minimal solution* to (10) if it satisfies (10) for almost all $t \in [t_0, t_1]$ and if there exists no other solution $\mathcal{Z}[t]$ to (10) such that $\mathcal{Z}_*[t] \supset \mathcal{Z}[t]$ for all $t \in [t_0, t_1]$ and $\mathcal{Z}_*[t] \not\equiv \mathcal{Z}[t]$.

Equation (10) has a unique minimal solution under the conditions required for the existence and uniqueness of the solutions to (9). In this case $\mathcal{X}[t] \equiv \mathcal{Z}_*[t]$.

On the other hand, by [8] we have that

$$\lim_{\sigma \to +0} \sigma^{-1} h_-(\mathcal{W}[t+\sigma], [(I + A(t)\sigma)\mathcal{W}[t] \cap \mathcal{Y}(t) + \sigma\mathcal{P}(t)) = 0, \qquad t_0 \leq t \leq t_1, \tag{11}$$

$$\mathcal{W}[t_0] = \mathcal{X}_0,$$

has a unique maximal solution $\mathcal{W}^*[t]$ — defined analogously to the minimal solutions to (10) — if, for example, $\mathcal{Y}(t)$ is "upper semicontinuous" in t, [1]. and then $\mathcal{X}[t] \equiv \mathcal{W}^*[t]$.

The *Guaranteed State Estimation* problem may now be solved as follows. Suppose the measurement function $y[t], t_0 \leq t \leq t_1$, of (6) is given and

$$\mathcal{Y}(t) = y[t] - \mathcal{Q}(t), \qquad t_0 \leq t \leq t_1 \tag{12}$$

Denote $\mathbf{X}[t] = \mathbf{X}(t, t_0, \mathcal{X}_0)$ the attainability domain for system (1), (2), (3), (12). Then $\mathbf{X}[t]$ is known as the "informational domain " [3], (the "domain of consistency", the "feasibility domain", etc. [15], [12], [17]) for the state estimation problem (1), (2), (4), (5). In other words it is the set of states $x[t]$ of system (1) at time t that are consistent with the constraints (2), (6), $y(t)$ being given.

If measurement $y[t] = y_*[t]$ is generated by an unknown triplet $\zeta_*(t) = \{x_{0*}, u_*(t), v_*(t)\}$ due to system (1), (4), that is now

$$y_*[t] = G(t)x_*[t] + v_*(t), \qquad t_0 \leq t \leq t_1 \tag{13}$$

where

$$\dot{x}_*[t] = A(t)x_*[t] + u_*(t), \quad t_0 \leq t \leq t_1, \qquad x_{0*} \in \mathcal{X}_{0*}, \tag{14}$$

then the tube $\mathbf{X}_*[t]$ generated by (1), (3), (12), $y[t] \equiv y_*[t]$ does *always* contain the unknown actual trajectory $x_*[t]$ of the system. The tube $\mathbf{X}_*[t]$ therefore gives a *"guaranteed estimate"* of the state of system (1) on the basis of a measurement $y[t]$ of (4).

In this paper we presume that $y[t]$ is Lipschitz-continuous in t, in order to conform with the assertions of the above. The situation however allows a generalization to the case when $y[t]$ is a function measurable on $[t_0, t_1]$. The respective mathematical details are beyond the scope of this paper.

The solutions to the *Estimation Problems* of this paper are therefore given through the evolution equations (9), (10). The objective is now to devise an algorthmic scheme for solving these equations.

Equations (9), (10) yield a natural discrete-time scheme.

3 The Discrete-time Scheme

The discrete-time scheme can be given in two versions reflecting (9), (10) and (11), these are "first-order" schemes.

$$\mathcal{X}[t + \sigma] = [(I + \sigma A(t))\mathcal{X}[t] + \sigma\mathcal{P}(t)] \cap \mathcal{Y}(t + \sigma) \tag{15}$$

$$\mathcal{X}[t + \sigma] = [I + \sigma A(t)]\mathcal{X}[t] \cap \mathcal{Y}(t) + \sigma\mathcal{P}(t) \tag{16}$$

that yield a convergence to the continuous-time solutions. The main problem, however, is that the $\mathcal{X}[t]$'s are arbitrary convex compact sets being mathematically described through infinite-dimensional elements (e.g. through their support functions $\rho(\ell \mid \mathcal{X}[t]) = g(t,\ell)$). Our objective is to give a constructive scheme for their description by approximating them through finite-dimensional elements which, in this paper, are taken as ellipsoids and further, through ellipsoidal-valued functions.

4 The Ellipsoidal Techniques

Denote a nondegenerate ellipsoid as

$$\mathcal{E}(a,P) = \{x|(x-a)'P^{-1}(x-a) \le 1\}$$

where a is its center and the symmetic matrix $P > 0$ determines its configuration. From here we have

$$\rho(\ell \mid \mathcal{E}(a,P)) = (\ell,a) + (\ell,P\ell)^{1/2}$$

where the latter description also allows $\det P = 0$.

