NEW SOCIETAL EQUATIONS

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I. Introduction

In a recent paper, W. Häfele /1/ established a number of phenomenological equations describing the behavior of a model society. The state variables of this model society were gross national product, population, energy consumption and risk acceptance.

In this paper, the state of the discussion within the IIASA energy project at the time being shall be fixed. Several new sets of equations will be established which extend the set given by Häfele and Manne /2/ in the following sense:

- capital will be included as another state variable
- a finite asymptotic population will be assumed
- there are several primary energy sources (fossil and nuclear)

In the following sections, we will outline three different approaches, namely

- an approach where a complete system of equations, including one primary energy source, is established and where the topological features (separatrices, fix points, etc.) can be studied in detail,

- a "control theoretical approach", including two primary energy sources, where we limit the number of state variables in such a way that there remains only one "control variable" subject to optimization with respect to an appropriate objective function,

- a "linear programming approach" where we introduce the same number of energy supply variables as in the work of Häfele and Manne /2/, and where we optimize the (more than one) free state variables according to different objective functions. The total energy demand is either taken from a model of the first kind or is assumed to be an independent control variable subject to optimization.

2. Complete System of Equations for one Sort of Energy

We consider the following state variables:

Total gross national product G
Total population P
Per capita gross national product g, i.e.,

$$G = g \cdot P \qquad (2\text{-}1)$$

Total energy demand E
Risk acceptance r
Total energy operating costs K
Total energy investment costs i
Total capital M
Total consumption C
Per capita consumption c,

$$C = c \cdot P \qquad (2\text{-}2)$$

The following equations are assumed to describe the development of the state variables with time.

We consider a special Cobb-Douglas production function

$$G = A \cdot P^{\alpha} \cdot M^{\beta} \cdot E^{\gamma} \text{ , } A = \text{const} \qquad (2\text{-}3)$$

where

$$\alpha + \beta + \gamma = 1, \qquad (2\text{-}3')$$

which means that we consider a function without economies of scale.

The assumption that A is independent of the time may be questioned. In Appendix I a different objective function with time dependent A is considered.

We assume operating and investment costs to be inversely proportional to the risk acceptance r for all energy resources:

$$\frac{K}{K_o} = \frac{r_o}{r} \qquad (2\text{-}4)$$

$$\frac{i}{i_o} = \frac{r_o}{r} \qquad (2\text{-}5)$$

The risk acceptance r and the per capita consumption are assumed to be related in the following way

$$\frac{r}{r_o} = \left(\frac{c_o}{c}\right)^2 \quad . \qquad (2\text{-}6)$$

We describe the population growth in the following way:

$$\frac{1}{P} \cdot \frac{dP}{dt} = a_p \cdot \left(1 - \frac{P}{P_o}\right) - a_c \cdot a_v \cdot g \qquad (2\text{-}7)$$

The per capita gross national product g is assumed to develop as follows:

$$\frac{dg}{dT} = \mu \cdot g \cdot \left(1 - \frac{g}{g_a}\right) \qquad (2\text{-}8)$$

In the spirit of a CD function which includes P, M and E as production factors, M should not include capital invested in the energy sector. Thereofre, the total gross national product is assumed to be distributed as:

$$G = C + (K-K_o)E + (i-i_o)\frac{dE}{dt} + \frac{dM}{dt} + i_o\frac{dE}{dt} \quad , \qquad (2\text{-}9)$$

where

$$C = a_v \cdot G \quad . \qquad (2\text{-}10)$$

We have separated the non-productive risk expenditures $(K-K_o) \cdot E$ and $(i-i_e) \cdot \frac{dE}{dt}$ from the pure investment costs $i_o \cdot \frac{dE}{dt}$. The corresponding term $K_o \cdot E$ does not occur for it would mean a double counting.

Eqs.(2-1)-(2-10) represent a complete set of equations for the ten variables G,P,g,E,r,K,i,C,c,M.

We can reduce the system of equations given above to the following system

$$\frac{dg}{dt} = \mu \cdot g \cdot \left(1 - \frac{g}{g_o}\right) \qquad (2\text{-}11)$$

$$\frac{1}{P} \cdot \frac{dP}{dt} = a_p \cdot (1 - \frac{P}{P_A}) - a_c \cdot c \qquad (2\text{-}12)$$

$$G = A \cdot P^{\alpha} \cdot M^{\beta} \cdot E^{\gamma} \qquad (2\text{-}13)$$

$$g \cdot P = c \cdot P + (K - K_o) \cdot E + (i - i_o) \frac{dE}{dt} + \frac{dM}{dt} \qquad (2\text{-}14)$$

$$K = K_o \cdot (\frac{g}{g_o})^2 \qquad (2\text{-}15)$$

$$i = i_o \cdot (\frac{g}{g_o})^2 \qquad (2\text{-}16)$$

$$c = a_v \cdot g \; . \qquad (2\text{-}17)$$

In this system, the risk acceptance r which has not been quantified anyhow, does not occur anymore. This means that we could proceed in such a way that we take from the very beginning only the equations (2-11) to (2-17) as a description of the model society.

Values or ranges for constants, and initial conditions have been fixed as follows:

$$\mu = 0.04$$

$$g_a = 12 * 10^3 \left[\frac{\$}{cap}\right]$$

$$0.03 \leq a_p \leq 0.06$$

$$3 * 10^8 \leq P_A \leq 8 * 10^8 \left[cap\right]$$

$$10^{-6} \leq a_c \leq 3 * 10^{-6} \left[\frac{cap}{\$}\right]$$

$$0.5 \leq a_v \leq 0.7$$

$$\left.\begin{array}{l} 0.65 \leq \alpha \leq 0.7 \\ 0.15 \leq \beta \leq 0.2 \\ 0.1 \leq \gamma \leq 0.15 \end{array}\right\} \text{ such that } \alpha + \beta + \gamma = 1$$

$$K_o = 10 \left[\frac{\$}{kWa}\right]$$

$$i_o = 160 \left[\frac{\$}{kW}\right]$$

$$g_Q = 6 * 10^3 \quad \left[\frac{\$}{cap}\right]$$

$$P_o = 2.1 * 10^8 \quad cap$$

$$\frac{E_o}{P_o} = e_o = 10 \quad \left[\frac{kWa}{cap}\right]$$

$$M_o = 4 * 10^{12} \quad [\$]$$

Preliminary results of the analysis of the system (2-11) to (2-17) are given in Appendix II .

3. Two Energy Options; Control Theoretical Approach

We consider two different primary energy sources, fossil energy E_f and nuclear energy E_n. Then the total energy demand E is given by

$$E = E_f + E_n \ . \qquad (3\text{-}1)$$

In addition, we assume that the use of fossil energy does not pose any risk, whereas the use of nuclear energy does. Therefore, we have instead of (2-14)

$$g \cdot P = c \cdot P + (K-K_o) \cdot E_n + (i-i_o) \frac{dE_n}{dt} + \frac{dM}{dt} \qquad (3\text{-}2)$$

All the other equations of section 2 we will keep. Furthermore we assume that only a finite amount of fossil energy can be spent:

$$\int_0^t E_f dt \leq V \ . \qquad (3\text{-}3)$$

Finally, we assume a growth restriction on the production of nuclear energy of the type

$$\dot{E}_n < N \qquad \text{or} \qquad \dot{E}_n / E_n < \nu \qquad (3\text{-}4)$$

The first constraint would represent limited abilities of industry to construct nuclear power plants, the second one, f.i.,inner growth limitations of a breeder economy. Due to the current abundance of plutonium, we prefer a constraint of the first type, which will not make any difference as to the qualitative features of the model.

