# Working Paper

## Isaacs' Equations for Value-Functions of Differential Games

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**Hélène Frankowska & Marc Quincampoix**

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# FOREWORD

The authors study value functions of a differential game with payoff which depends on the state at a given end time. They consider differential games with feedback strategies and with nonanticipating strategies. They prove that value-functions are solutions to some Hamilton-Jacobi-Isaacs equations in the viscosity and contingent sense. For these two notions of strategies, with some regularity assumptions, The authors prove that value-functions are the unique solution of Isaacs' equations.

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# Isaacs' equations for value-functions of differential games

**Hélène Frankowska & Marc Quincampoix**

## 1 Introduction

Let us consider the following differential game:

$$(1) \qquad \begin{cases} i) & x'(t) = f(t, x(t), u(t), v(t)), \ t \in [t_0, T] \\ ii) & u(t) \in U \quad v(t) \in V \end{cases}$$

The two players act on the state $x(\cdot)$ by choosing controls, $u$ for the first player and $v$ for the second one. The goal of the first player is to maximize at the given end time $T$ the payoff $g(x(T))$, the second player wants to minimize it.

Let us recall that the game with the following payoff:

$$(2) \qquad g(x(T)) + \int_0^T L(t, x(t), u(t), v(t))dt$$

may be reduced to the above one. In fact, by the simple change of variable $z := (x, y)$, we obtain the new game

$$
(3) \quad \begin{cases} i) & (x'(t), y'(t)) = (f(t, x(t), u(t), v(t)), L(t, x(t), u(t), v(t))) \\ ii) & u(t) \in U \\ iii) & v(t) \in V \end{cases}
$$

with the payoff $G(x(T), y(T)) := g(x(T)) + y(T)$ which is equal to (2).

Since Isaacs (cf [14]), it is well-known that the value-function satisfies a partial differential equation (the Isaacs' equation) when the game is regular enough. The solutions of this equation have been studied by Isaacs himself in $C^1$ case (see [14]), lipschitz solutions have been studied for example in [15], and in [9], [5], [17] for viscosity solutions. However such regularity is not always the case (see also for instance [12], [4], [6]... for control systems).

We introduce two notions of strategies and we prove that the associated value-functions are solutions to Isaacs' equation without any assumptions concerning the regularity of $g$. In this paper, we mainly state results (see [13] for more detailed proofs).

# 2 Feedback strategies of differential games

Consider a function $f : [0, T] \times \mathbb{R}^n \times U \times V \mapsto \mathbb{R}^n$ where $U$ and $V$ are two complete separable metric spaces. Let us denote by $t \mapsto x(t, t_0, x_0, u(\cdot), v(\cdot))$ the solution to (1) corresponding to controls $u(\cdot)$, $v(\cdot)$, starting from $x_0$ at time $t_0$ (i.e. such that $x(t_0) = x_0$). We shall need the following assumptions:

$$(4) \qquad \begin{cases} i) & f \text{ is continuous} \\ \\ ii) & \forall\,(t,x) \in [0,T] \times \mathbb{R}^n,\ \forall\,(u,v) \in U \times V \text{ the sets} \\ & f(t,x,u,V),\ f(t,x,U,v) \text{ are compact and convex} \\ \\ iii) & \forall\,(t,x,u,v) \in [0,T] \times \mathbb{R}^n \times U \times V, \\ & \text{functions } f(t,x,\cdot,v) \text{ and } f(t,x,u,\cdot) \text{ are} \\ & l-\text{Lipschitz, where } l > 0 \\ \\ iv) & \forall\, R > 0,\ \exists\, c_R \in L^1(0,T) \text{ such that for almost all} \\ & t \in [0,T] \text{ and for all } (u,v) \in U \times V,\ f(t,\cdot,u,v) \\ & \text{is } c_R(t) - \text{Lipschitz on } B(0,R). \\ \\ v) & \exists\, k \in L^1(0,T) \text{ such that for almost all } t \in [0,T] \\ & \sup_{u\in U} \sup_{v\in V} \|f(t,x,u,v)\| \leq k(t)(1 + \|x\|) \end{cases}$$

We call *feedback strategy* for the first player any function $\varphi : [0,T] \times X \mapsto U$ such that $(t,x) \mapsto f(t,x,\varphi(t,x),V)$ is upper semicontinuous with respect to $(t,x)$. We denote by $\Phi$ the set of feedback strategies for the first player. In a similar way we can define feedback strategies for the second player and $\Psi$ the set of such strategies. We denote by $\mathcal{U}$ (respectively $\mathcal{V}$) the set of measurable functions $[0,T] \mapsto U$ (respectively $[0,T] \mapsto V$).