Suppose the sets $\mathcal{X}_0, \mathcal{P}(t), \mathcal{Q}(t), \mathcal{Y}(t)$ $t_0 \le t \le t_1$, are ellipsoids, so that

$$\mathcal{X}_0 = \mathcal{E}(x_0, X_0),\ \mathcal{P}(t) = \mathcal{E}(p(t), P(t)),\ \mathcal{Q}(t) = \mathcal{E}(q(t), Q(t)),\ \mathcal{Y}(t) = \mathcal{E}(y(t), Y(t)), \tag{17}$$

and the matrices

$$X_0 \ge 0, \quad P(t) \ge 0, \quad Q(t) > 0, \quad Y(t) > 0.$$

The discrete-time schemes (15), (16) then make it necessary to handle the following operations.

$$[\mathcal{E}(a_1,Q_1) + \mathcal{E}(a_2,Q_2)] \cap \mathcal{E}(a_3,Q_3)$$

and

$$[\mathcal{E}(a_1,Q_1) \cap \mathcal{E}(a_2,Q_2)] + \mathcal{E}(a_3,Q_3)$$

where $\mathcal{E}(a_i,Q_i)$, are given ellipsoids, $Q_i \ge 0$, $i = 1,2,3$.

This could be done through a combination of the following relations:

(i) *The sum of ellipsoids*

Given ellipsoids $\mathcal{E}(a_i,Q_i)$, $i = 1,\ldots k$, their sum

$$\mathcal{E}_s = \sum_{i=1}^{k} \mathcal{E}(a_i,Q_i)$$

which need not be an ellipsoid, could be approximated from above as

$$\mathcal{E}_s \subset \mathcal{E}\left(\sum_{i=1}^{k} a_i, Q(\pi)\right), \tag{18}$$

where

$$Q(\pi) = \left(\sum_{i=1}^{k} \pi_i\right)\left(\sum_{i=1}^{k} \pi_i^{-1} Q_i\right), \qquad \pi_i > 0, \quad i = 1, \ldots k.$$

Lemma 4.1 *The inclusion (18) is true whatever are the coefficients $\pi_i > 0, \quad i = 1, \ldots k$. The following relation holds:*

$$\mathcal{E}_s = \bigcap \left\{ \mathcal{E}\left(\sum_{i=1}^{k} a_i, Q(\pi)\right) \mid \pi_i > 0, \quad i = 1, \ldots k \right\}. \tag{19}$$

(ii) *The intersection of ellipsoids*

The intersection

$$\mathcal{E}_i = \bigcap \{\mathcal{E}(a_i, Q_i) \mid \quad i = 1, \ldots k\} \tag{20}$$

could be approximated from above as

$$\mathcal{E}_i \subset \sum_{i=1}^{k} \mathcal{E}(B_i a_i, B_i Q B_i'), \tag{21}$$

$$\sum_{i=1}^{k} B_i = I, \tag{22}$$

where B_i is an $(n \times n)$-matrix and the prime stands for the transpose.

Lemma 4.2 *The inclusion (21) is true, whatever are the $(n \times n)$-matrices B_i, $i = 1, \ldots k$ that satisfy (22). The following equality is true*

$$\mathcal{E}_i = \bigcap \left\{ \sum_{i=1}^{k} \mathcal{E}(B_i a_i, B_i Q_i B_i') \mid \sum_{i=1}^{k} B_i = I \right\}. \tag{23}$$

A particular case of (22) is when the matrices B_i are selected in the form of $B_i = \alpha_i I$, $i = 1, \ldots k$. The equality (23) is then transformed into

$$\mathcal{E}_i \subset \bigcap \left\{ \sum_{i=1}^{k} \mathcal{E}(\alpha_i a_i, \alpha_i^2 Q_i) \mid \sum_{i=1}^{k} \alpha_i = 1 \right\}. \tag{24}$$

The combination of (19) and (23) gives an exact external approximation of $\mathcal{E}_i$ by a family of ellipsoids that can be simulated through parallelization. Among these one may also select an optimal ellipsoid.

A somewhat different scheme could be given along the lines of [15], [16], [2].

Under the constraints of (17) we come to the attainability problem for the system

$$\dot{x}(t) \in A(t)x(t) + \mathcal{E}(p(t), P(t)), \qquad t_0 \le t \le t_1, \tag{25}$$

$$x(t_0) \in \mathcal{E}(x_0, X_0), \tag{26}$$

$$G(t)x(t) \in \mathcal{E}(y(t), Y(t)), \qquad t_0 \leq t \leq t_1. \tag{27}$$

The set $\mathcal{X}[t]$ for (25), (26), (27) may be approximated both externally and internally by ellipsoidal-valued functions. We will further deal only with the former case. (The schemes of internal ellipsoidal approximation for various attainability problems could be found in [13], [16]).

Consider an evolution equation

$$\lim_{\sigma \to +0} \sigma^{-1} \cdot h_{-}(\mathcal{E}[t+\sigma], (I + A(t)\sigma)\mathcal{E}[t] \cap \mathcal{E}(y(t), Y(t)) + \sigma\mathcal{E}(p(t), \mathcal{P}(t)) = 0, \tag{28}$$

$$t_0 \leq t \leq t_1,$$

$$\mathcal{E}[t_0] = \mathcal{E}(x_0, X_0).$$

A function $\mathcal{E}[t]$ will be defined as a solution to (28) if it satisfies (28) for almost all $t \in [t_0, t_1]$ and is *ellipsoidal-valued.* Obviously the solution $\mathcal{E}[t]$ is *non-unique* and satisfies the inclusion

$$\mathcal{X}[t] \subset \mathcal{E}[t], \quad t_0 \leq t \leq t_1, \qquad \mathcal{X}[t_0] = \mathcal{E}(x_0, X_0).$$

Moreover

$$\mathcal{X}[t] = \bigcap \{\mathcal{E}[t] \mid \mathcal{E}[t] \text{ is a solution to (28) } \}, \qquad t_0 \leq t \leq t_1.$$

The ellipsoidal solutions $\mathcal{E}_{-}[t] = \mathcal{E}(x_{-}(t), X_{-}(t))$ to (27) allow explicit representations through appropriate systems of ODE's for the centers $x_{-}(t)$ and the matrices $X_{-}(t) > 0$ of these ellipsoids, (see for example [2], [16], [9], [10], [11]).