Compared to the set of equations in section 2, we have one additional variable, but no additional equation. We will use this situation to introduce an optimization criterion with the help of which we optimize an appropriately chosen "control variable", e.g., E_n/E.

We now remark that equ. (3.3) express the limitations of control we can exert over the evolution of society according to the model. When all fossil fuel resources are used up, the deterministic evolution according to sec. 2 and appendix III takes over. At least from this time-denoted by T_f-, we can talk about the model in terms of separatrix, fixed point etc. and these will be the same as in the deterministic case.

With respect to the selection of the appropriate optimization criterion, we may proceed as follows: one would try as a first approach to take an nonlinear function of the per capita consumption as a preference function, i.e.

$$W = \int_0^\infty c(t)^\beta \cdot e^{-\rho t}\, dt$$

where W is the level of preference function, c(t) is per capita consumption at time t, β is the elasticity of the preference function with respect to consumption, and ρ is a discount factor used to relate the weighting of consumption of different generations.

This would become trivial in our model since, according to equ.(2-11) and (2-17 c(t) is given deterministically. We therefore have to look for other objective functions. A possible candidate would be the total discounted energy production costs, like in the model by Häfele and Manne /2/. But in the spirit of the resilience discussion, we can also introduce a resilience measure of the following kind as an objective function (see app. II for further discussion).

$$R_{av} = 1/\int_{T_f}^\infty \frac{dt}{|\dot{x}(t)|\, d(x,S)} \quad \text{or} \quad R_{min} = \inf_{t \geq T_f} d(x(t),S) \qquad (3\text{-}6)$$

or, for simplification, even $R_f = d(x(T_f),S)$.

The integrals, resp. infima in R_{av}, resp. R_{min} could also be taken from 0 instead of T_f. $d(x(t),S)$ denotes the distance suitably scaled, from the system state $x(t)$ at time t to the spearatrix S of the deterministic model. We avoid, at least partially, the conceptual difficulties of the question "distance from which separatrix", since as remarked, from T_f on, the model is deterministic.

Finally, a suitable, possibly linear combination of these two types of objective functions could be tried. This would avoid the artificiality of pure maximization of distance to the separatrix. However, it is a difficult value judgment to find the right scaling for this combination.

An outline of a dynamic programming optimization procedure is given in Annex III.

4. Coupling the Häfele-Manne Model to the New Societal Equations

by

Carlos Winkler

Introduction:

In the Häfele-Mann model the energy demand over time is an exogenous variable that has to be met at a minimum cost. The new societal equations are an attempt at relaxing these conditions. It is assumed that they govern the development of society and that from them we can obtain the energy demand, and as long as there are some degrees of freedom, the demand itself could respond to then adapt best to the objective to be minimized.

The highly non-linear nature of the new societal equations constitutes an apparent drawback, since it seems to foreclose the use of the powerful linear programming techniques. A closer inspection of the equations reveals that this is not the case, and that with a slight modification in the assumptions we can get away with a linear programming optimization. Moreover it can be argued that the change of assumptions generalizes the societal equations instead of restricting their application.

The Societal-Häfele-Manne Equations.

As mentioned in Avenhaus, Grümm, Häfele, et al. the system of equations for the society is given by

$$\frac{dg}{dt} = \mu g(1 - \frac{g}{gO}) \qquad (2\text{-}11)$$

$$\frac{1}{P}\frac{dP}{dt} = a_p(1 - \frac{P}{P_A}) - a_c \cdot c \qquad (2\text{-}12)$$

$$gP = G = A\ P^{\alpha}\ M^{\beta}\ E^{\gamma} \qquad (2\text{-}13)$$

$$gP = cP + \sum_j \left\{(k^j - k_O^j)E^j + (i^j - i_O^j)\frac{dE^j}{dt}\right\} + \frac{dM}{dt} + \sum_j i_O^j \frac{dE^j}{dt} \qquad (4\text{-}4)$$

$$k^j = k_O^j \cdot (^g/_{gO})^2 \quad j\varepsilon J \quad , \quad k^j = k_O^j \quad j\varepsilon\bar{J} \qquad (2\text{-}15)$$

$$i^j = i_O^j \cdot (^g/_{gO})^2 \quad j\varepsilon J \quad , \quad i^j = i_O^j \quad j\varepsilon\bar{J} \qquad (2\text{-}16)$$

$$c = a_V \cdot g \qquad (2\text{-}17)$$

where J is the set of indices for high risk energies and $\bar{J}$ is its complement.

$$E = \sum_{j\varepsilon J\cup\bar{J}} E^j \qquad (3\text{-}1)$$

In addition we have that if we define by y^j the reserves of the j-th type of energy we then have

$$y^j = y_O^j - \int_O^t E^j dt \geq 0 \quad . \qquad (3\text{-}3)$$

Observe that the above equations, together with the initial conditions give:

from (2-11)	$\Longrightarrow$	$g(t)$	R-1
2-12 and R-1	$\Longrightarrow$	$P(t)$	R-2
2-17 and R-1	$\Longrightarrow$	$c(t)$	R-3
2-15 and R-1	$\Longrightarrow$	$K^j(t)\ \forall_j$	R-4
2-16 and R-1	$\Longrightarrow$	$i^j(t)\ \forall_j$	R-5

Thus we can remove equations 2-11, 2-12, 2-15, 2-16 and 2-17 from our optimization model and introduce $g(t)$, (Pt), $c(t)$, $k^j(t)$ and $i^j(t)$ as known exogenous functions of time in the remaining equations.

Using discrete time intervals (and using $\hat{g}_t$, $\hat{P}_t$, etc. to denote the known exogenous values of g, P, etc. during time period t) we are left for period t with

$$\hat{g}_t\hat{P}_t = A\ \hat{P}_t^{\alpha}\ M_t^{\beta}\ E_t^{\gamma} \tag{2-13)'}$$

$$\hat{g}_t\hat{P}_t = \hat{c}_t\hat{P}_t + \sum_j \left\{(\hat{k}_t^j - k_0^j)E_t^j + \hat{i}_t^j\ (E_{t+1}^j - E_t^j)\right\} + M_{t+1} - M_t \tag{4-4)'}$$

$$E_t = \sum_{\forall_j} E_t^j \tag{3-1)'}$$

$$Y_{t+1}^j = Y_t^j - E_t^j \qquad \forall_j \tag{3-3)'}$$

plus the usual non-negativity constraints.

