Working Paper

Static and Dynamic Issues in Economic Theory III. Dynamical Economies

Jean-Pierre Aubin

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IIASA
International Institute for Applied Systems Analysis □ A-2361 Laxenburg □ Austria
Telephone: +43 2236 715210 □ Telex: 079 137 iiasa a □ Telefax: +43 2236 71313

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Static and Dynamic Issues in Economic Theory

III. Dynamical Economies

Contents

FOREWORD

This the third part of STATIC AND DYNAMIC ISSUES IN ECONOMIC THEORY devoted to dynamical economies and the applications of viability theory.

Section 1 is devoted to the nontâtonnement model, leaving aside a further study of the tâtonnement process for its lack of viability. It relies on the Viability Theorem[1], the dynamical analogue of the Fixed Point Theorem. The analogy is even stronger, since the assumptions which characterize the viability property, together with convexity assumptions, provide the existence of an equilibrium! The proof of the Viability Theorem is provided, but can be omitted. Only the statement of the Viability Theorem will be used later on in the book. The recent viability theorem for stochastic differential equations obtained by in [4, Aubin & Da Prato] is also presented.

Section 2 deals with the issue of selecting **feedback mechanisms** from the regulation map, which can be regarded as **planning procedures**. The key tool here is the concept of **Selection Procedure**, which allows to choose feedback prices associating with each allocation prices in the regulation set which are solutions to (spot) optimization or game-theoretical mechanisms, involving then a **myopic behavior**.

Section 3 takes another road. It assumes that a bound to inflation is set in the model, and we look for feedback prices which regulate viable evolutions under bounded inflation.

These maps, characterized as solutions to first-order systems of partial differential inclusions, can then be differentiated, using the calculus of set-valued maps[2].

Then, **by differentiating the regulation law**, a differential inclusion governing the evolution of prices does emerge.

By using selection procedures (presented in the preceding section), we may obtain **dynamical feedbacks**. Among then, the minimal selection provides the **heavy evolutions** (in the sense of heavy trends), for which the prices

[1]We refer to [?, Aubin] for an exhaustive presentation of VIABILITY THEORY, which was motivated by the dynamical behavior of economic systems and an attempt to provide a mathematical metaphor for Darwinian evolution of biological and cognitive systems. For the sake of self-countenance, some of the results of this book are reproduced here, but adapted to specific economic themes.

[2]We refer to [5, Aubin & Frankowska] for a presentation of SET-VALUED ANALYSIS, a mandatory tool box for mathematical economists.

evolve with minimal velocity. Heavy evolution provides the simplest example of evolutions satisfying the **Inertia Principle.**

1 Dynamical Economies

Introduction

We have introduced the nontâtonnement decentralized model of allocation of scarce resources in the last part and showed that under convexity assumptions, there exists at least an equilibrium whenever the regulation map $\Pi_M(\cdot)$ built from the knowledge of the set of scarce resources and the change functions of the consumers had nonempty values $\Pi_M(x)$ for every allocation x.

We had also characterized this property by **budgetary rules** and evidenced that in certain cases, instantaneous Walras laws (individual and even collective ones) warrant that this condition is satisfied.

In this section, we prove that this condition is also sufficient (and necessary) to imply that **the set of allocations is viable in the sense that starting from any initial allocation, there exists at least prices $p(t)$ and thus, consumptions $x_i(t)$ of the consumers, which constitute at each instant an allocation of scarce resources.**

Furthermore, we know exactly what are the prices which regulate such allocations: their evolution is regulated by the **regulation law**

$$\text{for almost all } t \geq 0, \ \ p(t) \in \Pi_M(x(t))$$

By the way, this makes sense since we shall prove that the necessary and sufficient condition for the viability of this nontâtonnement process is that, as we have said, the images $\Pi_M(x(t))$ are not empty.

In the same way than the existence of an equilibrium of this non tâtonnement model was derived from a general Equilibrium Theorem in the preceding part, the above viability property is inferred from a general Viability Theorem which is stated in the second section and proved in the third one. This proof is quite involved and can be overlooked in the same manner than the proof of the Brouwer Fixed-Point Theorem is often skipped by its users.

Instead of describing this Viability Theorem in this most abstract form, we shall introduce an intermediate framework, which we call a **dynamical economy** (P, c) governing the evolution of an abstract **commodity** and an abstract **price** (playing the role of a regulatory control) according to the laws

$$\begin{cases} i) & x'(t) = c(x(t), p(t)) \\ ii) & p(t) \in P(t)) \end{cases}$$

where the commodity $x(\cdot)$ ranges over a finite dimensional vector-space X and the price $p(\cdot)$ ranges over another finite dimensional vector-space Z.

Here, the first equation describes how the price — regarded as an *input* to the system — yields the commodity of the dynamical economy[3] — regarded as an **output** — whereas the second inclusion shows how the commodity-output "feeds back" to the price-input. The set-valued map $P : X \rightsquigarrow Z$ may be called an "a priori pricing map".

It describes some **commodity-dependent constraints on the prices.** A solution to this system is a function $t \to x(t)$ satisfying this system for some price $t \to p(t)$.

Viability or scarcity constraints are described by a closed subset[4] K of the commodity space: These are intended to describe the "viability" of the dynamical economy.

A subset K **is viable** under the dynamical economy described by c and P if from every initial commodity $x_0 \in K$ starts at least one solution to the dynamical economy which is **viable** in the sense that

$$\forall\, t \geq 0, \quad x(t) \in K$$

The first task is to characterize the subsets having this property. To be of value, this task must be done without solving the system and then, without checking the existence of viable solutions from each initial commodity.

An immediate intuitive idea jumps to the mind: at each point on the boundary of the viability set, where the viability of the dynamical economy is at stake, there should exist a velocity which is in some sense **tangent** to the viability domain and serves to allow the solution to bounce back and remain inside it. This is, in essence, what the Viability Theorem states. But, first, the mathematical implementation of the concept of tangency must be made.

We cannot be content with viability sets that are smooth manifolds, because most of the sets defined through inequality constraints which we need in economics would thereby be ruled out.

[3]once the initial commodity is fixed.

[4]One can naturally investigate the cases when K depends upon the time, the commodity, the history of the evolution of the commodities. We shall also cover the case of solutions which improve a reference preorder when time evolves.

We have seen already how to adapt the definition of tangent directions in the case of convex sets. But we can "implement" the concept of a direction v tangent to any subset K at $x \in K$, which should mean that starting from x in the direction v, we do not go too far from K.

To convert this intuition into mathematics, we shall choose from among the many ways[5] that have been designed to translate what it means to be "not too far" the one suggested by Bouligand fifty years ago: a direction v is *contingent to K at $x \in K$* if it is a limit of a sequence of directions v_n such that $x + h_n v_n$ belongs to K for some sequence $h_n \to 0+$. The collection of such directions, which are in some sense "inward", constitutes a closed cone $T_K(x)$, called the **contingent cone**[6] to K at x. Naturally, except if K is a smooth manifold, we lose the fact that the set of contingent vectors is a vector-space.

We then associate with the dynamical economy (described by c and P) and with the viability constraints (described by K) the *(set-valued) regulation map* Π_K. It maps any commodity x to the subset $\Pi_K(x)$ consisting of prices $p \in P(x)$ which are *viable* in the sense that

$$c(x,p) \text{ is contingent to } K \text{ at } x$$

If, for every $x \in K$, there exists at least one viable price $p \in \Pi_K(x)$, we then say that K is a **viability domain** of the dynamical economy described by both c and P.

The Viability Theorem we mentioned earlier holds true for a rather large class of systems, called **Marchaud systems**: Beyond imposing some weak technical conditions, the only severe restriction is that, for each commodity x, the set of velocities $c(x,p)$ when p ranges over $P(x)$ is **convex**[7]. From now on, we assume that the dynamical economies under investigation are Marchaud systems.

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[5]For a presentation of the ménagerie of tangent cones, we refer to Chapter 4 of [5, Aubin & Frankowska].

[6]replacing the linear structure underlying the use of tangent spaces by the contingent cone is at the root of Set-Valued Analysis.

[7]This happens for the class of dynamical economies of the form

$$x'(t) = c(x(t)) + G(x(t))p(t)$$

where $G(x)$ are linear operators from the price space to the commodity space and when the pricing map P has convex images $P(x)$.

The basic viability theorem states that for such systems,

a closed subset K is viable under a Marchaud dynamical economy if and only if K is a viability domain of this economy.

Many of the traditional interesting subsets such as **equilibrium points, trajectories of periodic solutions, ω-limit sets of solutions, are examples of closed viability domains.** Actually, equilibrium points $\overline{x}$, which are solutions to

$$c(\overline{x},\overline{p}) = 0 \text{ for some } \overline{p} \in P(\overline{x})$$

are the smallest viability domains, the ones reduced to a single point. This is because being **stationary states,** the velocities $c(\overline{x},\overline{p})$ are equal to zero. Furthermore, we have seen in the preceding part that there exists a basic and curious link between viability theory and general equilibrium theory: the General Equilibrium Theorem — an equivalent version of the 1910 **Brouwer Fixed Point Theorem,** the cornerstone of nonlinear analysis — states that

every compact convex viability domain
contains an equilibrium point.

It finds here a particularly relevant formulation: **viability implies stationarity.**

The Viability Theorem also provides a **regulation law** for regulating the dynamical economy in order to maintain the viability of a solution: The viable solutions $x(t)$ are regulated by viable prices $p(t)$ through the **regulation law:**

$$\text{for almost all } t, \quad p(t) \in \Pi_K(x(t))$$

The multivaluedness of the regulation map — this means that several prices $p(t)$ may exist in $\Pi_K(x(t))$ — is an indicator of the "robustness" of the dynamical economy: **The larger the set $\Pi_K(x(t))$, the larger the set of disturbances which do not destroy the viability of the economy !**

Observe that solutions to a dynamical economy are solutions to the differential inclusion $x'(t) \in F(x(t))$ where, for each commodity x, $F(x) := c(x, P(x))$ is the subset of feasible velocities[8]. This is in the general framework of differential inclusions that the Viability Theorem is stated and proved.

We conclude this section by extending to the stochastic case Nagumo's Theorem on viability properties of closed subsets with respect to a differential equation. In VIABILITY THEORY, [?, Aubin], only invariance theorems

[8]Conversely, a differential inclusion is an example of an economy in which the prices are the velocities ($c(x,p) = p$ & $P(x) = F(x)$.)

were presented. Here, we proved that under adequate stochastic tangential conditions, from any closed random variable $\mathcal{K}$ starts a stochastic process which is viable (remains) in $\mathcal{K}$.

1.1 Dynamical Allocation of Resources

We now address the problem of finding viable allocations, i.e., solutions to the controlled system

$$\forall\, t \geq 0, \quad \begin{cases} i) & x_i'(t) \;=\; c_i(x_i(t), p(t)) \quad (i = 1, \ldots, n) \\ ii) & p(t) \;\in\; P(x(t)) \end{cases} \tag{1.1}$$

satisfying

$$\forall\, t \geq 0, \quad \begin{cases} i) & x_i(t) \;\in\; L_i \quad (i = 1, \ldots, n) \\ ii) & \sum_{i=1}^n x_i(t) \;\in\; M \end{cases}$$

Recall that the **regulation map** Π_M is defined from the set of scarce resources M and the behavior of the consumers by

$$\forall\, x \in K,\ \Pi_M(x) \;:=\; \left\{ p \in P(x) \mid \sum_{i=1}^n c_i(x_i, p) \;\in\; T_M(\sum_{i=1}^n x_i) \right\}$$

The Viability Theorem 1.8 that we shall state and prove next implies that whenever this regulation map is strict, the **the allocation set is viable under this system**. This means that from any initial allocation $x_0 = (x_{01}, \ldots, x_{0n})$ starts at least one viable allocation:

We recall then for the convenience of the reader:

$$\begin{cases} i) & M = M - \mathbf{R}^l_+ \text{ is a closed convex subset} \\ ii) & \forall\, i = 1, \ldots, n,\ \ L_i = L_i + \mathbf{R}^l_+ \text{ is closed and convex} \\ iii) & 0 \;\in\; \mathrm{Int}\left(\sum_{i=1}^n L_i - M \right) \\ iv) & M \subset \underline{y} - \mathbf{R}^l_+ \ \&\ \forall\, i = 1, \ldots, n,\ \ L_i \subset \underline{x_i} + \mathbf{R}^l_+, \end{cases} \tag{1.2}$$

$$\begin{cases} i) & \mathrm{Graph}(P) \text{ is closed and the images of } P \text{ are convex} \\ ii) & \forall\, x \in K,\ \ N_M\left(\sum_{i=1}^n x_i \right) \cap S^l \;\subset\; P(x) \end{cases} \tag{1.3}$$

and

$$\begin{cases} i) & c_i(x,p) := c_i(x) + G_i(x)p \text{ is affine, where} \\ ii) & c_i : L_i \mapsto Y \text{ is continuous} \\ iii) & G_i : L_i \mapsto \mathcal{L}(Y^\star, Y) \text{ is continuous} \\ iv) & \forall\, x_i \in L_i,\ p \in \operatorname{Im}(P),\ c_i(x_i,p) \in T_{L_i}(x_i) \end{cases} \quad (1.4)$$

As a corollary (in the case when , we obtain the "dynamical version" of the Arrow-Debreu Theorem.

Theorem 1.1 *We posit assumptions (1.2), (1.3) and (1.4) of the Equilibrium Theorem. If the change functions c_i obey the collective instantaneous Walras law*

$$\forall\, p \in S^l,\ \sum_{i=1}^{n} \langle p, c_i(x_i,p)\rangle \leq 0$$

then, from any initial allocation $x_0 \in K$ starts at least one allocation evolving according

$$x_i'(t) = c_i(x_i(t), p(t)) \quad (i = 1, \ldots, n)$$

Recall that under these conditions, there exists at least a viable equilibrium $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ by the Equilibrium Theorem.

As it was the case with the existence of the equilibrium, the instantaneous Walras law guarantees that the images of the regulation map Π_M are not empty, without the knowledge of the set M of resources and without the knowledge of the behavior of other consumers (in the case of the individual instantaneous Walras law.) Collective instantaneous Walras law allows balnaced "monetary transactions" at each instant.

But the existence of viable allocations from every initial allocation holds true under the assumption that the regulation map has nonempty values:

Theorem 1.2 *We posit assumptions (1.2), (1.4) and*

$$\operatorname{Graph}(P) \text{ is closed and the images of } P \text{ are convex} \quad (1.5)$$

Then the three following conditions are equivalent:

$$\begin{cases} a) & \forall\, x \in K,\ \Pi_M(x) \neq \emptyset \\ b) & \sup_{q \in N_M(\sum_{i=1}^n x_i)} \inf_{p \in P(x)} \langle q, \sum_{i=1}^n c_i(x_i,p)\rangle \leq 0 \\ c) & \forall\, x_0 \in K, \text{ starts one allocation evolving according (1.1)} \end{cases}$$

In this case, the viable allocations are governed by the regulation law

$$\text{for almost all } t \geq 0, \quad p(t) \in \Pi_M(x(t)) \tag{1.6}$$

We recall again that under one of these equivalent conditions, there exists at least a viable equilibrium $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ by the Equilibrium Theorem.

Remark — Naturally, we can extend this basic result in many directions and relax some of the assumptions.

For instance, if we are not interested in the existence of an equilibrium, we can dispense of the convexity assumptions. In this case, we replace the tangent cone to a convex subset by the contingent cone and assume instead that

$$\begin{cases} i) & M = M - \mathbf{R}^l_+ \text{ is closed and sleek} \\ ii) & \forall\, i = 1, \ldots, n, \;\; L_i = L_i + \mathbf{R}^l_+ \text{ is closed and sleek} \\ iii) & \forall\, x \in K, \; \sum_{i=1}^n T_{L_i}(x_i) - T_M(\sum_{i=1}^n x_i) \; = \; Y \\ iv) & M \subset \underline{y} - \mathbf{R}^l_+ \;\; \& \;\; \forall\, i = 1, \ldots, n, \;\; L_i \subset \underline{x}_i + \mathbf{R}^l_+ \end{cases}$$

(See next Section) The first part of the theorem still holds true.

We observe also that condition (1.2)iv) is one among many which implies the compactness of K. Again, this compactness property is needed to obtain the existence of an equilibrium. For the first part of the theorem, we can relax it by assuming only that the functions $c_i : L_i \mapsto Y$ has linear growth and $G_i : L_i \mapsto \mathcal{L}(Y^\star, Y)$ is bounded. □

More generally, we can assume that the change functions c_i are replaced by set-valued change maps $C_i : L_i \times S^l \leadsto Y$.

Then the dynamics of the evolution of the consumption is described by the set-valued controlled system

$$\forall\, t \geq 0, \quad \begin{cases} i) & x_i'(t) \; \in \; C_i(x_i(t), p(t)) \quad (i = 1, \ldots, n) \\ & \\ ii) & p(t) \; \in \; P(x(t)) \end{cases} \tag{1.7}$$

We recall that the regulation map Π_M is defined by

$$\forall\, x \in K, \; \Pi_M(x) \; := \; \left\{ p \in P(x) \mid 0 \; \in \; T_M(\sum_{i=1}^n x_i) \; - \; \sum_{i=1}^n C_i(x_i, p) \right\}$$

Viability Theorem 1.8 implies

Theorem 1.3 *We posit assumptions (1.2), (1.5) and*

$$\begin{cases} i) & C_i(x,p) \text{ is a closed convex set-valued map with respect to } p \\ ii) & C_i \text{ is upper hemicontinuous with convex compact images} \\ iii) & \forall\, x_i \in L_i,\ p \in \mathrm{Im}(P),\ C_i(x_i,p) \subset T_{L_i}(x_i) \end{cases} \tag{1.8}$$

Then the three following conditions are equivalent:

$$\begin{cases} a) & \forall\, x \in K,\ \Pi_M(x) \neq \emptyset \\ b) & \sup_{q\in N_M(\sum_{i=1}^n x_i)} \inf_{p\in P(x)} \sigma^\flat_{C_i(x_i,p)}(q) \leq 0 \\ c) & \forall\, x_0 \in K, \text{ starts one allocation evolving according (1.1)} \end{cases}$$

In this case, the viable allocations are governed by the regulation law

$$\text{for almost all } t \geq 0, \quad p(t) \in \Pi_M(x(t)) \tag{1.9}$$

Recall that under one these equivalent conditions, there exists at least a viable equilibrium $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ by the Equilibrium Theorem.

1.2 The Viability Theorem

We now present the Viability Theorem in the general case. It can be regarded as a dynamical pendant of the general Equilibrium Theorem. Contrary to the Equilibrium Theorem, convexity of the viability domain K is no longer required, answering a long awaited demand of economists (but they have to forgo their demand for an equilibrium, a stationary solution. There is no such thing as a free lunch). This requires to adapt the definition of tangents to any subset.

But convexity of the images of the set-valued map F is imperative as we shall see in the example below.

1.2.1 Definition of Viability Domains

We consider initial value problems (or Cauchy problems) associated with the differential inclusion

$$\text{for almost all } t \in [0,T], \quad x'(t) \in F(x(t)) \tag{1.10}$$

satisfying the initial condition $x(0) = x_0$.

Definition 1.4 (Viability and Invariance Properties) *Let K be a subset of the domain of F. A function $x(\cdot) : I \mapsto X$ is said to be* viable in K on the interval I *if and only if*

$$\forall\, t \in I,\;\; x(t) \;\in\; K$$

We shall say that K is viable under F *if from any initial state x_0 in K starts* at least *a solution on $[0, \infty[$ to differential inclusion (1.10) which is* viable *in K.*

The subset K is said to be invariant under F *if starting from any initial state x_0 of K,* all *solutions to differential inclusion (1.10) are* viable in K on $[0, \infty[$.

Contrary to theorems on existence of an equilibrium, we do not need to assume anymore that the set K is convex. However, we need to implement the concept of tangency.

When K is a subset of X and x belongs to K, we recall that the contingent cone $T_K(x)$ to K at x is the closed cone of elements v

$$\begin{cases} v \;\in\; T_K(x) \;\text{ if and only if } \exists\, h_n \to 0+ \;\text{ and }\; \exists\, v_n \to v \\ \text{such that } \;\forall\, n,\;\; x + h_n v_n \in K \end{cases}$$

It is very convenient to use the following characterization of this contingent cone in terms of distances: the contingent cone $T_K(x)$ to K at x is the closed cone of elements v such that

$$\liminf_{h \to 0+} \frac{d(x + hv, K)}{h} \;=\; 0$$

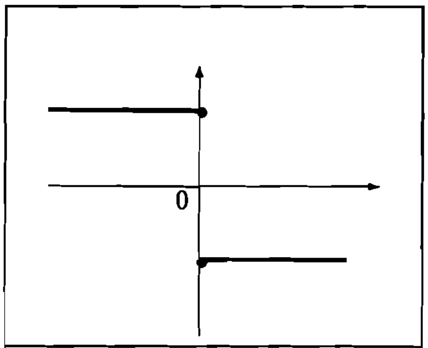
We also observe that

$$\text{if }\; x \in \operatorname{Int}(K),\; \text{ then }\; T_K(x) \;=\; X$$

Definition 1.5 (Viability Domain) *Let $F : X \leadsto X$ be a nontrivial set-valued map. We shall say that a subset $K \subset \operatorname{Dom}(F)$ is a* viability domain *of F if and only if*

$$\forall\, x \in K,\quad F(x) \cap T_K(x) \neq \emptyset$$

Figure 1: Example of a Map without Convex Values



There is no solution starting at 0

The simplest example of a differential inclusion we can think of does not solutions starting from some point:

Example

Let us consider $X := \mathbf{R}$, $K := [-1, +1]$ and the set-valued map $F : K \rightsquigarrow \mathbf{R}$ defined by

$$F(x) \;:=\; \begin{cases} -1 & \text{if } \; x > 0 \\ \{-1, +1\} & \text{if } \; x = 0 \\ +1 & \text{if } \; x < 0 \end{cases}$$

Obviously, no solution to the differential inclusion $x'(t) \in F(x(t))$ can start from 0, since 0 is not an equilibrium of this set-valued map!

We note however that

- The graph of F is closed
- F is bounded
- K is convex and compact
- K is a viability domain of F.

But the value $F(0)$ of F at 0 is not convex. Observe that if we had set $F(0) := [-1, +1]$, then 0 would have been an equilibrium.

This example shows that upper semicontinuity is not strong enough to compensate the lack of convexity. □

Therefore, we have to introduce the class of **Marchaud maps**:

1.2.2 Marchaud Maps

We set

$$\|F(x)\| := \sup_{y \in F(x)} \|y\|$$

and we say that F has **linear growth** if there exists a positive constant c such that

$$\forall\, x \in \mathrm{Dom}(F), \ \ \|F(x)\| \ \leq \ c(\|x\| + 1)$$

Definition 1.6 (Marchaud Map) *We shall say that F is a* Marchaud map *if it is nontrivial, upper hemicontinuous, has compact convex images and linear growth.*

We deduce the following result:

Corollary 1.7 *If Y is a finite dimensional vector-space, to say that a nontrivial set-valued map F is a Marchaud map amounts to saying that*

$$\begin{cases} i) & \text{the graph and the domain of } F \text{ are closed} \\ ii) & \text{the values of } F \text{ are convex} \\ iii) & \text{the growth of } F \text{ is linear} \end{cases}$$

1.2.3 The Viability Theorem

Theorem 1.8 (Viability Theorem) *Consider a* Marchaud map $F : X \rightsquigarrow X$ *and a* closed *subset $K \subset \mathrm{Dom}(F)$ of a finite dimensional vector space X.*

If K is a viability domain, *then for any initial state $x_0 \in K$, there exists a* viable *solution on $[0, \infty[$ to differential inclusion (1.10.) More precisely, if we set*

$$c_K \ := \ \sup_{x \in K} \frac{\|F(x)\|}{\|x\| + 1}$$

then every solution $x(\cdot)$ starting at x_0 satisfies the estimates

$$\begin{cases} \forall\, t \geq 0, \ \ \|x(t)\| \ \leq \ (\|x_0\| + 1)e^{c_K t} \\ \text{and} \\ \text{for almost all } t \geq 0, \ \ \|x'(t)\| \ \leq \ c_K(\|x_0\| + 1)e^{c_K t} \end{cases}$$

1.2.4 Dynamical Economy

We have seen that our dynamical model of allocation of scarce resources could be written is a semi-abstract form, between the explicit description and its translation as a differential inclusion.

As it was advocated in the introduction of this chapter, it would be convenient to choose a middle ground inspired from systems theory, more informational than plain differential inclusions, but simpler to handle, and which also contain other economical models than the one of allocation of scarce resources.

We translate the viability theorems in the language of Economic Theory by introducing two finite dimensional vector-spaces:

1. — the (abstract) commodity space X

2. — the (abstract) price space Z and a *pricing set-valued map* $P : X \rightsquigarrow Z$ associating with any commodity x the (possibly empty) subset $P(x)$ of feasible prices associated with the commodity x. In other words, we assume that the available *prices of the system are required to obey constraints which may depend upon the commodity.* We shall investigate later the cases when the prices depend also upon the time and/or the history of the solution to the system.