5 Guaranteed State Estimation as a Tracking Problem

The center $x_{+}(t)$ of the tube $\mathcal{E}_{+}[t]$ allows a representation

$$\dot{x}_{+}(t) = A(t)x_{+}(t) + f(t, y_{*t}(\cdot), M_t(\cdot)), \qquad t_0 \leq t \leq t_1, \tag{29}$$

$$x_{+}(t_0) = x_0, \tag{30}$$

where $f(t, y_t(\cdot), M_t(\cdot))$ is a nonlinear functional with memory on the actual measurement $y_{*}[t]$ of (12) and the parametrization $M(t)$. (For a given function $h(.)$, the index t in $h_t(.)$ refers to the function defined by $h_t(s) = h(t+s), \quad t_0 - t \leq s \leq 0$).

According to (13), the actual trajectory to be estimated is $x_{*}[t]$. The result of the (approximate) estimation procedure is that vector $x_{-}(t)$ tracks the actual trajectory $x_{*}(t)$ on the basis of the measurement $y_{*}[\tau]$ with $t_0 \leq \tau \leq t$. This procedure is similar in nature with a *differential* *game of observation*, [3]. (A feedback duality theory for differential games of observation and control was indicated by [4]).

What follows are the results of numerical simulations for the estimation problems of the above, including the "tracking type" representation for the solutions.

6 Numerical Examples

We study a 4 dimensional system (1) over the time interval $[0, 5]$ considering first the *attainability problem under state constraints.*

The initial state is bounded by the ellipsoid $\mathcal{X}_0 = \mathcal{E}(x_0, X_0)$ at the initial moment $t_0 = 0$ with

$$x_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

and

$$X_0 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We consider a case when the right hand side is constant:

$$A(t) \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 0 \end{pmatrix},$$

describing the position and velocity of two independent oscillators. Inputs $u(t)$ are also bounded by time independent constraints $\mathcal{P}(t) = \mathcal{E}(p(t), P(t))$:

$$p(t) \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$P(t) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.01 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.01 \end{pmatrix},$$

(this form of the bounding sets makes the system coupled). State constraint (3) is defined by the data

$$G(t) \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{31}$$

a projection, and $\mathcal{Y}(t) = \mathcal{E}(y(t), Y(t))$ with $y(t) \equiv 0$ and

