Working Paper

PARADOXES WHEN COMPUTING LIFE EXPECTANCY OVER AGGREGATED SUBPOPULATIONS

Deanna B. Haunsperger

WP-91-19
July 1991

IIASA
International Institute for Applied Systems Analysis □ A-2361 Laxenburg □ Austria
Telephone: +43 2236 715210 □ Telex: 079 137 iiasa a □ Telefax: +43 2236 71313

PARADOXES WHEN COMPUTING LIFE EXPECTANCY OVER AGGREGATED SUBPOPULATIONS

Deanna B. Haunsperger

WP-91-19
July 1991

IIASA International Institute for Applied Systems Analysis □ A-2361 Laxenburg □ Austria
Telephone: +43 2236 715210 □ Telex: 079 137 iiasa a □ Telefax: +43 2236 71313

ABOUT THE AUTHOR

Deanna B. Haunsperger received her Ph.D. in mathematics from Northwestern University, Chicago, Illinois, USA in 1991. She is currently Assistant Professor of Mathematics at St. Olaf College in Minnesota, USA. Her interests lie in voting theory and the study of nonlinear functions. This report was prepared in part while she was participating in IIASA's Young Scientist Summer Program, 1990, and in part in 1991 under Peccei Scholarship. Contact address: Department of Mathematics, St. Olaf College, Northfield, Minnesota 55057, USA.

ACKNOWLEDGEMENTS

The author would like to thank Nathan Keyfitz and Wolfgang Lutz for their comments on an earlier version of this paper, and Stephen Kennedy for his suggestion of a geometric approach and without whose comments this would not have been possible. The financial support of the U. S. National Science Foundation and the IIASA Peccei Scholarship is gratefully acknowledged.

ABSTRACT

The calculated life expectancy (whether by subpopulation-specific characteristics or by age-specific mortality rates) for aggregated subpopulations need not lie within the limits of the individual subpopulations' life expectancies. For two subpopulations and two age groups, we present a geometric interpretation of exactly what happens and indicate how this generalizes. Also, we give a complete algebraic characterization of when paradoxes can and cannot occur for the case of two subpopulations and two age groups.

TABLE OF CONTENTS

PARADOXES WHEN COMPUTING LIFE EXPECTANCY OVER AGGREGATED SUBPOPULATIONS

Deanna B. Haunsperger

1. INTRODUCTION

In a recent paper Andreev, Lutz, and Scherbov (1989) report on a paradox that can occur when computing life expectancy over aggregated subpopulations. Specifically, they discuss several occurrences of data for which the life expectancy of aggregated subpopulations, calculated by weighting the given age-specific mortality rates, need not lie within the range of the life expectancies of the subpopulations. They cite an example where, in the Soviet Republic of Azerbaydzhan, the life expectancy is 64.13 years for the male population, and the Baltic Republic of Estonia has a life expectancy of 64.23 years. Yet, when the two republics are combined into one, the life expectancy is 63.76 years, below that of either population. This aggregation problem – aggregated data sets giving unexpected (and seemingly impossible) conclusions when compared with the individual data sets – is not one peculiar to life expectancy calculations. It occurs in many aspects of the social and natural sciences (recent literature shows examples in decision theory (Haunsperger and Saari 1991) and nonparametric statistics (Haunsperger, to appear)) where non-linear methods are used when aggregating data sets.

In this paper I will present a method of mathematically investigating when this phenomenon can occur. Specifically, in section three, I examine the case of two subpopulations separated into two age groups and characterize precisely when this paradox will and will not occur for this situation.

2. WHEN AN AGGREGATION PARADOX CAN OCCUR

Is the phenomenon which Andreev, Lutz, and Scherbov (1989) observed in the life expectancy calculations for the Soviet republics merely an anomaly? If this is the case then it should be of no concern when calculating life expectancies in this manner; however, if it is not an anomaly could further subclassification of the populations give an even lower life expectancy when aggregated? To begin to answer this question, assume one is given a set of k mortality vectors, one for each subpopulation, each mortality vector having as its i^{th} component the mortality rate for the i^{th} age group. Also, assume one is given a set of probability vectors, one vector p_α for each age group α, where $p_\alpha = (p_\alpha^{(1)}, p_\alpha^{(2)}, \cdots, p_\alpha^{(k)})$, $\sum_{j=1}^{k} p_\alpha^{(j)} = 1$, and $p_\alpha^{(j)}$ is the fraction of the total number of people in age group α that are in subpopulation j. For this given set of mortality vectors and probability vectors, say a *paradox* occurs if the life expectancy of the aggregated subpopulations computed with the probability vectors lies outside the range of life expectancies computed for the subpopulations individually. Furthermore, say a mortality vector *bounds from above (below)* another mortality vector for the same age groups if each component of the mortality vector for the first is greater (less) than or equal to the corresponding component of the mortality vector for the second.

Now, given a set of mortality vectors for subpopulations, a simple test to check if there exist probability vectors that give a paradox is the following:

Theorem 1. Given a set of mortality vectors, one for each subpopulation of a population, there do not exist probability vectors (one for each age group) that give a paradox when computing the life expectancy of the aggregated subpopulations if and only if there exists a subpopulation whose mortality vector bounds all others from above and there exists a subpopulation whose mortality vector bounds all others from below.