We assume furthermore the following crucial condition which allows to define the value function of the game:

There exists a pair of feedback strategies $(\varphi^\star, \psi^\star) \in \Phi \times \Psi$ such that for any measurable control $u(\cdot)$, there exists an unique solution to

$$x'(t) = f(t, x(t), u(t), \psi^\star(t, x(t)))$$

such that $x(t_0) = x_0$ and we denote by $x(\cdot, t_0, x_0, u(\cdot), \psi^\star(\cdot,\cdot))$ this solution. In a similar way, we assume also the existence and unicity of $x(\cdot, t_0, x_0, \varphi^\star(\cdot,\cdot), v(\cdot))$ and $x(\cdot, t_0, x_0, \phi^\star(\cdot,\cdot), \psi^\star(\cdot,\cdot))$.

$$(5) \qquad \begin{cases} \forall\,(t_0, x_0),\ \forall\,(u(\cdot), v(\cdot)) \in \mathcal{U} \times \mathcal{V} \\ g(x(T, t_0, x_0, u(\cdot), \psi^\star(\cdot,\cdot))) \leq g(x(T, t_0, x_0, \phi^\star(\cdot,\cdot), \psi^\star(\cdot,\cdot))) \\ \leq g(x(T, t_0, x_0, \phi^\star(\cdot,\cdot), v(\cdot))) \end{cases}$$

**Definition 2.1** *If (5) is satisfied, we call*

$$W(t_0, x_0) := g(x(T, t_0, x_0, \phi^\star(\cdot,\cdot), \psi^\star(\cdot,\cdot)))$$

*the value function*[1] *of the differential game with feedback strategy.*

# 3 Contingent solutions to Isaacs' equations

Consider the following contingent[2] inequalities:

$$(6) \qquad \begin{cases} \Theta(T,\cdot) = g(\cdot) \text{ and } \forall\,(t,x) \in Dom(\Theta) \\ i) \text{ if } t \in [0,T[, \text{ then} \\ \sup_{u\in U} \inf_{v\in V} D_\uparrow \Theta(t,x)(1, f(t,x,u,v)) \le 0 \\ ii) \text{ if } t \in [0,T[, \text{ then} \\ \sup_{u\in U} \inf_{v\in V} D_\downarrow \Theta(t,x)(1, f(t,x,u,v)) \ge 0 \end{cases}$$

$$(7) \qquad \begin{cases} \Theta(T,\cdot) = g(\cdot) \text{ and } \forall\,(t,x) \in Dom(\Theta) \\ i) \text{ if } t \in [0,T[, \text{ then} \\ \inf_{v\in V} \sup_{u\in U} D_\uparrow \Theta(t,x)(1, f(t,x,u,v)) \le 0 \\ ii) \text{ if } t \in [0,T[, \text{ then} \\ \inf_{v\in V} \sup_{u\in U} D_\downarrow \Theta(t,x)(1, f(t,x,u,v)) \ge 0 \end{cases}$$

A such $\Theta : [0,T] \times I\!R^n \mapsto \overline{I\!R}$ is called contingent solution to the Isaacs' equation. We can prove *without any assumptions on* $g$ the following:

**Proposition 3.1** *Assume that (4) and (5) hold true, then the value function satisfies (6)i) and (7)ii).*

[1]This definition is very related to the one of Pierre Bernhard (see [7]).