The dynamics of the system are further described by a (single-valued) change map $c : \text{Graph}(P) \mapsto X$ which assigns to each commodity-price pair $(x, p) \in \text{Graph}(P)$ the *velocity* $c(x, p)$ with which the commodity evolves.

Hence the set

$$F(x) \ := \ \{c(x,p)\}_{p \in P(x)}$$

is the set of available velocities to the system when its commodity is x.

Definition 1.9 (Dynamical Economy) *A* dynamical economy (P, c) *is defined by*

— *a "pricing" set-valued map* $P : X \rightsquigarrow Z$

— *a map* $c : \text{Graph}(P) \mapsto X$ *describing the dynamics of the system.*

The evolution of the commodity and of the price is governed by the differential inclusion

$$\begin{cases} i) & \text{for almost all } t, \ x'(t) \ = \ c(x(t), p(t)) \\ ii) & \text{where } p(t) \ \in \ P(x(t)) \end{cases} \tag{1.11}$$

We associate with any subset $K \subset \text{Dom}(P)$ *the* regulation map Π_K :

$K \rightsquigarrow Z$ defined by

$$\forall x \in K, \;\; \Pi_K(x) \; := \; \{p \in P(x) \mid c(x,p) \in T_K(x)\}$$

We observe that K *is a viability domain if and only if the regulation map* Π_K *is strict (has nonempty values).*

It is convenient to introduce the following definition:

Definition 1.10 *We shall say that the dynamical economy (P,c) is a* Marchaud dynamical economy *if it satisfies the following conditions:*

$$\left\{\begin{array}{ll} i) & \text{Graph}(P) \;\; \textit{is closed} \\ ii) & c \;\; \textit{is continuous} \\ iii) & \textit{the velocity subsets } F(x) \;\; \textit{are convex} \\ iv) & c \;\; \textit{and} \;\; P \;\; \textit{have linear growth} \end{array}\right. \tag{1.12}$$

and that it is an affine dynamical economy *if furthermore*

$$\left\{\begin{array}{ll} i) & c \text{ is affine with respect to } p \\ ii) & \textit{the images of } P \textit{ are convex} \end{array}\right. \tag{1.13}$$

Hence Viability Theorem 1.8 can be restated in the following form:

Theorem 1.11 *Let us consider a Marchaud dynamical economy (P,c). Then a closed subset $K \subset \text{Dom}(P)$ is viable[9] under F if and only if it the regulation map $\Pi_K(\cdot)$ is strict.*

Furthermore, any price $p(\cdot)$ regulating a viable solution $x(\cdot)$ in the sense that

$$\text{for almost all } t, \;\; x'(t) = c(x(t), p(t))$$

obeys the regulation law

$$\text{for almost all } t, \;\; p(t) \in \Pi_K(x(t)) \tag{1.14}$$

Remark — The Filippov Measurable Selection Theorem[10] actually allows us to choose price functions obeying the regulation law (1.14) which are *measurable.* We shall also provide in Chapter 5 conditions implying the existence of continuous prices. □

[9] This means that for any initial commodity $x_0 \in K$, there exists a solution on $[0,\infty[$ to the dynamical economy (1.11) *viable in K*.

[10] See Theorem 8.2.10 of SET-VALUED ANALYSIS, [5, Aubin & Frankowska] for instance.

1.3 Proof of the Viability Theorem

We provide a (new[11]) proof for the sake of completeness.

It can be omitted by the non professional mathematician who is more interested to its applications. Like the Brouwer Fixed Point Theorem, the proof of the Viability Theorem is quite involved, and uses most of the theorems of functional analysis.

Since viable absolutely continuous functions $x(\cdot) : [0,T] \mapsto K$ satisfy $x'(t) \in T_K(x(t))$ for almost all $t \in [0,T]$, we could be tempted to derive viability theorems from existence theorems of solutions to differential inclusion $x'(t) \in R_K(x(t))$ where we set $R_K(x) := F(x) \cap T_K(x)$. Unfortunately, this is not possible because $T_K(\cdot)$ may be neither upper semicontinuous nor lower semicontinuous[12]. For instance, it is not upper semicontinuous as soon as inequality constraints are involved: take for example $K := [-1,+1]$. The graph of $T_K(\cdot)$, equal to

$$\{-1\} \times \mathbf{R}_+ \cup]-1,+1[\times \mathbf{R} \cup \{+1\} \times \mathbf{R}_-$$

is not closed, and not even locally compact.

So we have to devise a specific proof of Theorem 1.8.

Although the proof of the necessary condition is quite simple, we postponed it at the end because it is less important naturally than the sufficient condition.

As it is the case for proving many existence theorems of a solution to a problem, we proceed in three steps:

1. Construct approximate solutions

2. Prove that these approximate solutions converge to some limit

3. Check that this limit is a solution to the problem

We shall not use Euler's method to build approximate solutions, but use instead Zorn's Lemma (i.e., the axiom of choice) to prove the existence of approximate viable solutions in a given time interval.

Then Ascoli's and Alaoglu's Theorem will be used to make the approximate solutions and their derivatives converge. Unfortunately, contrary to differential equations, the convergence of the derivatives is not obtained

[11] due to Hélène Frankowska (personal communication).

[12] See Section 4.1., p. 178 of DIFFERENTIAL INCLUSIONS for an example of subset K such that $T_K(\cdot)$ is neither upper semicontinuous nor lower semicontinuous.

from the convergence of solutions. We have to use a priori estimates to infer that the derivatives of the approximate solutions converge weakly. But this convergence is too weak to check easily that the limit of the approximate solutions is a solution. To answer this question, specific to differential inclusions, we use the Convergence Theorem, based on Mazur's Theorem, permitting to pass from weak convergence to strong convergence of "convex combinations of tails of the sequences", and from that, an almost everywhere convergence of the derivatives.

The proof of the Viability Theorem shows at least that Functional Analysis is useful !

1.3.1 Sufficient Conditions

Construction of Approximate Solutions We begin by proving that there exist approximate viable solutions to the differential inclusion.

Lemma 1.12 *Assume that $K \subset X$ is a viability domain of $F : X \rightsquigarrow X$. Then, for any $\varepsilon > 0$, the set $S_\varepsilon(x_0)$ of a continuous functions $x(\cdot) \in \mathcal{C}(0,1;X)$ satisfying*

$$\begin{cases} i) & x(0) = x_0 \\ ii) & \forall\, t \in [0,1],\ d(x(t),K) \ \leq\ \varepsilon \\ iii) & \forall\, t \in [0,1],\ d((x(t),x'(t)),\mathrm{Graph}(F) \ \leq\ \varepsilon \end{cases} \tag{1.15}$$

is not empty.

Proof — We denote by $\mathcal{A}_\varepsilon(x_0)$ the set of pairs $(T_x, x(\cdot))$ where $T_x \in [0,1]$ and $x(\cdot) \in \mathcal{C}(0,T_x;X)$ is a continuous functions satisfying

$$\begin{cases} i) & x(0) = x_0 \\ ii) & d(x(T_x),K) \ \leq\ \varepsilon T_x \\ iii) & \forall\, t \in [0,T_x],\ d(x(t),K) \ \leq\ \varepsilon \\ iv) & \forall\, t \in [0,T_x],\ d((x(t),x'(t)),\mathrm{Graph}(F) \ \leq\ \varepsilon \end{cases}$$

The set $\mathcal{A}_\varepsilon(x)$ is not empty: take $T_x = 0$ and $x(0) \equiv x_0$.

It is an inductive set for the order relation

$$(T_{x_1}, x_1(\cdot)) \ \preceq\ (T_{x_2}, x_2(\cdot))$$

if and only if

$$T_{x_1} \ \leq\ T_{x_2} \ \ \&\ \ x_2(\cdot)|_{[0,T_{x_1}]} \ =\ x_1(\cdot)$$

Zorn's Lemma implies that there exists a maximal element $(T_x, x(\cdot)) \in \mathcal{A}_\varepsilon(x_0)$. The Lemma follows from the claim that for such a maximal element, we have $T_x = 1$.

If not, we shall extend $x(\cdot)$ by a solution $\widehat{x}(\cdot)$ on an interval $[T_x, S_x]$ where $S_x > T_x$, contradicting the maximal character of $(T_x, x(\cdot))$.

Let us take $\widehat{x} \in K$ achieving the distance between $x(T_x)$ and K:

$$d(x(T_x), \widehat{x}) \;=\; d(x(T_x), K)$$

We then choose a direction $\widehat{u} \in F(\widehat{x}) \cap T_K(\widehat{x})$, which exists by assumption.

We set

$$\forall\, t \in [T_x, 1],\;\; \widehat{x}(t) \;:=\; x(T_x) + ((t - T_x)\widehat{u}$$

and

$$\alpha \;:=\; \min\left(\varepsilon, \frac{\varepsilon(1 - T_x)}{\|\widehat{u}\|}\right)$$

By the definition of the contingent cone, there exists $h_x \in]0, \alpha]$ such that

$$d(\widehat{x} + h_x\widehat{u}, K) \;\leq\; \varepsilon h_x$$

We then set $S_x := T_x + h_x > T_x$.

We obtain

$$\begin{cases} d(\widehat{x}(S_x), K) \;=\; d(x(T_x) + h_x\widehat{u}, K) \\ \leq\; d(\widehat{x} + h_x\widehat{u}, K) + d(x(T_x), \widehat{x}) \\ \leq\; \varepsilon(S_x - T_x) + \varepsilon T_x \;=\; \varepsilon S_x \end{cases}$$

We observe that for any $t \in [T_x, S_x]$,

$$\begin{cases} d(\widehat{x}(t), K) \;\leq\; d(\widehat{x}(t), \widehat{x}) \;\leq\; d(\widehat{x}(t), x(T_x)) + d(x(T_x), \widehat{x}) \\ \leq\; (t - T_x)\|\widehat{u}\| + \varepsilon T_x \;\leq\; \alpha\|\widehat{u}\| + \varepsilon T_x \;\leq\; \varepsilon \end{cases}$$

from the very choice of α.

Finally, we note that for any $t \in [T_x, S_x[$, $\widehat{x}'(t) = \widehat{u}$. Therefore, for all $t \in [T_x, S_x]$,

$$d((\widehat{x}(t), \widehat{x}'(t)), \mathrm{Graph}(F) \;\leq\; d((\widehat{x}(t), \widehat{x}'(t)), (\widehat{x}, \widehat{u})) \;\leq\; d(\widehat{x}(t), \widehat{x}) \;\leq\; \varepsilon$$

Therefore, we have extended the maximal solution $(T_x, x(\cdot))$ on the interval $[0, S_x]$ and obtained the desired contradiction. □

Convergence of Approximate Solutions Consider now a sequence of ε- approximate solutions $x_\varepsilon(\cdot)$, which exist thanks to Lemma 1.19.

They satisfy the following a priori estimates:

$$\|x_\varepsilon(t)\| \leq \left(\|x_0\| + 1 + \varepsilon\frac{c+1}{c}\right)e^{ct} \ \& \ \|x'_\varepsilon(t)\| \leq c\left(\|x_0\| + 1 + \varepsilon\frac{c+1}{c}\right)e^{ct} \tag{1.16}$$

Indeed, the function $t \to \|x_\varepsilon(t)\|$ being locally Lipschitz, it is almost everywhere differentiable. Therefore, for any t where $x_\varepsilon(t)$ is different from 0 and differentiable, we have

$$\frac{d}{dt}\|x_\varepsilon(t)\| \;=\; \left\langle \frac{x_\varepsilon(t)}{\|x_\varepsilon(t)\|}, x'_\varepsilon(t)\right\rangle \;\leq\; \|x'_\varepsilon(t)\|$$

Since there exist elements $u_t \in \varepsilon B_X$ and $v_t \in \varepsilon B_X$ such that

$$x'_\varepsilon(t) \;\in\; F(x_\varepsilon(t) + u_t) + v_t$$

we obtain

$$\|x'_\varepsilon(t)\| \;\leq\; c(\|x_\varepsilon(t)\| + 1 + \varepsilon) + \varepsilon$$

Setting $\varphi(t) := \|x_\varepsilon(t)\| + 1 + \varepsilon\frac{c+1}{c}$, we infer that $\varphi'(t) \leq c\varphi(t)$, and thus

$$\varphi(t) \;\leq\; \varphi(0)e^{ct}$$

from which we deduce the estimates (1.16).

Estimates (1.16) imply that for all $t \in [0,T]$, the sequence $x_\varepsilon(t)$ remains in a bounded set and that the sequence $x_\varepsilon(\cdot)$ is equicontinuous, because the derivatives $x'_\varepsilon(\cdot)$ are bounded. We then deduce from Ascoli's Theorem that it remains in a compact subset of the Banach space $\mathcal{C}(0,1;X)$, and thus, that a subsequence (again denoted) $x_\varepsilon(\cdot)$ converges uniformly to some function $x(\cdot)$.

Furthermore, the sequence $x'_\varepsilon(\cdot)$ being bounded in the dual of the Banach space $L^1(0,1;X)$, which is equal to $L^\infty(0,1;X)$, it is weakly relatively compact thanks to Alaoglu's Theorem[13]. The Banach space $L^\infty(0,1;X)$ is contained in $L^1(0,1;X)$ with a stronger topology[14]. The identity map being

[13] Alaoglu's Theorem states that any bounded subset of the dual of a Banach space is weakly compact.

[14] Since the Lebesgue measure on $[0,1]$ is finite, we know that

$$L^\infty(0,1;X) \subset L^1(0,1;X)$$

continuous for the norm topologies, is still continuous for the weak topologies. Hence the sequence $x'_\varepsilon(\cdot)$ is weakly relatively compact in $L^1(0,1;X)$ and a subsequence (again denoted) $x'_\varepsilon(\cdot)$ converges weakly to some function $v(\cdot)$ belonging to $L^1(0,1;X)$. Equations

$$x_\varepsilon(t) - x_\varepsilon(s) \quad = \quad \int_s^t x'_\varepsilon(\tau)d\tau$$

imply that this limit $v(\cdot)$ is actually the weak derivative $x'(\cdot)$ of the limit $x(\cdot)$.

In summary, we have proved that

$$\begin{cases} i) & x_\varepsilon(\cdot) \text{ converges uniformly to } x(\cdot) \\ ii) & x'_\varepsilon(t) \text{ converges weakly to } \; x'(\cdot) \text{ in } L^1(0,T;X) \end{cases}$$

The Limit is a Solution Condition (1.15)ii) implies that

$$\forall\, t \in [0,T], \;\; x(t) \, \in \, K$$

i.e., that $x(\cdot)$ is viable. The Convergence Theorem 1.13 below and properties (1.15)iii) imply that

$$\text{for almost all } \; t \in [0,T], \;\; x'(t) \, \in \, F(x(t))$$

i.e., that $x(\cdot)$ is a solution to differential inclusion (1.10). □

The Convergence Theorem Let $a(\cdot)$ be a measurable strictly positive real-valued function from an interval $I \subset \mathrm{R}$ to R_+. We denote by $L^1(I,Y;a)$ the space of classes of measurable functions from I to Y integrable for the measure $a(t)dt$.

with a stronger topology. The weak topology $\sigma(L^\infty(0,1;X),\, L^1(0,1;X))$ (weak-star topology) is stronger than the weakened topology $\sigma(L^1(0,1;X),\, L^\infty(0,1;X))$ since the canonical injection is continuous. Indeed, we observe that the seminorms of the weakened topology on $L^1(0,1;X)$, defined by finite sets of functions of $L^\infty(0,1;X)$, are seminorms for the weak-star topology on $L^\infty(0,1;X)$), since they are defined by finite sets of functions of $L^1(0,1;X)$.

Theorem 1.13 (Convergence Theorem) *Let F be a nontrivial set-valued map from X to Y. We assume that F* is upper hemicontinuous with closed convex images.

Let I be an interval of R *and let us consider* measurable *functions $x_m(\cdot)$ and $y_m(\cdot)$ from I to X and Y respectively, satisfying:*

for almost all $t \in I$ and for all neighborhood $\mathcal{U}$ of 0 in the product space $X \times Y$, there exists $M := M(t, \mathcal{U})$ such that

$$\forall\, m > M, \ (x_m(t), y_m(t)) \in \text{Graph}(F) + \mathcal{U} \tag{1.17}$$

If we assume that

$$\begin{cases} i) & x_m(\cdot) \text{ converges almost everywhere to a function } x(\cdot) \\ ii) & y_m(\cdot) \in L^1(I, Y; a) \text{ converges weakly in } L^1(I, Y; a) \\ & \text{to a function } y(\cdot) \in L^1(I, Y; a) \end{cases} \tag{1.18}$$

then

$$\text{for almost all } \ t \in I, \ \ y(t) \in F(x(t)) \tag{1.19}$$

Proof — Let us recall that in a Banach space ($L^1(I, Y; a)$, for instance), the closure (for the normed topology) of a set coincides with its weak closure (for the weakened topology[15]

$$\sigma(L^1(I, Y; a), L^\infty(I, Y^\star; a^{-1}))$$

We apply this result: for every m, the function $y(\cdot)$ belongs to the weak closure of the convex hull $\text{co}(\{y_p(\cdot)\}_{p \geq m}.)$ It coincides with the (strong) closure of $\text{co}(\{y_p(\cdot)\}_{p \geq m})$. Hence we can choose functions

$$v_m(\cdot) := \sum_{p=m}^{\infty} a_m^p y_p(\cdot) \in \text{co}(\{y_p(\cdot)\}_{p \geq m})$$

(where the coefficients a_m^p are positive or equal to 0 but for a finite number of them, and where $\sum_{p=m}^{\infty} a_m^p = 1$) which converge strongly to $y(\cdot)$ in $L^1(I, Y; a)$. This implies that the sequence $a(\cdot)v_m(\cdot)$ converges strongly to the function $a(\cdot)y(\cdot)$ in

[15] By definition of the weakened topology, the continuous linear functionals and the weakly continuous linear functionals coincide. Therefore, the closed half-spaces and weakly closed half-spaces are the same. The Hahn-Banach Separation Theorem, which holds true in Hausdorff locally convex topological vector spaces, states that closed convex subsets are the intersection of the closed half-spaces containing them. Since the weakened topology is locally convex, we then deduce that closed convex subsets and weakly closed convex subsets do coincide. This result is known as Mazur's theorem.

$L^1(I,Y)$, since the operator of multiplication by $a(\cdot)$ is continuous from $L^1(I,Y;a)$ to $L^1(I,Y)$.

Thus, there exists another subsequence (again denoted by) $v_m(\cdot)$ such that[16]

$$\text{for almost all } t \in I, \quad a(t)v_m(t) \text{ converges to } a(t)y(t)$$

Since the function $a(\cdot)$ is strictly positive, we deduce that

$$\text{for almost all } t \in I, \quad v_m(t) \text{ converges to } y(t)$$

— Let $t \in I$ such that $x_m(t)$ converges to $x(t)$ in X and $v_m(t)$ converges to $y(t)$ in Y. Let $p \in Y^\star$ be such that $\sigma(F(x(t)),p) < +\infty$ and let us choose $\lambda > \sigma(F(x(t)),p)$. Since F is upper hemicontinuous, there exists a neighborhood $\mathcal{V}$ of 0 in X such that

$$\forall\, u \in x(t) + \mathcal{V}, \quad \text{then } \sigma(F(u),p) \leq \lambda \tag{1.20}$$

Let N_1 be an integer such that

$$\forall\, q \geq N_1, \quad x_q \in x(t) + \frac{1}{2}\mathcal{V}$$

Let $\eta > 0$ be given. Assumption (1.17) of the theorem implies the existence of N_2 and of elements (u_q, v_q) of the graph of F such that

$$\forall\, q \geq N_2, \quad u_q \in x_q(t) + \frac{1}{2}\mathcal{V}, \quad \|y_q(t) - v_q\| \leq \eta$$

Therefore u_q belongs to $x(t) + \mathcal{V}$ and we deduce from (1.20) that

$$\begin{cases} <p, y_q(t)> & \leq \; <p, v_q> + \eta\|p\|_\star \\ & \leq \; \sigma(F(u_q),p) + \eta\|p\|_\star \\ & \leq \; \lambda + \eta\|p\|_\star \end{cases}$$

[16]Strong convergence of a sequence in Lebesgue spaces L^p implies that some subsequence converges almost everywhere. Let us consider indeed a sequence of functions f_n converging strongly to a function f in L^p. We can associate with it a subsequence f_{n_k} satisfying

$$\|f_{n_k} - f\|_{L^p} \leq 2^{-k}; \quad \cdots < n_k < n_{k+1} < \cdots$$

Therefore, the series of integrals

$$\sum_{k=1}^{\infty} \int \|f_{n_k}(t) - f(t)\|_Y^p dt < +\infty$$

is convergent. The Monotone Convergence Theorem implies that the series

$$\sum_{k=1}^{\infty} \|f_{n_k}(t) - f(t)\|_Y^p$$

converges almost everywhere. For every t where this series converges, we infer that the general term converges to 0.

Let us fix $N \geq \max(N_1, N_2)$, multiply the above inequalities by the nonnegative a_m^q and add them up from $q = 1$ to ∞. We obtain :

$$< p, v_m(t) > \leq \lambda + \eta \|p\|_\star$$

By letting m go to infinity, it follows that

$$< p, y(t) > \leq \lambda + \eta \|p\|_\star$$

Letting now λ converge to $\sigma(F(x(t)), p)$ and η to 0, we obtain:

$$< p, y(t) > \leq \sigma(F(x(t)), p)$$

Since this inequality is automatically satisfied for those p such that

$$\sigma(F(x(t)), p) \; = \; +\infty$$

it thus holds true for every $p \in Y^\star$. Hence, the images $F(x)$ being closed and convex, the Separation Theorem implies that $y(t)$ belongs to $F(x(t))$. The Convergence Theorem ensues. □

1.3.2 Necessary Condition

Actually, the tangential condition is necessary for K to be viable under F.

Proposition 1.14 (Necessary Condition) *Let us assume that*

$$\begin{cases} i) & F : X \rightsquigarrow X \text{ is upper hemicontinuous} \\ ii) & \text{the images of } F \text{ are convex and compact} \end{cases}$$

If K is viable under F, then it is a viability domain.

Proof — Let $x(\cdot)$ be a solution to the differential inclusion starting at x_0. Actually, it is enough to assume that there exists a sequence $t_n \to 0+$ such that $x(t_n) \in K$.

Since F is upper hemicontinuous at x_0, we can associate with any $p \in X^\star$ and $\varepsilon > 0$ an $\eta_p > 0$ such that

$$\forall \, \tau \in [0, \eta_p], \;\; < p, x'(\tau) > \; \leq \; \sigma(F(x(\tau)), p) \; \leq \; \sigma(F(x_0), p) + \varepsilon \|p\|_\star$$

By integrating this inequality from 0 to t_n, setting $v_n := \dfrac{x(t_n) - x_0}{t_n}$ and dividing by $t_n > 0$, we obtain for n larger than some N_p

$$\forall \, p \in X^\star, \;\; \forall \, n \geq N_p, \;\; < p, v_n > \; \leq \; \sigma(F(x_0), p) + \varepsilon \|p\|_\star$$

Therefore, v_n lies in a bounded subset of a finite dimensional vector space, so that a subsequence (again denoted) v_n converges to some $v \in X$ satisfying

$$\forall\, p \in X^\star,\ \ < p, v > \leq\ \sigma(F(x_0), p) + \varepsilon \|p\|_\star$$

By letting ε converge to 0, we deduce that v belongs to the closed convex hull of $F(x_0)$.

On the other hand, since for any n, $x(t_n) = x_0 + t_n v_n$ belongs to K, we infer that v belongs to the contingent cone $T_K(x_0)$. The intersection $F(x_0) \cap T_K(x_0)$ is then nonempty, so that the necessary condition ensues. □

1.3.3 Upper Hemicontinuity of the Solution Map

We shall also need some continuity property of the **solution map** $\mathcal{S}(\cdot)$ or by $\mathcal{S}_F(\cdot)$ associating with any initial state x_0 the (possibly empty) set $\mathcal{S}(x_0)$ or $\mathcal{S}_F(x_0)$ of solutions to differential inclusion (1.10.)

Theorem 1.15 *Let us consider a* Marchaud map $F : X \rightsquigarrow X$. *The graph of the restriction of* $\mathcal{S}|_L$ *to any compact subset* L *of* Dom(F) *is compact in* $L \times \mathcal{C}(0,\infty;X)$, *where* $\mathcal{C}(0,\infty;X)$ *is the space of continuous functions supplied with the topology of uniform convergence on compact intervals.*

Therefore, the solution map $\mathcal{S}$ *is upper hemicontinuous with compact images.*

Proof — We shall show that the graph of the restriction $\mathcal{S}|_L$ of the solution map $\mathcal{S}$ to a compact subset $L \subset$ Dom(F) (assumed to be nontrivial) is compact.