This condition on mortality vectors to avoid the possibility of a paradox is quite strict: only those sets of mortality vectors where one subpopulation has lower mortality rates in every age group than every other subpopulation and one subpopulation has higher mortality rates in every age group than every other subpopulation can avoid this paradox. As an example, consider the following set of mortality rates for five subpopulations, each with four age groups:

$$
\begin{aligned}
m^{(1)} &= (0.0001, 0.0010, 0.0011, 0.0300)\\
m^{(2)} &= (0.0020, 0.0050, 0.0057, 0.0400)\\
m^{(3)} &= (0.0005, 0.0036, 0.0039, 0.0310)\\
m^{(4)} &= (0.0010, 0.0049, 0.0031, 0.0380)\\
m^{(5)} &= (0.0015, 0.0021, 0.0030, 0.0320)
\end{aligned}
$$

For this set of mortality vectors no paradoxes are possible, as the mortality vector for subpopulation two bounds all others from above, and that for subpopulation one bounds all others from below. However, if

$$m^{(4)} = (0.0010, 0.0051, 0.0031, 0.0380)$$

instead, then a paradox would be possible. Perhaps the easiest way to see one is to let $p^{(2)}_{\alpha_1} = 1, p^{(4)}_{\alpha_2} = 1, p^{(2)}_{\alpha_3} = 1$, and $p^{(2)}_{\alpha_4} = 1$, (with all other components of the probability vectors equalling zero). Then the mortality vector for the aggregated population is at least as high as the mortality vector for the second subpopulation in every age group and strictly higher in one age group. Since increasing any one mortality rate decreases life expectancy, the life expectancy for the aggregated population will be less than that for any subpopulation.

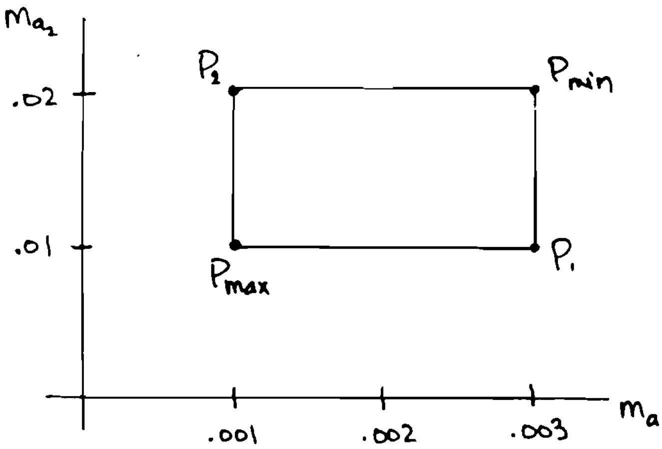
The probability vector created to give the example above may be the easiest to see quickly, but it is not the only probability vector to give a paradox with those mortality rates. On the contrary, if there does not exist a subpopulation whose mortality vector bounds all others from below or there does not exist a subpopulation whose mortality vector bounds all others from above, then there is a positive probability of obtaining counter-intuitive results (a paradox). This is proved for $k = 2$ subpopulations in Andreev, Lutz, and Scherbov (1989). An alternate proof for this is given in section three, and a geometric proof for any number of subpopulations and age groups is given in section four.

3. A GEOMETRIC INTERPRETATION FOR TWO SUBPOPULATIONS AND TWO AGE GROUPS

Consider the simplified situation where a population is split into two subpopulations, and mortality rates are given for each subpopulation for each of two age groups, a_1 and a_2. Let the first subpopulation (P_1) have mortality vector $(m_{a_1}^{(1)}, m_{a_2}^{(1)}) = (a, c)$ and the second subpopulation (P_2) have mortality vector $(m_{a_1}^{(2)}, m_{a_2}^{(2)}) = (b, d)$. If p is the fraction of all the people in the first age group who come from P_1, and q is the fraction of all the people in the second age group who come from P_2, then $1 - p$ is the fraction of all people in the first age group who come from P_2 and $1 - q$ is the fraction of all people in the second age group who come from P_1. The mortality vector for the whole population then is equal to $(m_{a_1}^{(0)}, m_{a_2}^{(0)}) = (pa + (1 - p)b, (1 - q)c + qd)$, for any $0 \leq p \leq 1$ and any $0 \leq q \leq 1$. Therefore, the possible mortality rates for the whole population define a rectangle in the positive quadrant of $\mathbf{R}^2$, where the mortality rates for a_1 are on the horizontal axis, and the mortality rates for a_2 are on the vertical axis.

For example, let P_1 have the mortality vector $m^{(1)} = (0.003, 0.010)$, and let P_2 have mortality vector $m^{(2)} = (0.001, 0.020)$. Then the rectangle in $\mathbf{R}^2$ is as in Figure 1.

FIGURE 1



The rectangle of possible mortality rates for the aggregated subpopulations.

There exists a one-to-one correspondence between the coordinates of points in the rectangle and the possible mortality vectors for the total population when weighting by age groups: the point (r_1, r_2) in the rectangle corresponds to $(p_0 a + (1-p_0)b, (1-q_0)c + q_0 d)$ for exactly one p_0 and one q_0. Also, there exists a one-to-one correspondence between the set of mortality vectors for the aggregated subpopulation and points of the rectangle. Therefore the point (r_1, r_2) corresponds to using the mortality rate r_1 for the first age group in the aggregated population and using r_2 for the second. Associate with every point of the rectangle the life expectancy calculated for the whole population using the coordinates as the mortality rates. The maximum life expectancy occurs when the mortality rate used for each age group is at a minimum, so at the point closest to the origin. Call this vertex P_{max} and the life expectancy calculated there E_{max}. The minimum life expectancy, E_{min}, occurs when the mortality rate used for each age group is at a maximum, so it occurs at the vertex farthest from the origin, P_{min}.

Life expectancy is a continuous function of mortality rates, $E = f(m_{a_1}, m_{a_2})$, so its value must vary continuously as a point moves through the rectangle. Given any continuous curve $\gamma(r_1) = r_2$ from E_{max} to E_{min} where $d\gamma/dr_1 \geq 0$ for $0 \leq r_1 \leq 1$, the life expectancy decreases continuously along it. Let E_i equal the life expectancy of P_i, $i = 1, 2$. Continuous functions γ cover the rectangle, and life expectancy E is defined for every pair of mortality rates, so consider the level sets $L_i, i = 1, 2$ for life expectancy where L_i corresponds to $E = E_i, i = 1, 2$.

Proposition 1. The level curves $L_i, i = 1, 2$ are continuous, non-intersecting curves that have the property: if (r_1, r_2) and (r_1', r_2') are two points on L_i, and if $r_1 > r_1'$, then $r_2 < r_2'$.