[2]Recall the definition of the contingent epiderivative of $\Theta : I\!R^n \mapsto \overline{I\!R}$ at $x_0 \in Dom(\Theta)$,

$$D_\uparrow \Theta(x_0)(u) = \liminf_{h \to 0+,\, v \to u} \frac{\Theta(x_0 + hv) - \Theta(x_0)}{h}$$

or equivalently $EpiD_\uparrow\Theta(x_0) = T_{Epi\Theta}(x_0, \Theta(x_0))$, where $Epi$ states for the epigraph. In a similar way for the contingent hypoderivative of $\Theta$ at $x_0 \in Dom(\Theta)$ is defined by $D_\downarrow\Theta(x)(u) := -D_\uparrow(-\Theta)(x)(u)$, and the contingent derivative of $\Theta$ at $x_0 \in Dom(\Theta)$ is defined by:

$$GraphD\Theta(x_0) = T_{Graph\Theta}(x_0, \Theta(x_0)).$$

**Proof** — Let us prove (6)i). Fix $\bar{u} \in U$ and $h > 0$. Define $x_h$ a solution of $x'(t) = f(t, x(t), \bar{u}, \psi^\star)$ $t \in [t_0, t_0 + h]$ such that $x(t_0) = x_0$. Let us introduce

$$u_h(t) := \begin{cases} \bar{u} \ \text{ if } \ t \in [t_0, t_0 + h] \\ \varphi^\star(t, x(t, t_0 + h, x_h(t_0 + h), \varphi^\star(\cdot,\cdot), \psi^\star(\cdot,\cdot)))) \\ \text{if } \ t > t_0 + h \end{cases}$$

Then by the very definition of $W$,

$W(t_0 + h, x_h(t_0 + h) = g(x(T, t_0 + h, x(t_0 + h), \phi^\star(\cdot,\cdot), \psi^\star(\cdot,\cdot)))$ which is equal to $g(x(T, t_0, x_0, u_h(\cdot), \psi^\star(\cdot,\cdot)))$ and by (5), $g(x(T, t_0, x_0, u_h(\cdot), \psi^\star(\cdot,\cdot))) \leq g(x(T, t_0, x_0, \phi^\star(\cdot,\cdot), \psi^\star(\cdot,\cdot))) = W(t_0, x_0)$. Hence

$$\liminf_{h \longrightarrow 0^+} \frac{W(t_0 + h, x_h(t_0 + h)) - W(t_0, x_0))}{h} \leq 0$$

Thanks to our assumptions and by the Mean Value Theorem, there exists $h_n \longrightarrow 0^+$ such that:

$$\frac{x_{h_n}(t_0 + h_n) - x_0}{h_n} \longrightarrow w \in f(t_0, x_0, \overline{u}, V)$$

This yields $\inf_{v \in V} D_\uparrow W(t_0, x_0)(1, f(t, x, \overline{u}, v)) \leq 0$ and proves (6)i). □

We can obtain other contingent inequalities with suitable regularity assumptions concerning strategies

**Proposition 3.2** *Assume that (4) and (5) hold true. If $(t, x) \mapsto f(t, x, \varphi^\star(t, x), V)$ is continuous at $(t_0, x_0)$, then the value function $W$ satisfies (6)ii). If $(t, x) \mapsto f(t, x, U, \psi^\star(t, x))$ is continuous at $(t_0, x_0)$, then the value function $W$ satisfies (7)i).*

Now we shall state an unicity result[3]:

**Theorem 3.3** *Assume that (4), (5) hold true and that $W$ is continuous.*

- *If $(t, x) \mapsto f(t, x, \varphi^\star(t, x), V)$ is continuous, then any lower semicontinuous (l.s.c.) function $\Theta$ satisfying (6)i) is larger or equal than $W$*

  .