Let us choose a sequence of elements $(x_{0_n}, x_n(\cdot))$ of the graph of the solution map $\mathcal{S}$. They satisfy:

$$x_n'(t) \in F(x_n(t)) \ \ \& \ \ x_n(0) = x_{0_n} \in L$$

A subsequence (again denoted) x_{0_n} converges to some $x_0 \in L$ because L is compact.

Then inequalities

$$\text{for almost all } t \geq 0, \quad \frac{d}{dt}\|x_n(t)\| \leq \|x_n'(t)\| \leq c(\|x_n(t)\| + 1)$$

imply that

$$\forall\, n \geq 0,\ \ \|x_n(t)\|\ \leq\ (\|x_{0_n}\| + 1)e^{ct}\ \ \&\ \ \|x_n'(t)\|\ \leq\ c(\|x_{0_n}\| + 1)e^{ct}$$

Therefore, by Ascoli's Theorem, the sequence $x_n(\cdot)$ is relatively compact in the Fréchet space $\mathcal{C}(0,\infty;X)$ and by Alaoglu's Theorem, the sequence $x_n'(\cdot)e^{-ct}$ is weakly relatively compact in $L^\infty(0,\infty;X)$.

Let us take $b > c$. Since the multiplication by $e^{-(b-c)t}$ is continuous from $L^\infty(0,\infty;X)$ to $L^1(0,\infty;X)$, it remains continuous when these spaces are supplied with weak topologies[17].

We have proved that the sequence $x_n'(\cdot)$ is *weakly relatively compact* in the weighted space $L^1(0,\infty;X;e^{-bt}dt)$.

We thus deduce that a subsequence (again denoted) x_n converges to x in the sense that:

$$\begin{cases} i) & x_n(\cdot) \text{ converges uniformly to } x(\cdot) \text{ on compact sets} \\ ii) & x_n'(\cdot) \text{ converges weakly to } x'(\cdot) \text{ in } L^1(0,\infty;X;e^{-bt}dt) \end{cases}$$

Inclusions

$$\forall\, n > 0, \quad (x_n(t), x_n'(t)) \in \text{Graph}(F)$$

imply that

$$\text{for almost all } t > 0, \quad x'(t) \in F(x(t))$$

thanks to the Convergence Theorem 1.13.

We thus have proved that a subsequence of elements $(x_{0_n}, x_n(\cdot))$ of the graph of $\mathcal{S}|_L$ converges to an element $(x_0, x(\cdot))$ of this graph. This shows that it is compact, and thus, that the solution map $\mathcal{S}$ is upper hemicontinuous with compact images. □

1.4 Stochastic Viability Theorem

Let us consider a σ-complete probability space $(\Omega, \mathcal{F}, P)$, an increasing family of σ-sub- algebras $\mathcal{F}_t \subset \mathcal{F}$ and a finite dimensional vector-space $X := \mathbf{R}^n$.

We shall study a stochastic differential equation

$$d\xi \;=\; f(\xi(t))dt + g(\xi(t))dW(t) \tag{1.21}$$

the solution of which is given by the formula

$$\xi(t) \;=\; \xi(0) + \int_0^t f(\xi(s))ds + \int_0^t g(\xi(s))dW(s)$$

[17] If u_n converges weakly to u in $L^\infty(0,\infty;X)$, then $e^{-(b-c)t}u_n$ converges weakly to $e^{-(b-c)t}u$ in $L^1(0,\infty;X)$, because, for every $\varphi \in L^\infty(0,\infty;X) = L^1(0,\infty;X)^\star$, the values

$$< u_n, e^{-(b-c)t}\varphi > := \int_0^\infty e^{-(b-c)t}u_n(t)\varphi(t)dt$$

converge to

$$< u, e^{-(b-c)t}\varphi > := \int_0^\infty e^{-(b-c)t}u(t)\varphi(t)dt$$

since $e^{-(b-c)t}\varphi(\cdot)$ belongs to $L^1(0,\infty;X)$.

when one of the following conditions is satisfied:

- f and g are Lipschitz functions
- f and g are uniformly continuous and monotone

We say that a stochastic process $\xi(t)$ is a solution to the stochastic differential equation (1.21) if the functions f and g satisfy:

$$\text{for almost all } \omega \in \Omega, \quad f(\xi(\cdot)) \in L^1(0,T;X) \;\; \& \;\; g(\xi(\cdot)) \in L^2(0,T;X)$$

1.4.1 Stochastic Tangent Sets

The constraints are defined by closed subsets $K_\omega \subset X$, where the set-valued map

$$K : \omega \in \Omega \mapsto K_\omega \subset X$$

is assumed to be $\mathcal{F}_0$- measurable (which can be regarded as a random set-valued variable).

We denote by $\mathcal{K}$ the subset

$$\mathcal{K} \;:=\; \{u \in L^2(\Omega, \mathcal{F}, P) \mid \text{for almost all } \omega \in \Omega, \;\; u_\omega \in K_\omega\}$$

For simplicity, we restrict ourselves to scalar $\mathcal{F}_t$-Wiener processes $W(t)$.

Definition 1.16 (Stochastic Contingent Set) *Let us consider a $\mathcal{F}_t$-random variable $x \in K$ (i.e., a $\mathcal{F}_t$-measurable selection of K).*

Definition 1.17 *We shall say that a stochastic process $x(\cdot)$ is* viable in K *if and only if*

$$\forall\, t \in [0,T], \;\; x(t) \;\in\; \mathcal{K} \tag{1.22}$$

i.e., if and only if

$$\forall\, t \in [0,T], \;\; \text{for almost all } \omega \in \Omega, \;\; \xi_\omega(t) \;\in\; K_\omega$$

We shall say that K enjoys the (stochastic) viability property *with respect to the pair (f,g) if for any random variable x in K, there exists a solution ξ to the stochastic differential equation starting at x which is* viable in K.

In order to characterize this stochastic viability property, we define the **stochastic contingent set** $\mathcal{T}_K(t,x)$ *to* K *at* x *(with respect to* $\mathcal{F}_t$*) as the set of pairs* (γ, v) *of* $\mathcal{F}_t$*-random variables satisfying the following property: For any* α, $\rho > 0$, *there exist* $h \in]0,\alpha[$ *and* $\mathcal{F}_{t+h}$*-random variables* a^h *and* b^h *such that*

$$\begin{cases} i) & \mathbf{E}(\|a^h\|^2) \le \rho^2 \\ ii) & \mathbf{E}(\|b^h\|^2) \le \rho^2 \\ iii) & \mathbf{E}(b^h) = 0 \\ iv) & b^h \text{ is independent of } \mathcal{F}_t \end{cases} \tag{1.23}$$

and satisfying

$$x + v(W(t+h) - W(t)) + h\gamma + ha^h + \sqrt{h}b^h \in \mathcal{K} \tag{1.24}$$

For instance, this condition means that for every $\mathcal{F}_t$-random variable x viable in K

- $$f(x) \in K \ \& \ g(x) \in K$$

  when K is a vector subspace,

- $$\langle x, g(x)\rangle = 0 \ \& \ \langle x, f(x)\rangle + \frac{1}{2}\|g(x)\|^2 = 0$$

  when K is the unit sphere

- $$\langle x, g(x)\rangle = 0 \ \& \ \langle x, f(x)\rangle + \frac{1}{2}\|g(x)\|^2 \le 0$$

  when K is the unit ball.

We mention that an elementary calculus of stochastic tangent sets to direct images, inverse images and intersections of closed subsets can be found in [3, Aubin & Da Prato].

1.4.2 Stochastic Viability

Theorem 1.18 (Stochastic Viability) *Let K be a closed subset of X. We assume that either*

- *the maps f and g are Lipschitz*
- *the maps f and g are uniformly continuous and monotone in the sense that there exists $\nu \in \mathbf{R}$ such that*

$$\forall\, x, y \in X, \;\; 2\langle f(x) - f(y), x - y\rangle + \|g(x) - g(y)\|^2 \leq \nu\|x - y\|^2,$$

Then the following conditions are equivalent:

1. — From any initial stochastic process $\xi_0 \in \mathcal{K}$ starts a solution to the stochastic differential equation which is viable in $\mathcal{K}$.

2. — for every $\mathcal{F}_t$-random variable x in $\mathcal{K}$,

$$(f(x), g(x)) \;\in\; \mathcal{T}_{\mathcal{K}}(t, x) \tag{1.25}$$

Sufficient Condition We begin by constructing approximate viable solutions to the stochastic differential equation.

Lemma 1.19 *Let K be a closed subset of X. We assume that the maps f and g are uniformly continuous. Then, for any $\varepsilon > 0$, the set $\mathcal{S}_\varepsilon(\xi_0)$ of stochastic processes $\xi(\cdot)$ on $[0,1]$ satisfying $\xi(0) = \xi_0$ and*

$$\begin{cases} i) & \forall\, t \in [0,1],\ \mathbf{E}(d^2(\xi(t), \mathcal{K})) \leq \varepsilon^2 \\ ii) & \forall\, t \in [0,1],\ \mathbf{E}\left(\left\|\xi(t) - \xi(0) - \int_0^t f(\xi(s))ds - \int_0^t g(\xi(s))dW(s)\right\|^2\right) \leq \varepsilon^2 \end{cases} \tag{1.26}$$

is not empty.

Proof — Let us fix $\varepsilon > 0$. Since f and g are uniformly continuous with concave uniform continuity modulus[18] ω, we choose $\eta \in]0, \varepsilon]$ such that

$$\omega(\eta^2) \;\leq\; \frac{\varepsilon^2}{4}.$$

[18] Set $\omega(t) = \sup_{\|x-y\|^2 \leq t} \|f(x) - f(y)\|^2$. Then ω is a non decreasing, subadditive continuity modulus of f. One can check that the concave envelope of ω is still a uniform continuity modulus.

We denote by $\mathcal{A}_\varepsilon(\xi_0)$ the set of pairs $(T_\xi, \xi(\cdot))$ where $T_\xi \in [0,1]$ and $\xi(\cdot)$ is a stochastic process satisfying $\xi(0) = \xi_0$ and

$$\begin{cases} i) & \forall\, t \in [0, T_\xi],\ \mathbf{E}d^2(\xi(T_\xi), \mathcal{K}) \leq \eta^2 T_\xi \\ ii) & \forall\, t \in [0, T_\xi],\ \mathbf{E}d^2(\xi(t), \mathcal{K}) \leq \eta^2 \\ iii) & \forall\, t \in [0, T_\xi],\ \mathbf{E}\left(\left\| \xi(t) - \xi(0) - \int_0^t f(\xi(s))ds - \int_0^t g(\xi(s))dW(s) \right\|^2 \right) \leq \varepsilon^2 \end{cases} \tag{1.27}$$

The set $\mathcal{A}_\varepsilon(\xi)$ is not empty: take $T_\xi = 0$ and $\xi(0) \equiv \xi_0$.

It is an inductive set for the order relation

$$(T_{\xi_1}, \xi_1(\cdot)) \preceq (T_{\xi_2}, \xi_2(\cdot))$$

if and only if

$$T_{\xi_1} \leq T_{\xi_2} \ \&\ \xi_2(\cdot)|_{[0, T_{\xi_1}]} = \xi_1(\cdot)$$

Zorn's Lemma implies that there exists a maximal element $(T_\xi, \xi(\cdot)) \in \mathcal{A}_\varepsilon(\xi_0)$. The Lemma follows from the claim that for such a maximal element, we have $T_\xi = 1$.

If not, we shall extend $\xi(\cdot)$ by a stochastic process $\widehat{\xi}(\cdot)$ on an interval $[T_\xi, S_\xi]$ where $S_\xi > T_\xi$, contradicting the maximal character of $(T_\xi, \xi(\cdot))$.

Since K_ω and $\xi_\omega(T_\xi)$ are $\mathcal{F}_{T_\xi}$ measurable, the projection map $\Pi_{K_\omega}(\xi_\omega(T_\xi))$ is also $\mathcal{F}_{T_\xi}$-measurable (see [5, Theorem 8.2.13, p. 317]). Then there exists a $\mathcal{F}_{T_\xi}$-measurable selection $y_\omega \in \Pi_{K_\omega}(\xi_\omega(T_\xi))$, which we call a **projection of the random variable $\xi(T_\xi)$ onto the random set-valued variable $\mathcal{K}$**. For simplicity, we set $x = \xi(T_\xi)$ and thus choose a projection $y \in \Pi_{\mathcal{K}}(x)$.

We take

$$\rho := \frac{\eta\sqrt{1 - T_\xi}}{2} > 0$$

and we set

$$c^2 := \max(\mathbf{E}(\|f(y)\|^2), \mathbf{E}(\|g(y)\|^2)) < +\infty \tag{1.28}$$

We then introduce

$$\alpha := \min\left(\eta, \frac{(1 - T_\xi)\eta^2}{\eta^2 + 4c^2}\right) > 0$$

which is positive whenever $T_\xi < 1$.

We know that $(f(y), g(y))$ belongs to the stochastic contingent set $\mathcal{T}_{\mathcal{K}}(T_x, y)$: There exist $h_x \in]0, \alpha]$ and $\mathcal{F}_{T_x + h_x}$-random variables a^{h_x} and b^{h_x} such that

$$\begin{cases} i) & \mathbf{E}(\|a^{h_x}\|^2) \leq \rho^2 \\ ii) & \mathbf{E}(\|b^{h_x}\|^2) \leq \rho^2 \\ iii) & \mathbf{E}(b^{h_x}) = 0 \\ iv) & b^{h_x} \text{ is independent of } \mathcal{F}_t \end{cases} \tag{1.29}$$

and satisfying

$$y + g(y)(W(T_x + h_x) - W(T_x)) + h_x f(y) + h_x a^h + \sqrt{h_x} b^{h_x} \in \mathcal{K} \tag{1.30}$$

We then set $S_x := T_x + h_x > T_x$ and we define the stochastic process $\widehat{\xi}(t)$ on the interval $[T_x, S_x]$ by

$$\widehat{\xi}(t) := x + (t - T_x)f(y) + (W(t) - W(T_x))g(y)$$

Therefore, setting $h := t - T_x$,

$$\begin{cases} d_{\mathcal{K}}^2(\widehat{\xi}(t)) - d_{\mathcal{K}}^2(\widehat{\xi}(T_x)) \leq \left\| x - y - ha^h - \sqrt{h} b^h \right\|^2 - \|x - y\|^2 =: \\ \|ha^h + \sqrt{h} b^h\|^2 - 2\left\langle x - y, ha^h \right\rangle - 2\left\langle x - y, \sqrt{h} b^h \right\rangle \end{cases}$$

We take the expectation in both sides of this inequality and estimate each term of the right hand-side. First, we use estimate

$$\mathbf{E}(\|ha^h + \sqrt{h} b^h\|^2) \leq 2h(h\mathbf{E}(\|a^h\|^2) + \mathbf{E}(\|b^h\|^2))$$

because

$$\mathbf{E}\left(\left\| \int_0^t \varphi(s) ds \right\|^2\right) \leq t \int_0^t \mathbf{E}(\|\varphi(s)\|^2) ds$$

and

$$\mathbf{E}\left(\left\| \int_0^t \varphi(s) dW(s) \right\|^2\right) = \int_0^t \mathbf{E}(\|\varphi(s)\|^2) ds$$

Next,

$$\mathbf{E}\left(\left\langle x - y, a^h \right\rangle\right) \leq \mathbf{E}\left(\|x - y\|^2\right)^{\frac{1}{2}} \left(\mathbf{E}\left(\left\|a^h\right\|^2\right)\right)^{\frac{1}{2}}$$

and we observe that

$$\mathbf{E}\left\langle x - y, \frac{1}{\sqrt{h}} b^h \right\rangle = 0$$

since b^h is independent of $x - y$ and $\mathbf{E}(b^h) = 0$.

We obtain, by the very choice of ρ,

$$\left\{\begin{array}{l} \mathbf{E}(\boldsymbol{d}^2(\widehat{\xi}(S_x),\mathcal{K})) \;=\; \mathbf{E}(\boldsymbol{d}^2(\widehat{\xi}(T_x+h_x),\mathcal{K})) \\ \\ \leq\; \mathbf{E}\boldsymbol{d}^2(\widehat{\xi}(T_x),\mathcal{K}) + 2h_x\mathbf{E}\left(\|x-y\|^2\right)^{\frac{1}{2}}\left(\mathbf{E}\left(\left\|a^{h_x}\right\|^2\right)\right)^{\frac{1}{2}} \\ \\ +2h_x(h_x\mathbf{E}(\|a^{h_x}\|^2) + \mathbf{E}(\|b^{h_x}\|^2)) \\ \\ \leq\; \mathbf{E}\boldsymbol{d}^2(\widehat{\xi}(T_x),\mathcal{K}) + h_x\left(\mathbf{E}\left(\|x-y\|^2\right) + 3\mathbf{E}(\|a^{h_x}\|^2) + \mathbf{E}(\|b^{h_x}\|^2)\right)\right) \\ \\ \leq\; \eta^2T_x + h_x(\eta^2T_x + 4\rho^2) \;\leq\; \eta^2T_x + h_x\eta^2 \;=\; \eta^2S_x \end{array}\right.$$

by (1.27)i).

Hence $\widehat{\xi}(\cdot)$ satisfies (1.27)i) for S_x.

We observe also that for any $t \in [T_x, S_x]$,

$$\boldsymbol{d}_{\mathcal{K}}^2(\widehat{\xi}(t)) \;\leq\; \|\widehat{\xi}(t) - y\|^2$$

and that

$$\left\{\begin{array}{l} \|\widehat{\xi}(t)-y\|^2 \;=\; \|x-y+(t-T_x)f(y)+(W(t)-W(T_x))g(y)\|^2 \\ \\ =\; \boldsymbol{d}_{\mathcal{K}}^2(x) + 2\langle x-y,(t-T_x)f(y)+(W(t)-W(T_x))g(y)\rangle \\ \\ +\|(t-T_x)f(y)+(W(t)-W(T_x))g(y)\|^2 \end{array}\right.$$

By taking the expectations, we obtain

$$\left\{\begin{array}{l} \mathbf{E}(\|\widehat{\xi}(t)-y\|^2) - \mathbf{E}(\boldsymbol{d}_{\mathcal{K}}^2(\widehat{\xi}(T_x))) \\ \\ \leq\; (t-T_x)(\mathbf{E}(\boldsymbol{d}_{\mathcal{K}}^2(\widehat{\xi}(T_x)) + (1+2(t-T_x))\mathbf{E}(\|f(y)\|^2) + \mathbf{E}(\|g(y)\|^2)) \end{array}\right.$$

Therefore, since $\max(\mathbf{E}(\|f(y)\|^2), \mathbf{E}(\|g(y)\|^2)) \;=\; c^2$ by (1.28), we deduce that

$$\mathbf{E}(\|\widehat{\xi}(t)-y\|^2) \;\leq\; \eta^2T_x + (t-T_x)(\eta^2T_x + 4c^2) \;\leq\; \eta^2T_x + \alpha(\eta^2+4c^2) \;\leq\; \eta^2 \tag{1.31}$$

since, by the choice of α, we have $\alpha(\eta^2+4c^2) \leq (1-T_x)\eta^2$. Therefore,

$$\mathbf{E}(\boldsymbol{d}_{\mathcal{K}}^2(\widehat{\xi}(t))) \;\leq\; \mathbf{E}(\|\widehat{\xi}(t)-y\|^2) \;\leq\; \eta^2$$

Hence $\widehat{\xi}(\cdot)$ satisfies (1.27)ii) for S_x.

We also observe that

$$\begin{cases} \mathbf{E}\left(\left\|\widehat{\xi}(t) - x - \int_{T_x}^t f(\widehat{\xi}(s))ds - \int_0^t g(\widehat{\xi}(s))dW(s)\right\|^2\right) \\ = \mathbf{E}\left(\left\|\int_{T_x}^t (f(y) - f(\widehat{\xi}(s)))ds + \int_{T_x}^t (g(y) - g(\widehat{\xi}(s)))ds\right\|^2\right) \\ \leq 2\left(\mathbf{E}\left(\int_{T_x}^t \left\|f(y) - f(\widehat{\xi}(s))\right\|^2 ds\right) + \mathbf{E}\left(\int_{T_x}^t \left\|g(y) - g(\widehat{\xi}(s))\right\|^2 ds\right)\right) \end{cases}$$

Since the functions f and g are uniformly continuous, we deduce from the concavity of the continuous modulus $\omega(\cdot)$ that

$$\begin{cases} \mathbf{E}\left(\left\|\widehat{\xi}(t) - x - \int_{T_x}^t f(\widehat{\xi}(s))ds - \int_{T_x}^t g(\widehat{\xi}(s))dW(s)\right\|^2\right) \\ \leq 2\left(\mathbf{E}\left(\int_{T_x}^t \omega\left(\left\|y - \widehat{\xi}(s)\right\|^2\right) ds\right) + \mathbf{E}\left(\int_{T_x}^t \omega\left(\left\|y - \widehat{\xi}(s)\right\|^2\right) ds\right)\right) \\ \leq 4\mathbf{E}\left(\int_{T_x}^t \omega\left(\left\|y - \widehat{\xi}(s)\right\|^2\right) ds\right) \leq 4\omega\left(\int_{T_x}^t \mathbf{E}\left(\|y - \widehat{\xi}(s)\|^2\right)\right) \\ \leq 4\omega(\eta^2) \leq \varepsilon^2. \end{cases}$$

since we have already proved that

$$\mathbf{E}(\|\widehat{\xi}(t) - y\|^2) \leq \eta^2$$

so that $\widehat{\xi}(\cdot)$ satisfies (1.27)iii). Therefore, we have extended the maximal solution $(T_\xi, \xi(\cdot))$ on the interval $[0, S_x]$ and obtained the desired contradiction. Hence the proof of Lemma 1.19 is completed. □

It remains now to prove that the limit of the sequence of approximate solutions to a viable stochastic process exists and is a solution to the stochastic differential equation.

Let us choose for every ε an approximate solution ξ_ε which can be written in the form

$$\xi_\varepsilon(t) = \xi_0 + \int_0^t f(\xi_\varepsilon(s))ds + \int_0^t g(\xi_\varepsilon(s))dW(z) + \zeta_\varepsilon(t)$$

where $\sup_{t\in[0,1]} \mathbf{E}(\|\zeta_\varepsilon(t)\|^2) \leq \varepsilon^2$. Then for any $\varepsilon, \eta > 0$,

$$\mathbf{E}(\|\xi_\varepsilon(t) - \xi_\eta(t)\|^2)$$

$$\leq \mathbf{E}\left(\left\|\int_0^t (f(\xi_\varepsilon(s)) - f(\xi_\eta(s)))ds + \int_0^t (g(\xi_\varepsilon(s)) - g(\xi_\eta(s))_d W(z) + \zeta_\varepsilon(t) - \zeta_\eta(t)\right\|^2\right)$$

It follows that

Lipschitz case

$$\mathbf{E}\left(\|\xi_\varepsilon(t) - \xi_\eta(t)\|^2\right) \leq 4l^2 \left(\int_0^t \mathbf{E}\left(\|\xi_\varepsilon(s) - \xi_\eta(s)\|^2\right) ds\right) + 2(\varepsilon^2 + \eta^2)$$

Gronwall's Lemma implies that

$$\mathbf{E}\left(\|\xi_\varepsilon(t) - \xi_\eta(t)\|^2\right) \;\leq\; 2(\varepsilon^2 + \eta^2)e^{4l^2 t}$$

Monotone case We use Ito formula for the function $\|\xi_\varepsilon - \xi_\eta\|^2$ to obtain

$$\begin{cases} \mathbf{E}\left(\|\xi_\varepsilon(t) - \xi_\eta(t)\|^2\right) \\ \leq \mathbf{E}\left(\int_0^t \left(\langle \xi_\varepsilon(s) - \xi_\eta(s), f(\xi_\varepsilon(t)) - f(\xi_\eta(s))\rangle + \|g(\xi_\varepsilon(t)) - g(\xi_\eta(s))\|^2\right) ds\right) \\ +2(\varepsilon^2 + \eta^2) \\ \leq \; \nu^2 \int_0^t \mathbf{E}\left(\|\xi_\varepsilon(s) - \xi_\eta(s)\|^2\right) ds + 2(\varepsilon^2 + \eta^2) \end{cases}$$

Gronwall's Lemma implies that

$$\mathbf{E}\left(\|\xi_\varepsilon(t) - \xi_\eta(t)\|^2\right) \;\leq\; 2(\varepsilon^2 + \eta^2)e^{\nu^2 t}$$

In both cases we deduce that the above Cauchy sequences converge to some $\xi(\cdot)$:

$$\forall\, t \in [0,1], \;\; \lim_{\varepsilon \to 0} \mathbf{E}(\|\xi_\varepsilon(t) - \xi(t)\|^2) \;=\; 0$$

Furthermore, inequalities (1.26)ii) imply that

$$\mathbf{E}\left(d_{\mathcal{K}}^2(\xi(t))\right) \;=\; 0$$

so that the solution is viable in $\mathcal{K}$. □.

Necessary Condition Let K be a set-valued random variable.