$$Y(t) \equiv \begin{pmatrix} 16 & 0 \\ 0 & 25 \end{pmatrix}.$$

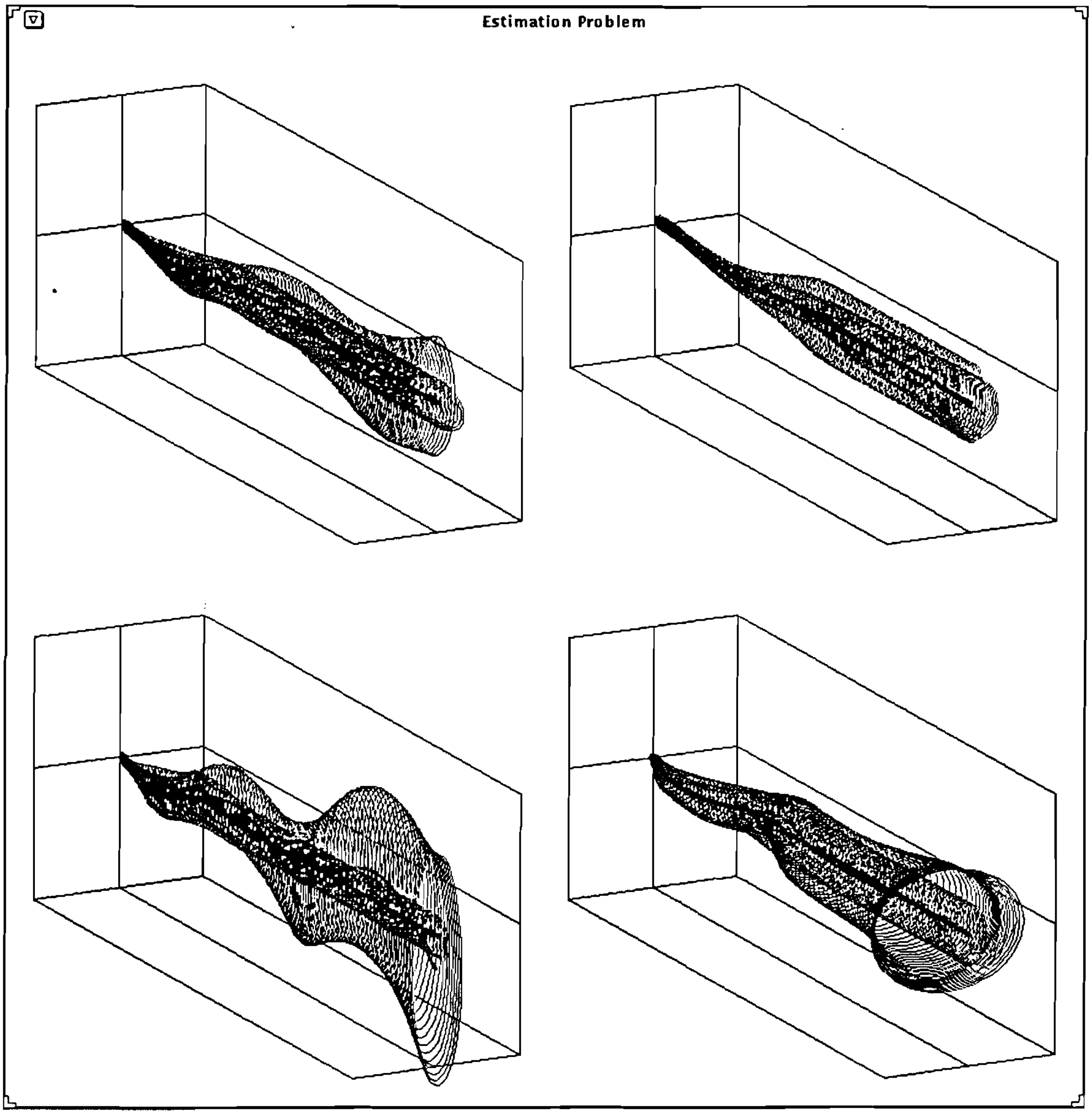

Figure 1: Tube of external ellipsoidal estimates of attainability sets.

In Figure 1 we show the graph of external ellipsoidal estimates of the *system outputs* — with and without constraints — presenting them in four *windows*, according to H of (7) being equal to projections onto the planes spanned by the first and second, third and fourth, first and third, and second and fourth coordinate axes, in a clockwise order starting from bottom left. The drawn segments of coordinate axes corresponding to the output variables range from -30 to 30. The skew axis in Figure 1 is time, ranging from 0 to 5.

Calculations are based on the discretized version of (25) and scheme (15), as well as analogous estimates in the case when $\mathcal{Y}(t) \equiv \mathcal{R}^n$, that is in the absence of state constraints. Trajectories of the centers are also drawn, the thick line corresponding to estimates of the nonconstrained outputs.

