Working Paper

Ellipsoidal Calculus, Singular Perturbations and the State Estimation Problems for Uncertain Systems

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Foreword

One of the basic elements of dynamic modelling of complex systems is the linkage and synchronization of subsystems that develop in different time scales. The relevant techniques applied here are related to a singular perturbation theory for differential systems. A more complicated issue arises for uncertain systems described by differential inclusions, for which an appropriate theory is being developed. In order to make the theory constructive, some further steps are necessary. These are presented in this paper, where a computer-implementable 'ellipsoidal' version is given.

The results are particularly relevant to the linkage of models related to environmental, demographic and economic problems. They were derived within the Activity Plan of the SDS Program of IIASA.

Contents

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1 Introduction

The paper deals with problems of guaranteed state estimation for dynamic systems described by linear differential inclusions with state constraints, namely

$$\dot{x}(t) \in A(t)x(t) + \mathcal{P}(t), \tag{1.1}$$

$$x(t_0) \in \mathcal{X}_0, \tag{1.2}$$

$$G(t)x(t) \in \mathcal{Q}(t), \tag{1.3}$$

$$t_0 \leq t \leq \vartheta.$$

Here $x \in \mathcal{R}^n$, $t_0 \leq \tau \leq \vartheta$, $A(\cdot) \in \mathcal{M}_{n,n}[t_0, \vartheta]$, $G(\cdot) \in \mathcal{M}_{m,n}[t_0, \vartheta]$, $\mathcal{X}_0 \in \text{conv}\mathcal{R}^n$ with $\mathcal{R}^n$ standing for the n-dimensional Euclidean space, $\mathcal{M}_{k,l}[t_0, \vartheta]$ for the space of continuous $k \times l$ matrix valued functions defined on $[t_0, \vartheta]$ and conv $\mathcal{R}^n$ for the space of all convex and compact subsets of $\mathcal{R}^n$. The multifunctions $\mathcal{P} : [t_0, \vartheta] \rightarrow \text{conv } \mathcal{R}^n$ and $\mathcal{Q} : [t_0, \vartheta] \rightarrow \text{conv } \mathcal{R}^m$ are assumed to be continuous.

The relations of type 1.1, 1.2, 1.3 may be considered as a mathematical model for an uncertain dynamic process with set-membership description of the unknown parameters, [10], [2], [4], [8], [9] and [7]. To solve an optimization or estimation problem through the theory of control and observation under uncertainty conditions, the main point to start with is to construct the *attainability* (reachability) *sets* for the system. In the theory of observation under uncertainty, when the state constraints 1.3 are due to incomplete state measurements, these attainable sets are also known as the *informational domains* [3].

There has been much activity in studying the attainability sets for the system 1.1, 1.2 without state constraints and also for the more complicated case that also involves 1.3. We should indicate that there exists a close relation of these problems with those of *viability theory* [1]. since the reachable set of 1.1, 1.2, 1.3 at instant τ is precisely the τ-section of the tube of all the solutions to 1.1, 1.2 that are viable on $[t_0, \tau]$ with respect to constraint 1.3. Here we follow an earlier version of this approach that appeared in [5] in order to find a precise description of these sets. The principal ideal of those papers was to avoid the procedures of constructing tangent cones to the map $\mathcal{Q}$ that defines the restriction 1.3 and also to consider a broader class of set-valued functions $\mathcal{Q}$ (for example, those that are semicontinuous or even measurable in time t).

Much of the basic material for this presentation is derived from the theory of *ellipsoidal approximations* for the problem that has been treated in [8], [9] and [11]. The aim of this paper is to combine the ellipsoidal calculus techniques that are effective in computer simulations with those of the above-mentioned approaches by introducing the techniques of *singular perturbations* for the system 1.1, 1.2, 1.3.

2 Estimation via Ellipsoids

Preliminary Results

We will first investigate the system 1.1 assuming the data to be ellipsoidal-valued.

$$\dot{x}(t) \in A(t)x(t) + \mathcal{E}(p(t), P(t)), \tag{2.4}$$

$$x(t_0) \in \mathcal{E}(x_0, X_0), \tag{2.5}$$

$$t_0 \leq t \leq \vartheta.$$

Here $\mathcal{E}(q, Q)$ is an ellipsoid in $\mathcal{R}^n$ with center q and a symmetric positively definite matrix Q representing its "shape". The support function $\rho(\ell|\mathcal{E}(q, Q))$ of the set $\mathcal{E}(q, Q)$ has the form

$$\rho(\ell|Q(q, Q)) = \ell' q + (\ell' Q \ell)^{\frac{1}{2}}, \quad \ell \in \mathcal{R}^n$$

where the prime stands for the transpose. For the convenience of the reader we will indicate some basic results in ellipsoidal calculus [9], [11].

Let us consider the Minkowski-sum $\mathcal{E}_1 + \mathcal{E}_2$ of two ellipsoids

$$\mathcal{E}_1 = \mathcal{E}(q_1, Q_1), \quad \mathcal{E}_2 = \mathcal{E}(q_2, Q_2).$$

The following Lemma gives external estimates for $\mathcal{E}_1 + \mathcal{E}_2$ with respect to the inclusion of sets.