Theorem 1.20 *If the random set-valued variable K is invariant by the pair (f,g), then for every $\mathcal{F}_t$-random variable x viable in K,*

$$(f(x),g(x)) \in \mathcal{T}_K(t,x) \tag{1.32}$$

Proof — We consider the viable stochastic process $\xi(t)$

$$\xi(h) = x + \int_0^h f(\xi(s))ds + \int_0^h g(\xi(s))dW(s) \tag{1.33}$$

which is a solution to the stochastic differential equation (1.21) starting at x.

We can write it in the form

$$\xi(t) = \xi(0) + hf(\xi(0)) + g(\xi(0))W(h) + \int_0^h a(s)ds + \int_0^h b(s)dW(s)$$

where

$$\begin{cases} a(s) &= f(\xi(s)) - f(\xi(0)) \\ b(s) &= g(\xi(s)) - g(\xi(0)) \end{cases}$$

converge to 0 with s.

We set

$$a^h := \frac{1}{h}\int_t^{t+h} a(s)ds$$

and

$$b^h := \frac{1}{\sqrt{h}}\int_t^{t+h} b(s)dW(s)$$

and we observe that

$$\begin{cases} \mathbf{E}\left(\left\|a^h\right\|^2\right) = \frac{1}{h^2}\mathbf{E}\left(\left\|\int_t^{t+h} a(s)ds\right\|^2\right) \\ \leq \frac{1}{h}\int_t^{t+h}\mathbf{E}\left(\|(a(s)\|^2)\right) ds \end{cases}$$

converges to 0 because $\mathbf{E}\left(\left\|\int_0^t \varphi(s)ds\right\|^2\right) \leq t\int_0^t \mathbf{E}(\|\varphi(s)\|^2)ds$.

In the same way,

$$\begin{cases} \mathbf{E}\left(\left\|b^h\right\|^2\right) = \frac{1}{h}\mathbf{E}\left(\left\|\int_t^{t+h} b(s)dW(s)\right\|^2\right) \\ = \frac{1}{h}\int_t^{t+h}\mathbf{E}\left(\|b(s)\|^2\right) ds \end{cases}$$

also converges to 0 because $\mathbf{E}\left(\left\|\int_0^t \varphi(s)dW(s)\right\|^2\right) = \int_0^t \mathbf{E}(\|\varphi(s)\|^2)ds$.

The expectation of b^h is obviously equal to 0 and b^h is independent of $\mathcal{F}_t$. Since $\xi(h)_\omega$ belongs to K_ω for almost all ω, we deduce that the pair $(f(x), g(x))$ belongs to $\mathcal{T}_K(t, x)$. □

2 Myopic Behavior

Introduction

Consider a dynamical economy (P, c) described by

$$\begin{cases} i) & x'(t) = c(x(t), p(t)) \\ ii) & p(t) \in P(x(t)) \end{cases}$$

We have associated with each viability domain K the **regulation map** $\Pi_K \subset P$ associating with every state $x \in K$ the set

$$\Pi_K(x) := \{p \in P(x) \mid c(x,p) \in T_K(x)\}$$

of **viable prices**. We did prove that under adequate assumptions on P and c that K is a viability domain if and only if the images $\Pi_K(x)$ of the regulation map are not empty for all $x \in K$. In this case, the price functions $p(\cdot)$ which regulate viable solutions obey the **regulation law**

$$\text{for almost all } t \geq 0, \ p(t) \in \Pi_K(x(t))$$

But how can one find prices $p(t) \in \Pi_K(x(t))$?

We then borrow to systems theory the concept of "feedback" prices[19], i.e., single-valued maps $\varpi(\cdot)$ which are **selections** of the regulation map in the sense that $\varpi(x) \in \Pi_K(x)$ for all $x \in K$.

They can be regarded as **planning mechanisms**, since they associate prices to commodities. Since the prices chosen this way depend only upon the commodities in a timeless manner, involving neither the past[20] nor the future (and this seems highly reasonable), one can say that such mechanisms describe a **myopic behavior** of the economic agents.

[19]The terminology comes from systems theory.

[20]that is a flaw of this model which ca be corrected by using hystory-dependent models later on.

Then the solutions to the differential equation

$$x'(t) = c(x(t), \varpi(x(t)))$$

(when they exist) are viable since the implemented prices $p(t) := \varpi(x(t))$ obey the regulation law by construction. Existence is guaranteed when $\varpi(\cdot)$ is continuous, but may still holds true for discontinuous, but explicit, feedback feedbacks.

Hence, we have to find **selection procedures** of the regulation map which provide either continuous selections or discontinuous selections for which the above differential equation still has solutions (section 1.)

We shall provide in next section a class of such feedback prices obtained as solutions of system of first-order partial differential inclusions in the case of evolution under bounded inflation, and even other type of feedbacks, called **dynamical feedbacks.**

In this Section, we concentrate on feedback prices obtained by selection procedures of the regulation map.

The first one, turning out continuous selections, is obtained in a nonconstructive way, and thus, has no economic relevance.

But, by assuming a myopic behavior of price-takers in some models, we can assume that she chooses prices in the regulation map with minimal norms, or, more generally, minimizing a given function on the subset $\Pi_K(x)$ or solving a game on such a subset. By doing that, we obtain discontinuous feedback prices. But we shall overcome this difficulty by observing that such selections can be obtained from the subset $\Pi_K(x)$ by "slicing out" the piece we desire with a **selection procedure** $S_{\Pi_K}(x)$, where the map $S_{\Pi_K}(\cdot)$ has "good" properties that the selection does not have[21]. This is the topic of the first section.

We shall see that some explicit selection procedures based on optimization and game theory require that the **regulation map should be lower semicontinuous with convex values.** Providing sufficient conditions for the regulation map to be lower semicontinuous is thus the second preliminary task.

Observe that this is not at all desperate, since we know that the set-valued map $x \rightsquigarrow T_K(x)$ which is involved in the definition of the regulation map is lower semicontinuous with convex values whenever K is **convex.** A nonconvex subset K satisfying this property will be called **sleek.** Then, if we add the assumption that pricing map $P(\cdot)$ is also **lower semicontinuous,** one

[21]The ideal of introducing and using these selection procedures is due to Hélène Frankowska.

can expect Π_K to be lower semicontinuous as well. This statement is true under further adequate assumptions (constraint qualification or transversality), as we show in the lower semicontinuity criteria that we prove in the second section.

Finally, we build feedback prices in the third section. Michael's Continuous Selection Theorem provides the existence of continuous feedback prices, but, being proved in a nonconstructive way, **does not furnish algorithmic ways to construct them.**

On the other hand, we can think of explicitly selecting some prices of the regulation map, for instance, the price $\varpi^\circ(x) \in \Pi_K(x)$ with minimal norm. Viable solutions obtained with this feedback price are called **slow viable solutions.** Unfortunately, lower semicontinuity of the regulation map is not sufficient for implying the continuity of this minimal norm feedback price. But since the minimal selection can be obtained by selection procedures, as well as other selection procedures involving optimization or game theoretical mechanisms, we can provide many instances of evolution obtained under a myopic behavior of the price-takers.

Section 4 provides examples when the set K is defined through explicit constraints. It recalls the calculus of contingent cones to **closed sleek** subsets and Section 5 studies the regulation maps in these more explicit cases.

The actual computation of slow solutions involving minimization of quadratic norms, we then devote the last section to the presentation of pseudo-inverses and quadratic minimization problems. These tools are recalled in Sectin 6.

2.1 Selections of the Regulation Map

A selection $S(F(\cdot))$ of a set-valued map F is a set-valued map contained in F. We consider the class of selections obtained through a selection procedure $S_F(\cdot)$, which allows to obtain the desired selection of F by "cut pieces" out of its images $F(x)$.

Definition 2.1 (Selection Procedure) *A* selection procedure of a set-valued map *$F : X \rightsquigarrow Y$ is a set-valued map $S_F : X \rightsquigarrow Y$ satisfying*

$$\begin{cases} i) & \forall x \in \mathrm{Dom}(F),\ \ S(F(x)) := S_F(x) \cap F(x) \neq \emptyset \\ ii) & \textit{the graph of } S_F \textit{ is closed} \end{cases}$$

The set-valued map $S(F) : x \rightsquigarrow S(F(x))$ is called the selection *of F.*

Then the selection is a single-valued map denoted $s(F(\cdot))$ whenever

$$\forall x \in \mathrm{Dom}(F), \quad s(F(x)) := S_F(x) \cap F(x) \text{ is a singleton}$$

We derive from the Equilibrium Theorem selection procedures yielding equilibria in $F(x)$.

Proposition 2.2 *Let us assume that a set-valued map $F : X \rightsquigarrow Y$ has nonempty compact convex values. Let us consider an upper semicontinuous set-valued map E with nonempty compact convex values from* Graph(F) *to Y satisfying:*

$$\forall (x,y) \in \mathrm{Graph}(F), \quad E(x,y) \cap T_{F(x)}(y) \neq \emptyset$$

Then the set-valued map S_F defined by

$$S_F(x) := \{y \in Y \mid 0 \in E(x,y)\}$$

is selection procedure of F. The selection map $S(F)$ associates with any $x \in$ Dom(F) *the set*

$$S(F)(x) := \{y \in F(x) \mid 0 \in E(x,y)\}$$

of equilibria of $E(x,\cdot)$ in $F(x)$.

We shall provide other examples of selection procedures of lower semicontinuous in the next section.

Theorem 2.3 *Consider a Marchaud dynamical economy (P,c) and suppose that K is a viability domain. Let S_{Π_K} be a selection of the regulation map Π_K. Suppose that the values of S_{Π_K} are convex. Then, for any initial commodity $x_0 \in K$, there exist a viable solution starting at x_0 and a viable price to dynamical economy (1.11) which are regulated by the selection $S(\Pi_K)$ of the regulation map Π_K, in the sense that*

$$\begin{cases} \text{for almost all } t \geq 0, \\ p(t) \in S(\Pi_K)(x(t)) := \Pi_K(x(t)) \cap S_{\Pi_K}(x(t)) \end{cases}$$

Proof — Since the convex selection procedure S_{Π_K} has a closed graph and convex values, we can replace the dynamical economy (1.11) by the dynamical economy

$$\begin{cases} i) & x'(t) = c(x(t), p(t)) \\ ii) & \text{for almost all } t, \ p(t) \in P(x(t)) \cap S_{\Pi_K}(x(t)) \end{cases} \tag{2.1}$$

which satisfies the assumptions of the Viability Theorem 1.11. It remains to check that K is still a viability domain for this smaller system. But by construction, we know that for all $x \in K$, there exists $p \in S(\Pi_K)(x)$, which belongs to the intersection $P(x) \cap S_{\Pi_K}(x)$ and which is such that $c(x,p)$ belongs to $T_K(x)$.

Hence the new dynamical economy (2.1) enjoys the viability property, so that, from all initial states $x_0 \in K$ starts one viable solution and a viable price to the dynamical economy (2.1) which, for almost all $t \geq 0$, are related by

$$\begin{cases} i) & p(t) \in P(x(t)) \cap S_{\Pi_K}(x(t)) \\ ii) & c(x(t), p(t)) \in T_K(x(t)) \end{cases}$$

Therefore, for almost all $t \geq 0$, $p(t)$ belongs to the intersection of $\Pi_K(x(t))$ and $S_{\Pi_K}(x(t))$, i.e., to the selection $S(\Pi_K)(x(t))$ of the regulation map Π_K.
□

When the selection is single-valued, we derive from Theorem 2.3 the existence of single-valued feedback prices:

Theorem 2.4 *Consider a Marchaud dynamical economy (P, c) and suppose that K is a viability domain. Let S_{Π_K} be a selection procedure of the regulation map Π_K. Suppose that the values of S_{Π_K} are convex and that the selection map*

$$s(\Pi_K(\cdot)) := S_{\Pi_K}(\cdot) \cap \Pi_K(\cdot) \text{ is single-valued}$$

Then the selection $s(\Pi_K)(\cdot)$ is a feedback price *regulating viable solutions of the dynamical economy (1.11).*

As a first example, we state:

Proposition 2.5 *Consider a Marchaud dynamical economy (P, c) and suppose that the images of the regulation map are nonempty, convex and compact. Let us consider an upper hemicontinuous set-valued map E with nonempty closed convex values from* Graph(P) *to Z satisfying:*

$$\forall (x,p) \in \text{Graph}(\Pi_K), \quad E(x,p) \cap T_{\Pi_K(x)}(p) \neq \emptyset$$

Then, for any initial commodity $x_0 \in K$, there exists a viable solution $x(\cdot)$ to the dynamical economy (1.11) regulated by an open loop price $p(\cdot)$ satisfying for almost all $t \geq 0$,

$$p(t) \in \Pi_K(x(t)) \ \& \ 0 \in E(x(t), p(t))$$

Proof — We apply Theorem 2.3 and Proposition 2.2. □

In order to provide other examples of selection procedures, and in particular, continuous selections or minimal selections, we have to assume (and thus, to check) that the regulation map Π_K is lower semicontinuous.

Next section is devoted to present lower semicontinuous maps, their selections and lower semicontinuity criteria which we need to provide more examples of myopic behavior.

2.2 Lower Semicontinuous Maps

2.2.1 Definitions and Example

To proceed further, we need the regulation map to be lower semicontinuous with convex compact values.

Definition 2.6 *A set-valued map $F : X \rightsquigarrow Y$ is called* lower semicontinuous *at $x \in$* Dom(F) *if and only if for any $y \in F(x)$ and for any sequence of elements $x_n \in$* Dom(F) *converging to x, there exists a sequence of elements $y_n \in F(x_n)$ converging to y.*

It is said to be lower semicontinuous *if it is lower semicontinuous at every point $x \in$* Dom(F).

We refer to SET-VALUED ANALYSIS, [5, Aubin & Frankowska] for more details, although we provide some of the proofs for the convenience of the reader.

Example: Parametrized Set-Valued Maps

Let us consider three finite dimensional vector-space X, Y and Z, a set-valued map

$$P : X \rightsquigarrow Z$$

and a single-valued map

$$c : \text{Graph}(P) \mapsto Y$$

We associate with these data the set-valued map $F : X \rightsquigarrow Y$ defined by

$$\forall\, x \in X, \quad F(x) := (c(x,p))_{p \in P(x)}$$

Proposition 2.7 *Assume that c is* continuous *from* Graph(P) *to Y. If P is lower semicontinuous, so is F.*

Proof — Let us consider a sequence $x_n \in \text{Dom}(F)$ converging to $x \in \text{Dom}(F)$ and take $y := c(x,p)$ belonging to $F(x)$, where $p \in P(x)$. Since P is lower semicontinuous, there exists a sequence $p_n \in P(x_n)$ converging to p. Then the sequence $q_n = c(x_n, p_n)$, where q_n belongs to $F(x_n)$, converges to y because c is continuous. Hence F is lower semicontinuous. □

Definition 2.8 (Marginal Functions) *Let us consider a set-valued map $F : X \rightsquigarrow Y$ and a function $c :$ Graph(F) $\mapsto \mathbf{R}$. We associate with them the* marginal function $g : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ *defined by*

$$g(x) := \sup_{y \in F(x)} c(x,y)$$

Theorem 2.9 (Maximum Theorem) *Let a set-valued map $F : X \rightsquigarrow Y$ and a function $c :$ Graph(F) $\mapsto \mathbf{R}$ be given. If c and F are* lower semicontinuous, *so is the marginal function.*

Proof — Let us consider a sequence x_n converging to x, fix $\lambda < g(x)$ and choose $y \in F(x)$ such that $\lambda \leq c(x,y)$. Then there exist elements $q_n \in F(x_n)$ converging to y (because F is lower semicontinuous) and we know that $c(x_n, q_n) \leq g(x_n)$. Since c is lower semicontinuous, we infer that

$$\lambda \;\leq\; c(x,y) \;\leq\; \liminf_{n\to\infty} c(x_n, q_n) \;\leq\; \liminf_{n\to\infty} g(x_n)$$

By letting λ converge to $g(x)$, the claim ensues. □

2.2.2 Selections of Lower Semicontinuous Maps

If the values of a set-valued map F are closed and convex, we can take, for instance, the **minimal selection** defined by

$$\begin{array}{ll} F^{\circ}(x) & := m(F(x)) \\ & := \{u \in F(x) \mid \|u\| = \min_{y \in F(x)} \|y\|\} \end{array} \qquad (2.2)$$

The upper semicontinuity of F, even when it is closed convex valued, is not strong enough to imply the continuity of the minimal selection[22].

However, we can still prove the following

Proposition 2.10 *Let us assume that $F : X \rightsquigarrow Y$ is closed and lower semicontinuous with convex values. Then the graph of the minimal selection is closed*[23].

Proof — The projection of 0 onto the closed convex set $F(x)$ is the element $u := m(F(x)) \in F(x)$ such that

$$\|u\|^2 + \sigma(-F(x), u) = \sup_{y \in F(x)} < u - 0, u - y > \leq 0 \qquad (2.3)$$

(It is actually equal to 0). Let us introduce the set-valued map $S_F : X \rightsquigarrow Y$ defined by

$$u \in S_F(x) \text{ if and only if } \|u\|^2 + \sigma(-F(x), u) \leq 0 \qquad (2.4)$$

Therefore, the graph of the minimal selection is equal to:

$$\mathrm{Graph}(m(F)) = \mathrm{Graph}(F) \cap \mathrm{Graph}(S_F)$$

Since F is lower semicontinuous, the function $(x, u) \mapsto \sigma(-F(x), u)$ is lower semicontinuous, so that the graph of S_F, and thus, of $m(F(\cdot))$, is closed. □

---

[22] Consider the set-valued map $F : \mathbf{R} \rightsquigarrow \mathbf{R}$ defined by

$$F(x) := \begin{cases} \{2\} & \text{if} \quad x \neq 0 \\ [1,2] & \text{if} \quad x = 0 \end{cases}$$

It is upper semicontinuous with compact convex values and its minimal selection is obviously not continuous.

[23] If moreover F is upper hemicontinuous, then the minimal selection is continuous. See Theorem 9.3.4 of SET-VALUED ANALYSIS, [5, Aubin & Frankowska].

The set-valued map defined by (2.3) is naturally a selection procedure of a set-valued map with closed convex values which provides the minimal selection. We could also have used the selection procedure S_F° defined by

$$S_F^\circ(x) \ := \ \{y \in Y \mid \|y\| \ \leq \ d(0, F(x))\}$$

We can easily provide other examples of selection procedures through optimization thanks to the Maximum Theorem.

Proposition 2.11 *Let us assume that a set-valued map $F : X \rightsquigarrow Y$ is lower semicontinuous with compact values. Let V : Graph$(F) \mapsto \mathbf{R}$ be continuous. Then the set-valued map S_F defined by:*

$$S_F(x) := \left\{ y \in Y \mid V(x,y) \leq \inf_{z \in F(x)} V(x,z) \right\}$$

is a selection procedure of F which yields the selection $S(F)$ equal to:

$$S(F(x)) = \left\{ y \in F(x) \mid V(x,y) \leq \inf_{z \in F(x)} V(x,z)) \right\}$$

Proof — Since F is lower semicontinuous, the function

$$(x,y) \mapsto V(x,y) + \sup_{z \in F(x)} (-V(x,z))$$

is lower semicontinuous thanks to the Maximum Theorem. Our proposition follows from :

$$\text{Graph}(S_F) = \left\{ (x,y) \mid V(x,y) + \sup_{z \in F(x)} (-V(x,z)) \leq 0 \right\} \quad \square$$

Most selection procedures through game theoretical models or equilibria are instances of this general selection procedure based on Ky Fan's Inequality (Theorem ??).

Proposition 2.12 *Let us assume that a set-valued map $F : X \rightsquigarrow Y$ is lower semicontinuous with convex compact values. Let $\varphi : X \times Y \times Y \mapsto \mathbf{R}$ satisfy*

$$\begin{cases} i) & \varphi(x,y,z) \text{ is lower semicontinuous} \\ ii) & \forall (x,y) \in X \times Y, \ \ z \mapsto \varphi(x,y,z) \text{ is concave} \\ iii) & \forall (x,y) \in X \times Y, \ \ \varphi(x,y,y) \leq 0 \end{cases}$$

Then the map S_F associated with φ by the relation

$$S_F(x) := \left\{ y \in Y \mid \sup_{z \in F(x)} \varphi(x, y, z) \le 0 \right\}$$

is a selection procedure of F yielding the selection map $x \mapsto S(F(x))$ defined by

$$S_F(x) := \left\{ y \in F(x) \mid \sup_{z \in F(x)} \varphi(x, y, z) \le 0 \right\}$$

Proof — Ky Fan's inequality states that the subsets $S_F(x)$ are not empty since the subsets $F(x)$ are convex and compact. The graph of S_F is closed thanks to the assumptions and the Maximum Theorem because it is equal to the lower section of a lower semicontinuous function:

$$\text{Graph}(S_F) = \left\{ (x, y) \mid \sup_{z \in F(x)} \varphi(x, y, z) \le 0 \right\} \quad \square$$

Proposition 2.13 *Assume that $Y = Y_1 \times Y_2$, that a set-valued map $F : X \rightsquigarrow Y$ is lower semicontinuous with convex compact values and that $a : X \times Y_1 \times Y_2 \to \mathbf{R}$ satisfies*

$$\begin{cases} i) & a \text{ is continuous} \\ ii) & \forall (x, y_2) \in X \times Y_2, \;\; y_1 \mapsto a(x, y_1, y_2) \text{ is convex} \\ iii) & \forall (x, y_1) \in X \times Y_1, \;\; y_2 \mapsto a(x, y_1, y_2) \text{ is concave} \end{cases}$$

Then the set-valued map S_F associating to any $x \in X$ the subset

$$\begin{array}{l} S_F(x) \;:=\; \{(y_1, y_2) \in Y_1 \times Y_2 \text{ such that} \\ \forall (z_1, z_2) \in F(x), \;\; a(x, y_1, z_2) \le a(x, z_1, y_2)\} \end{array}$$

is a selection procedure of F (with convex values). The selection map $S(F(\cdot))$ associates with any $x \in X$ the subset

$$\begin{array}{l} S(F)(x) \;:=\; \{(y_1, y_2) \in F(x) \text{ such that} \\ \forall (z_1, z_2) \in F(x), \;\; a(x, y_1, z_2) \le a(x, y_1, y_2) \le a(x, z_1, y_2)\} \end{array}$$

of saddle-points *of $a(x, \cdot, \cdot)$ in $F(x)$.*

Proof — We take

$$\varphi(x, (y_1, y_2), (y_1', y_2')) \;:= a(x, y_1, y_2') - a(x, y_1', y_2)$$

and we apply the above theorem. $\square$

2.2.3 Michael's Selection Theorem

We shall now state the celebrated Michael's theorem stating that lower semicontinuous convex-valued maps do have **continuous selections.**

Theorem 2.14 (Michael's Theorem) *Let F be a lower semicontinuous set-valued map with closed convex values from a compact metric space X to a Banach space Y. It does have a continuous selection.*

In particular, for every $\overline{y} \in F(\overline{x})$ there exists a continuous selection f of F such that $f(\overline{x}) = \overline{y}$.

We refer to Section 9.1 of SET-VALUED ANALYSIS for the proof of this Theorem. □

2.2.4 Sleek Subsets

The main example of lower semicontinuous set-valued maps is provided by the map $x \rightsquigarrow T_K(x)$ associating with any x belonging to a closed convex set K its tangent cone $T_K(x)$ at x.

This "regularity property" of a set K at a point $x \in K$ is not always true for any subset. It is convenient to introduce the following definition:

Definition 2.15 *We shall say that a subset K of X is* **sleek** *at $x \in K$ if the set-valued map*

$$K \ni x' \rightsquigarrow T_K(x') \text{ is lower semicontinuous at } x$$

and that it is **sleek** *if and only if it is sleek at every point of K.*

We define the **(Clarke) tangent cone** *(or* **circatangent cone***) $C_K(x)$ by*

$$C_K(x) := \{v \mid \lim_{h \to 0+, K \ni x' \to x} \frac{d_K(x' + hv)}{h} = 0\}$$

Therefore, with this definition, we obtain:

Theorem 2.16 *Any closed convex subset of a finite dimensional vector-space X is sleek.*

We refer to Theorem 4.2.2 of SET-VALUED ANALYSIS for the proof of this Theorem. □

We see at once that $C_K(x) \subset T_K(x)$ and that if x belongs to Int(K), then $C_K(x) = X$.

It is very convenient to observe that when x belongs to $\overline{K}$,

$$\begin{cases} v \in C_K(x) \text{ if and only if } \forall\ h_n \to 0+,\ \forall\ K \ni x_n \to x, \\ \exists\ v_n \to v \text{ such that } \forall\ n,\ x_n + h_n v_n \in K \end{cases}$$

The charm of the tangent cone C_K at x is that it is always convex[24]. Unfortunately, the price to pay for enjoying this convexity property of the Clarke tangent cones is that they may often be reduced to the trivial cone $\{0\}$.

For convex subsets K, the Clarke tangent cone and the contingent cone coincide with the closed cone spanned by $K - x$, the tangent cone of convex analysis.