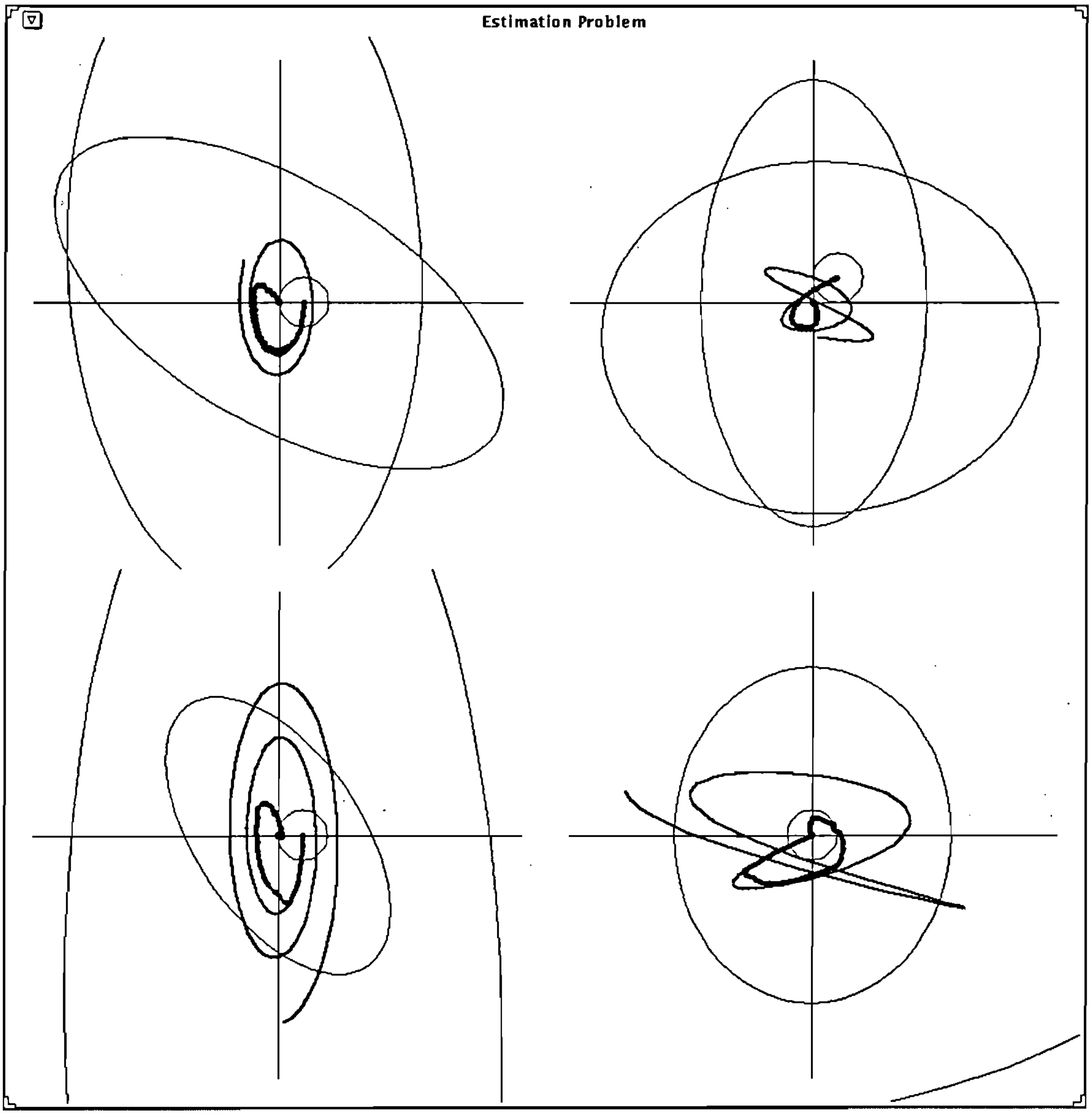

Figure 2: Trajectories of the centers and final estimates in phase space

Approximating a general convex set by an ellipsoid means a compromise. This is seen in the right top window, where contrary to one's expectation, the constrained estimate is "bigger" than the nonconstrained. Note however that it is exactly these coordinates where the phase constraints are inactive, see (31). A modification of the scheme (15) or (16) would allow us to eliminate this "anomaly", but then we would loose accuracy in other directions.

Figure 2 shows the trajectory of the centers, initial sets and the ellipsoidal estimates of the outputs in phase space, with the coordinate axes ranging from -10 to 10.

We turn now to the *guaranteed state estimation problem* interpreted as a *tracking problem*, of the form (13), (14) and (7). We keep the above parameter values of the time interval, $A(t)$, $\mathcal{E}(p(t), P(t))$ and $\mathcal{E}(x_0, X_0)$.

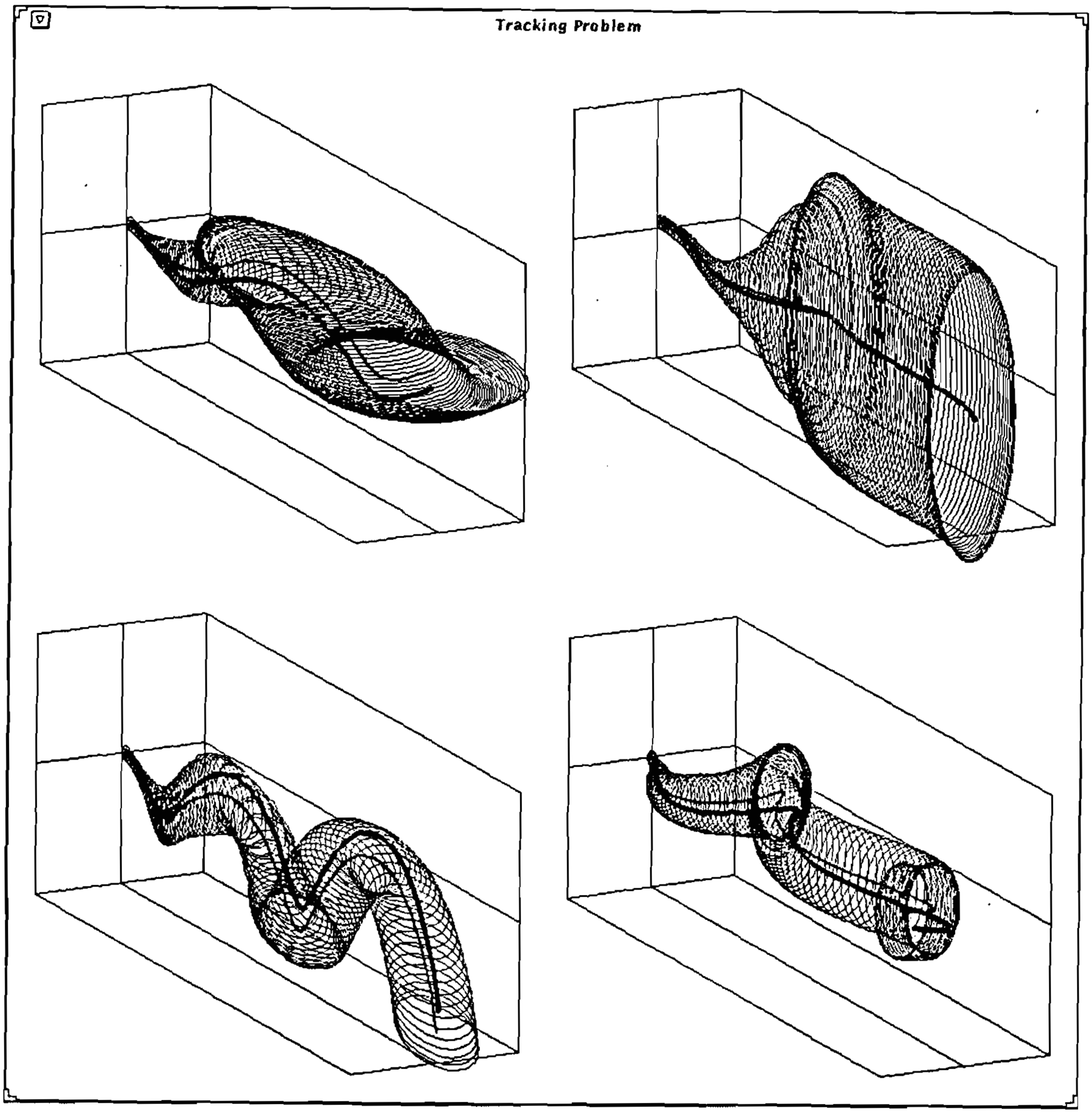

Figure 3: Time representation of ellipsoidal tracking — "worst" noise.