[3]cf the proof in [13]

- *If $(t,x) \mapsto f(t,x,U,\psi^\star(t,x))$ is continuous, then any upper semicontinuous (u.s.c.) function $\Theta$ satisfying (7)ii) is lower or equal than $W$.*

**Corollary 3.4** *When (4), (5) hold true, and $W$ is continuous and when $(t,x) \mapsto f(t,x,U,\psi^\star(t,x))$ and $(t,x) \mapsto f(t,x,\varphi^\star(t,x))$ are continuous, the value function $W$ is the unique continuous solution to Hamilton-Jacobi-Isaacs inequalities (6) and (7).*

# 4 Viscosity solutions to Isaacs' equation

We first define the lower and upper Hamiltonians of the differential (1):

$$\begin{array}{l} H_-(t,x,p) := \max_{v\in V} \min_{u\in U} < p, f(t,x,u,v) > \\ H_+(t,x,p) := \min_{u\in U} \max_{v\in V} < p, f(t,x,u,v) >, \end{array}$$

Consider two Hamilton-Jacobi equations:

$$(8) \qquad \left\{ \begin{array}{l} -\frac{\partial\Theta}{\partial t}(t,x) + H_+(t,x,-\frac{\partial\Theta}{\partial x}(t,x)) = 0 \\ \Theta(T,\cdot) = g(\cdot) \end{array} \right.$$

$$(9) \qquad \left\{ \begin{array}{l} -\frac{\partial\Theta}{\partial t}(t,x) + H_-(t,x,-\frac{\partial\Theta}{\partial x}(t,x)) = 0 \\ \Theta(T,\cdot) = g(\cdot) \end{array} \right.$$

In this section, we give some results concerning viscosity solutions to Hamilton-Jacobi-Isaacs equations. First, we recall the definition of viscosity solution by using sub and super differentials[4]:

**Definition 4.1** *Consider $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ Let us recall that the function $\Theta : [0,T] \times \mathbb{R}^n \mapsto \overline{\mathbb{R}}$ is a viscosity supersolution to the following Hamilton-Jacobi equation $-\frac{\partial\Theta}{\partial t}(t,x) + H(t,x,-\frac{\partial\Theta}{\partial x}(t,x)) = 0$ if and only if:*
$\forall\, (t,x) \in Dom(\Theta),\ \forall\, (p_t,p_x) \in \partial_-\Theta(t,x),\ -p_t + H(t,x,-p_x) \geq 0$
*The function $\Theta$ is a viscosity subsolution if and only if:*
$\forall\, (t,x) \in Dom(\Theta),\ \forall\, (p_t,p_x) \in \partial_+\Theta(t,x),\ -p_t + H(t,x,-p_x) \leq 0$
*A function $\Theta$ is a viscosity solution if it is a supersolution and a subsolution.*

[4]Recall the definition of the subdifferential of $\phi : \mathbb{R}^n \mapsto \overline{\mathbb{R}}$, at $x_0 \in Dom(\phi)$
$\partial_-\phi(x_0) := \{ p \in \mathbb{R}^n \mid \liminf_{x \longrightarrow x_0} \frac{\phi(x)-\phi(x_0)-<p,x-x_0>}{\|x-x_0\|} \geq 0 \}$ and the super differential of $\phi$ at $x_0$ is given by: $\partial_+\phi(x_0) := -\partial_-(-\phi)(x_0)$.

We can prove *without any assumptions* concerning $g$ the following existence result:

**Proposition 4.2** *Assume that (4) holds true, then the value-function $W$ is a supersolution to (8) and a subsolution to (9).*

But when the value-function $W$ is continuous, we have the more precise

**Proposition 4.3** *Assume (4). Then if $W$ is continuous and $(t,x) \mapsto f(t,x,U,\psi^\star(t,x))$ is continuous, then $W$ is a viscosity solution to (8).*
*If $W$ is continuous and $(t,x) \mapsto f(t,x,\varphi^\star(t,x),V)$ is continuous, then $W$ is a viscosity solution to (9).*

**Theorem 4.4** *Let assumptions of Corollary 3.4 hold true. Let $\Theta : [0,T] \times \mathbb{R}^n \mapsto \mathbb{R}$ be continuous. Then $\Theta$ is the value function of the game if and only if it is a viscosity supersolution to (8) and a viscosity subsolution to (9).*

**Corollary 4.5** *Let us assume (4), (5) and let $\Theta : [0,T] \times \mathbb{R}^n \mapsto \mathbb{R}$ be a continuous function. If we assume the following Isaacs' condition:*

$$\forall\,(t,x,p),\;\; H_-(t,x,p) = H_+(t,x,p), \tag{10}$$

*then the value function is the* unique *viscosity solution to (9) (or equivalently(8)).*

These results follow from results of the previous section and from the following section.