Lemma 2.1 *The following equality is true*

$$\mathcal{E}_1 + \mathcal{E}_2 = \bigcap\{\mathcal{E}(q_1 + q_2, Q(\nu))|\nu > 0\} \tag{2.6}$$

where

$$Q(\nu) = (1 + \nu^{-1})Q_1 + (1 + \nu)Q_2.$$

Formula 2.6 for non-degenerate ellipses $\mathcal{E}_1$ and $\mathcal{E}_2$, as well as the following theorem have been proved in [11].

Let us denote the attainability set at instant $t \in [t_0, \vartheta]$ for the system 1.1, 1.2 by $\mathcal{X}(t)$ and the set of all continuous positive real valued functions defined on $[t_0, \vartheta]$ by the symbol $C[t_0, \vartheta]$.

Theorem 2.1 *For every $t \in [t_0, \vartheta]$ the following equality is fulfilled*

$$\mathcal{X}(t) = \bigcap\{\mathcal{E}(z(t), Z(t, \sigma))|\sigma \in C[t_0, \vartheta]\} \tag{2.7}$$

where $z : [t_0, \vartheta] \to \mathcal{R}^n$, $Z(\cdot, \sigma) : [t_0, \vartheta] \to \mathcal{R}^{n\times n}$ are the solutions of the following differential equations

$$\begin{aligned}
\dot{z}(t) &= A(t)z(t) + p(t) \\
z(t_0) &= x_0 \\
\dot{Z}(t) &= A(t)Z(t) + Z(t)A'(t) + \sigma^{-1}(t)Z(t) + \sigma(t)P(t) \\
Z(t_0) &= X_0.
\end{aligned}$$

The result of the above lemma, however, is also true in a more general situation.

Lemma 2.2 *Let $\mathcal{E}_1 = \mathcal{E}(q_1, Q_1)$, $\mathcal{E}_2 = \mathcal{E}(q_2, Q_2)$ where*

$$Q_1 = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 0 & 0 \\ 0 & A_2 \end{pmatrix},$$

$A_1 \in \mathcal{R}^{k\times k}$, $A_2 \in \mathcal{R}^{l\times l}$ symmetric positive definite matrices with $k + l = n$.

Then

$$\mathcal{E}_1 + \mathcal{E}_2 = \bigcap\{\mathcal{E}(q_1 + q_2, Q(\nu))|\nu > 0\}, \tag{2.8}$$

with

$$Q(\nu) = \begin{pmatrix} (1 + \nu^{-1})A_1 & 0 \\ 0 & (1 + \nu)A_2 \end{pmatrix}. \tag{2.9}$$

Proof. The upper estimate

$$\mathcal{E}_1 + \mathcal{E}_2 \subset \mathcal{E}(q_1 + q_2, Q(\nu)), \tag{2.10}$$

for $\nu > 0$ can be obtained on the basis of comparing the support functions, along the lines of the proof of Lemma 2.2.

Consider now an arbitrary vector $w = \{b, c\} \in \mathcal{R}^n$, $b \in \mathcal{R}^k$, $c \in \mathcal{R}^l$ such that $b \neq 0$, $c \neq 0$. It is not difficult to demonstrate that $\rho(w|\mathcal{E}_1 + \mathcal{E}_2) = w'(q_1 + q_2) + (b'A_1b)^{\frac{1}{2}} + (c'A_2c)^{\frac{1}{2}} = w'(q_1 + q_2) + (w'Q(\nu)w)^{\frac{1}{2}}$ for

$$\nu = \nu(w) = \frac{(b'A_1b)^{\frac{1}{2}}}{(c'A_2c)^{\frac{1}{2}}}.$$

This yields

$$\rho(w|\mathcal{E}_1 + \mathcal{E}_2) = \rho(w|\mathcal{E}(q_1 + q_2, Q(\nu))) \tag{2.11}$$

for every direction $w = \{b, c\}$ with $b \neq 0$, $c \neq 0$. From 2.8, 2.9 it follows that

$$\rho(w|\mathcal{E}_1 + \mathcal{E}_2) = \rho(w|\bigcap\{\mathcal{E}(q_1 + q_2, Q(\nu))|\nu > 0\}) \tag{2.12}$$

Indeed from relation 2.11 and the continuity of the support functions of the convex compact sets $\mathcal{E}_1 + \mathcal{E}_2$ and $\bigcap\{\mathcal{E}(q_1 + q_2, Q(\nu))|\nu > 0\}$ we conclude that the last equality is true for all $w \in \mathcal{R}^n$. Then relation 2.4 is also true. Q.E.D.

We now indicate a slight modification of this theorem related to the result of Lemma 2.2. Consider the system 1.1, 1.2 with data of the form

$$\begin{aligned} \mathcal{X}_0 &= \mathcal{E}(x_0, X_0), \\ \mathcal{P}(t) &= \mathcal{E}_k(s(t), S(t)) \times \mathcal{E}_l(r(t), R(t)) \end{aligned}$$

where $\mathcal{E}_k(s(t), S(t)) \subset \mathcal{R}^k$, $\mathcal{E}_l(r(t), R(t)) \subset \mathcal{R}^l$, $k + l = n$. Let us keep the notation $\mathcal{X}(t)$ for the attainability set of the above system at time t.