We model the trajectory $x_*(t)$ of (14) – the one to be tracked – by using the following construction for the triplet $\zeta_*(t) = \{x_{0*}, u_*(t), v_*(t)\}$. The initial value x_{0*} is a (randomly selected) element at the boundary of the initial set $\mathcal{X}_0 = \mathcal{E}(x_0, X_0)$, and the input $u_*(t)$ and the measurement error $v_*(t)$ are so called *extremal bang-bang type* feasible disturbances. The construction for $u_*(t)$ is the following. The time interval is divided into subintervals of constant lengths. A value u is chosen randomly at the boundary of the respective bounding set, that is here of $\mathcal{P}(t) = \mathcal{E}(p(t), P(t))$ and its value is then defined by $u_*(t) = u$ over all the first interval and by $u_*(t) = -u$ over the second. Then a new value for u is selected and the above procedure is repeated for the next pair of intervals, etc. In equation (13) we take, for technical reasons, $G(t)$ to be a 4 dimensional identity matrix. In the case of $u_*(t)$ we chose this interval to have a length of 0.25.

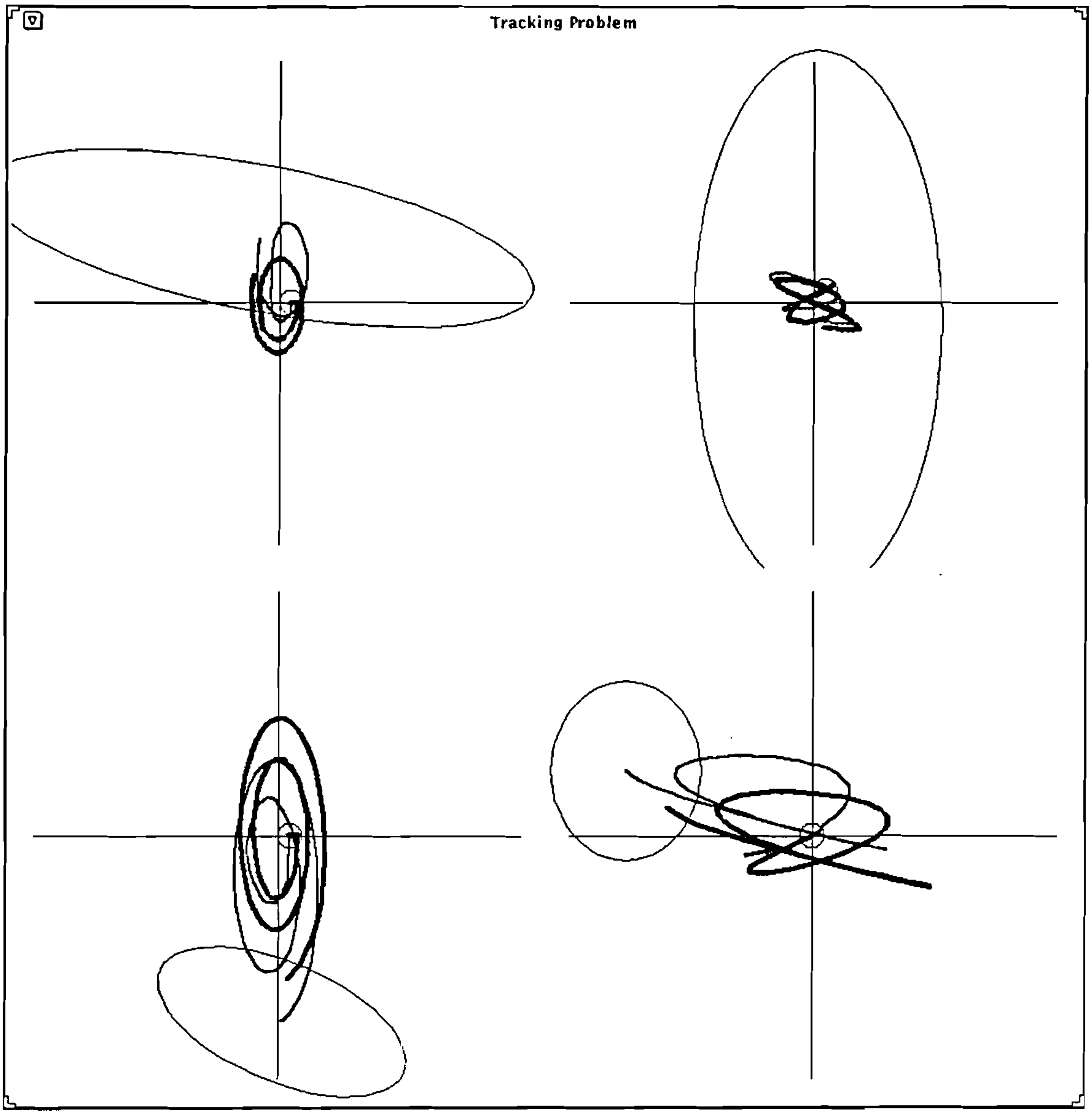

Figure 4: Phase space representation of ellipsoidal tracking — "worst" noise.

