Working Paper

Source Localization Problem for Parabolic Systems

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Foreword

Two problems for the linear distributed parameter systems of parabolic type motivated by environmental monitoring are discussed:

1. *Nonlinear localization problem:* recover the location of an unknown single source on the basis of available observations. In general, the solution of this problem is set-valued and disconnected.

2. *Identifiability problem:* what types of observations are able to ensure enough information to restore the location point?

An approach is given, based on the introduction of a suitable space of test-functions: in order to determine the unknown location, one has to analyse a proper system of algebraic equations. The latter can be constructed in advance. Sufficient conditions for identifiability are derived and the duality relations between the above nonlinear problems and the problems of open loop control and controllability for an associated adjoint linear system are established.

Source Localization Problem for Parabolic Systems

A.Yu. Khapalov

1. Introduction and Problem Formulation.

Let Ω be an open bounded domain of an n-dimensional Euclidean space R^n with a boundary $\partial\Omega$. Consider the following homogeneous problem for the linear parabolic equation

$$\frac{\partial u(x,t)}{\partial t} = Au(x,t) + f(x,t), \tag{1.1}$$

$$t \in T = (0,\theta), \;\; x \in \Omega \subset R^n, \;\; Q = \Omega \times T, \;\; \Sigma = \partial\Omega \times T,$$

$$u|_\Sigma = 0, \quad u(x,0) = u_0(x).$$

In the above:

$$A = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}) - a(x), \quad a(x) \geq 0,$$

$$v\sum_{i=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij}\,(x)\xi_i\xi_j, \;\; \forall\xi_i \in R, \quad a.e. \text{ in } \Omega, \quad v = \text{const} > 0,$$

$$a_{ij}(x) = a_{ji}(x), \quad a_{ij}(\cdot),\, a(\cdot) \in L^\infty(\Omega),\; i,j = 1,2,\ldots n.$$

We assume that the distributed process (1.1) is excited by a *single source* concentrated either at an unknown spatial point of $\bar{\Omega}$ (where "–" stands for closure) or over some neighborhood of this point. Below we consider two cases. In the first it is supposed that the source acts only at the initial instant of time. In other words, we consider the system

$$\frac{\partial u(x,t)}{\partial t} = Au(x,t), \tag{1.2}$$

$$u|_\Sigma = 0,$$

$$u_0(x) = \phi_l(x, x^0), \quad l = 1 \text{ or } 2, \quad x^0 \in \bar{\Omega},$$

where

$$\phi_1(x, x^0) = \delta(x - x^0), \tag{1.3}$$

$$\phi_2(x, x^0) = \text{meas}^{-1}\{S_h(x^0) \cap \Omega\} \begin{cases} 1, \text{ if } x \in S_h(x^0) \cap \Omega, \\ 0, \text{ if } x \bar{\in} S_h(x^0) \cap \Omega, \end{cases} \tag{1.4}$$

h is a given (and therefore we exclude it from the list of variables) positive parameter characterizing the effective zone of source and $S_h(x^0)$ is the Euclidean neighborhood of radius h of point x^0,

$$S_h(x^0) = \{x \mid \| x - x^0 \|_{R^n} < h\}.$$

Then we consider the case when the system is exited by a single source of type (1.3) or (1.4) *acting in time,* namely,

$$\frac{\partial u(x,t)}{\partial t} = Au(x,t) + \phi_l(x, x^0), \quad l = 1 \text{ or } 2, \tag{1.5}$$

$$u(x, 0) = 0, \quad u|_\Sigma = 0.$$

It is supposed that an output of the system in question may be represented in the form

$$y(t) = \mathbf{G}(t)u(\cdot, t), \quad t \in T, \tag{1.6}$$

where $y(\cdot)$ is an r-dimensional output and $\mathbf{G}(\cdot)$ stands for an observation operator with images in $L^2_r(T) = \underbrace{L^2(T) \times \ldots \times L^2(T)}_{r}$.

In the present paper we study two problems.

Nonlinear localization problem: recover the location x^0 of the unknown source in (1.2) or (1.5) on the basis of available *finite-dimensional at every instant of time* observations (1.6).