From this characterization of these level curves comes an alternate proof to the result of Andreev, Lutz, and Scherbov (1989):

Theorem 2. Given mortality vectors for two subpopulations, each with two age groups, of which neither is bounded above (or below) by the other. There is a positive probability of obtaining a paradox. That is, the Lebesgue measure of the set of probability vectors for the mortality functions which give rise to a paradox is positive.

Proof. Let L_{max} be the level curve where the life expectancy function has the value $max\{E_1, E_2\}$. Similarly, let L_{min} be the level curve where the life expectancy function has the value $min\{E_1, E_2\}$. Neither mortality vector bounds the other either below or above; therefore, $E_{max} \not\in \{E_1, E_2\}, E_{min} \not\in \{E_1, E_2\}$. This implies $L_{max} \neq \{P_{max}\}$ and $L_{min} \neq \{P_{min}\}$. Now L_{max} being a one-dimensional, smooth, nontrivial curve in the rectangle must separate the two-dimensional box into two nontrivial pieces of two-space: one which corresponds to life expectancies greater than that on L_{max} and one which corresponds to life expectancies less than that on L_{max}. The former, in particular, must have positive Lebesgue measure. A similar statement is true for L_{min}.

The life expectancy varies throughout the rectangle, and as each point of the rectangle uniquely identifies values for p and q, one could think of the value of life expectancy as being a function of p and q, $E(p,q)$. If $E(p,q) = E_1$, where E_1 is the life expectancy for the first subpopulation, this produces a level curve of the life expectancy function. Projecting this onto the plane of the rectangle, one sees a smooth curve emanating from P_1 that shows precisely the values of p and q that will give a joint life expectancy of E_1. Similarly, one sees a smooth curve emanating from P_2 in the rectangle from projecting $E(p,q) = E_2$ onto the plane of the rectangle.

This allows for a geometric proof of the following result of Andreev, Lutz, and Scherbov (1989):

Proposition 2. The Lebesgue measure of the set of pairs (p,q) that give rise to E_1 in the joint life expectancy calculation is zero.

Proof. The Lebesgue measure of a smooth, one-dimensional curve in $\mathbf{R}^2$ is zero.

More can be shown about the two level curves, $E(p,q) = E_1$ and $E(p,q) = E_2$. In particular, the concavity of their graphs follows from an investigation of the functions.

Proposition 3. Thinking of q as a function of p (that is, $q(p)$), $E(p,q) = E_1$ and $E(p,q) = E_2$ are both concave up.

That is, the only graphs of these two level curves possible are those similar to the graphs of Figure 2.

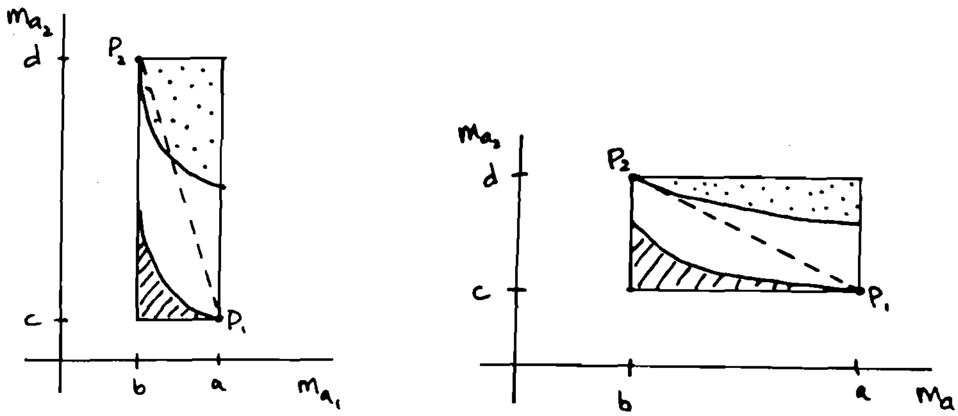
Assume for the moment that $E_1 > E_2$. This implies the level curve $E(p,q) = E_2$ lies above and/or to the right of the level curve $E(p,q) = E_1$, (as in Case 1 of Figure 2). Any point of the rectangle that lies above and/or to the right of the level curve $E(p,q) = E_2$ has a life expectancy associated with it that is less than E_2, and, hence gives rise to a paradox. Any point of the rectangle that lies below and/or to the left of the level curve $E(p,q) = E_1$ has a life expectancy associated with it that is greater than E_1, and, hence, gives rise to a paradox. Similar statements are true if $E_2 > E_1$, (as in Case 2 of Figure 2). Thus, the only points of the rectangle that do not give rise to a paradox are those between or on the two level curves $E(p,q) = E_1$ and $E(p,q) = E_2$ (which corresponds to the white area in the graphs of Figure 2).

When weighting life expectancy by subpopulation-specific characteristics such as birthrate, available resources, industrial development of the area, etc., the life expectancies that are calculated fall into a (possibly) more-restricted range. In particular, weighting by subpopulation-specific characteristics corresponds, geometrically, to the convex-hull of (that is, the line segment joining) the points P_1 and P_2.

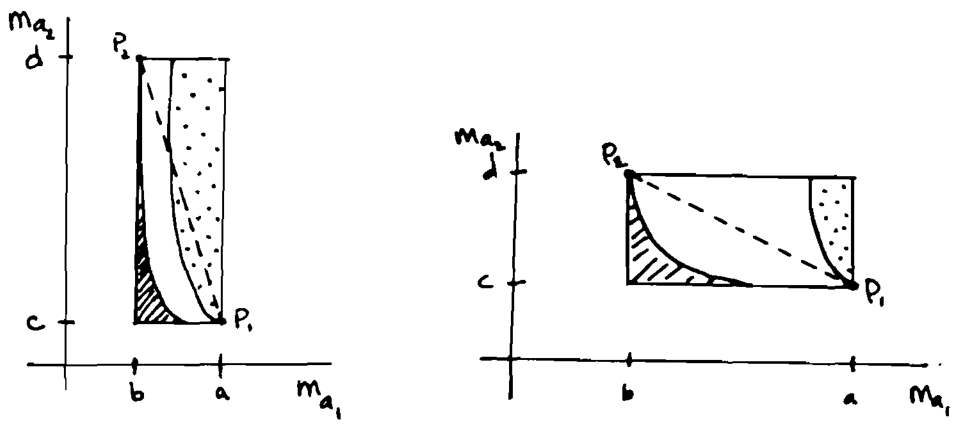
One might wonder if weighting life expectancy by subpopulation-specific characteristics would avoid this paradox. However, from the graphs in Figure 2, one can see that

FIGURE 2

Case 1. $E_2 < E_1$.