The disturbance $v_*(t)$ on the measurement $y_*[t]$ is constructed in an analogous way, to be bounded by $\mathcal{Q}(t) = \mathcal{E}(q(t), Q(t))$, with $q(t) \equiv 0$ and

$$Q(t) \equiv \begin{pmatrix} 10000 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 \\ 0 & 0 & 10000 & 0 \\ 0 & 0 & 0 & 36 \end{pmatrix}.$$

According to the terminology used in identification theory, the set $\mathbf{X}^*[t]$ is the error set of the estimation process, its size clearly depending on the nature of the measurement noise $v_*(t)$. As is expounded in [6], if we chose it in such a way that it takes a constant value at the boundary of $\mathcal{E}(q(t), Q(t))$ over all the time interval under study, then it corresponds to the "worst case". This results in "large" error sets.

In Figures 3 and 4 we see the same four cases of system outputs as before, in the respective four windows. The trajectory drawn with the thick line is of $x_*(t)$. The thin line represents the trajectory of the centers $x_+(t)$ of the tracking ellipsoids. Figure 3 shows the process developing over time, - the drawn segments of coordinate axes corresponding to the output variables range from -20 to 20. and Figure 4, displaying the initial sets of uncertainty (appearing as circles) and the confidence region at the final moment only, in phase space. Coordinate axes range here from -10 to 10.

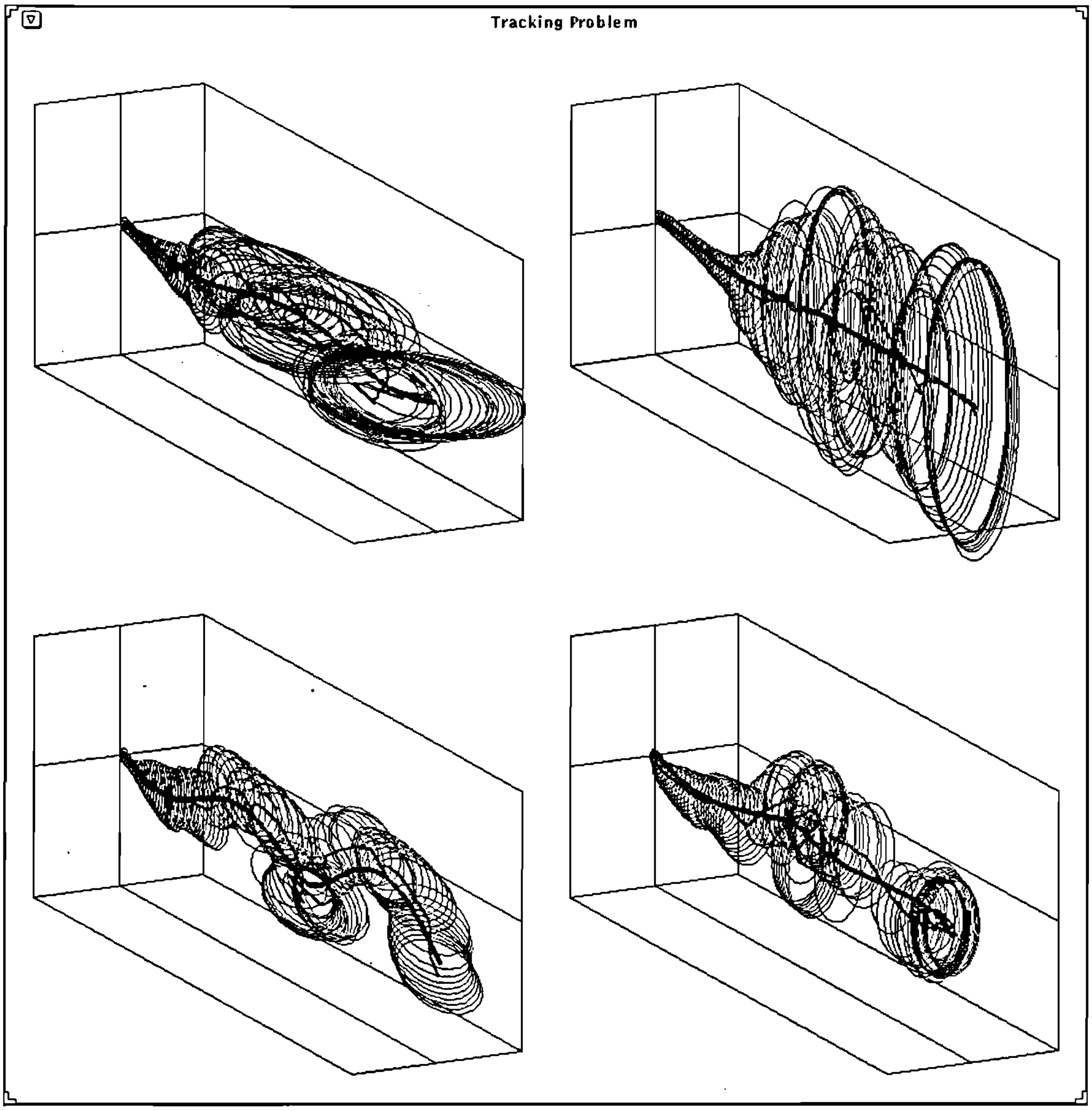

Figure 5: Phase space representation of ellipsoidal tracking — "better" noise.