In general, the solution of this problem is *set-valued* and *disconnected.*

Identifiability problem: what types of observation operators are able to ensure enough information in order to restore the location of source in a unique way?

The paper is organized as follows. The next section deals with preliminary results concerning the regularity of solutions of systems (1.2) and (1.5). Section 3 introduces a number of observation operators that we study below and their correctness is also discussed. In Section 4 we show first how the localization problem in question may be transformed into the system consisting of *continuum nonlinear algebraic equations* associated with a suitable space of *test-functions.* Then we introduce the definitions of identifiability and ε-identifiability distinguishing those classes of observation operators that allow to reduce the problem to a *finite number* of algebraic equations. This section is concluded by examples illustrating the non-redundancy of such definitions. Sufficient conditions for identifiability for the case of one dimensional parabolic systems with stationary observations are derived in Section 5. Section 6 is devoted to the general (non-smooth) case with sources of type (1.4). Then, in Section 7 we introduce the class of associated *linear* control systems and establish the duality relations between them and the *nonlinear* localization problems in question. In fact, these relations state the coincidence of the sets of test-functions with either the attainable sets or the specified images of the sets of solutions of associated linear control systems. We show how the latter can serve as a tool in the construction of desirable set of test-functions that enables us to solve the localization problem in question.

Remark 1.1. In the present paper we assume that the intensity of the unknown source is given and equal to 1 (in the general case we have $\alpha\ \phi_l(x, x^0)$). Under the assumption that the locations of several individual sources are given in advance and observations are corrupted by unknown deterministic errors the problem of estimation of unknown intensities on the basis of the value of their total emission has been considered in [7]. We stress that the intensities estimation problem, in fact, is a linear one, whereas the localization problem is nonlinear.

Remark 1.2. If available observations are corrupted by unknown *additive* disturbances, the expression (1.6) turns into

$$y(t) = \mathbf{G}(t)u(\cdot,t) + \xi(t), \quad t \in T, \tag{1.7}$$

where $\xi(\cdot)$ stands for a measurement "noise" subjected to prescribed a priori constraints. Due to the nonlinearity of localization problem, the observations of type (1.7) make the latter be quite different from the case of precise observations. The problem (1.2) or (1.5), (1.7) might be a subject for a separate investigation.

2. Preliminaries.

Let $\lambda_i, \ \omega_i(\cdot) \ (i = 1, 2, \ldots)$ denote sequences of eigenvalues and respective orthonormalized (in the norm of $L^2(\Omega)$) eigenfunctions for the spectral problem

$$A\omega_i(\cdot) = -\lambda_i\omega_i(\cdot), \quad \omega_i(\cdot) \in H_0^1(\Omega),$$

$$< \omega_i(\cdot), \omega_j(\cdot) >= \delta_{ij},$$

so that

$$\lambda_{i+1} \geq \lambda_i; \ \lambda_i \to +\infty, \ \ i \to +\infty\, ; \ \ \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

It is well-known that the initial-boundary value problem (1.1) with $u_0(\cdot) \in L^2(\Omega), \ \ f(\cdot,\cdot) \in L^2(Q)$ admits a unique solution in the Banach space $\overset{o}{V}{}_2^{1,0}(Q) = H^{1,0}(Q) \bigcap C([0,\theta]; L^2(\Omega))$ [9, 10] that may be represented in the following general form,

$$u(\cdot,t) = \mathbf{S}_1(t)u_0(\cdot) + \mathbf{S}_2(t)f(\cdot,\cdot), \tag{2.1}$$

$$\mathbf{S}_1(t)u_0(\cdot) = \sum_{i=1}^{\infty} e^{-\lambda_i t} < u_0(\cdot), \, \omega_i(\cdot) > \omega_i(\cdot), \tag*{(2.1)$'$}$$

$$\mathbf{S}_2(t)f(\cdot,\cdot) \ = \ \sum_{i=1}^{\infty} \int_0^t \ e^{-\lambda_i(t-\tau)} < f(\cdot,\tau), \omega_i(\cdot) > d\tau\omega_i(\cdot), \tag*{(2.1)$''$}$$

where the symbol $< \cdot, \cdot >$ stands for the standard scalar product in $L^2(\Omega)$. Here and below we use the standard notations for the Sobolev spaces.