However, we shall show that the Clarke tangent cone and the contingent cone do coincide at those points x where K is sleek:

Theorem 2.17 *Let K be a closed subset of a finite dimensional vector-space X. Consider a set-valued map $F : K \rightsquigarrow X$ satisfying*

$$\begin{cases} i) & F \text{ is lower semicontinuous at } x \\ ii) & \exists\ \delta > 0 \quad \text{such that } \forall\ z \in B_K(x, \delta),\ \ F(z) \subset T_K(z) \end{cases}$$

Then $F(x) \subset C_K(x)$.

In particular, if K is sleek at $x \in K$, then $T_K(x) = C_K(x)$ is a closed convex cone.

Before proving this theorem, we need to establish Lemma 2.18 We introduce the following notation:

$$D_\uparrow d_K(x)(v) \ := \ \liminf_{h \to 0+} \frac{d_K(x + hv) - d_K(x)}{h}$$

which will be justified later[25]. We observe that when $x \in K$, a direction v is contingent to K at x if and only if $D_\uparrow d_K(x)(v) \leq 0$.

[24] Let v_1 and v_2 belong to $C_K(x)$. To prove that $v_1 + v_2$ belongs to this cone, let us choose any sequence $h_n > 0$ converging to 0 and any sequence of elements $x_n \in K$ converging to x. There exists a sequence of elements v_{1n} converging to v_1 such that the elements $x_{1n} := x_n + h_n v_{1n}$ do belong to K for all n. But since x_{1n} does also converge to x in K, there exists a sequence of elements v_{2n} converging to v_2 such that

$$\forall\ n,\ \ x_{1n} + h_n v_{2n} \ = \ x_n + h_n(v_{1n} + v_{2n}) \ \in \ K$$

This implies that $v_1 + v_2$ belongs to $C_K(x)$ because the sequence of elements $v_{1n} + v_{2n}$ converges to $v_1 + v_2$.

[25] this is the contingent epiderivative of the distance functions d_K. See SET-VALUED ANALYSIS, [5, Aubin & Frankowska]) for further details.

Lemma 2.18 *Let K be a closed subset of a finite dimensional vector-space and $\Pi_K(y)$ be the set of projections of y onto K, i.e., the subset of $z \in K$ such that $\|y - z\| = d_K(y)$. Then the following inequalities:*

$$D_\uparrow d_K(y)(v) \;\leq\; d(v, T_K(\Pi_K(y)))$$

hold true. Therefore,

$$T_K(\Pi_K(y)) \subset T_K(y)$$

Proof — Choose $z \in \Pi_K(y)$ and $w \in T_K(z)$. Then

$$\begin{cases} \dfrac{d_K(y+hv) - d_K(y)}{h} \leq \dfrac{\|y-z\| + d_K(z+hv) - d_K(y)}{h} \\[2ex] = \dfrac{d_K(z+hv)}{h} \leq \dfrac{d_K(z+hw)}{h} + \|v - w\| \end{cases}$$

Since z belongs to K and $w \in T_K(z)$, the above inequality implies that

$$D_\uparrow d_K(y)(v) \;\leq\; d(v, T_K(z)) \;\square$$

Proof of Theorem 2.17

Let us take $x \in K$ and $v \in F(x)$, assumed to be different from 0. Since F is lower semicontinuous at x, we can associate with any $\varepsilon > 0$ a number $\eta \in]0, \delta[$ such that $d(v, F(z)) \leq d(v, F(x)) + \varepsilon = \varepsilon$ for any $z \in B_K(x, \eta)$ (because $d(v, F(x)) = 0$). Therefore, for any $y \in B(x, \eta/4)$ and $\tau \leq \eta/4\|v\|$, the inequality

$$\forall\, z \in \Pi_K(y + \tau v),\ \|z - x\| \leq 2\|y + \tau v - x\| \leq 2\|x - y\| + 2\tau\|v\| \leq \eta$$

implies that

$$\begin{cases} d(v, T_K(\Pi_K(y+\tau v))) \;\leq\; d(v, F(\Pi_K(y+\tau v))) \\[1ex] \qquad\qquad\qquad\qquad\qquad\;\; \leq\; d(v, F(x)) + \varepsilon \;=\; \varepsilon \end{cases}$$

We set $g(\tau) := d_K(y + \tau v)$. By Lemma 2.18, we obtain

$$\begin{cases} \liminf_{h \to 0+} (g(\tau + h) - g(\tau))/h \;=\; D_\uparrow d_K(y + \tau v)(v) \\ \leq\; d(v, T_K(\Pi_K(y + \tau v))) \;\leq\; \varepsilon \end{cases}$$

The function g being Lipschitz, it is almost everywhere differentiable, so that $g'(t) \leq \varepsilon$ for almost all t small enough. Integrating this inequality from 0 to h, we obtain

$$d_K(y+hv) \;=\; g(h) \;=\; g(h)-g(0) \;\leq\; \varepsilon h$$

for any $y \in B(x, \eta/4)$ and $\tau \leq \eta/4\|v\|$. This shows that v belongs to $C_K(x)$.

By taking $F(x) = T_K(x)$, we deduce that $T_K(x) \subset C_K(x)$ whenever K is sleek at $x \in K$, and thus, that they coincide. □

2.2.5 Lower Semicontinuity Criteria

In order to prove that the regulation map Π_K, we can always assume that the pricing map P is lower semicontinuous and we know that the set-valued map $T_K(\cdot)$ is lower semicontinuous whenever the viability set is sleek (and, in particular, smooth or convex).

We thus need lower semicontinuity criteria to derive that the regulation map $\Pi_K(\cdot)$ is lower semicontinuous.

Proposition 2.19 *Consider a two set-valued maps T and P from X to Y and Z respectively and a (single-valued) map c from $X \times Z$ to Y satisfying the following assumptions:*

$$\begin{cases} i) & T \text{ and } P \text{ are lower semicontinuous with convex values} \\ ii) & c \text{ is continuous} \\ iii) & \forall\, x, \;\; p \mapsto c(x,p) \text{ is affine} \end{cases}$$

We posit the following condition:
$\forall\, x \in X,\ \exists\, \gamma > 0,\ \delta > 0,\ c > 0,\ r > 0$ *such that* $\forall\, x' \in B(x, \delta)$ *we have*

$$\gamma B_Y \;\subset\; c(x', P(x') \cap rB_Z) - T(x')$$

Then the set-valued map $\Pi : X \rightsquigarrow Z$ defined by

$$\Pi(x) \;:=\; \{p \in P(x) \mid c(x,p) \in T(x)\} \tag{2.5}$$

is lower semicontinuous with nonempty convex values.

Proof — Let us fix $p \in \Pi(x)$ and a sequence x_n converging to x. Since P and T are lower semicontinuous and c is continuous, there exist sequences $p_n \in P(x_n)$ and

$q_n \in T(x_n)$ converging to p and $c(x,p)$ respectively. Let us set $\varepsilon_n := \|c(x_n, p_n) - q_n\|$ and $\theta_n := \frac{\gamma}{\gamma+\varepsilon_n} \in]0,1[$. Then ε_n converges to 0. Since

$$\theta_n \varepsilon_n = (1-\theta_n)\gamma$$

we deduce that

$$\begin{cases} \theta_n(c(x_n,p_n) - q_n) & \in & \theta_n \varepsilon_n B \\ & = & (1-\theta_n)\gamma B \\ & \subset & (1-\theta_n)(c(x_n, P(x_n) \cap rB_Z)) - T(x_n)) \end{cases}$$

Therefore, there exist $\widehat{p}_n \in P(x_n) \cap rB_Z$ and $\widehat{q}_n \in T(x_n)$ such that

$$c(x_n, \theta_n p_n + (1-\theta_n)\widehat{p}_n) = \theta_n q_n + (1-\theta_n)\widehat{q}_n$$

This implies that

$$q_n := \theta_n p_n + (1-\theta_n)\widehat{p}_n$$

belongs to $P(x_n)$ and that

$$p_n - q_n = (1-\theta_n)(p_n - \widehat{p}_n) \in (1-\theta_n)(r + \|u\| + 1)B$$

because $\|\widehat{p}_n\| \leq r$, $\|p_n\|$ is bounded, $P(x_n)$ and $T(x_n)$ are convex and c is affine with respect to p. Hence the elements $q_n \in \Pi(x_n)$ converge to p given in $\Pi(x)$. □

We state now another condition which is less symmetric.

Proposition 2.20 *Consider a metric space X, two normed spaces Y and Z, two set-valued maps T and P from X to Y and Z respectively and a (single-valued) map c from $X \times Z$ to Y such that*

$$\begin{cases} i) & P \text{ is lower semicontinuous with convex values} \\ ii) & f \text{ is continuous} \\ iii) & \forall\, x,\ \ p \mapsto c(x,p) \text{ is affine} \\ iv) & \forall\, x,\ \ T(x) \text{ is convex and its interior is nonempty} \\ v) & \text{the graph of the map } x \rightsquigarrow \mathrm{Int}(T(x)) \text{ is open} \end{cases}$$

We posit the following condition:

$$\forall\, x \in X,\ \exists\, p \in P(x) \text{ such that } c(x,p) \in \mathrm{Int}(T(x)) \tag{2.6}$$

Then the set-valued map Π defined by (2.5) is lower semicontinuous with convex values.

Proof

1. — We introduce the set-valued map $S : X \rightsquigarrow Z$ defined by

$$S(x) := \{p \in P(x) \mid c(x,p) \in \text{Int}(T(x))\} \subset \Pi(x)$$

Assumption (2.6) implies that $S(x)$ is not empty. We claim that S is lower semicontinuous. Indeed, if $x_n \to x$ and if p belongs to $S(x) \subset P(x)$, there exists $p_n \in P(x_n)$ which converges to p because P is lower semicontinuous. Since

$$(x_n, c(x_n, p_n)) \text{ converges to } (x, c(x,p)) \in \text{Graph}(\text{Int}(T(\cdot)))$$

by continuity of c and since the graph of $\text{Int}(T(\cdot))$ is open, the elements $c(x_n, p_n)$ belong to $\text{Int}(T(x_n))$ for n large enough and thus, the elements p_n belong to $S(x_n)$ and converge to p.

2. — Convexity of $P(x)$ and $T(x)$ implies that $\overline{S(x)} = \Pi(x)$. Indeed, let us fix $p \in \Pi(x)$ and $p_0 \in S(x)$. Then $q_\theta := \theta p_0 + (1-\theta)p$ belongs to $S(x)$ when $\theta \in]0,1[$, because $T(x)$ is convex and $c(x,p_0)$ belongs to the interior of $T(x)$, so that for every $\theta \in]0,1[$,

$$c(x,p) + \theta(c(x,p_0) - c(x,p)) = c(x, u + \theta p_0 - \theta p) = c(x, q_\theta)$$

belongs to the interior of $T(x)$. Then p is the limit of q_θ when $\theta > 0$ converges to 0.

3. — The theorem ensues because the closure of any lower semicontinuous set-valued map is still lower semicontinuous. □

2.2.6 Lower Semicontinuity of the Pricing Map

We are now able to proof that the regulation map is lower semicontinuous:

Theorem 2.21 *Assume that the dynamical economy is affine and that K is a closed* sleek *viability domain. Then the regulation map Π_K has compact convex values.*

Let us assume furthermore that the set-valued map P is lower semicontinuous and that

$$\begin{cases} \forall x \in K,\ \exists \gamma > 0, \delta > 0 \ \textit{such that}\ \forall x' \in B_K(x,\delta), \\ \gamma B \ \subset \ c(x', (P(x') \cap c_K B) - T_K(x') \end{cases}$$

or that the interior of the contingent cones are not empty and

$$\forall x \in K,\ \exists p \in P(x) \cap c_K B \ \mid \ c(x,p) \in \text{Int}T_K(x)$$

Then the regulation map is lower semicontinuous.

2.3 Myopic Behavior of Price-Takers

2.3.1 Continuous Feedback Controls

Viable solutions to the dynamical economy (1.11) are regulated by the prices whose evolution is governed by the regulation law (1.11).

Continuous single-valued selections ϖ_K of the regulation map Π_K are **viable feedback prices,** since the Viability Theorem states that the differential equation

$$x'(t) = c(x(t), \varpi_K(x(t)))$$

enjoys the viability property.

Indeed, by construction, K is a viability domain of the single-valued map $x \in K \mapsto c(x, \varpi_K(x))$. Hence, when the regulation map is lower semicontinuous with convex values, we deduce from Michael's Theorem 2.14 the existence of viable continuous feedback prices.

Proposition 2.22 *Consider a Marchaud dynamical economy (P, c). If its regulation map is lower semicontinuous with nonempty convex values, then the dynamical economy can regulate viable solutions in K by* **continuous feedback prices.**

2.3.2 Slow Viable Solutions

This result is not useful in practice, since Michael's Selection Theorem does not provide constructive ways to find those continuous feedback prices.

Therefore, we are tempted to use explicit selections of the regulation map Π_K, such as the minimal selection ϖ_K° (see (2.2)). Unfortunately, since there is no hope of having continuous regulation maps Π_K in general (as soon as we have inequalities constraints), this minimal selection is not continuous. But the minimal selection ϖ_K° being obtain through a selection procedure, the differential equation

$$x'(t) = c(x(t), \varpi_K^\circ(x(t))) \tag{2.7}$$

has viable solutions.

Definition 2.23 *The solutions to differential equation (2.7) are called* **slow** *viable solutions to dynamical economy (1.11).*

We derive from Theorem 2.4 the existence of slow viable solutions:

Theorem 2.24 *Consider a Marchaud dynamical economy (P, c). If the regulation map is lower semicontinuous with nonempty convex values, then the dynamical economy (1.11) has slow viable solutions.*

We can now multiply the possible corollaries, since we have given several instances of selection procedures of set-valued maps.

2.3.3 Other Examples of Myopic Behavior

We shall just mention some of the examples. We begin by selecting viable solutions through minimization procedures:

Proposition 2.25 *Consider a Marchaud dynamical economy (P, c) and suppose that the regulation map is lower semicontinuous with nonempty convex images. Let us consider a loss function*

$$V : (x, p) \in \text{Graph}(P) \mapsto V(x, p) \in \mathbf{R}$$

be continuous and convex with respect to p. Then, for any initial commodity $x_0 \in K$, there exists a viable solution $x(\cdot)$ to the dynamical economy (1.11) regulated by an open loop price $p(\cdot)$ satisfying for almost all $t \geq 0$,

$$p(t) \ \in \Pi_K(x(t)) \ \ \& \ \ V(x(t), p(t)) \ = \ \inf_{q \in \Pi_K(t)} V(x(t), q)$$

Proof — This is a consequence of Theorem 2.3 and Proposition 2.11. □

When the price space $Z := Z_1 \times Z_2$ is the product of two price spaces, viable prices can be required to be **saddle-points** of two-person games:

Proposition 2.26 *Consider a Marchaud dynamical economy (P, c), where $P(x) = P_1(x) \times P_2(x)$ is the product of two price sets and $c(x, p) := c(x) + g_1(x)p_1 + g_2(x)p_2$. Assume that the regulation map*

$$\Pi_K(x) \ := \ \{(p_1, p_2) \in P_1(x) \times P_2(x) \mid g_1(x)p_1 + g_2(x)p_2 \in T_K(x) - c(x)\}$$

is lower semicontinuous with nonempty convex values. Let $a : X \times Z_1 \times Z_2 \to \mathbf{R}$ satisfy

$$\begin{cases} i) & a \text{ is continuous} \\ ii) & \forall (x, p_2) \in X \times Y_2, \ \ p_1 \mapsto a(x, p_1, p_2) \ \text{ is convex} \\ iii) & \forall (x, p_1) \in X \times Y_1, \ \ p_2 \mapsto a(x, p_1, p_2) \ \text{ is concave} \end{cases}$$

Then, for any initial commodity $x_0 \in K$, there exist a viable solution $x(\cdot)$ and open loop prices $p_1(\cdot)$ & $p_2(\cdot)$ satisfying for almost all $t \geq 0$,

$$\left\{ \begin{array}{ll} i) & x'(t) \;=\; c(x(t)) + g_1(x(t))p_1(t) + g_2(x(t))p_2(t) \\ ii) & p_1(t) \in P_1(x(t)) \;\; \& \;\; p_2(t) \in P_2(x(t)) \\ iii) & \forall (q_1, q_2) \in \Pi_K(x), \\ & a(x(t), p_1(t), q_2) \leq a(x(t), p_1(t), p_2(t)) \leq a(x(t), q_1, p_2(t))\} \end{array} \right.$$

Proof — The proof follows from Theorem 2.3 and Proposition 2.14. □

2.4 Calculus of Contingent Cones

Tangent cones to sleek subsets enjoys almost the same calculus than the tangent cones to closed convex subsets. The results are proved in [5, Aubin & Frankowska], chapter 4.

2.4.1 Contingent Cones to Closed Sleek Subsets

We summarize in Table 1 the calculus of contingent cones. Formulas (1) to (4) are straightforward. The other properties are valid when K is sleek, and are a consequence of the Constrained Inverse Function Theorem, which we shall prove later[26].

Proposition 2.27 *Assume that the resource set is closed and sleek and the consumption sets satisfy*

$$\forall\, x \in K,\; \sum_{i=1}^{n} T_{L_i}(x_i) - T_M(\sum_{i=1}^{n} x_i) \;=\; Y$$

Then

$$T_K(x) \;:=\; \left\{ v \;:=\; (v_1, \ldots, v_n) \;\in \prod_{i=1}^{n} T_{L_i}(x) \;\middle|\; \sum_{i=1}^{n} v_i \;\in\; T_M\left(\sum_{i=1}^{n} x_i\right) \right\} \tag{2.8}$$

[26] We mention that transversality condition of formula (5) implies the constraint qualification assumption $0 \in \mathrm{Int}(f(L) - M)$ and that the stronger transversality condition

$$\exists\, c > 0 \mid \forall\, x \in K,\; B_Y \subset f'(x)(T_L(x) \cap cB_X) - T_M(Ax)$$

implies that if L and M are sleek and f is continuously differentiable, then K is also sleek.

Table 1: **Properties of Contingent Cones to Sleek Subsets.**

(1)	▷	If $K \subset L$ and $x \in \overline{K}$, then $T_K(x) \subset T_L(x)$
(2)	▷	If $K_i \subset X$, $(i = 1, \cdots, n)$ and $x \in \overline{\bigcup_i K_i}$, then $T_{\bigcup_{i=1}^n K_i}(x) \; = \; \bigcup_{i \in I(x)} T_{K_i}(x)$ where $I(x) := \{i \mid x \in \overline{K_i}\}$
(3)	▷	If $K_i \subset X_i$, $(i = 1, \cdots, n)$ and $x_i \in \overline{K_i}$, then $T_{\prod_{i=1}^n K_i}(x_1, \ldots, x_n) \; \subset \; \prod_{i=1}^n T_{K_i}(x_i)$
(4)	▷	If $g \in C^1(X, Y)$, if $K \subset X$, $x \in \overline{K}$ and $M \subset Y$, then $g'(x)(T_K(x)) \; \subset \; T_{g(K)}(g(x))$ $T_{g^{-1}(M)}(x) \; \subset \; g'(x)^{-1} T_M(g(x))$
(5)	▷	If $L \subset X$ and $M \subset Y$ are **closed sleek** subsets, $f \in C^1(X, Y)$ is a continuously differentiable map and $x \in L \cap f^{-1}(M)$ satisfies the **transversality condition** $f'(x)T_L(x) - T_M(f(x)) = Y$, then $T_{L \cap f^{-1}(M)}(x) \; = \; T_L(x) \cap f'(x)^{-1} T_M(f(x))$
(5)a)	▷	If $M \subset Y$ is a *closed sleek* subset, $f \in C^1(X, Y)$ is a continuously differentiable map and $x \in f^{-1}(M)$ satisfies $\mathrm{Im}(f'(x)) - T_M(f(x)) = Y$, then $T_{f^{-1}(M)}(x) \; = \; f'(x)^{-1} T_M(f(x))$
(5)b)	▷	If K_1 and K_2 are **closed sleek** subsets contained in X and $x \in K_1 \cap K_2$ satisfies $T_{K_1}(x) - T_{K_2}(x) = X$, then $T_{K_1 \cap K_2}(x) \; = \; T_{K_1}(x) \cap T_{K_2}(x)$
(5)c)	▷	If $K_i \subset X$, $(i = 1, \ldots, n)$, are **closed sleek** and $x \in \bigcap_i K_i$ satisfies $\forall v_i = 1, \ldots, n, \quad \bigcap_{i=1}^n (T_{K_i}(x) - v_i) \neq \emptyset$ then, $T_{\bigcap_{i=1}^n K_i}(x) \; = \; \bigcap_{i=1}^n T_{K_i}(x)$

2.4.2 Inequality Constraints

We also state the following example of the contingent cone to a set defined by equality and inequality constraints[27]:

Theorem 2.28 *Let us consider a closed subset L of a finite dimensional vector-space X and two continuously differentiable maps $g := (g_1, \ldots, g_p) : X \mapsto \mathbf{R}^p$ and $h := (h_1, \ldots, h_q) : X \mapsto \mathbf{R}^q$ defined on an open neighborhood of L.*

Let K be the subset of L defined by the constraints

$$K := \{x \in L \mid g_i(x) \geq 0,\ i = 1, \ldots, p,\ \&\ h_j(x) = 0,\ j = 1, \ldots, q\}$$

We denote by $I(x) := \{i = 1, \ldots, p \mid g_i(x) = 0\}$ the subset of active constraints.

We posit the following transversality condition at a given $x \in K$:

$$\left\{ \begin{array}{ll} i) & h'(x)C_L(x) = \mathbf{R}^q \\ ii) & \exists\ v_0 \in C_L(x) \text{ such that } h'(x)v_0 = 0 \\ & \text{and } \forall\ i \in I(x),\ < g_i'(x), v_0 >> 0 \end{array} \right.$$

Then u belongs to the contingent cone to K at x if and only if u belongs to the contingent cone to L at x and satisfies the constraints

$$\forall\ i \in I(x),\ < g_i'(x), u > \geq 0\ \&\ \forall\ j = 1, \ldots, q,\ h_j'(x)u = 0$$

Unfortunately, the graph of $T_K(\cdot)$ is not necessarily closed. However, there exists a closed set-valued map $T_K^\circ(\cdot)$ contained in $T_K(\cdot)$ introduced by N. Maderner. Set

$$\gamma_K(x) := \min_{i \notin I(x)} \frac{g_i(x)}{\|g_i'(x)\|} \in]0, +\infty] \quad (2.9)$$

We observe that γ_K is upper semicontinuous whenever the functions g_i are continuously differentiable. Indeed, let $x_n \in K$ converge to x_0 and $a_n \leq \gamma_K(x_n)$ converge to a_0. Since $g_i(x_0) > 0$ whenever $i \notin I(x_0)$, we infer that $i \notin I(x_n)$ for n large enough. Hence inequalities $a_n\|g_i'(x_n)\| \leq g_i(x_n)$ hold true for any $i \notin I(x_0)$ and imply at the limit that $a_0 \leq \gamma_K(x_0)$.

[27] See Proposition 4.3.6 of SET-VALUED ANALYSIS, [5, Aubin & Frankowska].

The growth of the function γ_K is linear whenever we assume that there exists a constant $c > 0$ such that

$$\forall\, i = 1, \ldots, p, \quad \|g_i'(x)\| \;\geq\; c\, \frac{g_i(x)}{\|x\| + 1}$$

Theorem 2.29 (Maderner) *We posit the assumptions of Theorem 2.28. Then the set-valued map $T_K^\circ(\cdot) : K \rightsquigarrow X$ defined by:*
$u \in T_K^\circ(x)$ if and only if $u \in T_L(x)$ and

$$\begin{cases} \forall\, i = 1, \ldots, p, & g_i(x) + < g_i'(x), u > \;\geq\; 0 \\ \forall\, j = 1, \ldots, q, & h_j'(x)u \;=\; 0 \end{cases}$$

is contained in the contingent cone $T_K(x)$ and satisfy

$$T_K(x) \cap \gamma_K(x) B \;\subset\; T_K^\circ(x)$$

Its graph is closed whenever the graph of $x \rightsquigarrow T_L(x)$ is closed.