Observe that three of the four remaining equations are linear. The only nonlinear term appears in equation (2-13)' which defines

$$M_t = \left(\frac{\hat{g}_t P_t^{1-\alpha}}{A}\right)^{1/\beta} E_t^{-\frac{\gamma}{\beta}}$$

Notice that the expression on the right hand side is a convex function of E_t. Thus if instead of an equality in (2-13) we had an inequality

$$\hat{g}_t \hat{P}_t \leq A \, \hat{P}_t^{\alpha} \, M_t^{\beta} \, E_t^{\gamma}$$

$$\Longrightarrow M_t \geq \left(\frac{\hat{g}_t \hat{P}_t^{1-\alpha}}{A} \right)^{1/\beta} E_t^{-\gamma/\beta}$$

we could linearize the right hand side and use linear programming to obtain a global optimum (assuming the objective function is also linear). That is, if ${}_1E_t, \ldots, {}_ME_t$ are a discrete set of possible values for E (covering its range) we have the following linear model

for $t = 1, \ldots, T$

$$M_t \geq \left(\frac{\hat{g}_t \hat{P}_t^{1-\alpha}}{A} \right)^{1/\beta} \sum_{k=1}^{M} \left({}_kE_t^{-\gamma/\beta} \right) \cdot \alpha_k \qquad \text{(LP-1)}$$

$$\sum_{\forall j} E_t^j = \sum_{k=1}^{M} {}_kE_t \cdot \alpha_k \qquad \text{(LP-2)}$$

$$1 = \sum_{k=1}^{M} \alpha_k \qquad \text{(LP-3)}$$

$$\hat{g}_t \hat{P}_t = \hat{c}_t \hat{P}_t + \sum_j \left\{ (\hat{k}_t^j - k_O^j) E_t^j + \hat{i}_t^j \, (E_{t+1}^j - E_t^j) \right\} + M_{t+1} - M_t \qquad \text{(LP-4)}$$

$$Y_{t+1}^j = Y_t^j - E_t^j \quad \forall_j \qquad \text{(LP-5)}$$

Plus non-negativity constraints on all linear variables. To the above equations we have to add other linear constraints already in the Häfele-Manne model, which restrict the rate growth of some forms of energies, etc. They do not change the nature of the resulting model.

Conclusions:

It is possible to couple the New Societal Equations with the Häfele-Manne model. This is achieved by relaxing the eequality constraint in equation (2.13). In other words, instead of requiring an energy capacity and a capital stock to exactly achieve a certain per capita income, we require that they are at a level to at least achieve that per capita income. If they are at a higher level we can interpret it as unused capacity. In a minimizing cost optimization this latter case is unlikely to occur.

References

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ANNEX I

A Different Set of Societal Equations

Millendorfer and Gaspari /AI-1/ proposed the following per capita production function g for the gross national product of a society:

$$g = A(t)\cdot e^{\frac{1}{4}}\cdot \exp b\cdot \frac{1}{2}\cdot \left[\left(\frac{a_e\cdot e^{\frac{1}{4}}}{\exp b}\right)^{-\rho} + \left(\frac{\exp b}{a_e\cdot e^{\frac{1}{4}}}\right)^{-\rho} \right]^{-\frac{1}{\rho}} \qquad (1')$$

where

A(t) is a function of time which describes the technological progress,

e is the per capita energy demand of the population [kW/cap], and

b is the level of education.

We assume strict proportionality between total capital M and total energy demand E:

$$M = a_M\cdot E. \qquad (2)$$

This has the consequence that in the production function the total capital has the exponent $\frac{1}{4}$ which is in line with usual assumptions for Cobb-Douglas production functions and observations /AII-2/.

In the following we put $\rho = 1$ which is approximately correct.

We assume that on the long term educational politics is done in such a way that the production function is optimized.

As

$$\frac{1}{2}\left[\frac{a_e \cdot e^{\frac{1}{4}}}{\exp b} + \frac{\exp b}{a_e \cdot e^{\frac{1}{4}}}\right] = \begin{cases} 1 \text{ for } & a_e \cdot e^{\frac{1}{4}} = \exp b \\ <1 \text{ for } & a_e \cdot e^{\frac{1}{4}} \neq \exp b \end{cases}$$

we put

$$a_e \cdot e^{\frac{1}{4}} = \exp b \quad (\text{"equilibrium relation"})$$

(which corresponds to the differential equation

$$\frac{db}{dt} = \frac{1}{4} \cdot \frac{1}{e} \cdot \frac{de}{dt}).$$

Therefore, we obtain

$$g = A(t) \cdot e^{\frac{1}{2}} . \tag{1}$$

The technological progress A(t) is assumed to be the same for all the nations of a group of nations (e.g., North Western Europe) and depends on the effort for education and research of the group of nations, which goes parallel with the increase of capital and energy consumption:

$$A(t) = a_e \exp 2\bar{b} = a_e \cdot \frac{\exp 2\bar{b}}{\bar{e}^{\frac{1}{2}}} \cdot \bar{e}^{\frac{1}{2}} . \tag{3}$$

The efforts for research and education which stimulate innovation capacities are complementary:

$$\bar{b} = \left[\delta_1 (\bar{b}_r)^{-\rho} + \delta_2 (\bar{b}_i)^{-\rho}\right]^{-\frac{1}{\rho}} ,$$

where $\bar{b}_r$ and $\bar{b}_i$ are the research and the education efforts of the groups of nations, and where

$$\delta_1 + \delta_2 = 1.$$

In the long run one may assume an optimum ratio between these two efforts such that in the long run the total per capita education and research effort $\bar{b}$ of the group of nations remains the relevant variable. This variable may be assumed to depend on the gross national product in the following way:

$$\frac{d\bar{b}}{dt} = a_b \cdot g . \qquad (4)$$

If we insert eq.(3) into eq.(1) we get for the averages $\bar{g}$, $\bar{e}$ and $\bar{b}$ for the groups of nations

$$\bar{g} = a_a \cdot \frac{\exp 2\bar{b}}{\bar{e}^{\frac{1}{2}}} \cdot \bar{e} = a_a \cdot \gamma_e \cdot \bar{e}$$

where γ_e is the efficiency of the use of energy which depends on the "intellectual intensity" per energy unit.

Numerator and denominator of the fraction

$$\frac{\exp 2\bar{b}}{\bar{e}^{\frac{1}{2}}} = \left(\frac{\exp \bar{b}}{\bar{e}^{\frac{1}{4}}}\right)$$

are similar to the "equilibrium relation" of the nations. However, contrary to the fractions in eq.(1') which correspond to the equilibrium relations of the nations, the efficiency of the energy use of the group of nations is not assumed to be constant.

A higher intellectual intensity causes a shift of the energy use from primitive and not efficient to more sophisticated and efficient sorts of energy. This is described by the following equation:

$$\exp 2b = a_s \left(\frac{e_s}{e_p}\right)^{\alpha_s} \qquad e_s + e_p = e \qquad (5)$$

where

e_s are the more sophisticated and efficient sorts of energy,

e_p are the less sophisticated and therefore, less efficient sorts of energy, and

$\alpha_s \approx 0.5$ if e_s is identified with electricity and e_p is identified with the remaining energy.

Note:

Eq.(5) is a first attempt to describe the effect of using more or less sophisticated sorts of energy. This attempt may be used to take into account empirical studies on this subject (see, e.g., references AII-3 and -4).

If we divide eq.(5) by $e^{\frac{1}{2}}$ we get

$$\gamma_e := \frac{\exp 2b}{e^{\frac{1}{2}}} = a_s \cdot \left[\frac{\frac{e_s}{e_p}}{e}\right]^{\frac{1}{2}} \qquad (6a)$$

A different way of representing γ_e in such a way that the relation between more and less efficient sorts of energy is used results from the assumption of the complementarity between relation of sorts of energy and energy intensity,

$$\gamma_e = \left[\left[\frac{\left(\frac{e_s}{e_p}\right)^{\frac{1}{2}}}{e^{\frac{1}{2}}}\right]^{-\rho} + \left[\frac{e^{\frac{1}{2}}}{\left(\frac{e_s}{e_p}\right)^{\frac{1}{2}}}\right]^{-\rho}\right]^{-\frac{1}{\rho}} \qquad (6b)$$

Eqs.(6a) and (6b) may be modified for special questions in such a way that nuclear energy e_n is identified with e_s, e.g., $e_2 = e_n$.