Theorem 2.2 *For every $t \in [t_0, \vartheta]$ the following equality is true*

$$\mathcal{X}(t) = \bigcap\{\mathcal{E}(z(t), Z(t, \pi, \sigma))|\pi, \sigma \in \mathcal{C}[t_0, \vartheta]\} \tag{2.13}$$

where $z : [t_0, \vartheta] \to \mathcal{R}^n$, $Z(\cdot, \pi, \sigma) : [t_0, \vartheta] \to \mathcal{R}^{n \times n}$ are the solutions to the differential equations

$$\begin{aligned} \dot{z}(t) &= A(t)z(t) + w(t), \\ z(t_0) &= x_0, \\ \dot{Z}(t) &= A(t)Z(t) + Z(t)A'(t) + \sigma^{-1}(t)Z(t) + \sigma(t)K(t, \pi(t)), \\ Z(t_0) &= X_0 \end{aligned}$$

with

$$w(t) = \{s(t), r(t)\},$$
$$K(t, \pi(t)) = \begin{pmatrix} (1+\pi^{-1}(t))S(t) & 0 \\ 0 & (1+\pi(t))R(t) \end{pmatrix}.$$

Proof. This theorem is a combination of Theorem 2.1 and Lemma 2.2. Q.E.D.

3 Perturbation Techniques

In this section we do not further assume that our data for the system 1.1, 1.2, 1.3 are ellipsoidal-valued. We recall here some facts from the singular perturbations approach suggested in [5].

Consider the system of differential inclusions:

$$\dot{x}(t) \in A(t)x(t) + \mathcal{P}(t), \tag{3.14}$$
$$\varepsilon\dot{y}(t) \in -G(t)x(t) + \mathcal{Q}(t), \tag{3.15}$$
$$\{x(t_0), y(t_0)\} \in \mathcal{Z}_0, \tag{3.16}$$
$$t_0 \leq t \leq \tau.$$

Here $y \in \mathcal{R}^m$, $\mathcal{Z}_0 \in \text{conv}(\mathcal{R}^n) \times \text{conv}(\mathcal{R}^m)$, and $\varepsilon \in \mathcal{R}$ is positive. By $\mathcal{X}(\cdot, t_0, \mathcal{X}_0, \tau)$ let us denote the set of all the solutions $x(\cdot)$ to 1.1, 1.2 that satisfy 1.3 for all $t \in [t_0, \tau]$ — that is $\mathcal{X}(\cdot, t_0, \mathcal{X}_0, \tau)$ is the bundle of viable trajectories, [1], — and by $\mathcal{X}[\tau]$ its cross-section at time τ, so that

$$\mathcal{X}[\tau] = \mathcal{X}(\tau, t_0, \mathcal{X}_0, \tau), \quad \tau \in [t_0, \vartheta].$$

The symbol $\mathcal{Z}(\cdot, t_0, \mathcal{Z}_0, \tau, \varepsilon)$ will denote the tube of solutions $z(\cdot) = \{x(\cdot), y(\cdot)\}$ for the system 3.14, 3.15, 3.16 over $t_0 \leq t \leq \tau$.

Let $\mathcal{Z}(\tau, t_0, \mathcal{Z}_0, \varepsilon) = \mathcal{Z}(\tau, t_0, \mathcal{Z}_0, \tau, \varepsilon)$. We will also use the notation $\Pi_x \mathcal{W}$ for the projection of the set $\mathcal{W} \subset \mathcal{R}^n \times \mathcal{R}^m$ onto the space $\mathcal{R}^n$ of x-variables.

Theorem 3.1 *Assume that*

$$\mathcal{X}_0 \subset \Pi_x \mathcal{Z}_0. \tag{3.17}$$

Then for every trajectory $x(\cdot) \in \mathcal{X}[\cdot]$ of 1.1, 1.2, 1.3 there exists a vector $y_0 \in \mathcal{R}^m$ such that

$$\{x(t_0), y_0\} \in \mathcal{Z}_0$$

and for every $\tau \in [t_0, \vartheta]$

$$z(\tau) = \{x(\tau), y_0\} \in \mathcal{Z}(\tau, t_0, \mathcal{Z}_0, \varepsilon)$$

for all $\varepsilon > 0$.

Corollary 3.1 *Assume relation 3.17 to be true. Then for every $\tau \in [t_0, \vartheta]$*

$$\mathcal{X}[\tau] \subset \Pi_x(\bigcap\{\mathcal{Z}(\tau, t_0, \mathcal{Z}_0, \varepsilon) | \varepsilon > 0\}) \tag{3.18}$$

Let us now introduce another system of differential inclusions of type 3.14, 3.15, 3.16 but with a matrix time-dependent perturbation $L(t)$, $t_0 \leq t \leq \tau$, instead of the scalar $\varepsilon > 0$:

$$\dot{x}(t) \in A(t)x(t) + \mathcal{P}(t) \tag{3.19}$$

$$L(t)\dot{y}(t) \in -G(t)x(t) + \mathcal{Q}(t) \tag{3.20}$$

$$\{x(t_0), y(t_0)\} \in \mathcal{Z}_0, \tag{3.21}$$

$$t_0 \leq t \leq \tau.$$

The class of all continuous invertible matrix functions $L \in \mathcal{M}_{m,m}[t_0, \tau]$ will be denoted as $\mathcal{M}^0_{m,m}[t_0, \tau]$, and $\mathcal{Z}(\cdot, t_0, \mathcal{Z}_0, L) = \mathcal{Z}(\cdot, t_0, \mathcal{Z}_0, \tau, L)$ will stand for the solution tube to the system 3.19, 3.20 with initial condition 3.21 over the interval $t_0 \leq t \leq \tau$.