Proof — Let u belong to $T_K^\circ(x)$. Then, when $i \in I(x)$, we see that $\langle g_i'(x), u\rangle = g_i(x) + \langle g_i'(x), u\rangle \geq 0$, so that $u \in T_K(x)$.
Conversely, let us choose u in $T_K(x)$ satisfying $\|u\| \leq \gamma_K(x)$. Then either $i \in I(x)$ and $g_i(x) + \langle g_i'(x), u\rangle = \langle g_i'(x), u\rangle \geq 0$ or $g_i(x) > 0$ so that

$$i \notin I(x) \;\;\&\;\; g_i(x) + \langle g_i'(x), u\rangle \;\geq\; g_i(x) \;-\; \|g_i'(x)\|\|u\| \;\geq\; 0$$

because $\|u\| \leq \gamma_K(x) \leq g_i(x)/\|g_i'(x)\|$. Thus, in both cases, $g_i(x) + \langle g_i'(x), u\rangle \geq 0$, so that u belongs to $T_K^\circ(x)$. □

2.5 Scarcity Constraints

We consider the case when the viability domain $K := h^{-1}(M)$ is defined by more explicit constraints through a map h from X to a resource space Y: we introduce three finite dimensional vector spaces:

1. — the commodity space X
2. — the resource space Y
3. — the price space Z

and we define the viability subset by the constraints

$$K := h^{-1}(M)$$

where we assume that the production map h satisfies

$$\begin{cases} i) & h \text{ is a } C^1\text{-map from } X \text{ to } Y \\ ii) & \forall x \in K, \;\; Y = \mathrm{Im}(h'(x)) - T_M(h(x)) \end{cases}$$

Let us recall that in this case[28]:

$$T_K(x) \;\; = \;\; h(x)'^{-1}T_M(h(x))$$

The regulation map Π_K can be written:

$$\Pi_K(x) \;\; := \;\; \{p \in P(x) \mid h'(x)G(x)p \in T_M(h(x)) - h'(x)c(x)\}$$

By replacing K by M, $G(x)$ by $B(x) := h'(x)G(x)$ and $c(x)$ by $b(x) := h'(x)c(x)$, we obtain the following corollary:

Corollary 2.30 *Assume that the dynamical economy is affine and that the constraints satisfy*

$$\begin{cases} i) & M \text{ is a closed sleek subset of } Y \\ ii) & h \text{ is a } C^1\text{-map from } X \text{ to } Y \\ iii) & \forall x \in K, \;\; Y = \mathrm{Im}(h'(x)) - T_M(h(x)) \\ iv) & \forall x \in h^{-1}(M), \; \exists p \in P(x) \text{ such that} \\ & h'(x)G(x)p \in T_M(h(x)) - h'(x)c(x) \end{cases}$$

Then the regulation map Π_K has compact nonempty convex values. Let us assume furthermore that the set-valued map P is lower semicontinuous and that[29]

$$\begin{cases} \forall x \in K, \;\; \exists \gamma > 0, \delta > 0 \;\text{ such that } \forall x' \in B_K(x,\delta), \\ \gamma B \;\subset\; h'(x')c(x') + h'(x')G(x')(P(x') \cap c_K B) - T_M(h(x')) \end{cases}$$

Then the regulation map is lower semicontinuous *and its support function is equal to:*

$$\sigma(\Pi_K(x), r) = \inf_{q \in N_M^\circ(h(x))} (\sigma(P(x), r - G(x)^\star h'(x)^\star q) - < q, h'(x)c(x) >)$$

---

[28]If we assume furthermore that there exists a positive constant c such that

$$\forall\, x \in h^{-1}(M), \;\; B_Y \;\subset\; h'(x)(cB_X) - T_M(h(x))$$

then $h^{-1}(M)$ is also sleek.

[29]or that the interior of the contingent cones are not empty and

$$\forall x \in K, \; \exists p \in P(x) \cap c_K B \;\text{ such that }\; h'(x)(c(x) + G(x)p) \in \mathrm{Int}T_M(h(x))$$

We also remark that checking whether $h^{-1}(M)$ is a viability domain amounts to solving for all $x \in K$ the inclusions

$$\text{find } p \in P(x) \text{ satisfying } 0 \in h'(x)c(x) + h'(x)G(x)p - T_M(h(x))$$

Hence we can use the general Equilibrium Theorem to derive sufficient conditions for $h^{-1}(M)$ to be a viability domain.

Proposition 2.31 *Let us assume that the dynamical economy is affine, that the values of the feedback map P are compact and that*

$$\begin{cases} i) & M \text{ is a closed } \textbf{sleek} \text{ subset of } Y \\ ii) & h \text{ is a } C^1\text{-map from } X \text{ to } Y \\ iii) & \forall x \in K,\ Y = \mathrm{Im}(h'(x)) - T_M(h(x)) \\ iv) & h^{-1}(M) \subset \mathrm{Dom}(P) \end{cases}$$

Assume furthermore that

$$\begin{cases} \text{there exists a continuous map } B : \mathrm{Graph}(P) \mapsto \mathcal{L}(Z,Y) \\ \text{such that } \forall x \in K,\ \forall p \in P(x), \\ h'(x)(c(x) + G(x)p) \in T_M(h(x)) + B(x,p)T_{P(x)}(p) \end{cases}$$

Then $K := h^{-1}(M)$ is a viability domain.

Let us emphasize the fact that in this statement, the map $B : \mathrm{Graph}(P) \to \mathcal{L}(Z,Y)$ is a parameter. It thus provides many possibilities for checking whether a given subset is a viability domain.

2.5.1 Duality Criterion

Definition 2.32 *Let x belong to $K \subset X$. We shall say that the (negative) polar cone*

$$N_K^0(x) := T_K(x)^- = \{p \in X^\star \mid \forall\, v \in T_K(x),\ < p, v > \leq 0\}$$

is the subnormal cone *to K at x.*

We see at once that

$$N_K^0(x) := (\overline{co}(T_K(x)))^-$$

The subnormal cone is equal to the whole space whenever the tangent cone $T_K(x)$ is reduced to 0.

We shall now characterize viability domains through a dual formulation.

For that purpose, we associate with any subset $K \subset \mathrm{Dom}(P)$ the subnormal cone $N_K^\circ(x)$ and the function β_K defined by:

$$\forall\,(x,p) \in \mathrm{Graph}(N_K^\circ),\;\; \beta_K(x,p) \;:=\; \inf_{q \in P(x)} < p, G(x)q + c(x) >$$

Theorem 2.33 *Assume that the set-valued map $F : K \rightsquigarrow X$ is upper semicontinuous with convex compact values. Then the three following properties are equivalent:*

$$\begin{cases} i) & \forall\, x \in K,\;\; F(x) \cap T_K(x) \;\neq\; \emptyset \\ ii) & \forall\, x \in K,\;\; F(x) \cap \overline{co}\,(T_K(x)) \;\neq\; \emptyset \\ iii) & \forall\, x \in K,\;\; \forall\, p \in N_K^0(x),\;\; \sigma(F(x), -p) \;\geq\; 0 \end{cases} \qquad (2.10)$$

We deduce from Theorem 2.33 the following:

Proposition 2.34 *Let us assume that the dynamical economy is affine and that the values of the feedback map P are compact. Then a closed subset K is a viability domain if and only if*

$$\forall (x,p) \in \mathrm{Graph}(N_K^\circ),\;\; \beta_K(x,p) \leq 0$$

If we assume in particular that

$$\begin{cases} i) & M \text{ is a closed subset of } Y \\ ii) & h \text{ is a } C^1\text{-map from } X \text{ to } Y \\ iii) & \forall x \in K,\;\; Y = \mathrm{Im}(h'(x)) - T_M(h(x)) \end{cases}$$

then $K := h^{-1}(M)$ is a viability domain if and only if

$$\begin{cases} \forall\, x \in K,\, \forall\, q \in N_M^\circ(h(x)), \\ d_M(x,q) \;:=\; \inf_{p \in P(x)} < q, h'(x)G(x)p + h'(x)c(x) > \;\leq 0 \end{cases}$$

For instance, this condition holds true when the following abstract Walras law *holds true:*

$$\begin{cases} i) & Z \;=\; Y,\;\; P(x) \;\supset\; N_M^\circ(h(x)) \\ ii) & \forall q \in N_M^\circ(h(x)),\;\; < q, h'(x)c(x) + h'(x)G(x)q > \leq 0 \end{cases}$$

2.5.2 Decoupling the Regulation Map

Finally, let us mention that the calculus of the contingent cones can be transferred to a calculus of regulation maps. For instance, a quite common type of viability constraints are of the form $K := L \cap h^{-1}(M)$ where we assume that

$$\begin{cases} i) & L \subset X \text{ and } M \subset Y \text{ are sleek} \\ ii) & h \text{ is a } \mathcal{C}^1\text{-map from } X \text{ to } Y \\ iii) & \forall x \in K := L \cap h^{-1}(M), \;\; Y = h'(x)T_L(x) - T_M(h(x)) \end{cases}$$

Indeed, K is the inverse image of the product $L \times M$ by the map $1 \times h$ from X to $X \times Y$.

This a particular case of a more general situation when both X, Y and Z are product spaces. It may then be convenient to provide once and for all the explicit formulas of the regulation map when this is the case. Let us assume namely that

$$\begin{cases} i) & X := \prod_{i=1}^n X_i \\ ii) & Y := \prod_{j=1}^m Y_j, \quad M := \prod_{j=1}^m M_j \\ iii) & Z := \prod_{k=1}^l Z_k, \quad P(x) := \prod_{k=1}^l P_k(x) \end{cases} \tag{2.11}$$

and that

$$\begin{cases} i) & \forall x \in X, \; G(x)p := (g_1(x)u, \ldots, g_m(x)u) \;\&\; g_i(x)u := \sum_{k=1}^l g_i^k(x)p_k \\ ii) & c(x) := (c_1(x), \ldots, c_n(x)) \\ iii) & \forall x \in X, \; h(x) := (h_1(x), \ldots, h_m(x)) \text{ where} \\ & h_j(x) := \sum_{i=1}^n h_j^i(x_i) \end{cases} \tag{2.12}$$

Therefore, K is the intersection of the subsets K_j defined by:

$$K_j := \{x \in X \mid \sum_{i=1}^n h_j^i(x_i) \in M_j\} \tag{2.13}$$

Let us introduce the matrix $B(x) := h'(x)G(x)$ of operators

$$B_j^k(x) = \sum_{i=1}^n h_j^{i'}(x) g_i^k(x) \in \mathcal{L}(P_k, Y_j)$$

and the vector $b(x) := h'(x)c(x)$ of components

$$b_j(x) = \sum_{i=1}^n h_j^{i'}(x) c_i(x)$$

Corollary 2.35 *We posit the assumptions (2.11), (2.12) and (2.13). We assume also that*

$$\begin{cases} i) & \forall k, \ \text{Graph}(P_k) \text{ is closed and the images of } P_k \\ & \text{are convex} \\ ii) & \forall i, \ c_i : \text{Dom}(P) \mapsto X_i \text{ is continuous} \\ iii) & \forall k, i, \ g_k^i : \text{Dom}(P) \mapsto \mathcal{L}(Z_k, X_i) \text{ is continuous} \\ iv) & \forall k, i, \ c_i \text{ and } g_k^i \text{ are bounded and } P_k \\ & \text{have linear growth} \end{cases}$$

and that

$$\begin{cases} i) & \text{the subsets } M_j \text{ are closed and sleek} \\ ii) & \text{the maps } h_j^i \text{ are } \mathcal{C}^1 \\ iii) & \forall v_j \in Y_j \ (j = 1, \ldots, n), \ \exists p_i \in X_i \text{ such that} \\ & v_j \in \sum_{i=1}^n h_j^{i\prime}(x_i)p_j + T_M\left(\sum_{i=1}^n h_j^i(x_i)\right) \end{cases} \tag{2.14}$$

Then the regulation map Π_K *is defined by*

$$\begin{cases} i) & \Pi_K(x) \ = \ \bigcap_{j=1}^m \Pi_{K_j}(x) \text{ where} \\ ii) & \Pi_{K_j}(x) = \{p = (p_1, \ldots, p_l) \in \prod_{k=1}^l P_k(x) \text{ such that} \\ & \sum_{k=1}^l B_j^k(x)p_k \ \in \ T_{M_j}(\sum_{i=1}^n h_j^i(x_i) - b_j(x)\} \end{cases} \tag{2.15}$$

and has compact values. If it is strict, then K *is a viability domain of the system, and thus, for any initial commodity* $x_0 \in K$, *there exist* viable *solutions* $x_i(\cdot)$ *on* $[0, \infty[$ *starting at* x_0 *to the system of differential equations*

$$\forall i = 1, \ldots, m, \ \ x_i'(t) \ = \ c_i(x(t)) + \sum_{k=1}^l g_i^k(x(t))p_k(t)$$

and open loop *controls regulating this viable solution* $x(\cdot)$ *in the sense that the* regulation laws

$$\forall j = 1, \ldots, m, \ \ \text{for almost all } t, \ \ p(t) \in \Pi_{K_j}(x(t))$$

are satisfied.

Proof — Assumptions (2.14) imply that the subsets

$$K_j \ \text{ and } \ K := \bigcap_{j=1}^m K_j \quad \text{are sleek}$$

and that

$$\begin{cases} i) & T_K(x) = \bigcap_{j=1}^m T_{K_j}(x), \\ ii) & T_{K_j}(x) = \\ & \{v \in X \ | \ \sum_{i=1}^n h_j^{i\prime}(x_i)(v_i) \in T_{M_j}(\sum_{i=1}^n h_j^i(x_i)\} \end{cases}$$

This implies obviously formulas (2.15). □

Definition 2.36 (Decoupled Regulation Map) *We posit the assumptions (2.11), (2.12) and (2.13). We shall say that the regulation map is* decoupled *if*

$$Z = Y \text{ and } \forall j \neq k, \ B_j^k(x) = 0$$

Corollary 2.37 *We posit the assumptions of Corollary 2.35. If the regulation map is decoupled, then each "partial" viability domain K_j is regulated by the i^{th} component of the price in the sense that*

$$\Pi_{K_j}(x) \ = \ \{p_j \in P_j(x) \mid B_j^j(x)p_j \in T_{M_j}(\sum_{i=1}^{n} h_j^i(x_i) - b_j(x)\}$$

2.6 Pseudo-Inverses

2.6.1 Orthogonal Right Inverses

We shall begin by defining and characterize the orthogonal right-inverse of a surjective linear operator and then, the orthogonal left-inverse of an injective linear operator. These concepts depend respectively upon scalar products l and m defined on the finite dimensional vector-spaces X and Y, which are then given once and for all. We shall denote by $L \in \mathcal{L}(X, X^\star)$ and $M \in \mathcal{L}(Y, Y^\star)$ their **duality mappings** defined respectively by

$$\begin{cases} \forall \, x_1, \ x_2 \in X, \ < Lx_1, x_2 >:= l(x_1, x_2) \\ \\ \forall \, y_1, \ y_2 \in Y, \ < My_1, y_2 >:= m(y_1, y_2) \end{cases}$$

(The matrix $(l(e^i, e^j)_{i,j=1,\ldots,n})$ of the bilinear form l coincides with the matrix of the linear operator L).

We shall denote by

$$\lambda(x) := \sqrt{l(x,x)} \text{ and } \mu(y) := \sqrt{m(y,y)}$$

the norms associated with these scalar products.

The bilinear form $l_\star(p_1, p_2) :=< p_1, L^{-1}p_2 >= l(L^{-1}p_1, L^{-1}p_2)$ is then a scalar product on the dual of $X^\star$, called the **dual scalar product**.

It is quite important to keep options open for the choice of a scalar product. The choice of the Euclidian scalar product is not always wise despite its simplicity. We shall see later that we can associate with a scalar product and a linear operator initial and final scalar product.

Another instance of a choice of a scalar product is to make a given basis orthonormal.

Lemma 2.38 *Assume that* $\{e^1, \ldots, e^n\}$ *be a basis of a finite dimensional vector-space* X *and* $B \in \mathcal{L}(\mathbf{R}^n, X)$ *the associated operator defined by*

$$\forall\, x \in \mathbf{R}^n, \;\; Bx \;=\; \sum_{i=1}^{n} x_i e^i$$

Then

$$l(x,y) \;:=\; \langle B^{-1}x, B^{-1}y\rangle$$

is a scalar product for which the basis $\{e^1, \ldots, e^n\}$ *is orthonormal. The associated duality mapping is equal to* $L = (BB^\star)^{-1}$.

Proof — Indeed, to say that the sequence $\{e^1, \ldots, e^n\}$ is linearly independent amounts to saying that the associated linear operator B is bijective and maps the canonical basis of $\mathbf{R}^n$ onto the basis $\{e^1, \ldots, e^n\}$. Hence the bilinear form l defined above is a scalar product satisfying

$$l(e^i, e^j) \;:=\; \langle B^{-1}e^i, B^{-1}e^j\rangle \;=\; \delta_{i,j} \;\; \Box$$

Let us consider a surjective linear operator $A \in \mathcal{L}(X, Y)$. Then, for any $y \in Y$, the problem $Ax = y$ has at least a solution. We may select the solution $\bar{x}$ with minimal norm $\lambda(x)$, i.e., a solution to the minimization problem with linear equality constraints

$$A\bar{x} = y \;\;\&\;\; \lambda(\bar{x}) = \min_{Ax=y} \lambda(x)$$

The solution to this problem is given by the formula

$$\bar{x} = L^{-1}A^\star(AL^{-1}A^\star)^{-1}y$$

Indeed, for any $v \in \mathrm{Ker}(A)$ and h, we have $\lambda(\bar{x}) \le \lambda(\bar{x} + hv)$ so that, taking the limit when $h \longrightarrow 0$ of $(\lambda(\bar{x} + hv)^2 - \lambda(\bar{x})^2)/2h$, we infer that $l(\bar{x}, v) =< L\bar{x}, v >= 0$ for any $v \in \mathrm{Ker}(A)$. This means that $L\bar{x}$ belongs to the orthogonal of $\mathrm{Ker}(A)$, which is the image of $A^\star$. Thus there exists some $q \in Y^\star$ such that $L\bar{x} = A^\star q$ so that $A\bar{x} = AL^{-1}A^\star q = y$. But $AL^{-1}A^\star$ is positive-definite because $A^\star$ is injective (A being surjective): Indeed, for all $q \in Y^\star$,

$$< AL^{-1}A^\star q, q > = < L^{-1}A^\star q, A^\star q > = \lambda_\star(A^\star q)^2 \;=\; 0$$

is equivalent to

$$A^\star q \;=\; 0 \iff q \;=\; 0$$

We thus infer that $q = (AL^{-1}A^\star)^{-1}y$.

Definition 2.39 *If $A \in \mathcal{L}(X,Y)$ is surjective, we say that the linear operator $A^+ := L^{-1}A^\star(AL^{-1}A^\star)^{-1} \in \mathcal{L}(Y,X)$ is the* **orthogonal right-inverse** *of A (associated with the scalar product l on X).*

Indeed, A^+ is obviously an right-inverse of A because $AA^+y = y$ for any $y \in Y$. We observe that $1 - A^+A$ **is the orthogonal projector onto the kernel** of A.

2.6.2 Quadratic Minimization Problems

Hence, we can write explicitly the solution $\bar{x}$ to the quadratic optimization problem with linear equality constraints

$$A\bar{x} = y \;\; \& \;\; \frac{1}{2}\lambda(\bar{x} - u)^2 = \min_{Ax=y} \frac{1}{2}\lambda(x-u)^2$$

Proposition 2.40 *Let us assume that $A \in \mathcal{L}(X,Y)$ is surjective. Then the unique solution $\bar{x}$ to the above quadratic minimization problem is given by*

$$\begin{cases} \bar{x} \;=\; u - L^{-1}A^\star\bar{q} = u - A^+(Au - y) \\ \\ \text{where } \; \bar{q} := (AL^{-1}A^\star)^{-1}(Au - y) \; \text{ is the } \textbf{Lagrange multiplier} \end{cases}$$

i.e., a solution to the dual minimization problem

$$\min_{q \in Y^\star} \left(\frac{1}{2}\lambda_\star(A^\star q - Lu)^2 + < q, y > \right)$$

We observe easily that $\xi(q) := u - L^{-1}A^\star(q)$ minimizes over X the function $x \mapsto \frac{1}{2}\lambda(x-u)^2 + < q, Ax >$ and that the Lagrange multiplier minimizes $q \mapsto \frac{1}{2}\lambda_\star(\xi(q))^2 + < q, y >$.

2.6.3 Projections onto Cones

Let us consider now a closed convex cone $Q \subset Y$, regarded as the cone of non negative elements of the partial ordering associated with Q ($x \leq y \Longleftrightarrow y - x \in Q$). We supply the dual $Y^\star$ of Y with the scalar product defined by $l_\star(A^\star q_1, A^\star q_2)$ whose duality mapping is $AL^{-1}A^\star \in \mathcal{L}(Y^\star, Y)$.

Let us consider now the quadratic optimization problem with linear inequality constraints

$$A\overline{x}_+ \geq y \;\; \& \;\; \frac{1}{2}\lambda(\overline{x}_+ - u)^2 = \min_{Ax \geq y} \frac{1}{2}\lambda(x-u)^2$$

We observe that the dual problem can be written

$$\inf_{q \in Q^-} \left(\frac{1}{2} \lambda_\star (A^\star q - Lu)^2 - < q, y > \right)$$

So the solution $\overline{x}_+$ of the minimization problem with inequality constraints and the solution $\overline{q}_- \in Q^-$ of its dual problem, the Lagrange multiplier, are defined by

$$\begin{cases} \overline{x}_+ = u - L^{-1} A^\star \overline{q}_- \\ \text{where } \; < \overline{q}_-, A\overline{x}_+ - y >= 0 \end{cases}$$

Since $A\overline{x}_+ - y \in Q$, then $< A\overline{x}_+ - y, q > \leq 0$ for all $q \in Q^-$, so that we can write

$$\forall\, q \in Q^-, \;\; < A\overline{x}_+ - y, q - \overline{q}_- >=< Au - y - AL^{-1}A^\star \overline{q}_-, q - \overline{q}_- >$$

$$= < AL^{-1}A^\star(\bar{q} - \overline{q}_-), q - \overline{q}_- >= l_\star(A^\star(\bar{q} - \overline{q}_-), A^\star(q - \overline{q}_-)) \leq 0$$

Therefore, we have proved that **the Lagrange multiplier $\overline{q}_- \in Q^-$ of the minimization problem with inequality constraints is the orthogonal projection onto Q^- of the Lagrange multiplier $\bar{q} \in Y^\star$ of the minimization problem with equality constraints by the formulas.** Hence the solution $\bar{x}_+$ to the minimization problem with inequality constraints is given by the formulas

$$\begin{cases} \overline{x}_+ \; = \; u - L^{-1} A^\star \overline{q}_- \;\; \text{where} \\ \overline{q}_- \text{ is the orthogonal projection onto } Q^- \text{ of the Lagrange multiplier} \\ \bar{q} := (AL^{-1}A^\star)^{-1}(Au - y) \;\; \text{of the problem with equality constraints} \\ \text{when } Y^\star \text{ is supplied with the scalar product } \;\; l_\star(A^\star q_1, A^\star q_2) \end{cases}$$

2.6.4 Projections onto Inverse Images of Convex Sets

Consider now a closed convex subset $M \subset Y$ and the minimization problem:

$$\inf_{Ax \in M} \lambda(x - u)$$

When $A \in \mathcal{L}(X, Y)$ is surjective, we can supply the space Y with the **final scalar product**

$$m^A(y_1, y_2) \; := \; l(A^+ y_1, A^+ y_2)$$

its associated final norm[30]

$$\mu^A(y) := \lambda(A^+y) = \inf_{Ax=y} \lambda(x)$$

Its duality mapping is equal to $(AL^{-1}A^\star)^{-1} \in \mathcal{L}(Y,Y^\star)$.

In this case, we denote by π_M^A the projector of best approximation onto the closed convex subset M when Y is supplied with the norm μ^A.

Proposition 2.41 *Assume that $A \in \mathcal{L}(X,Y)$ is surjective and that $M \subset Y$ is closed and convex. Then the unique solution $\bar{x}$ to the minimization problem*

$$\inf_{Ax\in M} \lambda(x-u)$$

is equal to

$$\bar{x} = u - A^+(Au - \pi_M^A(Au))$$

Proof — Indeed, we can write

$$\begin{cases} \lambda(\bar{x}-u) = \inf_{Ax\in M} \lambda(x-u) \\ = \inf_{y\in M} \inf_{Ax=y} \lambda(x-u) \\ = \inf_{y\in M} \lambda(A^+(y-Au)) \\ \text{(thanks to Proposition 2.40)} \\ = \lambda(A^+(\pi_M^A(Au)) - Au) \end{cases}$$

Hence $\bar{x} = u - A^+(Au - \pi_M^A(Au))$. □

2.6.5 Orthogonal Left Inverses

Let us consider now an injective linear operator $B \in \mathcal{L}(X,Y)$. Since the problem $Bx = y$ may not have a solution (when y does not belong to the image of B), we shall project y onto this image and take for an approximated solution the inverse of this projection. In other words, the approximated solution to this problem is the element $\bar{x} \in X$ defined by

$$\mu(B\bar{x} - y) = \min_{x\in X} \mu(Bx - y)$$

[30]which may also be denoted by $\mu^A := A \cdot \mu$ by contrast with the initial norm $\lambda \cdot A$. We observe that $\mu \cdot A = \lambda \cdot A^+$.

Since the derivative of the convex function $\frac{1}{2}\mu(\cdot)^2$ at y is equal to $< My, \cdot >$, the Fermat rule states that $\bar{x}$ is a solution to the equation

$$\forall\, v \in X, \quad < M(B\bar{x} - y), Bv > = < B^\star M(B\bar{x} - y), v > = 0$$

The self-transposed operator $B^\star MB$ is positive-definite for B is injective (because $< B^\star MBx, x > = \mu(Bx)^2 = 0$ if and only if $Bx = 0$, i.e., if and only if $x = 0$). Therefore $B^\star MB$ is invertible and we derive that $\bar{x}$ is equal to $(B^\star MB)^{-1}B^\star My$.