Case 2. $E_1 < E_2$.



The dotted areas correspond to mortality rates whose calculated life expectancy is less than either E_1 or E_2. The shaded areas correspond to mortality rates whose calculated life expectancy is greater than either E_1 or E_2.

this is not the case: a paradox from weighting by subpopulation-specific characteristics (that is, along the convex hull) occurs whenever either of the two level curves $E(p, q) = E_1$ or $E(p, q) = E_2$ crosses (or "cuts through") the convex hull of $\{P_1, P_2\}$. For the following, one can assume, without loss of generality as neither the mortality vector for P_1 nor the mortality vector for P_2 bounds the other from above or from below, $a > b$ and $d > c$.

Theorem 3.
i) A paradox arises from $E(p,q) = E_1$ crossing the convex hull $\iff E_1 < E_2$ and $(d-c)(4-n^2a^2) > (a-b)(12+8nc+n^2c^2)$. If such a paradox does arise, the Lebesgue measure of the set of points along the convex hull that gives paradoxes is:

$$1 - \frac{(a-b)(12+4nc+4nd+n^2cd)-(d-c)(4-n^2ab)}{(a-b)(d-c)(4n+n^2c-n^2a)}.$$

ii) A paradox arises from $E(p,q) = E_2$ crossing the convex hull $\iff E_1 > E_2$ and $(d-c)(4-n^2b^2) < (a-b)(12+8nd+n^2d^2)$. If such a paradox does arise, the Lebesgue measure of the set of points along the convex hull that gives paradoxes is:

$$\frac{(a-b)(12+8nd+n^2d^2)-(d-c)(4-n^2b^2)}{(a-b)(d-c)(4n+n^2d-n^2b)}.$$

These two statements characterize precisely when paradoxes can and cannot occur along the convex hull, that is, when weighting using subpopulation-specific information. The first of the two statements ($3i$) says that a paradox can only occur from the convex hull crossing $E(p,q) = E_1$ if $E_1 < E_2$. This says that the only type of paradox that can occur in ($3i$) results in the life expectancy calculation being lower than either E_1 or E_2 (see the first and third graphs of Figure 2). The second statement ($3ii$) says that a paradox can only occur from the convex hull crossing $E(p,q) = E_2$ if $E_2 < E_1$. This implies that the only type of paradox that can occur in ($3ii$) results in the life expectancy calculation being lower than either E_1 or E_2. As the only types of paradoxes that can occur along the convex hull are the result of $E(p,q) = E_1$ or $E(p,q) = E_2$ crossing the convex hull, a corollary to Theorem 3 follows:

Corollary 3.1. If a paradox occurs from weighting according to subpopulation-specific information, then it must result in a life-expectancy calculated for the aggregated subpopulations that is *lower* than either individual life expectancy.

Further, as it not possible for both $E_1 > E_2$ and $E_2 > E_1$ at the same time,

Corollary 3.2. For fixed a, b, c, d it is not possible to get paradoxes both from $E(p,q) = E_1$ crossing the convex hull and from $E(p,q) = E_2$ crossing the convex hull.

Moreover, there are many examples of mortality vectors where neither ($3i$) nor ($3ii$) hold. In particular, the algebraic conditions of Theorem 3 require the following:

Corollary 3.3. A necessary condition for the existence of a paradox when weighting by subpopulation-specific characteristics is $(d-c)/(a-b) > 3$.

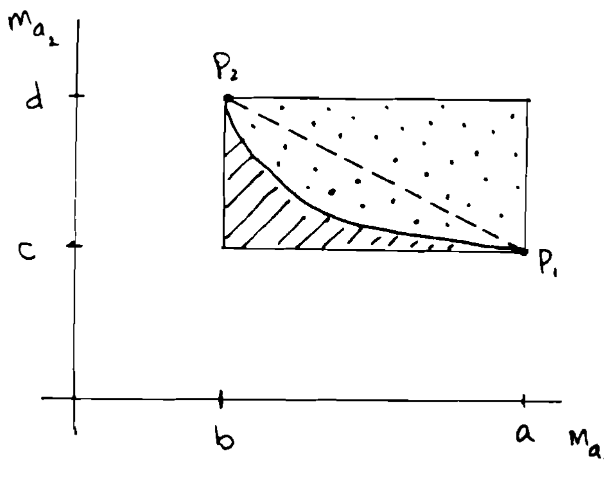
See, for example, the second and fourth graphs of Figure 2. If neither of the two mortality vectors bounds the other from above or from below however, then paradoxes can arise when calculating the life expectancy for the aggregated subpopulations. Therefore we have:

Corollary 3.4. There exist mortality rates whereby weighting with age-specific populations gives a paradox, yet weighting by subpopulation-specific characteristics (the convex hull) does not.

On the other hand, it follows from an earlier proposition, as the level curves are concave up (see Figure 3), that the level curve for $E(p,q) = E_1$ when $E_1 = E_2$ touches the convex hull only in the two points P_1 and P_2. Thus,

Corollary 3.5. If $E_1 = E_2$ and mortality vectors for the two subpopulations are different, then each point of the convex hull, except $\{P_1, P_2\}$ yields a paradox.

FIGURE 3



When $E_1 = E_2$ but P_1 and P_2 have different mortality vectors.