Note:

In eq.(1) and in eqs.(6a) and (6b) the two main problems of a future energy policy are formulated: Increase of energy consumption and/or increase of the efficiency of energy consumption.

The formulations (6a) and (6b) are only two out of many possible philosophies.

The change of the total capital is given by the fraction of the gross national product which is safed minus the extra current costs for reliability:

$$S_2 = (K-K_o)\cdot E_N$$

where E_N is the nuclear energy demand, minus the extra investment costs for reliability:

$$S_1 = (i-i_o)\cdot \frac{dE}{dt}$$

Therefore, we have

$$\frac{dM}{dt} = a_s\cdot G - (K-K_o)\cdot E + (i-i_o)\cdot \frac{dE}{dt}\ . \qquad (7)$$

Between costs K, i and the acceptable risk r there are the relations

$$\frac{K}{K_o} = \frac{i}{i_o}\ ,\quad \frac{K}{K_o} = \frac{r_o}{r}\ . \qquad (8)(9)$$

The acceptable risk and the per capita gross national product g are related by

$$\frac{r}{r_o} = \left(\frac{g_o}{g}\right)^{\lambda}\ . \qquad (10)$$

Finally, we assume that the total energy demand is given by the demand for fossil energy E_F and the demand for nuclear energy E_N:

$$E = E_F + E_N \qquad (11)$$

and that the increase of the population is described by the following relation:

$$\frac{1}{P} \cdot \frac{dP}{dt} = a_p \cdot (1 - \frac{P}{P_o}) - a_g \cdot g \qquad (12)$$

In addition, we have the following boundary condition: The change of the gross national product has to be greater zero:

$$\frac{dg}{dt} \geq 0 \qquad (13)$$

and the total consumption of fossil energy has to be limited:

$$V = V_o - \int_0^t E_F \cdot dt > 0 \ . \qquad (14)$$

This means, we have 8 equations ((1),(2), (7)-(12)) for the 9 time functions g, E, E_F, E_N, P, M, K, r and i.

The societal equations as given above have been established in view of easy tractability. For the determination of the numerical values of the constants and the initial conditions there exist empirical data which can be used.

If more time can be spent for the development of the equations listed above, an objective function should be introduced which corresponds to the concept of the health definition of the WHO /AII-5/. A simple approach in this direction was the objective function of the Bariloche model; this objective function should be developed further on the basis of new investigations. If one would introduce an objective function strong assumptions as equation (4) could be replaced by an appropriate optimization procedure.

References for Annex I

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Appendix II

On the topological structure of the "new societal equations".

by H.R. Grümm

I. Introduction

The purpose of this paper is a qualitative study of the phase portrait for the equation system (2.11)-(2.17). By the phase portrait we denote the totality of orbits (= evolution histories) as curves in phase space, disregarding the labelling of points on them by the "independent variable" time. This qualitative study is essential for the location of fixed points attractors, separatrices and basins; only after its completion numerical evoluation of their actual position can take place. The model is given by a causal differential equation of the form $\dot{x} = f(x)$, x denoting the state vector of the system, i.e. its components are the state variables.

We shall be looking especially for a separatix, i.e. a hypersurface in state space separating two basins of attraction. For discussion of this point, see /A1/, where separatrices were identified as stable manifolds of codimension one. Therefore, the interesting fixed point of the model will have just one unstable direction.

II. The reduced equations

As the model is written down, it is four dimensional: the phase-space coordinates are g, P, E, M. The Cobb-Douglas ansatz equ. 2.13, however, plays the role of a first integral of the differential equation:

$$\frac{\dot{G}}{G} = \alpha\frac{\dot{P}}{P} + \beta\frac{\dot{M}}{M} + \gamma\frac{\dot{E}}{E} \qquad (A1)$$

with a constant A to be determined from the initial conditions. Therefore, the 4-dimensional spaces is divided into invariant hypersurfaces on which we can use as coordinates g, P and E. After elimination of M, the equations look like:

$$\dot{g} = \mu g(1-\frac{g}{g_A}) \qquad (A2.1)$$

$$\dot{P} = P(a_p\ (1-\frac{P}{P_A}) - a_c\ a_v g) \qquad (A2.2)$$

$$\dot{E} = \frac{N(g,\ P,\ E)}{D(g,\ P,\ E)} \quad \text{with} \qquad (A2.3)$$

$$N_{(g,\ P,\ E)} = g\ P(1-a_v) - (K-K_o)\ E \qquad (A2.4)$$

$$- \frac{M}{\beta}\ [\mu(1-\frac{g}{g_A}) + (1-\alpha)\ (a_p(1-\frac{P}{P_A})-a_c a_v g)]$$

$$D = \frac{\gamma E}{\beta M} - i_o[(\frac{g}{g_o})^2 - 1] \qquad (A2.5)$$

In these equations, M denotes the function $(g\ P^{1-\alpha}/A\cdot E^{\gamma})^{\frac{1}{\beta}}$ of g, P and E. One notes that equ. (A2.1) and (A2.2) are independent of E, therefore all solution curves will lie on cylinders having as base curves the solutions of those two equations. We call these cylinders solution cylinders. In the (g-P) plane, Fig. 1 shows an example with the "canonical" choice of parameters $\mu = 0.04$, $a_p = 0.044$, $a_c = 3x10^{-6}$, $a_v = 0.7$.

One reorganizes immediately a fixed point of the restricted set of equations at:

$$g_{fix} = g_A,\ P_{fix} = P_A(1-\frac{a_c a_v g_A}{a_p}) \qquad (A3)$$

apart from other ones at $g = o$ or $P = o$ in regions which lie out of the validity domain of the model assumptions, and outside of realistic initial conditions). This fixed point is stable and attracts every trajectory in the region $\{g > o, P > o\}$. Its eigen values are given by

$$\lambda_1 = -\mu$$

$$\text{and} \quad \lambda_2 = -a_p + a_c\, a_v\, g_A \qquad \text{(A4)}$$

III. The divergence surface

The important fixed point of the whole set of model equations lies at

$$(g_{fix},\ P_{fix},\ E_{fix} = \frac{g_{fix}\ P_{fix}\ (1-a_v)}{K_o[(g/g_o)^2-1]}) \qquad \text{(A5)}$$

Before we discuss its stability, we have to point out a mathematical complication of the model, due to equ. (A2.3): at the zeroes of N, $\dot{E}$ is undefined. Indeed, at such points, the evolution of the system cannot be prolonged to future times. One way of looking at the situation is to realize that the condition $N = 0$ can be written as:

$$\frac{E(i-i_o)}{M} = \frac{\gamma}{\beta} \qquad \text{(A6)}$$

and represents therefore the condition for economic optimum, since γ and β are the elasticities for E and M resp. Thus, at zeroes of N, the model assumptions of prescribed economic growth and of equ. (2.14) are inconsistent[1].