Definition 2.42 *If $B \in \mathcal{L}(X, Y)$ is injective, we say that the linear operator $B^- := (B^\star MB)^{-1}B^\star M \in \mathcal{L}(Y, X)$ is the* **orthogonal left-inverse** *of B.*

Indeed, B^- is obviously a left-inverse of B because $B^- Bx = x$ for any $x \in X$ and BB^- is the orthogonal projector onto the image of B.

Proposition 2.43 *Let us assume that $B \in \mathcal{L}(X, Y)$ is injective. Then*

$$\begin{cases} i) & (B^-)^\star = (B^\star)^+ \\ ii) & B = (B^-)^+ \\ iii) & (B^\star MB)^{-1} = B^- M^{-1}(B^-)^\star \\ iv) & \text{If } V \in \mathcal{L}(Z, X) \text{ is invertible, then } (BV)^- = V^{-1}B^- \end{cases}$$

If $A \in \mathcal{L}(X, Y)$ is surjective, then

$$\begin{cases} i) & (A^+)^\star = (A^\star)^- \\ ii) & A = (A^+)^- \\ iii) & (AL^{-1}A^\star)^{-1} = (A^+)^\star LA^+ \\ iv) & \text{If } W \in \mathcal{L}(Y, Z) \text{ is invertible, then} (WA)^+ = A^+W^{-1} \end{cases}$$

Example Orthogonal Projector

Let $P := [b^1, \ldots, b^n]$ be the vector space spanned by n linearly independent vectors b^l of a finite dimensional vector-space Y, supplied with the scalar product $m(y_1, y_2)$. If we denote by $B \in \mathcal{L}(\mathbf{R}^n, Y)$ the linear operator defined by $Bx := \sum_{i=1}^n x_i b^i$, which is injective because the vectors b^i are independent, we infer that BB^- is the orthogonal projector on the subspace $P = \text{Im}(B)$.

The entries of the matrix of $B^\star MB$ are equal to $m(b^i, b^j)$ $(i, j = 1, \ldots, n)$. Let us denote by g_{ij} the entries of its inverse. We infer that $B^- y =$

$(\sum_{j=1}^{n} g_{ij} m(y, b^j))_{i=1,\dots,n}$ and therefore, that

$$BB^- y = \sum_{i,j=1}^{n} g_{ij} m(y, b^j) b^i$$

When the vectors are m-orthogonal, the formula becomes

$$BB^- y = \sum_{i=1}^{n} \frac{m(b^i, y)}{\mu(b^i)^2} b^i \quad \square$$

In particular, we can regard an element $y \in Y$ different from 0 as the injective linear operator $y \in \mathcal{L}(\mathbf{R}, Y)$ associating with α the vector αy. It transpose $y^\star \in \mathcal{L}(Y^\star, \mathbf{R}) = Y$ is the map $p \mapsto \langle p, y \rangle$ and its left-inverse $y^- \in \mathcal{L}(Y, \mathbf{R}) = Y^\star$ is equal to

$$y^- = \frac{My}{\lambda(y)^2} \in Y^\star$$

2.6.6 Pseudo-inverses

Let us consider now any linear operator $C \in \mathcal{L}(X, Y)$. It can be split as the product $C = BA$ of a surjective linear operator $A \in \mathcal{L}(X, Z)$ and an injective linear operator $B \in \mathcal{L}(Z, Y)$: we can take for instance $Z := \mathrm{Im}(C)$, $A := C \in \mathcal{L}(X, Z)$ and $B := I \in \mathcal{L}(Z, Y)$.

Take now two decompositions $C = B_1 A_1 = B_2 A_2$ where $A_i \in \mathcal{L}(X, Z_i)$ is surjective and where $B_i \in \mathcal{L}(Z_i, Y)$ is injective ($i = 1, 2$). Then the images of the B_i's are equal to $\mathrm{Im}(C)$ and the kernels of the A_i's are also equal to a vector subspace $\ker(C) \subset X$. Let $\varphi \in \mathcal{L}(X, X/\ker(C))$ denote the canonical surjection from X to the factor space $X/\ker(C)$. Hence the surjective operators A_i split as $\tilde{A}_i \varphi$ where $\tilde{A}_i \in \mathcal{L}(X/\ker(C), Z_i)$ are invertible ($i = 1, 2$). Then $V := \tilde{A}_1 \tilde{A}_2^{-1}$ is invertible and we have the relations

$$A_1 = VA_2 \;\; \& \;\; B_2 = B_1 V$$

We thus deduce that $A_1^+ B_1^- = A_2^+ B_2^-$ does not depend upon the decomposition of C as a product of an injective operator and an surjective operator because $A_1^+ B_1^- = A_2^+ V^{-1} V B_2^- = A_2^+ B_2^-$.

Definition 2.44 *Let $C = BA \in \mathcal{L}(X, Y)$ be any linear operator split as the product of an injective operator B and a surjective operator A. Then the operator $C^{\ominus 1} := A^+ B^- \in \mathcal{L}(Y, X)$ is called the* pseudo-inverse *of C.*

If $y \in Y$ is given, we can regard $\bar{x} = C^{\ominus 1}y$ as the closest solution with minimal norm. Indeed, by taking $A = C$ and $B = I$, we see that $\bar{x} = A^{+}\bar{y}$ where $\bar{y} := I^{-}(y)$ is the projection of y onto the image of C and that $\bar{x}$ is the solution with minimal norm to the equation $Cx = \bar{y}$. □

Naturally, if C is surjective, the pseudo-inverse $C^{\ominus 1} = C^{+}$ coincides with the orthogonal right-inverse, if C is injective, the pseudo-inverse $C^{\ominus 1} = C^{-}$ coincides with the orthogonal left-inverse and if C is bijective, the pseudo-inverse $C^{\ominus 1} = C^{-1}$ coincides with the inverse.

We list below the (obvious) properties of the pseudo-inverses.

$$\begin{cases} i) & C^{\ominus 1}CC^{\ominus 1} = C^{\ominus 1} \\ ii) & CC^{\ominus 1}C = C \\ iii) & C^{\ominus 1}C \text{ is the orthogonal projector onto the orthogonal of } \mathrm{Ker}(C) \\ iii) & CC^{\ominus 1} \text{ is the orthogonal projector onto } \mathrm{Im}(C) \end{cases}$$

We also observe that

$$\begin{cases} i) & (C^{\ominus 1})^{\ominus 1} = C \\ ii) & (C^{\ominus 1})^{\star} = (C^{\star})^{\ominus 1} \\ iii) & (C_2C_1)^{\ominus 1} = C_1^{\ominus 1}C_2^{\ominus 1} \end{cases}$$

2.6.7 Slow viable solutions on smooth subsets

When $K := h^{-1}(0)$ is smooth, one can obtain explicit differential equations yielding slow viable solutions.

Corollary 2.45 *Let us assume that $h : X \mapsto Y$ is a continuously differentiable map and that the viability subset is $K := h^{-1}(0)$, that $P(x) \equiv Z$ is constant, and that the system is affine, so that*

$$\forall\, x \in K,\ \ \Pi(x) := \{p \in Z \mid h'(x)c(x,p) = h'(x)(c(x)) + h'(x)G(x)p = 0\}$$

Then there exist slow solutions viable in K, which are the solutions to the system

$$\begin{cases} x'(t) = -G(x(t))^{\star}h'(x(t))^{\star} \\ (h'(x(t))G(x(t))G(x(t))^{\star}h'(x(t))^{\star})^{-1}h'(x(t))c(x(t)) \end{cases}$$

Proof — The element $q_0 \in \Pi(x)$ of minimal norm is the solution of the quadratic minimization problem under equality constraints

$$h'(x)G(x)p = -h'(x)c(x)$$

and is given explicitly by the formula

$$q_0 = -G(x)^\star h'(x)^\star (h'(x)G(x)G(x)^\star h'(x)^\star)^{-1} h'(x)c(x) \quad \Box$$

Slow viable solutions in affine spaces. Consider the case when $K := \{x \in X \mid Lx = y\}$ where $L \in \mathcal{L}(X,Y)$ is surjective. Then the differential equation yielding slow viable solutions is given by

$$x'(t) = -G(x(t))^\star L^\star (LG(x(t))G(x(t))^\star L^\star)^{-1} Lc(x(t))$$

When $Y := \mathbf{R}$ and $K := \{x \in X \mid < p, x >= y\}$ is a hyperplane, the above equation becomes

$$x'(t) = -\frac{< p, c(x(t)) >}{\|G(x(t))^\star p\|^2} G(x(t))^\star p$$

3 Bounded Inflation and Heavy Evolution

Introduction

Let us still consider the problem of regulating a dynamical economy

$$(i) \quad \text{for almost all } t \geq 0, \; x'(t) = c(x(t), p(t)) \; \text{ where } \; p(t) \in P(x(t))$$

where $P : K \rightsquigarrow Z$ associates with each commodity x the set $P(x)$ of feasible prices (in general commodity-dependent) and $c : \text{Graph}(P) \mapsto X$ is the change function.

For simplicity, we take for allocation set the domain $K := \text{Dom}(P)$ of P[31].

We have seen in the preceding section that viable price functions (which provide viable solutions $x(t) \in K := \text{Dom}(P)$) are the ones obeying the regulation law

$$\forall\, t \geq 0, \;\; p(t) \in \Pi_K(t) \;\; (\text{or } \; (x(t), p(t)) \in \text{Graph}(\Pi_K))$$

[31] or we replace P by its restriction to K.

where

$$\forall x \in K, \ \Pi_K(x) = \{ p \in P(x) \mid c(x,p) \in T_K(x) \}$$

But, as we have seen in the simple example of the Introduction, heavy evolutions (evolutions with constant price) have to be switched instantateneously when the boundary of the set of allocations is reached.

In order to avoid the use of such **impulses** (which are however observed in real economic systems in times of crisis), we may impose a bound on the velocity of the prices, i.e., a constraint of the form

$$\text{for almost all } t \geq 0, \quad \|p'(t)\| \leq \varphi(x(t),p(t))$$

and start anew the study of the evolution of the dynamical economy under this new constraint.

Therefore, in this section, we are looking for systems of differential equations or of differential inclusions governing **the evolution of both viable commodities and prices**, so that we can look for

— **heavy solutions**, which are evolutions where the prices evolve with minimal velocity

— **punctuated equilibria**, i.e., evolutions in which the price $\bar{p}$ remains constant whereas the commodity may evolve in the associated **viability cell**, which is the viability domain of $x \mapsto c(x,\bar{p})$,

The idea which allows us to achieve these aims is quite simple: **we differentiate the regulation law.**

This is possible whenever we know how to differentiate set-valued maps. Hence the first section is devoted to the definition and the elementary properties of the contingent derivative[32] $DF(x,y)$ of a set-valued map $F : X \rightsquigarrow Y$ at a point (x,y) of its graph: By definition, its graph is the contingent cone to the graph of F at (x,y). We refer to Chapter 5 of SET-VALUED ANALYSIS for further information on the differential calculus of set-valued maps.

In the second section, we differentiate the regulation law and deduce that

$$(ii) \quad \text{for almost all } t \geq 0, \ p'(t) \in D\Pi_K(x(t),p(t))(c(x(t),p(t)))$$

whenever the viable price $p(\cdot)$ is absolutely continuous,

This is the second half of the system of differential inclusions we are looking for.

[32] We set $Df(x) := Df(x,f(x))$ whenever c is single-valued. When f is Fréchet differentiable at x, then $Df(x)(v) = f'(x)v$ is reduced to the usual directional derivative.

Observe that this new differential inclusion has a meaning whenever the commodity-price pair $(x(\cdot), p(\cdot))$ remains viable in the graph of Π_K.

Fortunately, by the very definition of the contingent derivative, the graph of Π_K is a viability domain of the new system (i), (ii).

Unfortunately, as soon as viability constraints involve inequalities, there is no hope for the graph of the contingent cone, and thus, for the graph of the regulation map, to be closed, so that, the Viability Theorem cannot apply.

A strategy inspired from economic motivations to overcome the above difficulty is to bound the inflation, for instance

$$(iii) \text{ for almost all } t \geq 0, \quad \|p'(t)\| \leq \varphi(x(t), p(t))$$

In this case, we shall look for **graphs of closed set-valued maps** Π contained in Graph(P) which are viable under the system of differential inclusions. We already illustrated that in the simple economic example of Introduction.

Such set-valued maps Π are solutions to the **system of first-order partial differential inclusions**

$$\forall\, x \in K, \quad 0 \in D\Pi(x,p)(c(x,p)) - \varphi(x,p)B$$

satisfying the constraint

$$\forall\, (x,p) \in \mathrm{Graph}(\Pi), \quad \Pi(x) \subset P(x)$$

Since we shall show that such closed set-valued maps Π are all contained in the regulation map Π_K, we call them **subregulation maps** associated with the economy with bounded inflation $i), iii)$. In particular, there exists a largest subregulation map denoted Π^{φ}.

In particular, any **single-valued** $\varpi : K \mapsto Z$ with closed graph which is a solution to the **partial differential inclusion**

$$\forall\, x \in K, \quad 0 \in D\varpi(x)(c(x, \varpi(x))) - \varphi(x, \varpi(x))B$$

satisfying the constraint

$$\forall\, x \in K, \quad \varpi(x) \in P(x)$$

provides "feedback prices" regulating smooth allocations of the dynamical economy.

Set-valued and single-valued solutions to these partial differential inclusions are studied in Section 6 of Chapter 8 of VIABILITY THEORY, [?, Aubin].

Let us consider such a subregulation map Π. Viability Theorem 1.8 implies that **whenever the initial commodity x_0 is chosen in** $\text{Dom}(\Pi)$) **and the initial price p_0 in** $\Pi(x_0)$, **there exists a solution to the system of differential inclusions** (i), (iii) **viable in** $\text{Graph}(\Pi)$. The regulation law for the viable commodity-prices becomes

$$(iv) \quad p'(t) \in D\Pi(x(t), p(t))(c(x(t), p(t))) \cap \varphi(x(t), p(t))B$$

We call it the **metaregulation law** associated with the subregulation map Π.

This is how we can obtain **smooth viable commodity-price solutions** to our pricing problem by solving the system of differential inclusions (i), (iv).

Section 3 is devoted to selection procedures of **dynamical feedbacks**, which are selections $g(\cdot,\cdot)$ of the metaregulation map

$$(x,p) \rightsquigarrow D\Pi(x,p)(c(x,p)) \cap \varphi(x,p)B$$

They can be obtained through **selection procedures** introduced in the preceding section.

Naturally, under adequate assumptions, we shall check in Section 4 that Michael's Theorem implies the existence of a continuous dynamical feedback. But under the same assumptions, we can take as dynamical feedback the minimal selection $g^\circ(\cdot,\cdot)$ defined by $\|g^\circ(x,p)\| = \min_{v \in D\Pi(x,p)(c(x,p))} \|v\|$, which, in general, is not continuous.

However, we shall prove that this minimal dynamical feedback still yields smooth viable price-commodity solutions to the system of differential equations

$$x'(t) = c(x(t), p(t)) \quad \& \quad p'(t) = g^\circ(x(t), p(t))$$

called **heavy viable solutions**, (heavy in the sense of heavy trends.) They are the ones for which **the price evolves with minimal velocity**.

Heavy viable solutions obey the **inertia principle: "keep the prices constant as long as they provide viable allocations"**.

Indeed, if zero belongs to $D\Pi(x(t_1), p(t_1))(c(x(t_1), p(t_1)))$, then the price will remain equal to $p(t_1)$ as long as for $t \geq t_1$, a solution $x(\cdot)$ to the differential equation $x'(t) = c(x(t), p(t_1))$ satisfies the condition $0 \in D\Pi(x(t_1), p(t_1))(c(x(t_1), p(t_1)))$.

If at some time t_f, $p(t_f)$ is a "punctuated equilibrium", then the solution enters the viability cell associated to this price and may remain in this viability cell forever[33] and the price will remain equal to this punctuated equilibrium.

3.1 Contingent Derivatives

By coming back to the original point of view proposed by Fermat, we are able to geometrically define the derivatives of set-valued maps from the choice of tangent cones to the graphs, even though they yield very strange limits of differential quotients.

Definition 3.1 *Let $F : X \rightsquigarrow Y$ be a set-valued map from a normed space X to another normed space Y and $y \in F(x)$.*

The contingent derivative *$DF(x,y)$ of F at $(x,y) \in \text{Graph}(G)$ is the set-valued map from X to Y defined by*

$$\text{Graph}(DF(x,y)) \;:=\; T_{\text{Graph}(F)}(x,y)$$

When $F := f$ is single-valued, we set $Df(x) := Df(x, f(x))$.

We shall say that F is sleek *at $(x,y) \in \text{Graph}(F)$ if and only if the map*

$$(x',y') \in \text{Graph}(F) \rightsquigarrow \text{Graph}(DF(x',y'))$$

is lower semicontinuous at (x,y) (i.e., if the graph of F is sleek at (x,y).) The set-valued map F is sleek *if it is sleek at every point of its graph.*

Naturally, when the map is sleek at (x,y), the contingent derivative $DF(x,y)$ is a closed convex process.

We can easily compute the derivative of the inverse of a set-valued map F (or even of a noninjective single-valued map): The contingent derivative of the inverse of a set-valued map F is the inverse of the contingent derivative:

$$D(F^{-1})(y,x) \;=\; DF(x,y)^{-1}$$

If K is a subset of X and f is a single-valued map which is Fréchet differentiable around a point $x \in K$, then the contingent derivative of the restriction of f to K is the restriction of the derivative to the contingent cone:

$$D(f|_K)(x) = D(f|_K)(x, f(x)) = f'(x)|_{T_K(x)}$$

These contingent derivatives can be characterized by adequate limits of differential quotients:

[33] as long as the set of available resources does not change for external reasons which are not taken into account here.

Proposition 3.2 *Let* $(x,y) \in \mathrm{Graph}(F)$ *belong to the graph of a set-valued map* $F : X \rightsquigarrow Y$ *from a normed space* X *to a normed space* Y. *Then*

$$\left\{ \begin{array}{l} v \in DF(x,y)(u) \text{ if and only if} \\ \mathrm{liminf}_{h\to 0+, u'\to u} d\left(v, \frac{F(x+hu')-y}{h}\right) = 0 \end{array} \right.$$

If $x \in \mathrm{Int}(\mathrm{Dom}(F))$ *and* F *is Lipschitz around* x, *then*

$$v \in DF(x,y)(u) \text{ if and only if } \liminf_{h\to 0+} d\left(v, \frac{F(x+hu)-y}{h}\right) = 0$$

If moreover the dimension of Y *is finite, then*

$$\mathrm{Dom}(DF(x,y)) = X \text{ and } DF(x,y) \text{ is Lipschitz}$$

Proof — The first two statements being obvious, let us check the last one. Let u belong to X and l denote the Lipschitz constant of F on a neighborhood of x. Then, for all $h > 0$ small enough and $y \in F(x)$,

$$y \in F(x) \subset F(x+hu) + lh\|u\|B$$

Hence there exists $y_h \in F(x+hu)$ such that $v_h := (y_h - y)/h$ belongs to $l\|u\|B$, which is compact. Therefore the sequence v_h has a cluster point v, which belongs to $DF(x,y)(u)$. □

Remark — Lower Semicontinuously Differentiable Maps The lower semicontinuity of the set-valued map

$$(x,y,u) \in \mathrm{Graph}(F) \times X \rightsquigarrow DF(x,y)(u)$$

at some point (x_0, y_0, u_0) is often needed. Observe that it implies that F is sleek at (x_0, y_0). The converse needs further assumptions. We derive for instance the following criterion:

Proposition 3.3 *Assume that* X *and* Y *are Banach spaces and that* F *is sleek on some neighborhood* $\mathcal{P}$ *of* $(x_0, y_0) \in \mathrm{Graph}(F)$. *If the boundedness property*

$$\forall\, u \in X, \quad \sup_{(x,y)\in\mathcal{P}\cap\mathrm{Graph}(F)} \inf_{v\in DF(x,y)(u)} \|v\| < +\infty$$

holds true, then the set-valued map

$$(x,y,u) \in \mathrm{Graph}(F) \times X \rightsquigarrow DF(x,y)(u)$$

is lower semicontinuous on $(\mathcal{P} \cap \mathrm{Graph}(F)) \times X$

3.2 Bounded Inflation

3.2.1 Subregulation and Metaregulation Maps

Let us consider a dynamical economy (P, c) defined by a set-valued pricing map $P : X \leadsto Z$ and a single-valued change map $c : \mathrm{Graph}(P) \mapsto X$, where X is regarded as the commodity space, Z the price space. The evolution of a commodity-price solution $(x(\cdot), p(\cdot))$ viable in Graph(P) is governed by

$$x'(t) = c(x(t), p(t)), \; p(t) \in P(x(t)) \tag{3.1}$$

We shall look for viable solutions in $K := \mathrm{Dom}(P)$ which are smooth in the following sense:

Definition 3.4 (Smooth Commodity-Price) *We say that the pair* $(x(\cdot), p(\cdot))$ *is* smooth *if both* $x(\cdot)$ *and* $p(\cdot)$ *are absolutely continuous.*

It is said to be φ-smooth *if in addition for almost all* $t \geq 0$, $\|p'(t)\| \leq \varphi(x(t), p(t))$, *where* $\varphi : X \times Z \mapsto \mathbf{R}_+$.

We obtain smooth viable solutions by bounding inflation, i.e., setting a bound to the growth to the evolution of prices, as we did in the simple economic example of the Introduction.

For that purpose, we associate to this dynamical economy and to any nonnegative continuous function $p \to \varphi(x, p)$ with linear growth[34] the system of differential inclusions

$$\begin{cases} i) & x'(t) = c(x(t), p(t)) \\ ii) & p'(t) \in \varphi(x(t), p(t))B \end{cases} \tag{3.2}$$

Observe that any solution $(x(\cdot), p(\cdot))$ to (3.2) viable in Graph(P) is a φ-smooth solution to the dynamical economy (3.1).

Therefore, we are looking for closed set-valued feedback maps Π contained in P whose graphs are viable under the system (3.2).

We thus deduce from the Viability Theorem 1.8 applied to the system (3.2) on the graph of P the following

[34] which can be a constant ρ, or the function $(x, p) \to \rho\|p\|$, or the function $(x, p) \to \rho(\|p\| + \|x\| + 1)$. One could also take other dynamics $p' \in \Phi(x, p)$ where Φ is a Marchaud map.

Theorem 3.5 *Let us assume that the dynamical economy (3.1) satisfies*

$$\begin{cases} i) & \text{Graph}(P) \text{ is closed} \\ ii) & c \text{ is continuous and has linear growth} \end{cases} \quad (3.3)$$

Let $(x,p) \to \varphi(x,p)$ be a nonnegative continuous function with linear growth and $\Pi : Z \leadsto X$ a closed set-valued map contained in P. Then the two following conditions are equivalent:

a) — *Π regulates φ-smooth viable solutions in the sense that from any initial commodity $x_0 \in \text{Dom}(\Pi)$ and any initial price $p_0 \in \Pi(x_0)$ starts a φ-smooth commodity-price solution $(x(\cdot), p(\cdot))$ to the dynamical economy (3.1) viable in the graph of Π.*

b) — *Π is a solution to the* partial differential inclusion

$$\forall\,(x,p) \in \text{Graph}(\Pi),\ \ 0 \in D\Pi(x,p)(c(x,p)) - \varphi(x,p)B \quad (3.4)$$

satisfying the constraint: $\forall\, x \in K,\ \ \Pi(x) \subset P(x)$.

In this case, such a map Π is contained in the regulation map Π_K defined by

$$\forall\, x \in K,\ \Pi_K(x) = \{\, p \in P(x) \mid c(x,p) \in T_K(x)\,\}$$

*and is thus called a φ-*subregulation map of *P or simply a* subregulation map. *The* metaregulation law *regulating the evolution of commodity-price solutions viable in the graph of Π takes the form of the system of differential inclusions*

$$\begin{cases} i) & x'(t) = c(x(t), p(t)) \\ ii) & p'(t) \in G_\Pi(x(t), p(t)) \end{cases} \quad (3.5)$$

where the set-valued map G_Π defined by

$$G_\Pi(x,p) := D\Pi(x,p)(c(x,p)) \cap \varphi(x,p)B$$

is called the metaregulation map *associated with Π.*

Remark — One can also prove that there exists a largest φ-subregulation map denoted Π^φ contained in P[35] □

[35] The graph of Π^φ is the viability kernel of Graph(P) for the system of differential inclusions (3.2)). See VIABILITY THEORY, [?, Aubin].

Proof — Indeed, to say that Π is a regulation map regulating φ-smooth solutions amounts to saying that its graph is viable under the system (3.2).