Finally, as the calculation of life expectancy is continuous in the variables a, b, c, d, one can conclude:

Theorem 4. When weighting by characteristics specific to the subpopulations, the Lebesgue measure of the set of probability vectors that gives a paradox can be anything between 0 and 1 (depending on how close E_1 and E_2 are.)

4. MORE THAN TWO SUBPOPULATIONS AND MORE THAN TWO AGE GROUPS

Extend this idea of a rectangle containing the level curves of life expectancy to multiple subpopulations and multipe age groups. More subpopulations define a rectangle by letting the sides of the rectangle, or the length in each direction, be determined by the smallest and the largest mortality rate in each age group. No longer do subpopulations' mortality rates need to occur on the vertices or even on the edges of the rectangle. (By definition, however, every edge must contain at least one point.) Now, each axis of the plane holding the rectangle corresponds to an age group. Therefore, adding age groups corresponds to adding dimensions: for ω age groups, where the mortality rates in each age group are not constant, the rectangular box has ω-dimensions.

As in Proposition 1, the level sets $L_i = E_i$, $i = 1, 2, \cdots, k$ (where k is the number of subpopulations) are smooth (since the life expectancy function has infinitely many derivatives), non-intersecting hypersurfaces of co-dimension one in the ω-box. They, too, have the property that if lines parallel to all of the axes are drawn through any point r of level surface E_j, for any j, then they each touch E_j only in r.

Given k subpopulations and ω age groups, create the ω-dimensional box with points $P_1, P_2, \cdots, P_k$ and corresponding level sets through those points $L_1, L_2, \cdots, L_k$. Let L_{max} denote the level set that is closest to the origin and L_{min} denote the level set that is farthest from the origin. L_{max} and L_{min} are unique since level sets do not intersect.

A generalization of Theorem 2 to more than two subpopulations and more than two age groups gives the following:

Theorem 5. The probability of a paradox occuring when given subpopulations whose mortality vectors are not bounded from above and from below by mortality vectors has positive Lebesgue measure in the space of all combinations of mortality vectors possible for the aggregated subpopulations.

5. PROOFS

Proof of Theorem 1. Assume there exists a subpopulation whose mortality vector bounds all others from below and a subpopulation whose mortality vector bounds all others from above. The mortality vector for the aggregated subpopulations is computed in each component by a linear combination of the mortality rates from the subpopulations in that age group. For any particular age group, the minimum mortality rate for the aggregated subpopulations is attained by the subpopulation whose mortality vector bounds all others from below. Similarly, the maximum mortality rate for the aggregated subpopulations is attained by the subpopulation whose mortality vector bounds all others from above. Now, as the life expectancy function has negative first partial derivatives with respect to the mortality rates for the age groups, the minimum life expectancy for the aggregated subpopulation occurs when the mortality rate for each age group is at its maximum. However, the subpopulation whose mortality vector bounds all others from above attains this minimum life expectancy. Similarly, the subpopulation whose mortality vector bounds all others from below attains the maximum life expectancy possible for the aggregated subpopulations. Therefore, all life expectancies calculated for the aggregated subpopulations, no matter what probability vectors are used, fall between this maximum and minimum. Hence, no paradoxes can occur.

Assume now there does not exist both a subpopulation whose mortality vector bounds all others from below and a subpopulation whose mortality vector bounds all others from above. First, assume that no subpopulation has a mortality vector which bounds all others from below. Then assign probablities so that in each age group the subpopulation with the lowest mortality receives full weight. This yields a set of probability vectors which give a mortality vector for the aggregated subpopulations whose calculated life expectancy is higher than the life expectancy calculated for any one subpopulation. A similar result is obtained if one assumes that no subpopulation has a mortality vector which bounds all others from above.

The calculation of life expectancy using age-specific mortality rates when in the situation of two subpopulations and two age groups. Using standard notation, see Keyfitz (1977), life expectancy for two age groups is computed as follows:

$$\begin{aligned} E = e_0 = \frac{T_0}{l_0} = \frac{L_0 + L_n}{l_0} &= \frac{n}{l_0}(l_0 - \frac{d_0}{2} + l_n - \frac{d_n}{2}) \\ &= \frac{n}{l_0}(l_0 - \frac{d_0}{2} + l_0 - d_0 - \frac{q_n l_n}{2}) \\ &= \frac{n}{l_0}(2l_0 - \frac{3q_0 l_0}{2} - \frac{q_n(l_0 - d_0)}{2}) \\ &= \frac{n}{2l_0}(4l_0 - 3q_0 - l_0 - q_n(l_0 - q_0 l_0)) \\ &= \frac{n}{2}(4 - 3q_0 - q_n(1 - q_0)) \end{aligned}$$

where e_0 is the life expectancy for a person at age 0, T_0 is the total number of person-years lived, l_x is the number of people alive at age x, L_x is the total number of person-years lived in the interval from x to $x + n$, d_x is the number of people who die in the interval from x to $x + n$, and q_x is the probability of dying in the interval from x to $x + n$. Now, q_x is calculated from m_x by the following:

$$q_x = \frac{nm_x}{1 + \frac{n}{2}m_x}.$$

Therefore, if a population has mortality vector $(m_{a_1}, m_{a_2}) = (a, c)$, say P_1, then $q_0 = na/(1 + (n/2)a)$ and $q_n = nc/(1 + (n/2)c)$, so

$$\begin{aligned} E_1 &= \frac{n}{2}\Big(4 - 3\Big(\frac{na}{1 + (n/2)a}\Big) - \Big(\frac{nc}{1 + (n/2)c}\Big)\Big(1 - \frac{na}{1 + (n/2)a}\Big)\Big) \\ &= \frac{n}{2}\Big(4 - 3\Big(\frac{na}{1 + (n/2)a}\Big) - \Big(\frac{nc}{1 + (n/2)c}\Big)\Big(\frac{1 - (n/2)a}{1 + (n/2)a}\Big)\Big). \end{aligned}$$