The following analogy of Theorem 3.1 is true.

Theorem 3.2 *Assume relation 3.17 to be true. Then for every $x(\cdot) \in \mathcal{X}[\cdot]$, there exists a vector $y_0 \in \mathcal{R}^m$ such that*

$$\{x(t_0), y_0\} \in \mathcal{Z}_0$$

and for every $\tau \in [t_0, \vartheta]$

$$z(\tau) = \{x(\tau), y_0\} \in \mathcal{Z}(\tau, t_0, \mathcal{Z}_0, L)$$

whatever is the function $L \in \mathcal{M}^0_{m,m}[t_0, \tau]$.

Corollary 3.2 *Assume relation 3.17 to be true. Then for every $\tau \in [t_0, \vartheta]$*

$$\mathcal{X}[\tau] \subset \Pi_x(\bigcap\{\mathcal{Z}(\tau, t_0, \mathcal{Z}_0, L) | L \in \mathcal{M}^0_{m,m}[t_0, \tau]\}) \tag{3.22}$$

The principal result of the singular perturbations method applied to the problem under discussion is formulated as follows

Theorem 3.3 *Suppose*

$$\Pi_x \mathcal{Z}_0 \subset \mathcal{X}_0.$$

Then for every $\tau \in [t_0, \vartheta]$

$$\Pi_x(\bigcap\{\mathcal{Z}(\tau, t_0, \mathcal{Z}_0, L) | L \in \mathcal{M}^0_{m,m}[t_0, \vartheta]\}) \subset \mathcal{X}[\tau]. \tag{3.23}$$

In a slightly different form this result was announced in [6], its full proof will appear in [7].

From Corollary 3.1 and the Theorem 3.2 we obtain the exact description of the set $\mathcal{X}[\tau]$ by means of perturbed differential inclusions without state constraints.

Theorem 3.4 *Under the assumption*

$$\Pi_x \mathcal{Z}_0 = \mathcal{X}_0$$

the following formula is true for any $\tau \in [t_0, \vartheta]$

$$\mathcal{X}[\tau] = \Pi_x(\bigcap\{\mathcal{Z}(\tau, t_0, \mathcal{Z}_0, L) | L \in \mathcal{M}^0_{m,m}[t_0, \tau]\}).$$

4 The Principal Theorem

Consider the system 1.1, 1.2, 1.3 where all the sets involved are ellipsoids:

$$\begin{aligned} \dot{x}(t) &\in A(t)x(t) + \mathcal{E}(p(t), P(t)), && (4.24)\\ x(t_0) &\in \mathcal{E}(x_0, X_0) && (4.25)\\ G(t)x(t) &\in \mathcal{E}(q(t), Q(t)), && (4.26)\\ t_0 \leq & \; t \; \leq \tau. \end{aligned}$$

Here $p : [t_0, \vartheta] \to \mathcal{R}^n$, $q : [t_0, \vartheta] \to \mathcal{R}^m$, $P \in \mathcal{M}_{n,n}[t_0, \vartheta]$, $Q \in \mathcal{M}_{m,m}[t_0, \vartheta]$ $x_0 \in \mathcal{R}^n$, the matrices $X_0, P(t), \in \mathcal{R}^{n\times n}$ and $Q(t) \in \mathcal{R}^{m\times m}$ are symmetric and positive definite.

Our goal will be to find *the exact ellipsoidal estimate* to the attainable set $\mathcal{X}[\tau] = \mathcal{X}(\tau, t_0, \mathcal{X}_0)$ for the system 4.24, 4.25, 4.26.

After reviewing some preliminary results given in Sections 2 and 3 we are now in a position to respond to this aspiration.

Theorem 4.1 *Given an instant* $\tau \in [t_0, \vartheta]$*, the following exact formula is true for every* $\tau \in [t_0, \vartheta]$

$$\mathcal{X}[\tau] = \Pi_x(\cap\{\mathcal{E}(z(\tau, L), Z(\tau, L, \pi, \sigma)) | L \in \mathcal{M}^0_{m,m}[t_0, \tau], \pi, \sigma \in \mathcal{C}[t_0, \tau]\}), \quad (4.27)$$

where $x(t)$ *and* $y(t)$ *of*

$$z(t, L) = \{x(t), y(t)\}$$

are solutions to the system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + p(t),\\ L(t)\dot{y}(t) &= -G(t)x(t) + q(t),\\ x(t_0) &= x_0,\\ y(t_0) &= 0, \end{aligned}$$

and $Z_i(t)$*,* $i = 1, 2, 3$ *of*

$$Z(t, L, \pi, \sigma) = \begin{pmatrix} Z_1(t) & Z_2'(t) \\ Z_2(t) & Z_3(t) \end{pmatrix}$$