In this case, we deduce that for any $(x_0, p_0) \in \text{Graph}(\Pi)$, there exists a solution $(x(\cdot), p(\cdot))$ viable in the graph of P, so that $x(\cdot)$ is in particular viable in K. Since $x'(t) = c(x(t), p(t))$ is absolutely continuous, we infer that $c(x_0, p_0)$ is contingent to K at x_0, i.e., that p_0 belongs to $\Pi_K(x_0)$.

The regulation map for the system (3.2) associates with any $(x, p) \in \text{Graph}(\Pi)$ the set of pairs $(x', p') \in \{c(x, p)\} \times \varphi(x, p)B$ such that (x', p') belongs to the contingent cone to the graph of Π at (x, p), i.e., such that

$$p' \in D\Pi(x, p)(c(x, p)) \cap \varphi(x, p)B =: G_\Pi(x, p)$$

We can be particularly interested in **single-valued regulation maps** $\varpi : K \mapsto Z$, which are **feedback prices regulating** φ**-smooth viable solutions** and then be interpreted as **planning mechanisms:**

Proposition 3.6 *A closed single-valued continuous map* ϖ *is a feedback price regulating* φ*-smooth viable solutions to the price problem if and only if* ϖ *is a single-valued solution to the* inclusion

$$\forall\, x \in K, \;\; 0 \in D\varpi(x)(c(x, \varpi(x))) - \varphi(x, \varpi(x))B$$

satisfying the constraint

$$\forall\, x \in K, \;\; \varpi(x) \in P(x)$$

Then for any $x_0 \in K$*, there exists a solution to the differential equation* $x'(t) = c(x(t), \varpi(x(t)))$ *starting at* x_0 *such that*

$$\forall\, t \geq 0, \; p(t) := \varpi(x(t)) \in P(x(t))$$

and

$$\text{for almost all } t \geq 0, \; \|p'(t)\| \leq \varphi(x(t), \varpi(x(t)))$$

Remark — We observe that any φ-subregulation map remains a ψ-subregulation map for $\psi \geq \varphi$. □

Example: Equality Constraints Consider the case when $h : X \mapsto Y$ is a twice continuously differentiable map and when the viability domain is $K := h^{-1}(0)$.

Since $T_K(x) = \ker h'(x)$ when $h'(x)$ is surjective, we deduce that the regulation map is equal to

$$\Pi_K(x) = \{p \in P(x) \mid h'(x)c(x,p) = 0\}$$

Proposition 3.7 *Assume that $h'(x) \in \mathcal{L}(X,Y)$ is surjective whenever $h(x) = 0$, that the graph of P is sleek and that for any $y \in Y$ and $v \in X$, the subsets*

$$DP(x,p)(v) \cap (h'(x)c'_p(x,p))^{-1}(y - h''(x)(c(x,p),v) - h'(x)c'_x(x,p)v)$$

are not empty. Then the contingent derivative $D\Pi_K(x,p)(v)$ of the regulation map is equal to

$$DP(x,p)(v) \cap -(h'(x)c'_p(x,p))^{-1}(h''(x)(c(x,p),v) - h'(x)c'_x(x,p)v)$$

when $h'(x)v = 0$ and $D\Pi_K(x,v) = \emptyset$ if not. In particular, if $P(x) \equiv Z$, then it is sufficient to assume that $h'(x)c'_p(x,p)$ is surjective and we have in this case

$$D\Pi_K(x,p)(v) = -(h'(x)c'_p(x,p))^{-1}(h''(x)(c(x,p),v) - h'(x)c'_x(x,p)v)$$

when $h'(x)v = 0$ and $D\Pi_K(x,v) = \emptyset$ if not.

Proof — The graph of Π_K can be written as the subset of pairs $(x,p) \in \text{Graph}(P)$ such that $C(x,p) := (h(x), h'(x)c(x,p)) = 0$. Since the graph of P is closed and sleek, we know that the transversality condition

$$C'(x,p)T_{\text{Graph}(P)}(x,p) = C'(x,p)\text{Graph}(DP(x,p)) = Y \times Y$$

implies that the contingent cone to the graph of P is the set of elements $(v,w) \in \text{Graph}(DP(x,p))$ such that

$$\begin{cases} C'(x,p)(v,w) = \\ (h'(x)v, h'(x)c'_p(x,p)w + h'(x)c'_x(x,p)v + h''(x)(c(x,p),v)) = 0 \end{cases}$$

But the surjectivity of $h'(x)$ and the nonemptiness of the intersection imply this transversality condition. □

Therefore, the right-hand side of the metaregulation rule is equal to

$$\begin{cases} -(h'(x)c'_p(x,p))^{-1}(h''(x)(c(x,p),c(x,p)) - h'(x)c'_x(x,p)c(x,p)) \\ \cap DP(x,p)(c(x,p)) \cap \varphi(x,p)B \end{cases}$$

Example: Inequality Constraints Consider the case when

$$K := \{x \in X : \forall i = 1, \ldots, p, \; g_i(x) \geq 0\}$$

is defined by inequality constraints (for simplicity, we do not include equality constraints.)

We denote by $I(x) := \{i = 1, \ldots, p \mid g_i(x) = 0\}$ the subset of **active constraints** and we assume once and for all that for every $x \in K$,

$$\exists \, v_0 \in C_L(x) \quad \text{such that } \forall \, i \in I(x), \;\; < g_i'(x), v_0 >> 0$$

so that, by Theorem 2.28,

$$\Pi_K(x) := \{p \in P(x) \mid \forall \, i \in I(x), \langle g_i'(x), c(x,p)\rangle \geq 0\}$$

We set $g(x) := (g_1(x), \ldots, g_p(x))$.

We have seen that the graph of the set-valued map $x \rightsquigarrow \Pi_K(x)$ is not necessarily closed. However, we can find explicit subregulation maps by using Theorem 2.29. We thus introduce the set-valued map $\Pi_K^\circ : X \rightsquigarrow Z$ defined by

$$\Pi_K^\circ(x) := \{p \in P(x) \mid g(x) + g'(x)c(x,p) \geq 0\} \subset \Pi_K(x)$$

We can regulate solutions viable in K by smooth open-loop prices by looking for solutions to the system of differential inclusions (3.2) which are viable in the graph of Π_K°.

We thus need to compute the derivative of Π_K° in order to characterize the associated metaregulation map:

Proposition 3.8 *Assume that the stronger viability condition*[36]

$$\forall \, x \in K, \;\; \Pi_K^\circ(x) \neq \emptyset$$

is satisfied. We set

$$I(x,p) := \{i = 1, \ldots, p \mid g_i(x) + \langle g_i'(x), c(x,p)\rangle = 0\}$$

[36] which holds true whenever K is a viability domain for the dynamical economy and

$$\forall \, x \in K, \;\; \exists \, p \in P(x) \;\; \text{such that } \|c(x,p)\| \leq \gamma_K(x)$$

where the function γ_K is defined by (2.9) in Section 5.1. See Theorem 2.29.

Assume that P is sleek and closed and that for every $(x,p) \in \text{Graph}(\Pi_K^\circ)$, there exists $p_0' \in DP(x,p)(x_0')$ satisfying

$$\forall\, i \in I(x,p),\ \langle g_i'(x), x_0' + c_x'(x,p)x_0' + c_p'(x,p)p_0'\rangle + g_i''(x)(c(x,p),x_0') \geq 0$$

Then the contingent derivative $D\Pi_K^\circ(x,p)(v)$ of the subregulation map Π_K° is defined by: $p' \in D\Pi_K^\circ(x,p)(x')$ if and only if $p' \in DP(x,p)(x')$ and

$$\forall\, i \in I(x,p),\ \langle g_i'(x), x' + c_x'(x,p)x' + c_p'(x,p)p'\rangle + g_i''(x)(c(x,p),x') \geq 0$$

If $P(x) \equiv Z$, then it is sufficient to assume that $g'(x)c_p'(x,p)$ is surjective. We then have in this particular case

$$\left\{\begin{array}{l} D\Pi_K^\circ(x,p)(x') \ := \{p' \in Z \mid \forall\, i \in I(x,p), \\ \langle g_i'(x), c_p'(x,p)p'\rangle \ \geq \ -\langle g_i'(x), x' + c_x'(x,p)x'\rangle - g_i''(x)(c(x,p),x')\big\} \end{array}\right.$$

Proof — By Theorem 2.28 applied to $L := \text{Graph}(P)$ and to the constraints defined by $\tilde{g}_i(x,p) := g_i(x) + \langle g_i'(x), c(x,p)\rangle$, we deduce that $p' \in D\Pi_K^\circ(x,p)(x')$ if and only if $p' \in DP(x,p)(x')$ and

$$\forall\, i \in I(x,p),\ \langle g_i'(x), x' + c_x'(x,p)x' + c_p'(x,p)p'\rangle + g_i''(x)(c(x,p),x') \geq 0 \ \square$$

We then deduce from the above Proposition and the Regularity Theorem the following consequence:

Proposition 3.9 *We posit the assumptions of Proposition 3.8. If for any $(x,p) \in \text{Graph}(\Pi_K^\circ)$, there exists p' such that $\|p'\| \leq \varphi(x,p)$, then for any initial commodity x_0 and any $p_0 \in \Pi_K^\circ(x_0)$, there exists a solution $(x(\cdot),p(\cdot))$ to the dynamical economy (3.2) such that $x(\cdot)$ is viable in the set K defined by inequality constraints. The metaregulation law can then be written*

$$\left\{\begin{array}{lrcl} i) & x'(t) & = & c(x(t),p(t)) \\ \\ ii) & p'(t) & \in & G(x(t),p(t)) \end{array}\right. \tag{3.6}$$

where the metaregulation map G associated to Π_K°

$$G(x,p) \ := \ D\Pi_K^\circ(x,p)(c(x,p)) \cap \varphi(x,p)B$$

defined by:
$w \in G(x,p)$ if and only if $w \in DP(x,p)(c(x,p)) \cap \varphi(x,p)B$ and

$$\left\{\begin{array}{l} \forall\, i \in I(x,p),\ \ \langle g_i'(x), c_p'(x,p)p'\rangle \\ \geq\ -\langle g_i'(x), c(x,p) + c_x'(x,p)c(x,p)\rangle - g_i''(x)(c(x,p),c(x,p)) \end{array}\right.$$

Naturally, the graph of the metaregulation map G is not necessarily closed. However, we can still use Theorem 2.29 to obtain a "submetaregulation map" of this system of differential inclusions. We introduce the set-valued map G° defined by: $p' \in G^\circ(x,p)$ if and only if $\|p'\| \leq \varphi(x,p)$ and

$$\begin{cases} \forall\, i = 1,\ldots,p, \ \langle g_i'(x), c_p'(x,p)p'\rangle \\ \geq \ -g_i(x) - \langle g_i'(x), 2c(x,p) + c_x'(x,p)c(x,p)\rangle - g_i''(x)(c(x,p),c(x,p)) \end{cases}$$

Hence the system of differential inclusions

$$\begin{cases} i) & x'(t) \ = \ c(x(t),p(t)) \\ \\ ii) & p'(t) \ \in \ G^\circ(x(t),p(t)) \cap \varphi(x(t),p(t))B \end{cases} \tag{3.7}$$

regulates φ-smooth solutions which are viable in K.

3.2.2 Punctuated Equilibria

The case when the inflation bound φ is equal to 0 is particularly interesting, because the inverse N^0 of the 0-growth regulation map Π^0 determines the areas $N^0(p)$ regulated by constant price p.

One could call $N^0(p)$ the **viability cell or niche** of p. A price p is called a **punctuated equilibrium** if and only if its viability cell is not empty. Naturally, **when the viability cell of a punctuated equilibrium is reduced to a point, this point is an equilibrium.**

So, punctuated equilibria are constant prices which regulate the dynamical economies (in its viability cell):

Proposition 3.10 *The viability cell of a price p is the viability kernel of $P^{-1}(p)$ for the differential equation $x'(t) = c(x(t),p)$ parametrized by the constant price p.*

Proof — Indeed, viability cells describe the regions of $\mathrm{Dom}(P)$ which are regulated by the constant price p because for any initial commodity x_0 given in $N^0(p)$, there exists a viable solution $x(\cdot)$ to the differential inclusion

$$\begin{cases} i) & x'(t) \ = \ c(x(t),p(t)) \\ ii) & p'(t) \ = \ 0 \end{cases}$$

starting at (x_0,p), i.e., of the differential equation $x'(t) = c(x(t),p)$ which is **viable in the viability cell** $N^0(p)$ because $p \in \Pi^0(x(t))$ for every $t \geq 0$. □

3.3 Dynamical Feedbacks

Let us consider a dynamical economy (P, c), a regulation map $\Pi \subset P$ which is a solution to the partial differential inclusion (3.4) and the metaregulation map

$$(x,p) \rightsquigarrow G_\Pi(x,p) := D\Pi(x,p)(c(x,p)) \cap \varphi(x,p)B$$

regulating smooth commodity-price solutions viable in the graph of Π through the system (3.5) of differential inclusions.

The question arises as to whether we can construct selection procedures of the price component of this system of differential inclusions. It is convenient for this purpose to introduce the following definition.

Definition 3.11 *Let Π be a φ-growth subregulation map of P. We shall say that a selection g of the metaregulation map G_Π associated with Π mapping every $(x,p) \in \mathrm{Graph}(\Pi)$ to*

$$g(x,p) \in G_\Pi(x,p) := D\Pi(x,p)(c(x,p)) \cap \varphi(x,p)B \tag{3.8}$$

is a dynamical closed loop *of Π.*

The system of differential equations

$$\begin{cases} i) & x'(t) = c(x(t),p(t)) \\ ii) & p'(t) = g(x(t),p(t)) \end{cases} \tag{3.9}$$

is called the associated feedback differential system.

Clearly every solution to (3.9) is also a solution to (3.5). Therefore, a dynamical feedback being given, solutions to the system of ordinary differential equations (3.9) (if any) are smooth commodity-price solutions of the initial price problem (3.1).

In order to obtain dynamical feedback prices, we shall use selection procedures (See Definition 2.1) of the metaregulation map $G_\Pi(x,p)$.

Theorem 3.12 *Let us assume that the dynamical economy (3.1) satisfies*

$$\begin{cases} i) & \mathrm{Graph}(P) \text{ is closed} \\ ii) & c \text{ is continuous and has linear growth} \end{cases} \tag{3.10}$$

Let $(x,p) \to \varphi(x,p)$ be a nonnegative continuous function with linear growth and $\Pi : Z \rightsquigarrow X$ a closed set-valued map contained in P.

Let S_{G_Π} : Graph(Π) $\rightsquigarrow X$ *be a selection procedure with convex values of the metaregulation map* G_Π. *Then, for any initial commodity* $(x_0, p_0) \in$ Graph(Π), *there exists a commodity-price solution to the associated feedback system*

$$x' = c(x,p), \quad p' \in G_R(x,p) \cap S_{G_R}(x,p) \tag{3.11}$$

defined on $[0, \infty[$ *and starting at* (x_0, p_0). *In particular, if for any* $(x,p) \in$ Graph(Π), *the intersection*

$$G_R(x,p) \cap S_{G_R}(x,p) \;=\; \{s\,(G_\Pi(x,p))\}$$

is a singleton, then there exists a commodity-price solution defined on $[0, \infty[$ *and starting at* (x_0, p_0) *to the associated feedback system*

$$x'(t) \;=\; c(x(t), p(t)), \;\; p'(t) \;=\; s\,(G_\Pi(x(t), p(t)))$$

Proof — Consider the system of differential inclusions

$$x' \;= c(x,p), \;\; p' \;\in\; S_{G_R}(x,p) \cap \varphi(x,p)B \tag{3.12}$$

subject to the constraints

$$\forall\, t \geq 0, \;\; (x(t), p(t)) \;\in\; \text{Graph}(\Pi)$$

Since the selection procedure S_{G_R} has a closed graph and convex values, the right-hand side is an upper semicontinuous set-valued map with nonempty compact convex images and with linear growth. On the other hand Graph(Π) is a viability domain of the map $\{c(x,p)\} \times (S_{G_R}(x,p) \times \varphi(x,p)B)$. Therefore, the Viability Theorem can be applied. For any initial commodity-price $(x_0, p_0) \in$ Graph(Π), there exists a solution $(x(\cdot), p(\cdot))$ to (3.12) which is viable in Graph(Π). Consequently, for almost all $t \geq 0$, the pair $(x'(t), p'(t))$ belongs to the contingent cone to the graph of Π at $(x(t), p(t))$, which is the graph of the contingent derivative $D\Pi(x(t), p(t))$. In other words, for almost all $t \geq 0$, $p'(t) \in D\Pi(x(t), p(t))(c(x(t), p(t)))$. Since $\|p'(t)\| \leq \varphi(x(t), p(t))$, we deduce that $p'(t) \in G_R(x(t), p(t))$ for almost all $t \geq 0$. Hence, the commodity-price pair $(x(\cdot), p(\cdot))$ is a solution to (3.11). □

3.4 Heavy Viable Solutions

3.4.1 Continuous Dynamical Feedback

Such solutions do exist when g is continuous (and if such is the case, they will be continuously differentiable.) But they also may exist when g is no longer continuous. This is the case for instance when $g(x,p)$ is the element of minimal norm in $G_\Pi(x,p)$.

In both cases, we need to assume that the metaregulation map G_Π associated with Π is lower semicontinuous with closed convex images. By Proposition 3.3, it will be sufficient to assume that:

$$\begin{cases} i) & \Pi \text{ is sleek} \\ ii) & \sup_{(x,p)\in\text{Graph}(\Pi)} \|D\Pi(x,p)\| < +\infty \end{cases} \quad (3.13)$$

Indeed, assumptions (3.13)i) and ii) imply that the set-valued map $(x,p,v) \rightsquigarrow D\Pi(x,p,v)$ is lower semicontinuous. Since φ is continuous, we infer from Proposition 2.20 that the metaregulation map G_Π is also lower semicontinuous.

We thus begin by deducing from Michael's Theorem 2.14 the existence of continuously differentiable viable commodity-price solutions.

Theorem 3.13 *Assume that P is closed and that c, φ are continuous and have linear growth. Let $\Pi(\cdot) \subset P(\cdot)$ be a φ-growth subregulation map satisfying assumption (3.13). Then there exists a continuous dynamical feedback g associated with Π. The associated feedback differential system (3.9) regulates continuously differentiable commodity-price solutions to (3.1) defined on $[0,\infty[$.*

3.4.2 Heavy Solutions and the Inertia Principle

Since we do not know constructive ways to build continuous dynamical feedbacks, we shall investigate whether some explicit dynamical feedback provides feedback differential systems which do possess solutions.

The simplest example of dynamical feedback price is the minimal selection of the metaregulation map G_Π, which in this case is equal to the map g_Π° associating with each commodity-price pair (x,p) the element $g_\Pi^\circ(x,p)$ of minimal norm of $D\Pi(x,p)(c(x,p))$ because for all (x,p), $\|g_\Pi^\circ(x,p)\| \leq \varphi(x,p)$ whenever $G_\Pi(x,p) \neq \emptyset$.

Definition 3.14 (Heavy Viable Solutions) *Denote by $g_\Pi^\circ(x,p)$ the element of minimal norm of $D\Pi(x,p)(c(x,p))$. We shall say that the solutions to the associated feedback differential system*

$$\begin{cases} i) & x'(t) = c(x(t),p(t)) \\ ii) & p'(t) = g_\Pi^\circ(x(t),p(t)) \end{cases}$$

are **heavy viable solutions** *to the dynamical economy* (P, c) *associated with* Π.

Theorem 3.15 (Heavy Viable Solutions) *Let us assume that P is closed and that c, φ are continuous and have linear growth. Let $\Pi(\cdot) \subset P(\cdot)$ be a φ-growth subregulation map satisfying assumption (3.13). Then for any initial commodity-price pair (x_0, p_0) in Graph(Π), there exists a heavy viable solution to the dynamical economy (3.1).*

Remark — **Any heavy viable solution $(x(\cdot), p(\cdot))$ to the dynamical economy (3.1) satisfies the inertia principle:** Indeed, we observe that if for some t_1, the solution enters the subset $C_\Pi(p(t_1))$ where we set

$$C_\Pi(p) := \{x \in K \mid 0 \in D\Pi(x,p)(c(x,p))\}$$

the price $p(t)$ remains equal to $p(t_1)$ as long as $x(t)$ remains in $C_\Pi(p(t_1))$. Since such a subset is not necessarily a viability domain, the solution may leave it.

If for some $t_f > 0$, $p(t_f)$ is a punctuated equilibrium, then $p(t) = p_{t_f}$ for all $t \geq t_f$ and thus, $x(t)$ **remains in the viability cell** $N_1^0(p(t_f))$ **for all** $t \geq t_f$. □

Proof of Theorem 3.15 —

The reason why this theorem holds true is that the minimal selection is obtained through the selection procedure defined by

$$S^\circ_{G_\Pi}(x,p) := \|g^\circ_\Pi(x,p)\| B \tag{3.14}$$

By the Maximum Theorem 2.9 the map $(x,p) \mapsto \|g^\circ_\Pi(x,p)\|$ is upper semicontinuous. It has a linear growth on Graph(Π). Thus the set-valued map $(x,p) \rightsquigarrow \|g^\circ_\Pi(x,p)\| B$ is a selection procedure satisfying the assumptions of Theorem 3.12. □

Since we know many examples of selection procedures, it is possible to multiply examples of dynamical feedbacks as we did for usual feedbacks. We shall see some examples in the framework of differential games in Chapter 14.

3.4.3 Heavy Viable Solutions under Equality Constraints

Consider the case when $h : X \mapsto Y$ is a twice continuously differentiable map, when the viability domain is $K := h^{-1}(0)$ and when there are no constraints on the prices ($P(x) = Z$ for all $x \in K$). We derive from Proposition 3.7 the following explicit formulas for the dynamical feedback yielding heavy solutions.

Proposition 3.16 *We posit assumptions of Theorem 3.7. Assume further that* $P(x) \equiv Z$*, that the regulation map*

$$\Pi(x) \; := \; \{p \in Z \mid h'(x)c(x,p) = 0\}$$

has nonempty values, that $h(x)$ *is surjective whenever* $x \in K$ *and that* $h'(x)c'_p(x,p) \in \mathcal{L}(Z,Y)$ *is surjective whenever* $p \in \Pi(x)$.

Then there exist heavy solutions viable in K*, which are the solutions to the system of differential equations*

$$\begin{cases} i) & x' = c(x,p) \\ ii) & p' = -c'_p(x,p)^\star h'(x)^\star k(x,p) \text{ where} \\ & k(x,p) := (h'(x)c'_p(x,p)c'_p(x,p)^\star h'(x)^\star)^{-1} h'(x)c'_x(x,p)c(x,p) \end{cases}$$

Proof— The element $g(x,p) \in G(x,p)$ of minimal norm, being a solution to the quadratic minimization problem with equality constraints

$$h'(x)c'_p(x,p)w = -h'(x)c'_x(x,p)c(x,p) - h''(x)(c(x,p),c(x,p))$$

is equal to

$$\begin{array}{l} g(x,p) = -c'_p(x,p)^\star h'(x)^\star (h'(x)c'_p(x,p)c'_p(x,p)^\star h'(x)^\star)^{-1} \\ (h'(x)c'_x(x,p)c(x,p) + h''(x)(c(x,p),c(x,p))) \end{array}$$

because the linear operator $B := h'(x)c'_p(x,p) \in \mathcal{L}(Z,Y)$ is surjective.

Example: Heavy solutions viable in affine spaces. Consider the case when $K := \{x \in X \mid Lx = y\}$ where $L \in \mathcal{L}(X,Y)$ is surjective.

Let us assume that

$$\begin{cases} i) & \forall\, x \in K,\ \Pi(x) := \{p \in Z \ \text{ such that } \ Lc(x,p) = 0\} \neq \emptyset \\ ii) & \forall\, x \in K,\ \forall\, p \in \Pi(x),\ \ Lc'_p(x,p) \ \text{ is surjective} \end{cases}$$

Then, for any initial commodity $x_0 \in K$ and initial velocity p_0 satisfying $Lc(x_0,p_0) = 0$, there exists a heavy viable solution given by the system of differential equations

$$\begin{cases} i) & x' = c(x,p) \\ ii) & p' = -c'_p(x,p)^\star L^\star (Lc'_p(x,p)c'_p(x,p)^\star L^\star)^{-1} Lc'_x(x,p)c(x,p) \end{cases}$$

When $Y := \mathbf{R}$ and $K := \{x \in X \mid \ < p,x >= y\}$ is an hyperplane, the above assumption becomes

$$\begin{cases} i) & \forall\, x \in K,\ \Pi(x) := \{p \in Z \ \text{ such that } \ < p, c(x,p) >= 0\} \neq \emptyset \\ ii) & \forall\, x \in K,\ \forall\, p \in \Pi(x),\ \ c'_p(x,p)^\star p \neq 0 \end{cases}$$

and heavy viable solutions are solutions to the system of differential equations

$$\begin{cases} i) & x' = c(x,p) \\ ii) & p' = -\dfrac{< p, c'_x(x,p), c(x,p) >}{\|c'_p(x,p)^\star p\|^2} c'_p(x,p)^\star p \end{cases}$$

See Part I for a list of references.

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