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1) I am indebted for this observation to W. Nordhaus.

If we only want to describe the phase portrait then there is an easy remedy since it is not changed by multiplication with a scalar function: only the time scale and, possibly, the time direction is changed. For this discussion, we replace the defining equ. (A2) by

$$g' = \mu g(1-\frac{g}{g_a})\cdot D \tag{A7.1}$$

$$p' = P[a_p(a-\frac{P}{P_a}) - a_c\ a_v\ g] \cdot D \tag{A7.2}$$

$$E' = N(g,\ P,\ E) \tag{A7.3}$$

the ′ now denoting derivative w.r.t. some parameters s, defined by ds/dt = D. The right-hand-sides are now continuously differentiable in state space and a familiar theorem assures that the solution curves can be extended at each point in the state space {g, P, E > 0}. However, one has to be conscious about three facts:

1) As soon as a solution of (A7) crosses the divergence surface (defined by D = 0) it ceases to have realistic significance for the model, for reasons explained above.

2) "Above" the divergence surface (= for larger values of E), D < 0, so, as we follow the trajectory, time is running backward and stability and instability directions of fixed points become interchanged.

3) Spurious fixed-points will be introduced, i.e. fixed points which do not exist in equ. (A2). In fact, a whole fixed curve (= a curve consisting of fixed points) appears at the intersection of the surfaces D = 0 and N = 0. However, in one situation, a separatrix of the original model emerges from a spurious fixed point. The stability character of those points is determined in the following way: the fixed curve intersects each solution cylinder in at most two points. The

only intersection, or if there are two, the one at smaller g, is a stable focus; the other intersection, if it exists, is a saddle. Although the focus does not appear in equ. (A2), it attracts their solution curves as they are the same ones as those of equ. (A7).

IV The phase portrait

The stability character of the "real" fixed point is determined by the sign of:

$$\lambda_3 = \frac{\partial \dot{E}}{\partial E} = \frac{K-K_Q}{D}\text{ , taken at } E_{fix} \qquad (A8)$$

λ_3 is positive if and only if the fixed point lies below the divergence surface, i.e. if $E_{fix} < E_{div}$, where E_{div} denotes the intersection of the divergence surface with the line $\{g = g_{fix}, P = P_{fix}\}$. If $\lambda_3 > 0$, the fixed line intersects every solution cylinder exactly once. In the other case, there will be solution cylinders with two intersections and others with no intersection.

We know how to distinguish three differently structured phase portraits:

1) Fixed point below divergence surface. $E_{fix} < E_{div}$. In this case, the fixed point is a saddly point with two stable and one unstable direction, the latter coinciding with the direction of the E - axis. Its stable manifold therefore satisfies the conditions given in / / for a separatrix. The shape of this surface is shown in fig. 2. Points "below" it tend towards g_{fix} and P_{fix} at high values of M and low values of t; those above it are attracted by the spurious fixed line and cross the divergence surface; at the time of crossing, the given growth rate of GNP cannot be maintained any more and the solution curve cannot be extended into the

future; similarly for initial conditions above the divergence surface. Fig. 3 sketches the phase portrait restricted to one solution cylinder in this situation.

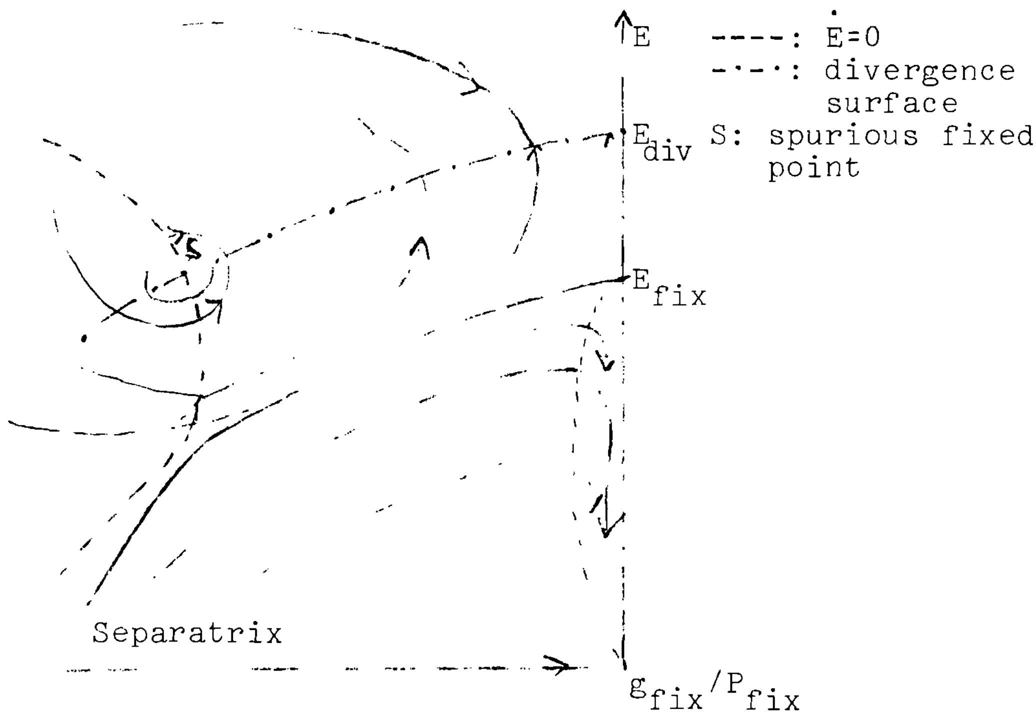


Fig. 3

The dotted arrows indicate time running backwards above the divergence surface.

2) Fixed point above divergence surface, $E_{fix} > E_{div}$. Fixed line intersects the solution cylinder. In this case, the fixed point is totally attractive as $\left.\frac{\partial \dot{E}}{\partial E}\right|_{E_{fix}} < o$. (i.e. repulsive in the phase portrait). The spurious fixed line intersects any cylinder twice; the intersection at smaller values of E is again an attracting focus as in situation 1) but the other one is a saddle. Its stable manifold is therefore again a separatrix but above the separatrix, due to the reversal of time direction, one has to take the other branch. The phase portrait on a cylinder therefore looks as in fig. 4.