Similarly, if, say, P_2 has mortality vector $(m_{a_1}, m_{a_2}) = (b, d)$, then

$$E_2 = \frac{n}{2}\Big(4 - 3\Big(\frac{nb}{1+(n/2)b}\Big) - \Big(\frac{nd}{1+(n/2)d}\Big)\Big(\frac{1-(n/2)b}{1+(n/2)b}\Big)\Big).$$

Recall that the fraction of people in the first age group that are in the subpopulation P_1 is p. This implies that the fraction of people in the first age group that are in the subpopulation P_2, is $1-p$. Also recall that the fraction of people in the second age group that are in P_1 is $1-q$. Similarly, then, the fraction of people in the second age group that are in P_2 is q. This gives age-specific mortality rates for the aggregated subpopulations of $(pa+(1-p)b, (1-q)c+qd)$. This implies:

$$q_0 = \frac{n(pa+(1-p)b)}{1+\frac{n}{2}(pa+(1-p)b)}, \quad q_n = \frac{n((1-q)c+qd)}{1+\frac{n}{2}((1-q)c+qd)}$$

Hence, $E(p,q)$ equals

$$\frac{n}{2}\Big(4 - 3\Big(\frac{2n(pa+(1-p)b)}{2+n(pa+(1-p)b)}\Big) - \Big(\frac{2n((1-q)c+qd)}{2+n((1-q)c+qd)}\Big)\Big(\frac{2-n(pa+(1-p)b)}{2+n(pa+(1-p)b)}\Big)$$

$$= n\Big(2 - 3\Big(\frac{n(pa+(1-p)b)}{2+n(pa+(1-p)b)}\Big) - \Big(\frac{n((1-q)c+qd)}{2+n((1-q)c+qd)}\Big)\Big(\frac{2-n(pa+(1-p)b)}{2+n(pa+(1-p)b)}\Big).$$

Proof of Proposition 1. This follows from the fact that the life expectancy function has negative first partial derivatives with respect to the mortality rate for any age group. That is, if $E = f(m_{a_1}, m_{a_2})$ then $df/dm_{a_1} < 0$ and $df/dm_{a_2} < 0$.

Lemma. Each of the mortality rates a, b, c, d is less than $\frac{2}{n}$.

Proof of Lemma. As q_0 and q_n are each between zero and one (they are probabilities), this implies, for example, that

$$0 \le \frac{na}{1+\frac{n}{2}a} \le 1.$$

This, in turn, implies that $0 \le a \le \frac{2}{n}$. The same is true for each of the other mortality rates.

Proof of Proposition 3. Without loss of generality, as there do not exist mortality vectors which bound from below or from above, assume $a > b$ and $d > c$. First, consider the level curve $E(p,q) = E_1$. That is,

$$n\Big(2 - 3\Big(\frac{n(pa+(1-p)b)}{2+n(pa+(1-p)b)}\Big) - \Big(\frac{n((1-q)c+qd)}{2+n((1-q)c+qd)}\Big)\Big(\frac{2-n(pa+(1-p)b)}{2+n(pa+(1-p)b)}\Big)$$

$$= n\left(2 - 3\left(\frac{na}{2+na}\right) - \left(\frac{nc}{2+nc}\right)\left(\frac{2-na}{2+na}\right)\right).$$

Letting $B = pa + (1-p)b$ and $D = (1-q)c + qd$ yields

$$3\left(\frac{B}{2+nB}\right) + \left(\frac{D}{2+nD}\right)\left(\frac{2-nB}{2+nB}\right) = 3\left(\frac{a}{2+na}\right) + \left(\frac{c}{2+nc}\right)\left(\frac{2-na}{2+na}\right).$$

Multiplying both sides by $(2+nB)(2+nD)(2+na)(2+nc)$ then simplifying reduces to:

$$(B-a)(12 + 4nc + 4nD + n^2cD) + (D-c)(4 - n^2aB) = 0.$$

After substituting back in for B and D and simplification, one can solve for q in terms of p:

$$q = \frac{(p-1)(a-b)(12 + 8nc + n^2c^2)}{p(d-c)(a-b)(n^2a - 4n - n^2c) + (d-c)(n^2ab - 4 + (a-b)(4n + n^2c))}.$$

Taking the derivative of this function, one sees that,

$$\frac{dq}{dp} = \frac{(d-c)(a-b)(12 + 8nc + n^2c^2)(n^2a^2 - 4)}{[p(d-c)(a-b)(n^2a - 4n - n^2c) + (d-c)(n^2ab - 4 + (a-b)(4n + n^2c))]^2}.$$

Note that $(n^2a^2 - 4)$ is negative, and all other terms are positive, so the first derivative of $q(p)$ is always negative. This implies that the function $q(p)$ is decreasing.

Taking another derivative and simplifying, one obtains the expression:

$$\frac{d^2q}{dp^2} = \frac{2(d-c)^2(a-b)^2(12 + 8nc + n^2c^2)(n^2a^2 - 4)(4n - n^2a + n^2c)}{[p(d-c)(a-b)(n^2a - 4n - n^2c) + (d-c)(n^2ab - 4 + (a-b)(4n + n^2c))]^3}$$

for the second derivative of q with respect to p.

The numerator of d^2q/dp^2 is always negative. (The factor $(n^2a^2 - 4)$ is negative by the lemma above and all other factors are positive). The second derivative can change sign, however, if the denominator is zero. We are interested only if the concavity changes within the boundaries of the rectangle.

Call the point where the function $q(p)$ crosses the line $q = 1$, $(p^*, 1)$. As the function $q(p)$ is always decreasing, the value of p that makes the denominator of the second derivative zero, call it p_{crit}, must lie between p^* and one for the concavity change to occur within the rectangle. Now,

$$p_{crit} = \frac{-(n^2ab - 4 + (a-b)(4n + n^2c))}{(a-b)(n^2a - 4n - n^2c)}.$$

and

$$p^* = \frac{-(d-c)(n^2ab - 4 + (a-b)(4n + n^2c)) - (a-b)(12 + 8nc + n^2c^2)}{(d-c)(a-b)(n^2a - 4n - n^2) - (a-b)(12 + 8nc + n^2c^2)}.$$

Thus, $p^* \leq p_{crit}$ implies, with simplification, $n^2a^2 \geq 4$, which is a contradiction by the lemma. Therefore, the second derivative does not change sign within the rectangle, and hence the function does not change concavity.