# 5 Comparison between viscosity and contingent solutions to Hamilton Jacobi Isaacs equations

**Proposition 5.1** *Consider $\Theta : [0,T] \times \mathbb{R}^n \mapsto \overline{\mathbb{R}}$ verifying (6) (respectively (7)). Then $\Theta$ is a viscosity solution to (8) (respectively to (9)).*

This result is a consequence of the following

**Lemma 5.2** *Consider $\Theta : [0,T] \times \mathbb{R}^n \mapsto \overline{\mathbb{R}}$.*

- *If $\Theta$ satisfies (6)i), then it is a supersolution of (8).*
- *If $\Theta$ satisfies (6)ii), then it is a subsolution of (8).*
- *If $\Theta$ satisfies (7)i), then it is a supersolution of (9).*
- *If $\Theta$ satisfies (7)ii), then it is a subsolution of (9).*

**Proof —** Let us prove the first statement. If $\Theta$ satisfying (6)i) then[5]:

$$(11) \qquad \left\{ \begin{array}{l} \forall\, (p_t, p_x) \in \partial_-\Theta(t,x),\ \forall\, (u,v) \in U \times V, \\ D_\uparrow\Theta(t,x)(1, f(t,x,u,v)) \geq p_t + < p_x, f(t,x,u,v) > \end{array} \right.$$

Then by taking the "supinf" of this inequality, we prove that $\Theta$ is a supersolution to (8). The proofs of the other statements are similar. □

When value functions are continuous, the notions of contingent and viscosity solutions of Isaacs' equations are equivalent.

**Theorem 5.3** *Let $\Theta : [0,T] \times I\!R^n \mapsto I\!R$ be a continuous function and let (4) hold true. Then $\Theta$ satisfies the contingent inequalities (6) (respectively (7)) if and only if it is a viscosity solution to the Hamilton-Jacobi-Isaacs equation (8) (respectively (9)).*

**Lemma 5.4** *If (4) holds true.*

- *Any l.s.c. function $\Theta$ is a supersolution of (8) if and only if it satisfies (6)i).*
- *Any l.s.c. function $\Theta$ is a supersolution of (9) if and only if it satisfies (7)i).*
- *Any u.s.c. function $\Theta$ is a subsolution of (8) if and only if it satisfies (6)ii).*

[5]Let us recall (see [2] chapter 6 for instance) that we have the following equivalent definition for the subdifferential of a function $\phi$

$$\partial_-\phi(x_0) = \{\, p \,|\, \forall\, q \in I\!R^n,\ D_\uparrow\phi(x_0)(q) \geq < p, q > \}$$

- *Any u.s.c. function $\Theta$ is a subsolution of (9) if and only if it satisfies (7)ii).*

**Proof of Lemma —** We already know, thanks to Proposition 5.1 and Lemma 5.2, that contingent solutions are viscosity solutions. Let us prove the converse implication.

Assume that $\Theta$ is a supersolution to (8), i.e.:

$$(12) \qquad \forall\,(p_t,p_x) \in \partial_-\Theta(t,x),\ \sup_{u\in U}\inf_{v\in V} p_t + < p_x, f(t,x,u,v) > \leq 0$$

Hence, for any $u \in U$, $\inf_{v\in V} p_t + < p_x, f(t,x,u,v) > \leq 0$. But we know, (cf [12]) that $(p_t,p_x) \in \partial_-\Theta(t,x)$ if and only if $(p_t,p_x,-1)$ belongs to the normal cone $(T_{Epi\Theta}(t,x,\Theta(t,x)))^-$. We claim that

$$(13) \quad \forall\, u \in U,\ \{1\} \times f(t,x,u,V) \times \{0\} \cap co(T_{Epi\Theta}(t,x,\Theta(t,x))) \neq \emptyset$$

Assume for a moment that is false, then, by the separation theorem we should have:

$$(14) \qquad \left\{ \begin{array}{l} \exists\,(p_t,p_x,q) \in (T_{Epi\Theta}(t,x,\Theta(t,x)))^-,\ \exists\, u \in U \text{ such that} \\ \forall\ v \in V,\ p_t + < p_x, f(t,x,u,v) >> 0 \end{array} \right.$$