to the matrix differential equations

$$\begin{aligned}
\dot{Z}_1(t) &= A(t)Z_1(t) + Z_1(t)A'(t) + \sigma^{-1}(t)Z_1(t) + \sigma(t)(1+\pi^{-1}(t))P(t),\\
L(t)\dot{Z}_2(t) &= -G(t)Z_1(t) + L(t)Z_2(t)A'(t) + L(t)\sigma^{-1}(t)Z_2(t),\\
L(t)\dot{Z}_3(t) &= -G(t)Z_2'(t) - L(t)Z_2(t)G'(t)L'^{-1}(t) +\\
&\quad \sigma^{-1}(t)L(t)Z_3(t) + \sigma(t)(1+\pi(t))Q(t)L'^{-1}(t),\\
Z_1(t_0) &= X_0,\\
Z_2(t_0) &= 0,\\
Z_3(t_0) &= I
\end{aligned}$$

with $I \in \mathcal{R}^{m\times m}$ being the identity matrix.

Proof. We first introduce the perturbed system

$$\dot{x}(t) \in A(t)x(t) + \mathcal{E}(p(t), P(t)) \tag{4.28}$$

$$L(t)\dot{y}(t) \in -G(t)x(t) + \mathcal{E}(q(t), Q(t)) \tag{4.29}$$

$$\{x_0, y_0\} \in \mathcal{E}(\{x_0, 0\}, \tilde{X}_0),$$

$$\tilde{X}_0 = \begin{pmatrix} X_0 & 0 \\ 0 & I \end{pmatrix}.$$

Applying consequently Theorems 3.3 and 2.2 to the systems 4.28, 4.29 and 4.24, 4.25, 4.26 we come to equality 4.27. Q.E.D.

Concluding this section we wish to emphasize that the proposed techniques may be extended to the case of measurable multivalued functions $\mathcal{P}, \mathcal{Q}$ appears to be especially important for the problems of observation for uncertain systems. The procedures presented here allows to construct effective algorithms for computer calculations and simulations on the basis of parallel visual representations of the solutions to the problem.

5 Numerical Examples

We take a 2 dimensional system (1.1), (1.2) over the time interval $[0, 5]$.

The initial state is bounded by the ellipsoid $\mathcal{X}_0 = \mathcal{E}(x_0, X_0)$ at the initial moment $t_0 = 0$ with

$$x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad X_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.30}$$

We consider a case when the right hand side is constant:

$$A(t) \equiv \begin{pmatrix} 0 & 1 \\ -8 & 0 \end{pmatrix}, \tag{5.31}$$

describing the position and velocity of an oscillator. Inputs $u(t)$ are also bounded by time independent constraints $\mathcal{P}(t) = \mathcal{E}(p(t), P(t))$ with

$$p(t) \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad P(t) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0.01 \end{pmatrix}. \tag{5.32}$$

State constraint (1.3) is 1 dimensional, and is defined by the data

$$G(t) \equiv (\, 0 \quad 1\,), \qquad q(t) \equiv 1, \qquad Q(t) \equiv (\,1\,). \tag{5.33}$$

Additionally we suppose the initial condition:

$$y(0) \in [-10^{-5}, 10^{-5}],$$

therefore we have that

$$\mathcal{Z}_0 = \mathcal{E}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \times [-10^{-5}, 10^{-5}] \subset \mathcal{R}^2 \times \mathcal{R}.$$

The time interval divided into 100 subintervals of equal lengths, calculations are based on the discretized version of the system (1.1), (1.2), (1.3).

We will illustrate Corollary 3.2 and Theorem 4.1 by calculating the ellipsoidal estimates in the inclusion

$$\mathcal{X}[\tau] \subset \Pi_x(\mathcal{E}(z(\tau, L_+), Z(\tau, L_+, \pi, \sigma) \cap \mathcal{E}(z(\tau, L_-), Z(\tau, L_-, \pi, \sigma))$$

for the following choices for the function L:

$$L_+(t) = \begin{cases} 1 & \text{if } t \in [0, 3.5] \\ 0.3 & \text{if } t \in (3.5, 5]\,, \end{cases} \qquad L_-(t) = \begin{cases} 1 & \text{if } t \in [0, 3.5] \\ -0.3 & \text{if } t \in (3.5, 5]\,. \end{cases} \tag{5.34}$$

Parameters π and σ are chosen according to the rule

$$\pi(t) = \frac{\mathrm{Tr}^{1/2}(P(t))}{\mathrm{Tr}^{1/2}(Q(t))}$$

and

$$\sigma(t) = \frac{\mathrm{Tr}^{1/2}(Z(t))}{\mathrm{Tr}^{1/2}(Q(t), \pi(t)))},$$

that are known to create so called locally Tr-minimal external estimates, (see [11]).