Thus, we may conclude that for $l = 2$ the solutions of systems (1.2) or (1.5) may be defined by $(2.1)'$ and $(2.1)''$.

In the case of the system (1.2), (1.3) we shall assume below that the following conditions are fulfilled.

Assumption 2.1:

$$\Omega = (0,1) \quad \text{and} \quad |\,\omega_i(x)\,| \leq const, \quad i = 1, \ldots, \quad \forall x \in \Omega. \tag{2.2}$$

We recall that under Assumption 2.1 from the embedding theorems [9, 10, 14] it follows that $\omega_i(\cdot) \in C(\bar{\Omega})$ and

$$< \phi_1(\cdot, x^0), \, \omega_i(\cdot) > \, = \lim_{h \to 0} < \phi_2(\cdot, x^0), \, \omega_i(\cdot) >, \quad i = 1, \ldots . \tag{2.3}$$

Hence, due to (2.3) and the asymptotics of the eigenvalues [1], we may define the solution of problem (1.2), (1.3) (as an element of $L^2(Q) \bigcap C((0,\theta]; H_0^1(\Omega)))$ as the limit of associated sequence of solutions of (1.2), (1.4) over $h \to 0$, so as the formula $(2.1)'$ holds in $L^2(Q)$.

For the solutions of both systems (1.2), (1.3) under Assumption 2.1 and (1.2), (1.4) the following identity holds

$$\int_Q u(x,t)(-\frac{\partial \varphi(x,t)}{\partial t} \; - \; A\varphi(x,t))dxdt \; = \; \int_\Omega \phi_l(x,x^0)\varphi(x,0)dx, \tag{2.4}$$

$$\forall \, \varphi(x,t) \in H^{2,1}(Q), \quad \varphi\,|_\Sigma = \; 0, \quad \varphi(x,\theta) = 0.$$

Assuming the coefficients of the operator A and the boundary $\partial\Omega$ to be sufficiently smooth, we may define for $n \leq 3$ the solution of the problem (1.5) with $l = 1$ as an element of $L^2(Q)$ [10]. Then for the solutions of both systems (1.5) with $l = 2$ and for $n \leq 3$ with $l = 1$, assuming below the needed regularity in the last case, we obtain,

$$\int_Q u(x,t)(-\frac{\partial \varphi(x,t)}{\partial t} \; - \; A\varphi)dxdt \; = \; \int_0^\theta \int_\Omega \phi_l(x,x^0)\varphi(x,t)dxdt, \tag{2.5}$$

$$\forall \, \varphi(x,t) \in H^{2,1}(Q), \quad \varphi\,|_\Sigma = \; 0, \quad \varphi(x,\theta) = 0.$$

The identities (2.4), (2.5) play an important role in deriving the duality relations in Section 7.

3. Observation operators.

Below we consider the following types of observations.

Spatially-averaged observations:

$$\mathbf{G}(t)u(\cdot,t) = \int_{\Omega} \chi\,(x,t)\; u(x,t)dx, \quad t \in T, \tag{3.1}$$

with $\chi(x,t) \in L_r^{\infty}(T; L_r^2(Q))$ given.

A special subclass of observation operators of the above type is

Zone observations:

$$\mathbf{G}(t)u(\cdot,t) = \int_{\Omega} \begin{pmatrix} \chi(x,S_1(t)) \\ \vdots \\ \chi(x,S_r(t)) \end{pmatrix} u(x,t)dx, \quad t \in T, \tag{3.2}$$

where

$$\chi(x,S_j(t)) = \quad \text{meas}^{-1}\{S_j(t)\} \begin{cases} 1, \text{ if } x \in S_j(t), \\ 0, \text{ if } x \bar{\in} S_j(t), \end{cases}$$

$S_j(t) \subset \Omega, \quad j = 1,\dots,r$ are effective sensing regions at the instant t. We assume that the set-valued maps: $t \to S_j(t)$ are continuous in time with respect to Lebesgue measure. When $S_j(t) = S_j, \quad j = 1,\dots,r$ we say about *stationary* zone observations [3], otherwise we have *dynamic* ones.