This is a contradiction with (12). So

$$\left\{ \begin{array}{l} \forall\,(t,x,y) \in Epi(\Theta),\ \text{for all } u \in U \\ \{1\} \times f(t,x,u,V) \times \{0\} \cap co(T_{Epi\Theta}(t,x,y)) \neq \emptyset \end{array} \right.$$

and we can deduce from[6] Theorem 3.2.4 in [1], that $\{1\} \times f(t,x,u,V) \times \{0\} \cap T_{Epi\Theta}(t,x,\Theta(t,x)) \neq \emptyset$, for any $(t,x) \in Dom\Theta$. This implies the following contingent equation:

$$\forall\,(t,x) \in Dom(\Theta),\ \forall\, u \in U,\ \inf_{v\in V} D_\uparrow\Theta(t,x)(1,f(t,x,u,v)) \leq 0$$

[6]Let us recall a duality result in viability theory (due to Ushakov see for instance Theorem 3.2.4 in [1]). Consider a closed set $K \subset \mathbb{R}^n$ and let $F$ be u.s.c set-valued map with compact convex values. Then the following two statements are equivalent:

$$\begin{array}{ll} i) & \forall\, x \in K,\ F(x) \cap T_K(x) \neq \emptyset \\ ii) & \forall\, x \in K,\ F(x) \cap co(T_K(x)) \neq \emptyset \end{array}$$

where $co$ is the closed convex hull.

Let us prove the third statement. Assume that $\Theta$ satisfies

$$(15) \qquad \forall\,(p_t,p_x) \in \partial_+\Theta(t,x),\ \sup_{u\in U}\inf_{v\in V} p_t + < p_x, f(t,x,u,v) > \geq 0$$

We claim that

$$(16) \quad \exists\, u \in U,\ \{\,1\,\} \times f(t,x,u,V) \times \{\,0\,\} \subset co(T_{Hypo\Theta}(t,x,\Theta(t,x)))$$

If (16) is not satisfied, by the separation Theorem

$$(17) \qquad \begin{cases} \forall\, u \in U,\ \exists\, v \in V \text{ such that} \\ \exists\,(p_t,p_x,q) \in (T_{Hypo\Theta}(t,x,\Theta(t,x)))^-, \\ p_t + < p_x, f(t,x,u,v) > < 0 \end{cases}$$

This is a contradiction with (15). Then, thanks to (4), and since (cf [2] p.130),

$$\liminf_{(t',x',y')\mapsto(t,x,\Theta(t,x))} co(T_{Hypo\Theta}(t',x',y')) \subset T_{Hypo\Theta}(t,x\Theta(t,x))$$

we can deduce that $\{\,1\,\} \times f(t,x,u,V) \times \{\,0\,\} \subset T_{Hypo\Theta}(t,x,\Theta(t,x))$ and consequently (7)ii) holds true. The proofs are similar for the other statements. □

# 6 Nonanticipating strategies

We shall define value-functions for a concept of strategy studied by Elliot-Kalton (see also [9]). We denote by $\mathcal{U}(t)$ (respectively by $\mathcal{V}(t)$ ) the set of measurable functions $u : [t,T] \mapsto U$ (respectively $v : [t,T] \mapsto V$ ).

Firstly let us recall the definition of *nonanticipating strategies.*

**Definition 6.1** *We call* nonanticipating strategy for the first player *any function* $\alpha : \mathcal{V}(t) \mapsto \mathcal{U}(t)$ *such that*

$$\begin{array}{l} \forall\, t \in [0,T],\ \forall\,(v,\bar{v}) \in \mathcal{V}(t),\ \forall\, s \in [0,T], \\ v \equiv \bar{v} \text{ a. e. in } [t,s] \ \Rightarrow\ \alpha(v) \equiv \alpha(\bar{v}) \text{ a. e. in } [t,s] \end{array}$$