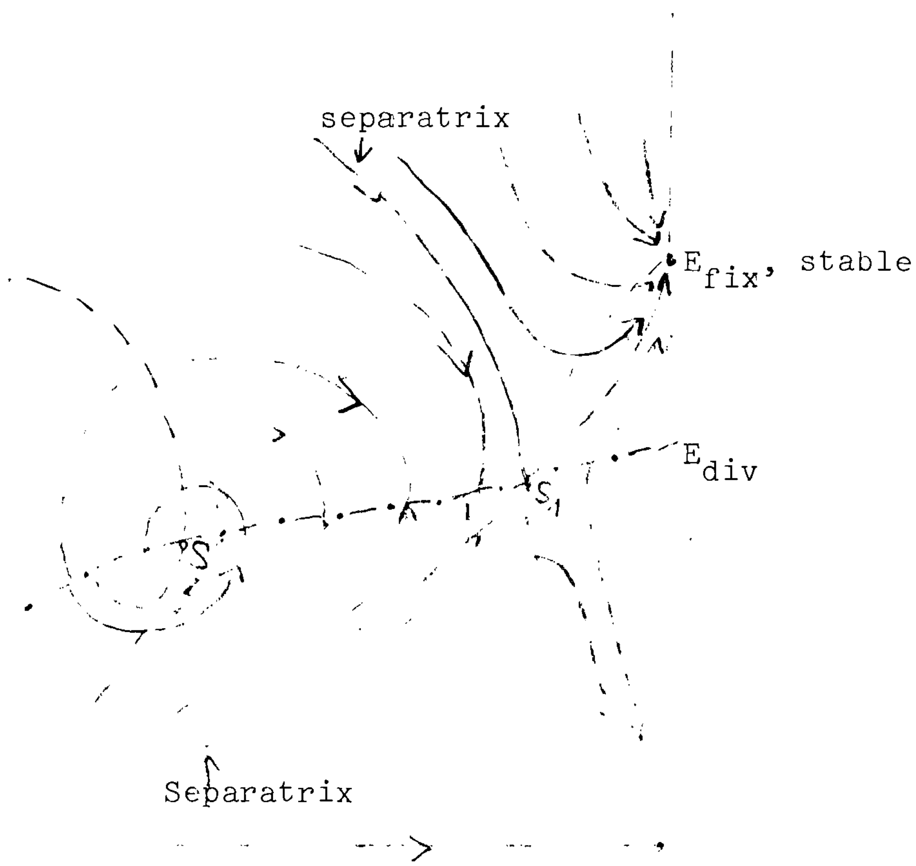
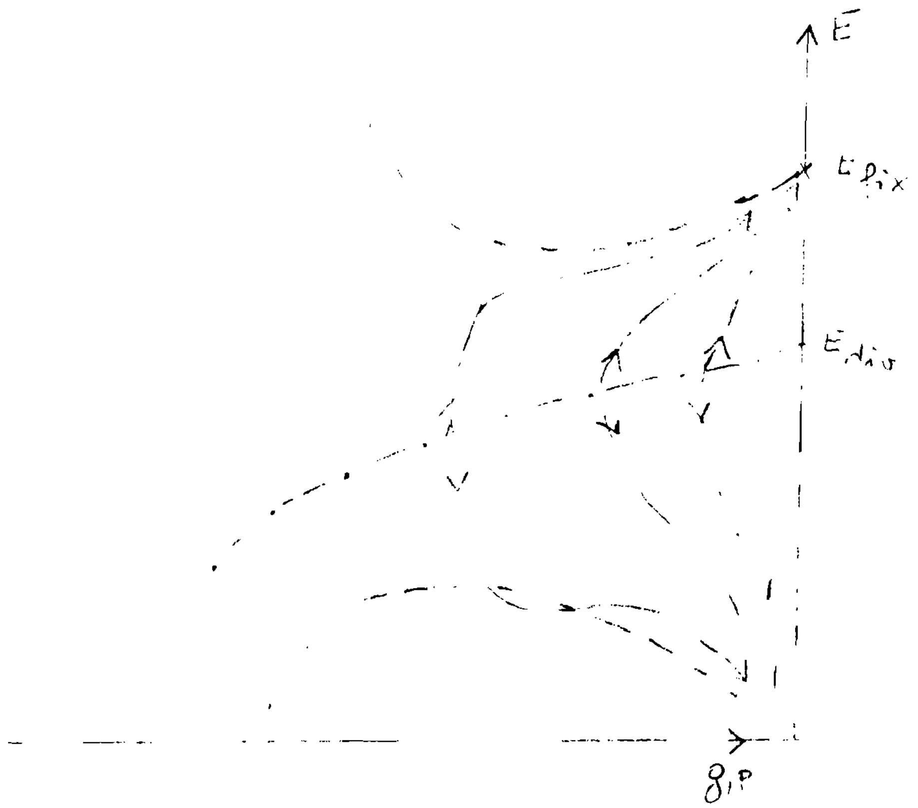


Fig. 4 ----: E=0
-.-.: divergence surface
S: spurious fixed points

Since S_1 is no real fixed point, the trajectory goes straight through and continues towards the stable E_{fix}. Points below the separatrix go again to low E-high M, points above it again end up on the divergence surface, but now points exactly <u>on</u> the separatrix end up at an attractor separated from the other trajectories. If E_{fix} would be taken as desirable, the system should there have be steered towards the separatrix since the only point to cross the divergence surface is on it !

3) $E_{fix} > E_{div}$, fixed line does not intersect the solution cylinder and the divergence surface plays the role of a "pseudo-separatrix": every point below it tends to low E-high M, every point above it to the now unique fixed point. See fig. 5



Inserting the "canonical choice" of parameters into the equations for E_{fix} and E_{div} shows that we are in situation 1 in this case. Further quantitative study of the system therefore will always assume the qualitative structure of situation 1.

V. Numerical results; the computation of the separatrix.

A true separatrix--a stable manifold of a fixed point--emerged in the first two situations discribed in IV [2]. Numerical evaluation of such stable manifolds is hampered by the fact that the separatrix cannot be defined by local data, e.g. a partial differential equation, from the given dynamical system; it depends on global features of the system if a given point will be on the separatrix.

The way chosen for numerical evaluation of the separatrix was the following: the tangent phase to the separatrix at the fixed point can be determined by the local stable manifold theorem /A4/: it is the hyperplane in tangent space belonging to the eigenvalues of the Jacobien $\left(\frac{\partial f_i}{\partial x_j}\right)$, evaluated at the fixed

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[2] The "pseudo-separatrix" of situation 3 is given in closed form by the equation D = 0.

point, with negative real parts. We give the formules for the Jacobian at (G_{fix}, P_{fix}, F_{fix}):

$$\frac{\partial\dot{g}}{\partial P} = \frac{\partial\dot{g}}{\partial E} = \frac{\partial\dot{P}}{\partial E} = 0 \qquad (A9.1)$$

$$\frac{\partial g}{\partial g} = -\mu, \ \frac{\partial\dot{P}}{\partial P} = - a_p + a_v a_c g_{fix}, \ \frac{\partial\dot{E}}{\partial E} = \frac{K-K_o}{D} \qquad (A9.2)$$

$$\frac{\partial\dot{P}}{\partial g} = - a_c . a_v . P_{fix}$$

$$\frac{\partial\dot{E}}{\partial g} = \frac{1}{D} \{P_{fix} (1-a_v) - 2 g_{fix} k_o / g_o{}^2 + M_{fix} *$$

$$* (\mu / g_{fix} + (1-\alpha) a_v . a_c) / ß\} \qquad (A9.3)$$

$$\frac{\partial\dot{E}}{\partial P} = \frac{1}{D} \{g_{fix} . (1-a_v) + M_{fix} (1-\alpha) a_p / ß \ P_{fix}\}$$

Using a theorem from /A1/, the separatrix can be approximated by starting on its largest phase at the fixe point a samll distance off the fixed point and evaluating the differential equation backwards in time. The numerical error of this approximation is kept small by the fact that, with time increasing points close to the manifold, f.i. on its tangent phase and close to the fixed point, will move away from it exponentially fast. 20 different starting points were taken; a plot of the results as viewed under an oblique angle is given in Fig. 2. As mentioned in the main paper, the knowledge of the separatrix is crucial in the non-deterministic situation of a dynamic programming, too.

Literature:

/A1/ H.R. Grümm, "Stable manifolds and separatrices", IIASA working paper, to appear.