Pointwise (stationary or dynamic) observations:

$$\mathbf{G}(t)u(\cdot,t) = \; (u(\bar{x}^1(t),t), \; \dots, \; u(\bar{x}^r(t),t))', \quad t \in T, \tag{3.3}$$

where measurements are taken at some spatial points or along specified trajectories in the domain

Ω. It is clear that this type of sensors requires a corresponding smoothness of solutions. We shall consider this type of observations only for the systems (1.2), (2.2), (1.4) or (1.5), (2.2), (1.3) and, also, (1.5), (1.4) with $n \leq 3$ assuming that the operator A and the boundary $\partial\Omega$ are sufficiently regular in order to ensure the enclosure of outputs into $L^2_r(T)$.

We stress that all the above observation operators at every instant of time provide *finite-dimensional* outputs.

Remark 3.1. Assuming the interval of observations to be equal (ε, θ) with $\varepsilon > 0$, we also may consider the localization problem (1.2), (1.3) with the pointwise observations of type (3.3).

4. Localization and Identifiability.

In this section we introduce the definitions of identifiability and ε-identifiability and show how on the basis of the introduction of suitable space of test-functions the localization problem in question can be transformed into the system of algebraic equations.

Denote by X^0_{ml} the set of all those x^0 that solve the localization problem in question, namely: $l = 1$ corresponds to the case of the sources of type (1.3), $l = 2$ – to (1.4), whereas $m = 1$ corresponds to the system (1.2) and $m = 2$ – to the system (1.5).

Consider first the problem (1.2), (1.6). By virtue of (2.1), we come to the following general representation for the unknown point x^0,

$$\mathbf{G}(t)\mathbf{S}_1(t)\phi_l(\cdot, x^0) \;=\; y(t), \;\; t \in T, \quad l = 1, 2. \tag{4.1}$$

In particular, for the problem (1.2), (2.2), (1.3), (3.1) we obtain

$$\sum_{i=1}^{\infty} e^{-\lambda_i t}\omega_i(x^0)\, \mathbf{G}(t)\omega_i(\cdot) \;=\; y(t), \;\; t \in T. \tag{4.2}$$

Due to Sections 2 and 3, (4.1), (4.2) are the equations in the functional space $L^2_r(T)$. Therefore, they are equivalent to the following set of algebraic equations

$$\int_0^\theta \lambda'(t)\, \mathbf{G}(t)\mathbf{S}_1(t)\phi_l(\cdot, x^0)dt \;=\; \int_0^\theta \lambda'(t)\; y(t)dt, \quad \forall\lambda(\cdot) \in L^2_r(T). \tag{4.3}$$

In turn, for the localization problem (1.5), (1.6) we have

$$\int_0^\theta \lambda'(t)\, \mathbf{G}(t)\mathbf{S}_2(t)\phi_l(\cdot, x^0)dt \;=\; \int_0^\theta \lambda'(t)\, y(t)dt, \quad \forall\lambda(\cdot) \in L^2_r(T). \tag{4.4}$$

Denote by $X^0_{ml}(\lambda(\cdot))$, $m, l = 1, 2$ the sets of all those x^0 that for a given $\lambda(\cdot)$ are the solutions of (4.3) for $m = 1$ and of (4.4) for $m = 2$. The index l has the same meaning as for X^0_{ml}. Then we obtain

Lemma 4.1. The following formula is fulfilled

$$X^0_{ml} = \bigcap_{\lambda(\cdot)\in L^2_r(T)} X^0_{ml}(\lambda(\cdot)), \quad m, l = 1, 2. \tag{4.5}$$

Under assumptions of Sections 2, 3

$$\mathbf{G}(\cdot)\mathbf{S}_m(\cdot) : L^2(\Omega) \to L^2_r(T).$$

Hence, we may set

$$\gamma_m(\cdot, \lambda(\cdot)) \;=\; \mathbf{S}^*_m(\cdot)\mathbf{G}^*(\cdot)\lambda(\cdot), \quad m = 1, 2,$$

$$\gamma_m(\cdot, \lambda(\cdot)) \in L^2(\Omega), \quad \forall\, \lambda(\cdot) \subset L^2_r(T).$$

Definition 4.1. Given $\lambda(\cdot)$, we shall say that the function $\gamma_m(\cdot, \lambda(\cdot))$ is *a test-function* for the localization problem associated with the index m.