*and we denote by* $\Gamma(t)$ *the set of such nonanticipating strategies.*

*We call* nonanticipating strategy for the sevond player *any function* $\beta$ : $\mathcal{U}(t) \mapsto \mathcal{V}(t)$ *such that*

$$\begin{array}{l}\forall\, t \in [0,T],\ \forall\, (u,\bar{u}) \in \mathcal{U}(t),\ \forall\, s \in [0,T],\\ u \equiv \bar{u} \text{ a. e. in } [t,s] \ \Rightarrow\ \beta(u) \equiv \beta(\bar{u}) \text{ a. e. in } [t,s]\end{array}$$

*and we denote by* $\Delta(t)$ *the set of such nonanticipating strategies.*

This notion of strategies enables us to define the two value-functions:

**Definition 6.2** *Consider the upper value-function of the game:*

$$\Phi(t_0,x_0) := \inf_{\beta \in \Delta(t_0)} \sup_{u(\cdot) \in \mathcal{U}(t_0)} g(x(T,t_0,x_0,u(\cdot),\beta(u)))$$

*and the lower value-function:*

$$\Psi(t_0,x_0) := \sup_{\alpha \in \Gamma(t_0)} \inf_{v(\cdot) \in \mathcal{V}(t_0)} g(x(T,t_0,x_0,\alpha(v),v(\cdot)))$$

**Proposition 6.3** *Assume that (4) holds true. If $g$ is continuous, then $\Psi$ and $\Phi$ are continuous.*

**Proof** — We shall prove that $\Psi$ is continuous[7] at some $t_1, x_1$. Consider $\varepsilon > 0$, , $x_2$ and $0 \le t_1 \le t_2 \le T$. By the very definition of the value-function $\Psi$, there exists $\alpha \in \Gamma(t_1)$ such that

$$\Psi(t_1,x_1) \le \inf_{v(\cdot) \in \mathcal{V}(t_1)} g(x(T,t_1,x_1,\alpha(v(\cdot)),v(\cdot))) + \varepsilon \tag{18}$$

Fix $\bar{v} \in V$. For any $v(\cdot) \in \mathcal{V}(t_2)$, we define $\underline{v}(s) =$

$$\begin{cases} \bar{v} & \text{if } s \in [t_1,t_2] \\ v(s) & \text{if } s \in [t_2,T] \end{cases}$$

and for any $\alpha$ we define $\underline{\alpha}(v) = \alpha(\underline{v})$.
Hence, there exists $v(\cdot) \in \mathcal{V}(t_2)$ such that $\Psi(t_2,x_2) \ge g(x(T,t_2,x_2,\underline{\alpha}(v),v))) - \varepsilon$ and according to (18), we have $\Psi(t_1,x_1) \le g(x(T,t_1,x_1,\alpha(v),v)) + \varepsilon$. On the other hand, from Gronwall's Lemma, there exists some $R > 0$ such that

$$\|x(T,t_1,x_1,\underline{\alpha}(v)(\cdot),v(\cdot)) - x(T,t_2,x_2,\alpha(v)(\cdot),v(\cdot))\| \le R(\|x_1-x_2\|+(t_1-t_2))$$

[7]It's easy to extend the proof when $g$ is uniformely continuous and then the value-functions are uniformely continuous too.

Since $g$ is continuous, there exists $\delta > 0$ such that for any $(t_2, x_2) \in R([0,1] \times B)$ we have

$$|g(x(T, t_1, x_1, \underline{\alpha}(v)(\cdot), v(\cdot))) - g(x(T, t_2, x_2, \alpha(v)(\cdot), v(\cdot)))| \leq \varepsilon$$

Hence $\Psi(t_1, x_1) - \Psi(t_2, x_2) \leq 3\varepsilon$. On the other hand for every $\alpha$: $\Psi(t_1, x_1) \geq \inf_{v(\cdot) \in \mathcal{V}(t_0)} g(x(T, t_1, x_1, \alpha(v), v(\cdot))) \geq \inf_{v(\cdot) \in \mathcal{V}(t_0)} g(x(T, t_2, x_2, \alpha(v), v(\cdot))) - \varepsilon$ Hence $\Psi(t_1, x_1) \geq \Psi(t_2, x_2) - \varepsilon$. We have similar result when $t_2 < t_1$ and for the value-function $\Phi$. $\square$