To discover the concavity of the function, it is enough to check the sign of the second derivative at one point within the interval $[p^*, 1]$. (It is easiest to check when $p = 1$.) One finds the sign of the second derivative positive, and thus, the graph of the level curve projected into the plane of the rectangle is concave up.

Now consider the level curve $E(p,q) = E_2$. That is,

$$n\Big(2 - 3\Big(\frac{n(pa + (1-p)b)}{2 + n(pa + (1-p)b)}\Big) - \Big(\frac{n((1-q)c + qd)}{2 + n((1-q)c + qd)}\Big)\Big(\frac{2 - n(pa + (1-p)b)}{2 + n(pa + (1-p)b)}\Big)$$

$$= n\Big(2 - 3\Big(\frac{nb}{2 + nb}\Big) - \Big(\frac{nd}{2 + nd}\Big)\Big(\frac{2 - nb}{2 + nb}\Big)\Big).$$

Making substitutions similar to above and simplifying, one can again solve for q in terms of p:

$$q = \frac{-p(a-b)(12 + 4nd + 4nc + n^2cd + n^2b(d-c)) + (d-c)(4 - n^2b^2)}{p(a-b)(d-c)(4n + n^2d - n^2b) + (d-c)(4 - n^2b^2)}.$$

Taking the derivative of this function, one obtains

$$\frac{dq}{dp} = \frac{-(a-b)(d-c)(4 - n^2b^2)(12 + 8nd + n^2d^2)}{[p(a-b)(d-c)(4n + n^2d - n^2b) + (d-c)(4 - n^2b^2)]^2}.$$

Note that all factors are always positive, so the first derivative of $q(p)$ is always negative, which implies that the function $q(p)$ is decreasing.

Taking another derivative and simplifying, one sees that the second derivative is given by:

$$\frac{d^2q}{dp^2} = \frac{2(a-b)^2(d-c)^2(4 - n^2b^2)(12 + 8nd + n^2d^2)(4 + n^2d - n^2b)}{[p(a-b)(d-c)(4n + n^2d - n^2b) + (d-c)(4 - n^2b^2)]^3}.$$

The numerator of d^2q/dp^2 is always positive. The second derivative can only change sign, therefore, when the denominator is zero. Again, we are only interested if the concavity changes within the boundaries of the rectangle. The point where the denominator is zero is

$$p_{crit} = \frac{-(4 - n^2b^2)}{(a-b)(4n + n^2d - n^2b)}.$$

This critical value is negative, however, and therefore does not correspond to a point within the rectangle.

One sees by substituting a point (say, p=0) into the second derivative that the sign of the second derivative is positive, and thus, the graph of this level curve projected into the plane of the rectangle is concave up.

Lemma. $E_1 > E_2 \iff (b-a)(12+4nc+4nd+n^2cd)+(d-c)(4-n^2ab)>0$

Proof of the Lemma. If $E_1 > E_2$ then

$$n\left(2-3\left(\frac{na}{2+na}\right)-\left(\frac{nc}{2+nc}\right)\left(\frac{2-na}{2+na}\right)\right)$$

$$> n\left(2-3\left(\frac{nb}{2+nb}\right)-\left(\frac{nd}{2+nd}\right)\left(\frac{2-nb}{2+nb}\right)\right)$$

Therefore,

$$3\left(\frac{na}{2+na}\right)+\left(\frac{nc}{2+nc}\right)\left(\frac{2-na}{2+na}\right)<3\left(\frac{nb}{2+nb}\right)+\left(\frac{nd}{2+nd}\right)\left(\frac{2-nb}{2+nb}\right).$$

Which yields, after simplification,

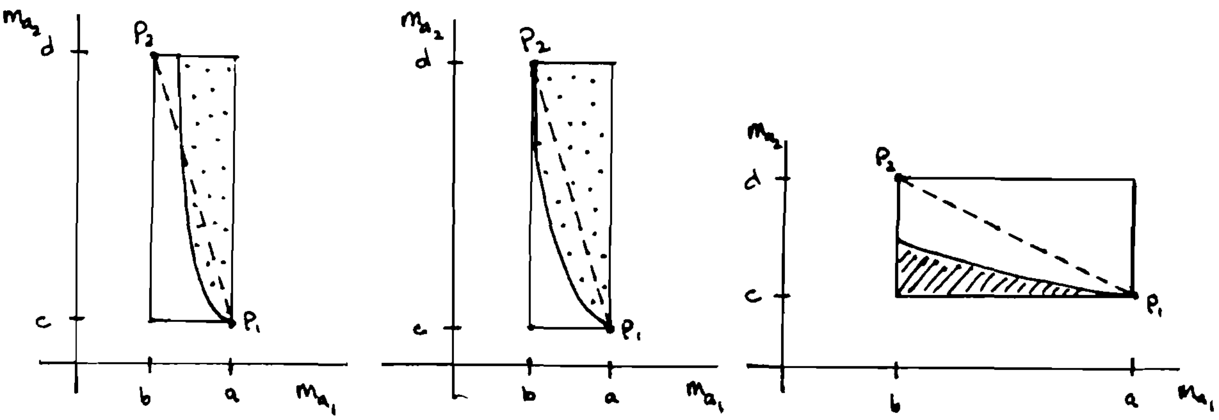
$$(b-a)(12+4nc+4nd+n^2cd)+(d-c)(4-n^2ab)>0.$$

If $E_1 < E_2$ then a similar simplification shows

$$(b-a)(12+4nc+4nd+n^2cd)+(d-c)(4-n^2ab)<0.$$

Proof of Theorem 3i). Consider the level curve $E(p,q)=E_1$ in the rectangle. It crosses the convex hull of $\{P_1,P_2\}$ in the point P_1 and in at most one other point. (At most because the level curve in concave up.) Thus, the three cases possible are as in Figure 4. Notice that if $E(p,q)=E_1$ touches the convex hull only in the point P_1, then no paradoxes arise along the convex hull from that level curve. Therefore we want to investigate what can happen when $E(p,q)=E_1$ touches the convex hull in two points.