Remark 4.1. Although in the general case $\gamma_m(\cdot, \lambda(\cdot))$ is an element of $L^2(\Omega)$, in the case of systems (1.2), (2.2) with the stationary observations of type (3.1) or (1.5), (2.2), (3.3) and for $n \leq 3$ (1.5), (3.1)

$$\gamma_m(\cdot, \lambda(\cdot)) \in C(\bar{\Omega}).$$

Set

$$\Gamma_m = \bigcup_{\lambda(\cdot)\in L^2_r(T)} \gamma_m(\cdot, \lambda(\cdot)), \quad m = 1, 2.$$

Then, in order to solve any of the above localization problems, one has to analyze the corresponding system containing *continuum set* of equations, namely,

$$\begin{cases} \int_\Omega \gamma_m(x,\lambda(\cdot))\,\phi_l(x,x^0)\,dx \;=\; \int_0^\theta \lambda'(t)\,y(t)dt, \quad \forall\lambda(\cdot)\in L_r^2(T), \\ x^0\in\bar{\Omega}. \end{cases} \tag{4.6}$$

Remark 4.2. When $\gamma_m(\cdot,\lambda(\cdot))\in C(\bar{\Omega})$ (see Remark 4.1) we obtain

$$\int_\Omega \gamma_m(x,\lambda(\cdot))\,\phi_1(x,x^0)\,dx \;=\; \gamma_m(x^0,\lambda(\cdot)), \tag{4.7}$$

otherwise we treat the formula (4.6) for the sources of type (1.3) as a formal record of (4.3). For the sources of type (1.4), we have

$$\int_\Omega \gamma_m(x,\lambda(\cdot))\,\phi_2(x,x^0)\,dx \;=\; \text{meas}^{-1}\{S_h(x^0)\cap\Omega\}\int_{S_h(x^0)\cap\Omega}\gamma_m(x,\lambda(\cdot))\,dx. \tag{4.8}$$

We rise here two questions:

1. Since x^0 is an element of the finite dimensional space R^n (and, accordingly, X^0 is a subset of $\bar{\Omega}\subset R^n$), can the set of equations in (4.6) be reduced to a finite number of equations?

2. How to chose properly functions $\lambda(\cdot)$ in order to obtain a "good" set of test-functions that enables us to determine x^0?

The first question is related to *nonlinear identifiability problem,* whereas the second leads to *open loop control problem.*

We begin with identifiability.

Definition 4.2. Let Ω^* be a subset of $\bar{\Omega}$ or coincide with it. We shall say that the problem (1.2), (1.6) or (1.5), (1.6) is identifiable in Ω^*, if there exists such a *finite* subset $\Gamma_m^*\subset\Gamma_m$ that the system

$$\begin{cases} \int_\Omega \gamma_m(x,\lambda(\cdot))\,\phi_l(x,x^0)\,dx \;=\; \int_0^\theta \lambda'(t)\,y(t)dt, \quad \forall\gamma_m(\cdot,\lambda(\cdot))\in\Gamma_m^*, \\ x^0\in\Omega^* \end{cases}$$

does not have more than one solution for any possible output $y(\cdot)$.