Noting that, in general, the relation

$$\Pi_x(\mathcal{E}_1 \cup \mathcal{E}_2) \subset \Pi_x(\mathcal{E}_1) \cup \Pi_x(\mathcal{E}_2)$$

is a *proper inclusion*, we will show the projections onto the space of state variables of the ellipsoidal estimates associated to L_+ and L_-, as well as the projection of their intersection. The above phenomenon is illustrated in Figure 1, where $\mathcal{E}_1 = \mathcal{E}(q_1, Q_1)$, $\mathcal{E}_2 = \mathcal{E}(q_2, Q_2)$ so that

$$q_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 12 & 7 & 0 \\ 7 & 8 & 0 \\ 0 & 0 & 10 \end{pmatrix},$$

$$q_2 = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 10 & 0 & 8 \\ 0 & 20 & 8 \\ 8 & 8 & 14 \end{pmatrix}.$$

The left upper window shows the projections onto the plane spanned by the first two variables, while on the right upper window we see the projection onto the plane of the first and third variable, and in the lower window onto that of the second and third.

To calculate the extreme point $x \in \mathcal{E}_1 \cap \mathcal{E}_2$, we need the following:

Lemma 5.1 *If $\ell \in \mathcal{R}^n$ is the normal vector to the supporting hyperplane containing the extreme point $x \in E(q_1, Q_1) \cap E(q_2, Q_2)$ then one of the following holds:*

(i) $x \in \partial E(q_i, Q_i) \cap \mathrm{int}(E(q_j, Q_j))$ for $i \neq j$, and then x satisfies

$$x = \frac{Q_i \ell}{\|Q_i^{1/2} \ell\|} + q_i$$

(ii) $x \in \partial\mathcal{E}_1 \cap \partial\mathcal{E}_2$. and then x satisfies

$$(x - q_1) Q_1^{-1} (x - q_1) = 1, \tag{5.35}$$

$$(x - q_2) Q_2^{-1} (x - q_2) = 1, \tag{5.36}$$

and

$$\ell = \alpha Q_1^{-1}(x - q_1) + \beta Q_2^{-1}(x - q_2) \tag{5.37}$$

Proof. In the more complicated case of (ii), the statement follows from the Lagrange necessary condition as we have

$$x = \mathrm{argmax}\{\ell' w \in \mathcal{R} | w \in \mathcal{E}_1 \cap \mathcal{E}_2.\}$$

Q.E.D.

In case (ii) we have to find α and $\beta \in \mathcal{R}$ such that $f(\alpha,\beta) = 1$ and $g(\alpha,\beta) = 1$, where

$$x = (\alpha Q_1^{-1} + \beta Q_2^{-1})^{-1}(\ell + \alpha Q_1^{-1} q_1 + \beta Q_2^{-1} q_2)$$

and

$$f(\alpha,\beta) = (x - q_1)^T Q_1^{-1}(x - q_1)$$

$$g(\alpha,\beta) = (x - q_2)^T Q_2^{-1}(x - q_2).$$

As the derivatives of the above functions can be calculated explicitely, we can use Newton's iteration to obtain thesolution to the above system.

Figure 2. shows the two estimates developing over time with the range of coordinate axes being -30 to 30. The left upper window shows the projections onto the plane spanned by the two state variables. Here they coincide as expected. In the right upper window we see the projection of the two estimating tubes onto the plane of the measurement variable and the first state variable, while in the lower window onto the plane of the measurement variable and the second state variable. In Figure 3. we see the estimates (in the same arrangement of the windows and in the same scale) at the moment $t = 4.25$, drawn by thin lines, and the *projection of their intersection*, drawn by a thicker line. It is to be noted here, that in the space of the first two variables, the projections of the two estimates coincide again, but the projection of their intersection is a proper subset.