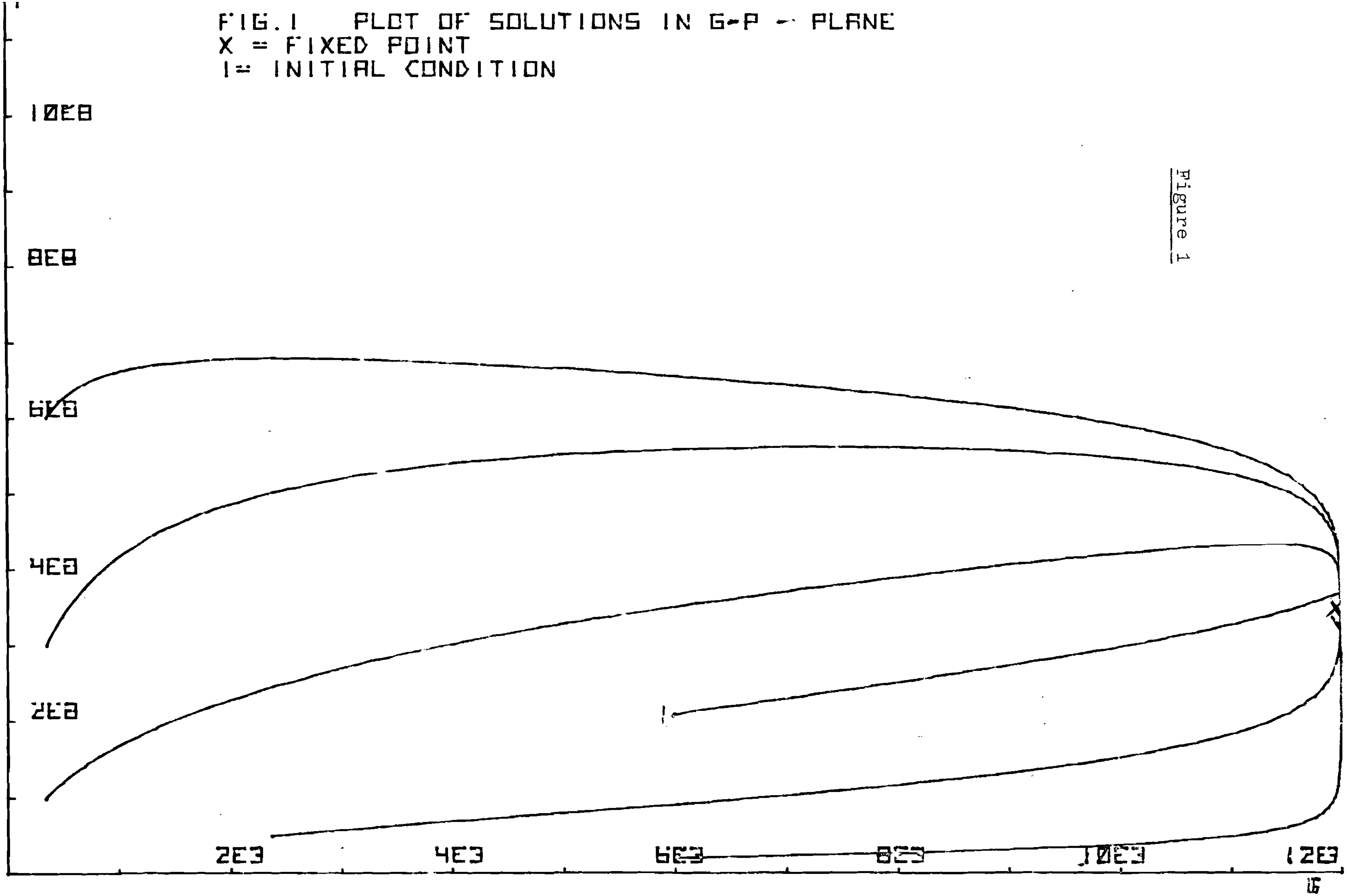
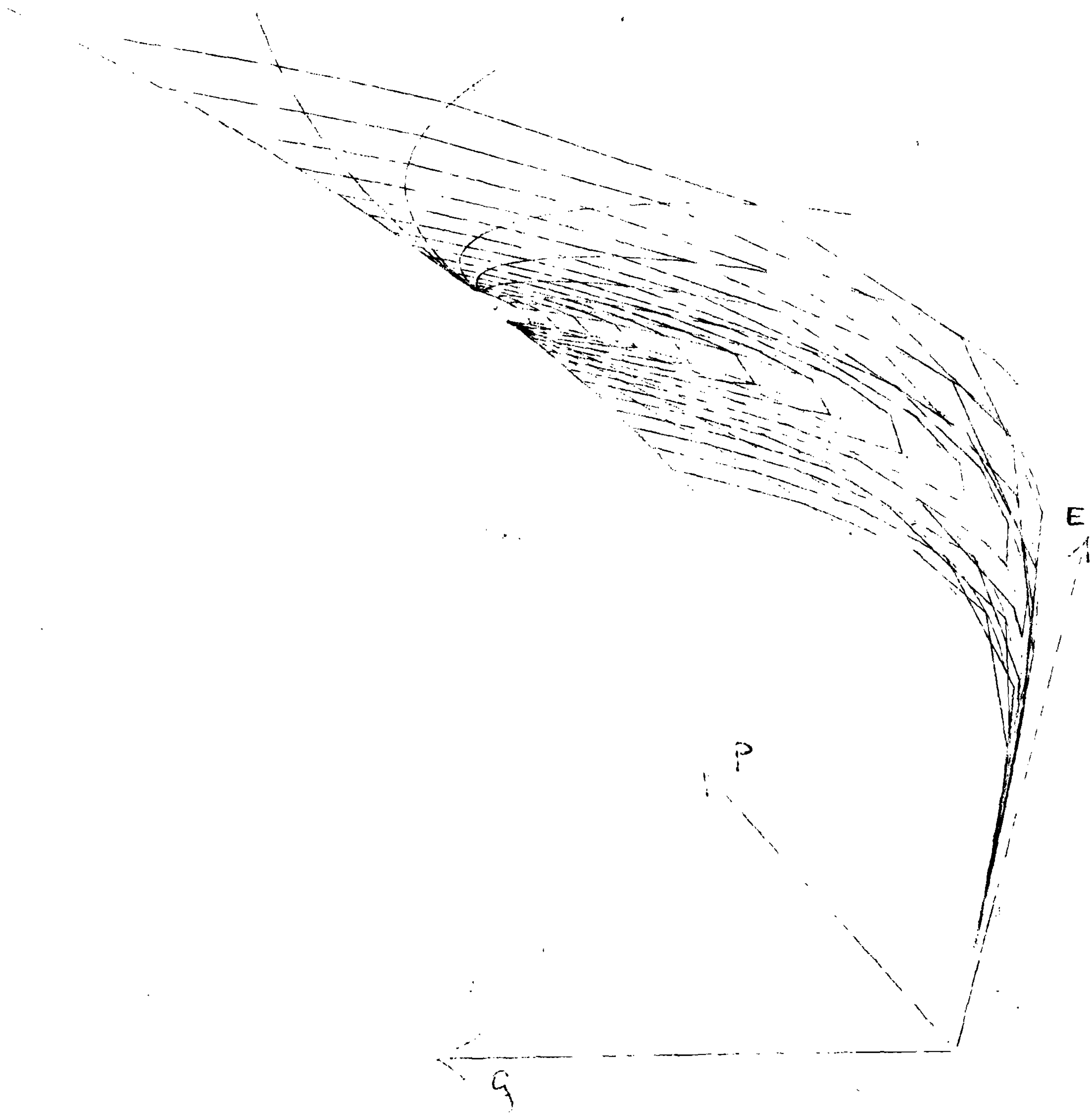