Definition 4.3'. We shall say that the problem (1.2) or (1.5), (1.6) is ε-identifiable in $\Omega^* \subseteq \bar{\Omega}$, if for an arbitrary positive ε there exists such a *finite* subset $\Gamma^*_{m\varepsilon} \subset \Gamma_m$ that for any possible output $y(\cdot)$ the corresponding set X^{0*}_{ml} consisting of all those x^0 that are the solutions of the system

$$\begin{cases} \int_\Omega \gamma_m(x,\lambda(\cdot))\,\phi_l(x,x^0)\,dx \;=\; \int_0^\theta \lambda'(t)\,y(t)dt, & \forall \gamma_m(\cdot,\lambda(\cdot)) \in \Gamma^*_{m\varepsilon}, \\ x^0 \in \Omega^*, \end{cases} \tag{4.6$'$}$$

satisfies the condition

$$diam\, X^{0*}_{ml} \;\leq\; \varepsilon, \tag{4.9$'$}$$

where $diam\, X^{0*}_{ml} = \inf\{\eta \mid X^{0*}_{m1} \subset S_\eta(x),\ \eta > 0,\ x \in R^n\}$. We assume that the empty set has zero-diameter.

In turn, for the sources of type (1.4) we introduce a modified (*"disturbed"*) version of this definition.

Definition 4.3''. We shall say that the problem (1.2) or (1.5), (1.4), (1.6) is εh-identifiable in $\Omega^* \subseteq \bar{\Omega}$, if for an arbitrary $\varepsilon > 0$ there exists such a *finite subset* $\Gamma^*_{m\varepsilon} \subset \Gamma_m$ that for any possible output $y(\cdot)$ the corresponding set X^{0*}_{m2} consisting of all those x^0 that are the solutions of the system

$$\begin{cases} \int_\Omega \gamma_m(x,\lambda(\cdot))\,\phi_2(x,x^0)\,dx \;=\; \int_0^\theta \lambda'(t)\,y(t)dt, & \forall \gamma(\cdot,\lambda(\cdot)) \in \Gamma^*_{m\varepsilon}, \\ x^0 \in \Omega^*, \end{cases} \tag{4.6$''$}$$

satisfies the condition

$$diam\, X^{0*}_{m2} \;\leq\; 2h + \varepsilon. \tag{4.9$''$}$$

Remark 4.3. In fact, Definition 4.3'' distinguishes the class of problems (1.2) or (1.5), (1.4), (1.6) only (regardless of Ω^*) with respect to the existence of a proper set of test-functions.

The following assertions establish the linkage between the property of "local" identifiability and the solutions of localization problems in Ω.

Proposition 4.1. Assume that the set Ω can be represented as an *infinite* union of a monotone with respect to inclusion sequence of subsets,

$$\Omega = \bigcup_{s=1}^{\infty} \Omega_s, \quad \Omega_1 \subseteq \ldots \subseteq \Omega_s \subseteq \ldots .$$

Then:

1. If the systems (1.2) or (1.5), (1.6) are identifiable in each Ω_s, then for any possible output the corresponding localization problems have unique solutions in Ω.

2. If the systems (1.2) or (1.5), (1.3), (1.6) are ε–identifiable in each Ω_s, then for any possible output the corresponding localization problems have unique solutions in Ω.

3. If the systems (1.2) or (1.5), (1.4), (1.6) are εh–identifiable in each Ω_s, then for any possible output the solutions X^0_{m2}, $m = 1, 2$ of the corresponding localization problems satisfy the condition

$$diam\,(X^0_{m2} \bigcap \Omega) \leq 2h. \tag{4.10}$$

The proof immediately follows from Definitions 4.2-4.3″.

The following examples show that Definitions 4.2-4.3″ are not redundant.

Examples. Consider the one-dimensional heat equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < 1, \quad t \in T, \tag{4.11}$$

$$u(t,0) = u(t,1) = 0, \quad u(x,0) = \delta(x - x^0)$$

with stationary observations of type (3.1),

$$y(t) = \mathbf{G}u(\cdot,t), \quad t \in T. \tag{4.12}$$

It is well-known that the eigenvalues and the (orthonormalized) eigenfunctions for problem (4.11) are given by

$$\lambda_k = -(\pi k)^2, \quad \omega_k(x) = \sqrt{2}\,\mathrm{Sin}\,\pi k x, \quad k = 1, 2, \ldots .$$

Expanding the output of system (4.11), (4.12) in a series of exponents we obtain

$$y(t) \;=\; \sum_{k=1}^{\infty} e^{-(\pi k)^2 t}\, \mathbf{G}\omega_i(\cdot)\, \omega_k(x^0), \quad t \in T. \tag{4.13}$$