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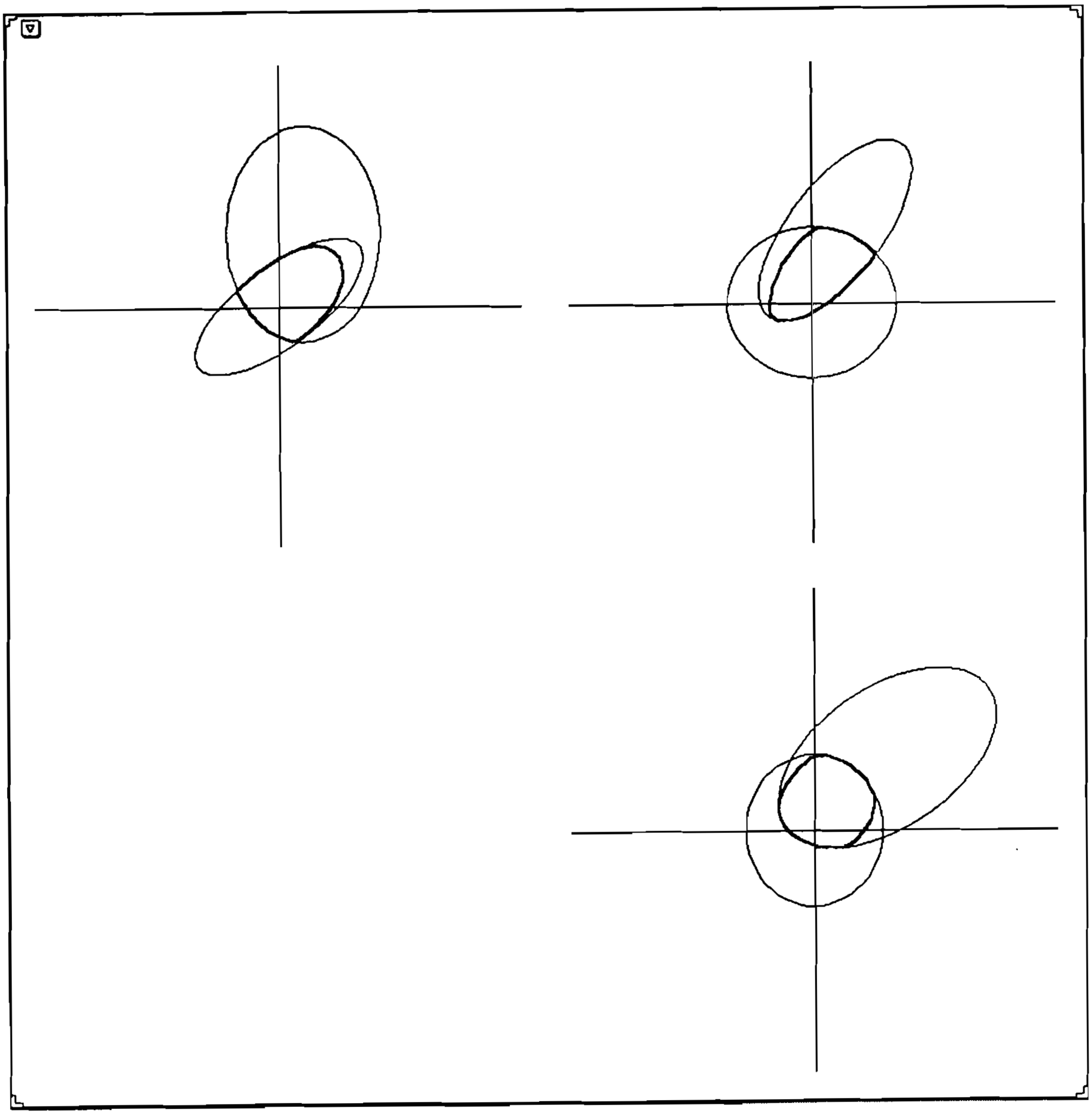

Figure 1: Proper inclusion in $\Pi_x(\mathcal{E}_1 \cup \mathcal{E}_2) \subset \Pi_x(\mathcal{E}_1) \cup \Pi_x(\mathcal{E}_2)$.