Figures 5 and 6 show how much the estimation can improve if the noise changes from worst to better — although we obtain here only external ellipsoidal estimates of the "true" error sets. Opposed to the above where the interval of "stationarity" of the noise was longer than the interval $[0, 5]$ under consideration, we chose its length to be 0.5 and 0.05, respectively. The range of coordinate axes is here again -20 to 20.

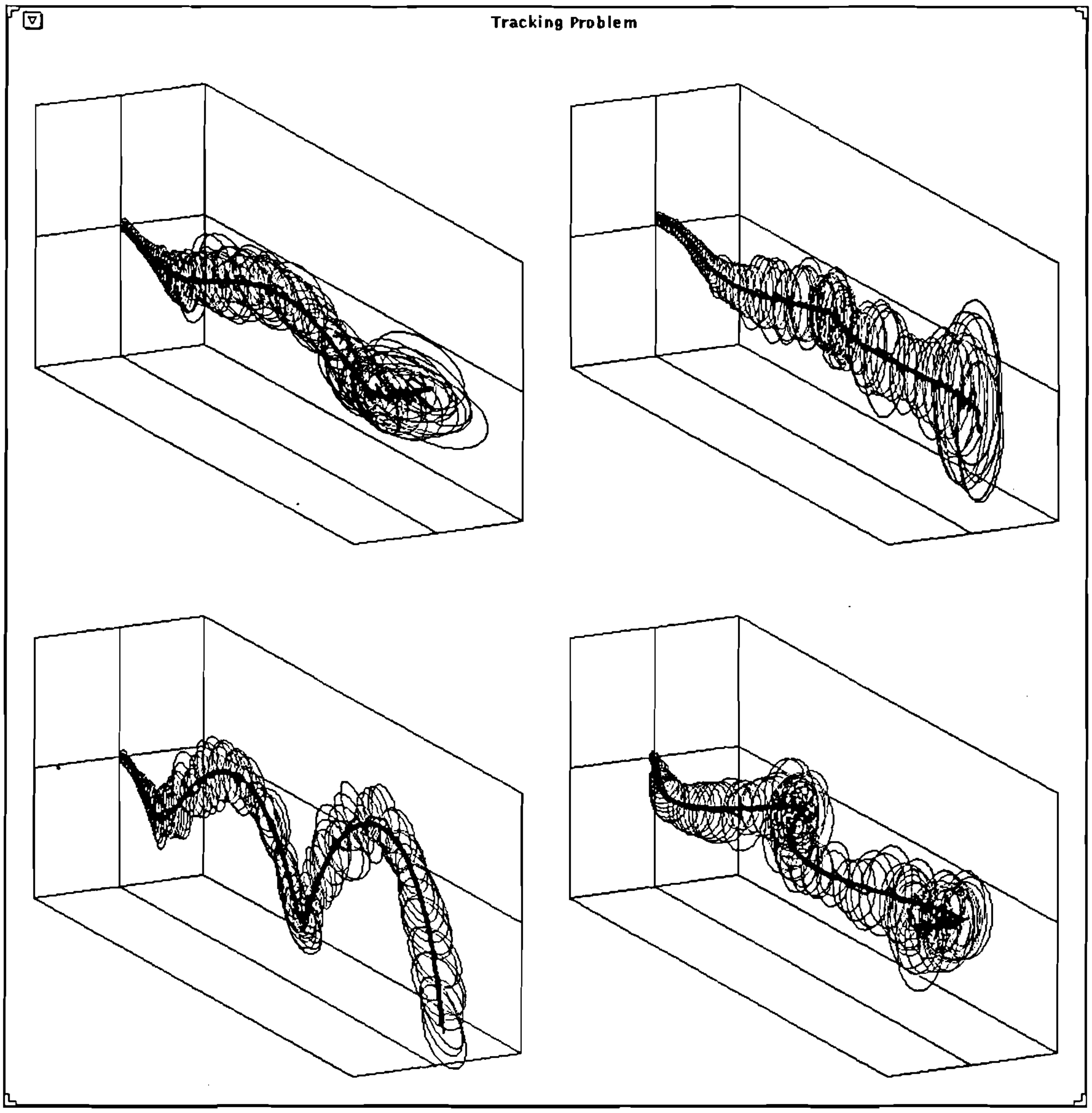

Figure 6: Phase space representation of ellipsoidal tracking — "even better" noise.

Conclusions

This paper indicates constructive algorithmic "ellipsoidal" procedures for the state estimation problem for dynamic systems under unknown errors bounded by given instantaneous constraints.

The "guaranteed" estimator may be presented as a tracking system that tracks the unknown actual trajectory of the system. The procedures allow effective graphic simulation that is demonstrated here on a system of order four.

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