# 7 Solutions to Isaacs equations with nonanticipating strategies

**Proposition 7.1** *If (4) holds true, then $\Phi$ satisfies (6)i) and $\Psi$ satisfies (7)ii).*

**Proof** — Fix $\bar{u} \in U$. Consider $\beta_h \in \Delta(t_0)$ such that

$$\sup_{u \in \mathcal{U}(t_0)} g(x(T, t_0, x_0, u(\cdot), \beta_h(u))) \leq \Phi(t_0, x_0) + h^2$$

Let define $\mathcal{U}_h(t_0)$ the subset of measurable controls $u(\cdot) \in \mathcal{U}(t_0)$ such that $u(s) = \bar{u}$ for almost every $s \in [t_0, t_0 + h]$. then

$$\sup_{u \in \mathcal{U}_h(t_0)} g(x(T, t_0, x_0, u(\cdot), \beta_h(u))) \leq \Phi(t_0, x_0) + h^2 \tag{19}$$

By the very definition of $\beta_h$, there exists some $v(\cdot) \in \mathcal{V}(t_0)$ such that for any $u(\cdot) \in \mathcal{U}_h$, $v(s) = \beta_h(u)(s)$ for almost every $s \in [t_0, t_0 + h]$.

Let $x_h(\cdot)$ denote the solution to $x'(t) = f(t, x(t), \bar{u}, v(t))$ on $[t_0, t_0 + h]$ such that $x_h(t_0) = x_0$. From (19), we deduce

$$\sup_{u \in \mathcal{U}_h(t_0)} g(x(T, t_0 + h, x_h(t_0 + h), u(\cdot), \beta_h(u))) \leq \Phi(t_0, x_0) + h^2$$

Define $\beta \in \Delta(t_0)$ such that for any $u(\cdot) \in \mathcal{U}(t_0)$ we have $\beta(u) := \beta_h(\underline{u})$ with

$$\underline{u}(s) := \begin{cases} \bar{u} \text{ if } s \in [t_0, t_0 + h] \\ u(s) \text{ if } s > t_0 + h \end{cases}$$

Hence $\sup_{u\in\mathcal{U}(t_0)} g(x(T,t_0,x_0,u(\cdot),\beta(u))) \le \Phi(t_0,x_0)+h^2$ and therefore $\inf_{\beta\in\Delta(t_0)}\sup_{u\in\mathcal{U}(t_0)} g(x($
$\Phi(t_0,x_0)+h^2$. This proves the following inequality

$$\Phi(t_0+h, x_h(t_0+h)) \le \Phi(t_0,x_0)+h^2.$$

On the other hand, there exists a sequence $h_i \longrightarrow 0$ and $\bar{v}\in V$ such that

$$\frac{x_{h_i}(t_0+h_i)-x_0}{h_i} \longrightarrow f(t_0,x_0,\bar{u},\bar{v})$$

this yields $D_\uparrow\Phi(t_0,x_0)(1,f(t_0,x_0,\bar{u},\bar{v})) \le 0$ and consequently (6)i). The proof is similar for the second statement. □

**Proposition 7.2** *If $g$ is continuous, then $\Phi$ satisfies (6)ii) and $\Psi$ satisfies (7)ii).*

It is possible to prove that $\Phi$ is a viscosity subsolution to (8) and thanks to results of section 5 that it is a contingent solution to (6)ii) (see [13] for the proof).

**Corollary 7.3** *If $g$ is continuous, then $\Phi$ is a viscosity solution to (8) and $\Psi$ is a viscosity solution to (9).*

Finally we just state an existence result

**Proposition 7.4** *Assume that (4) holds true and that $g$ is uniformely continuous. If we assume the Isaacs' condition (10), then $\Phi=\Psi$ and the value-function is the unique uniformely continuous viscosity solution to the Isaacs' equation.*

The proof is based on a theorem of Crandall-Lions concerning the unicity of bounded uniformely continuous solution of Hamilton-Jacobi equations (see [16]).

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