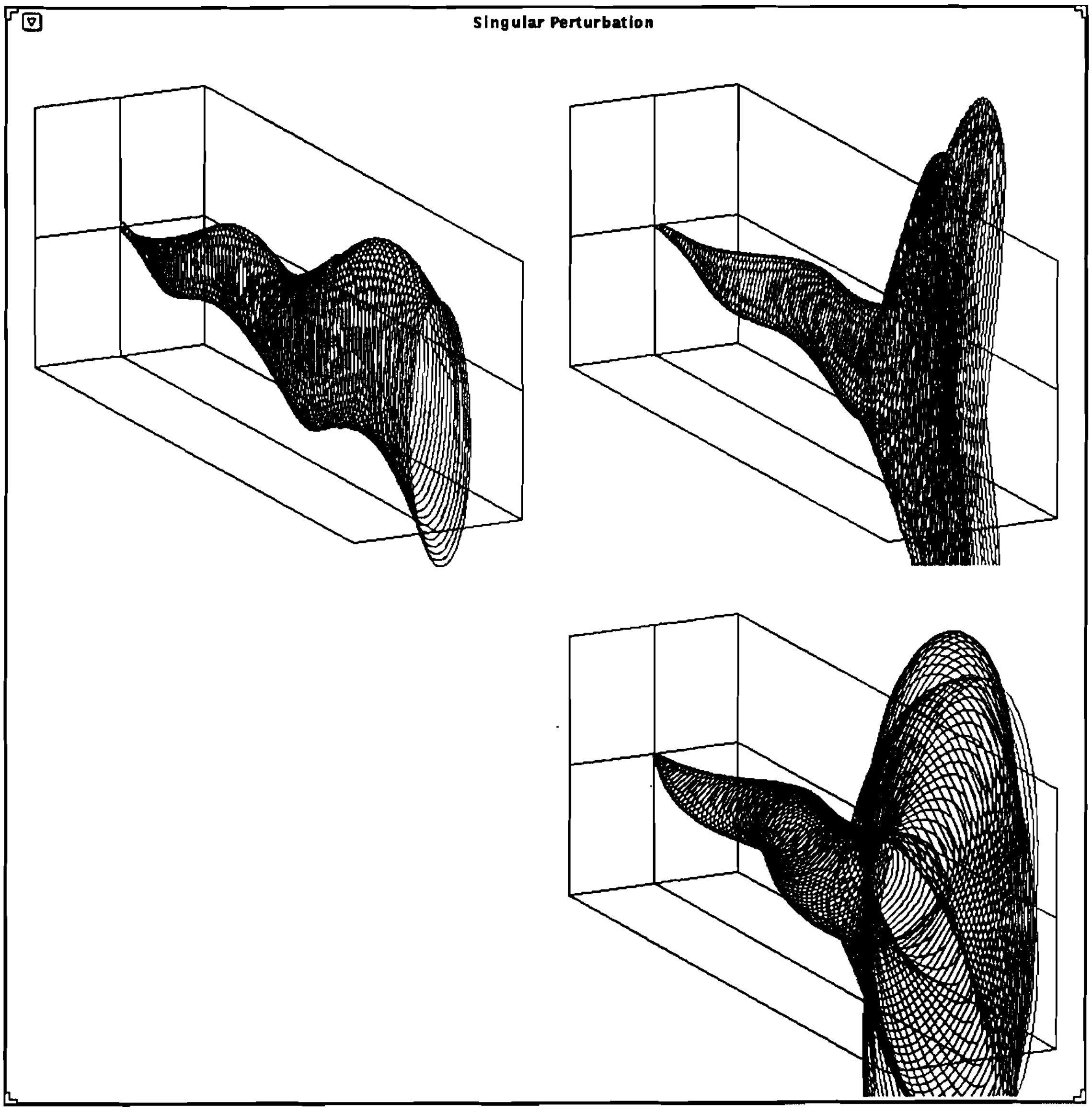

Figure 2: Ellipsoidal estimates developing over time.

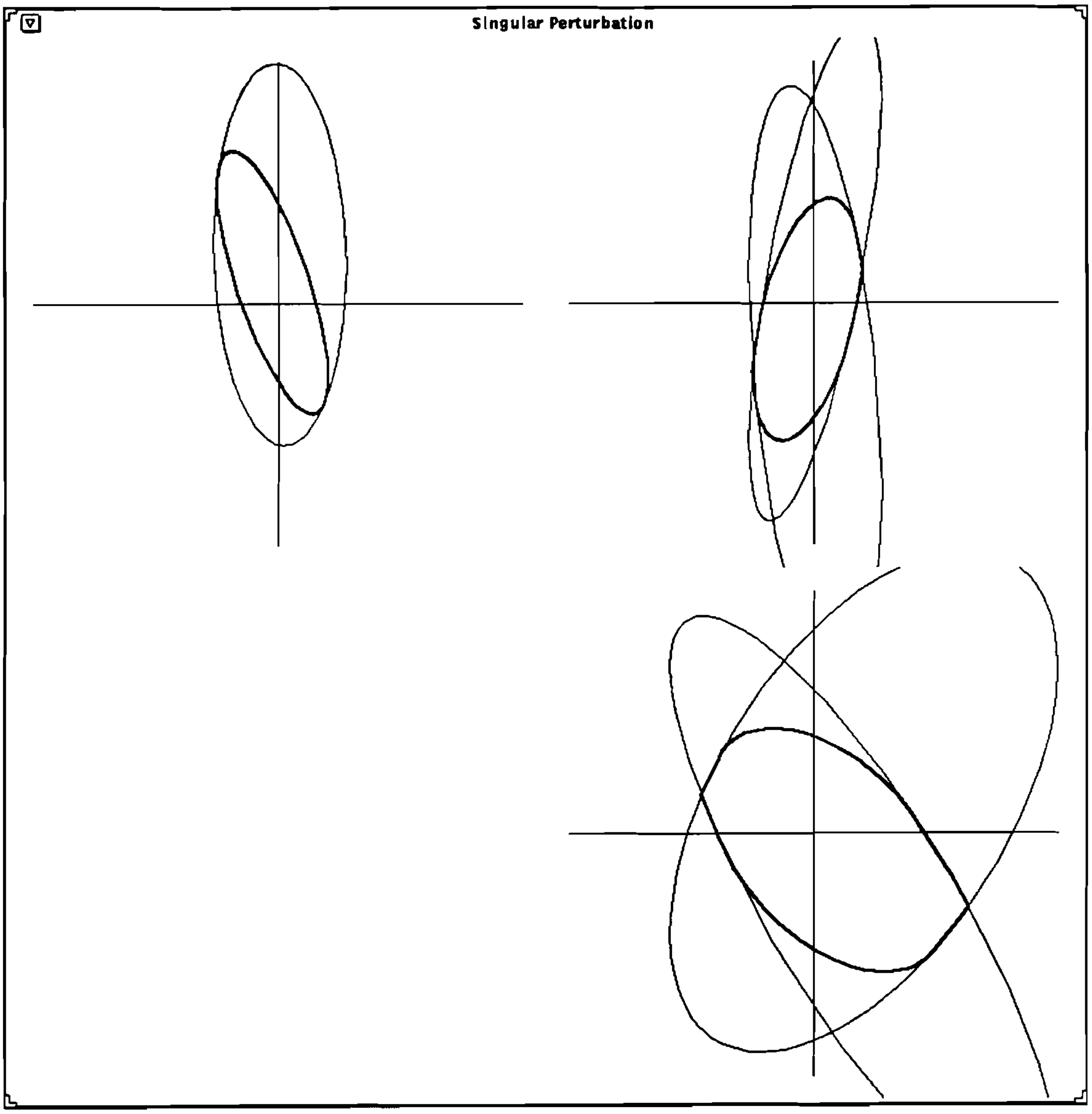

Figure 3: Ellipsoidal estimates and the projection of their intersection.