FIGURE 4



The three cases possible for $E(p,q) = E_1$.

The convex hull of $\{P_1, P_2\}$ is precisely the line segment connecting P_1 and P_2, or part of the line $q = 1 - p$. Therefore, these two curves cross when $E(p, 1-p) = E_1$. That implies

$$n\Big(2 - 3\Big(\frac{n(pa + (1-p)b)}{2 + n(pa + (1-p)b)}\Big) - \Big(\frac{n(pc + (1-p)d)}{2 + n(pc + (1-p)d)}\Big)\Big(\frac{2 - n(pa + (1-p)b)}{2 + n(pa + (1-p)b)}\Big)$$

$$= n\Big(2 - 3\Big(\frac{na}{2+na}\Big) - \Big(\frac{nc}{2+nc}\Big)\Big(\frac{2-na}{2+na}\Big)\Big).$$

Simplifying, one reduces it to:

$$(b-a)(1-p)(12 + 4nc + 4n(pc + (1-p)d) + n^2c(pc + (1-p)d))$$

$$+(d-c)(1-p)(4 - n^2a(pa + (1-p)b)) = 0.$$

Rearranging leads to:

$$p^2[(a-b)(d-c)(n^2a - 4n - n^2c)]$$

$$+p[(a-b)(d-c)(-n^2a + 4n + n^2c) + (a-b)(12 + 4nc + 4nd + n^2cd) - (d-c)(4 - n^2ab)]$$

$$+[(a-b)(12 + 4nc + 4nd + n^2cd) + (d-c)(4 - n^2ab)] = 0.$$

As $p = 1$ is a solution to this quadratic equation ($p = 1$ corresponds to the point P_1 which we know is on the convex hull of $\{P_1, P_2\}$), then long division yields the other solution:

$$p_{cross} = \frac{-(a-b)(12+4nc+4nd+n^2cd)+(d-c)(4-n^2ab)}{(a-b)(d-c)(n^2a-4n-n^2c)}.$$

Now p_{cross} corresponds to a point inside the rectangle if and only if $0 < p_{cross} < 1$, which is true if and only if:

$$-(a-b)(12+4nc+4nd+n^2cd)+(d-c)(4-n^2ab) < 0$$

and

$$(d-c)(n^2a^2-4) < (b-a)(12+8nc+n^2c^2).$$

Finally, if p_{cross} is a point within the rectangle, then the Lebesgue measure of the points of the convex hull of $\{P_1, P_2\}$ that gives paradoxes is $1 - p_{cross}$.

Proof of Theorem 3ii). Consider the level curve $E(p,q) = E_2$ in the rectangle. Similar to the proof of ($3i$), we note that this level curve crosses the convex hull of $\{P_1, P_2\}$ in the point P_2 and in at most one other point. Again, if $E(p,q) = E_2$ touches the convex hull only in the point P_2, then no paradoxes arise along the convex hull from that level curve. Therefore, we will again look at what happens when $E(p,q) = E_2$ touches the convex hull in two points.

The convex hull of $\{P_1, P_2\}$ and the projection of the level curve $E(p,q) = E_2$ into the rectangle cross precisely when $E(p, 1-p) = E_2$. That implies

$$n\Big(2-3\Big(\frac{n(pa+(1-p)b)}{2+n(pa+(1-p)b)}\Big)-\Big(\frac{n(pc+(1-p)d)}{2+n(pc+(1-p)d)}\Big)\Big(\frac{2-n(pa+(1-p)b)}{2+n(pa+(1-p)b)}\Big)$$

$$= n\Big(2-3\Big(\frac{nb}{2+nb}\Big)-\Big(\frac{nd}{2+nd}\Big)\Big(\frac{2-nb}{2+nb}\Big)\Big).$$

Simplifying and rearranging leads to:

$$p^2(a-b)(d-c)(-4n-n^2d+n^2b)+p[(a-b)(12+8nd+n^2d^2)-(d-c)(4-n^2b^2)] = 0.$$

As $p = 0$ is a solution to this quadratic equation ($p = 0$ corresponds to the point P_2 which we know is on the convex hull of $\{P_1, P_2\}$), then the other solution is:

$$p_{cross} = \frac{-(a-b)(12+8nd+n^2d^2)+(d-c)(4-n^2b^2)}{(a-b)(d-c)(-4n-n^2d+n^2b)}.$$

Now, p_{cross} corresponds to a point inside the rectangle if and only if $0 < p_{cross} < 1$, which is true if and only if:

$$(b-a)(12+4nc+4nd+n^2cd)+(d-c)(4-n^2ab) > 0$$

and

$$(d-c)(4-n^2b^2) < (a-b)(12+8nd+n^2d^2).$$

Now, if p_{cross} corresponds to a point inside the rectangle, then the Lebesgue measure of the set of points of the convex hull of $\{P_1, P_2\}$ that corresponds to paradoxes is p_{cross}.

REFERENCES

Andreev, E., W. Lutz, and S. Scherbov. 1989. Averaging Life Expectancy. WP-89-35. Laxenburg, Austria: International Institute for Applied Systems Analysis.

Haunsperger, D. B. To appear. Aggregation Paradoxes in Nonparametric Statistics.

Haunsperger, D.B. 1991. Projection and Aggregation Paradoxes in Nonparametric Statistical Tests. Ph.D. Dissertation. Chicago, Illinois: Northwestern University.

Haunsperger, D.B. and D. G. Saari. 1991. The Lack of Consistency for Statistical Decision Procedures. *The American Statistician* 45(3):251-254.

Keyfitz, N. 1977. *Introduction to the Mathematics of Population with Revisions.* Pages 3-16. Massachusetts: Addison-Wesley.