FIG.1 PLOT OF SOLUTIONS IN G-P - PLANE
X = FIXED POINT
I= INITIAL CONDITION
10E8
8E8
6E8
4E8
2E8
2E3
4E3
6E3
8E3
10E3
12E3
G

Figure 1

Figure 2



Annex III

A Dynamic Programming Optimization using the New Societal Equations

by

Carlos Winkler

Our aim here is to outline a dynamic programming procedure to optimize functions of the form

$$\int_0^\infty f(g,P,M,E)e^{-\rho t}dt$$

subject to the New Societal Equations. By New Societal Equations we refer to equations 2-11 through 2-17 and 3-1 through 3-3 in the handout of the same name by Avenhaus, Grümm, Häfele et al.

We start by noting that these equations together with the initial conditions have the following implications:

equation 2-11 gives $g(t)$ (R1)

eq. 2-12 with R1 $\Longrightarrow$ $P(t)$ (R2)

and similarly

2-15 and R1 $\Longrightarrow$ $k(t)$ (R3)

2-16 and R1 $\Longrightarrow$ $i(t)$ (R4)

2-17 and R1 $\Longrightarrow$ $c(t)$ (R5)

That is g, P, c, K and i can be viewed all as exogeneous functions of time which do not depend on the demand for electricity or Capital.

So we are left with

3-2) $g(t)P(t) = c(t)P(t) + (k(t) - k_0)\ E_n + (i(t) - i_0)\frac{dE_n}{dt} + \frac{dM}{dt} + i_0\ \frac{dE}{dt}$

2-13) $g(t)P(t) = AP(t)^{\alpha}M^{\beta}\ E^{\gamma}$

3-1) $E = E_n + E_F$

3-3) $\int_0^t E_F\ dt \leq V$

It will be convenient to work with discrete time. Also we define

$$y = V - \int_0^t E_F\ dt$$

Then

D-1 $g_tP_t = c_tP_t + (k_t - k_0)E_n^t + (i_t - i_0)(E_n^{t+1} - E_n^t) + M^{t+1} - M^t + i_0(E^{t+1} - E^t)$

D-2 $g_tP_t = AP_t^{\alpha}(M^t)^{\beta}(E^t)^{\gamma}$

D-3 $E^t = E_n^t + E_F^t$

D-4 $y^{t+1} = y^t - E_F^t$ (t subscript endogenous i.e. fixed) (t superscript variables

and non negativity constraints on all variables.

Notice that we have

9 variables (E_n^t, E_n^{t+1}, E^t, M^{t+1}, M^t, E_F^t, y^{t+1}, y^t, E^{t+1}) and 4 equations. That leaves 5 variables as state plus decision variables. Notice though that D-2 $\Rightarrow E^t = F(M^t) = \left(\frac{g_tP_t^{1-\alpha}}{A}\right)^{\frac{1}{\gamma}} M^{-\frac{\beta}{\gamma}}\ \forall t$. Substituting $E = F(M)$ in D-1 leaves 4 variables as state plus decision variables. Three

of them appear with indices t and t+1 so those can be quite naturally considered as state variables

$$\left.\begin{array}{l}\text{state variables}\\ \text{period t}\end{array}\right\} \quad \begin{array}{l}E_n^t\\ M^t\\ y^t\end{array}$$

Note all states are allowed. From

D-3 $\Longrightarrow$ $E_n^t \leq E^t = F(M^t)$

Also from D-3, D-4 and non-negativity

$$E^t - E_n^t = F(M^t) - E_n^t = E_f^t$$

and $$y^{t+1} = y^t - E_F = y^t - F(M^t) + E_n^t \geq 0 \quad .$$

Hence the allowable states satisfy

S-1 $\quad E_n^t \leq F(M^t)$

S-2 $\quad y^t \geq F(M^t) - E_n^t$

(these considerations should help in reducing the computational effort).

Assuming we have a Value function

$$V_{t+1} = V_{t+1}(E_n^{t+1}, M^{t+1}, y^{t+1})$$

($V_{t+1} = \infty$ for states that do not satisfy S-1 and S-2) and a cost function for period t

$$C_t = C_t(E_n^t, M^t, Y^t, \text{decisions})$$

then the dynamic programming recursion can be written

$$V_t(E_n^t, M^t, y^t) = \max \{C_t + \lambda V_{t+1}(E_n^{t+1}, M^t, y^t)\}$$

s.t. D-1 if E_n^t, M^t, y^t are feasible

$+ \infty$ otherwise.

Thus an outline for the Dynamic Programming Optimization for this problem would proceed as follows.

DO for $t = T, T-1, \ldots, 1$

DO for $M^t = M_1^t, \ldots, M_{N_m}^t$ (N_m possible discrete values of M^t)

DO for $E_n^t = E_{n_1}^t, \ldots, E_{n_{N_E}}^t$ (N_E possible discrete values of E_n^t)

if $E_{n_i}^t > F(M_j^t)$ let $V_t \ (E_{n_i}^t, M_j^t, \cdot) = +\infty$ and

← ← ← ← ← ← ← go to end of loop, otherwise

DO for $y^t = y_1^t, \ldots, y_{N_y}^t$

if $y_k^t < F(M_i^t) - E_{n_j}^t$ let $V_t \ (E_{n_j}^t, M_i^t, y_k^t) = \infty$ and

← ← ← ← ← ← ← go to end of loop, otherwise

let $y^{t+1} = y_k^t - F(M_i^t) - E_{n_j}^t$

and choose M^{t+1} to

$$\min \{C_t + \lambda V_{t+1}(E_n^{b+1}, M^{t+1}, y^{t+1})\}$$

s.t. $g_t P_t = c_t P_t + (k_t - k_0)E_n^t + (i_t - i_0)(E_n^{t+1} - E_n^t) + M^{t+1} - M^t$

$+ i_0(F(M^{t+1}) - F(M^t))$.

Observe that the minimization consists of a unidimensional search.

Let

$V_t(E_n^t, M^t, y^t)$ = minimum above

→ CONTINUE

→ CONTINUE

→ CONTINUE

→ CONTINUE

Other programming considerations:

T = number of time periods

N_M = number of grid points for M at which Pay-off functions are evaluated

N_E idem for E_n

N_y idem for y

Then the total number of evaluations will be

$$T \times N_M \times N_E \times N_y$$

and it can easily be seen to increase very rapidly with the number of grid points. For this reason it probably will become necessary to store the value functions out of core.

Even so probably no more than

$$N_M \approx N_E \approx N_y \underset{\approx}{\leq} 20$$

should be taken on a trial basis for the first runs.

Notice also that to start the optimization for t = T, we need to have a value function $V_{T+1}(E_n^{T+1}, M^{T+1}, y^{T+1})$, that for each possible state at period T+1 gives as the value or desirability of having ended it.