A. Assume that

$$\mathbf{G}\omega_i(\cdot) = \begin{cases} 1, & i = 1, \\ 0, & i \neq 1. \end{cases}$$

In this case (4.13) turns into

$$y(t) = \sqrt{2} e^{-\pi^2 t} \ \mathrm{Sin}\ \pi x^0, \quad t \in T.$$

It is not hard to see that for any given $y(\cdot)$ the set X^0_{11} is disconnected and consists of two points. Namely, $x^{01} = 1/\pi \arcsin \frac{y(t)}{\sqrt{2e^{-\pi^2 t}}}$ and $x^{02} = 1 - x^{01}$.

B. Assume that the observation operator is such that

$$\mathbf{G}\omega_i(\cdot) = 0, \quad i \neq 4k, \quad k = 1, \dots .$$

Then, if $y(\cdot) \equiv 0$, then the points $0,\ 1,\ 1/2$ and $1/4$ always belong to the set X^0_{11}.

5. Stationary Observations: One Dimensional Case.

In this section we consider the one dimensional parabolic systems and focus on the sources of type (1.3).

We begin by studying the sets of test-functions for the localization problem (1.2), (1.6) with stationary observations. Denote by $\beta_i,\ i = 1, \dots$ the distinct eigenvalues of the operator A and renumber the set of eigenfunctions, setting

$$\omega_{ik}(\cdot), \quad i = 1, \dots, \quad k = 1, \dots, k^i,$$

where k^i stands for the multiplicity of i-th eigenvalue. Then, we may write

$$\gamma_1(x, \lambda(\cdot)) = \sum_{i=1}^{\infty} \int_{o}^{\theta} e^{-\beta_i t} \lambda'(t)\, dt \sum_{k=1}^{k^i} \mathbf{G}\omega_{ki}(\cdot)\, \omega_{ik}(x). \tag{5.1}$$

From (5.1) it follows that, if observations are *stationary,* the sequence of exponentials $\{e^{-\beta_i t}\}_{i=1}^{\infty}$ plays a crucial role. We recall now for the well-known result in harmonic analysis, namely [11, 5]: if the sequence $\{\beta_i\}_{i=1}^{\infty}$ is such that

$$\sum_{i=1}^{\infty} \frac{1}{\beta_i} = \infty, \tag{5.2}$$

the sequence of exponentials $\{e^{-\beta_i t}\}_{i=1}^{\infty}$ spans all the spaces $C[0, \theta]$ and $L^p(T)$, $p \geq 1$, otherwise, when

$$\sum_{i=1}^{\infty} \frac{1}{\beta_i} < \infty, \tag{5.3}$$

in every of the above spaces all the distances from $e^{-\beta_i t}$, $i = 1, \dots$ to the closed span of $\{e^{-\beta_k t}\}_{k=1, k\neq i}^{\infty}$ are positive.

It is well-known that, due to the asymptotics of eigenvalues [1], (5.3) is fulfilled only for the one dimensional parabolic systems.

Lemma 5.1. Let Assumption 2.1 be fulfilled and all the eigenvalues $\{\lambda_i\}$ be simple (that is, $\lambda_i = \beta_i$). Then, for the stationary observations of type (3.1) the set Γ_1 contains all the functions of type

$$\sum_{i=1}^{I} u_k\, \omega_i(x), \quad I = 1, \dots , \tag{5.4}$$

if the following condition is fulfilled

$$\mathbf{G}\omega_i(\cdot) \neq 0, \quad i = 1, \dots . \tag{5.5}$$

Proof. We may restrict ourselves only by the case of scalar observations. The condition (5.3) implies ([5]) the existence of a biorthogonal sequence $\{q_i(t)\}_{i=1}^{\infty}$ for $\{e^{-\lambda_i t}\}_{i=1}^{\infty}$ such that

$$\int_0^\theta e^{-\lambda_i t} q_k(t)\, dt = \begin{cases} 1, \text{ if } & i = k, \\ 0, \text{ if } & i \neq k. \end{cases}$$

Therefore, in order to obtain all the functions in (5.4), namely,

$$\gamma_1(x, \lambda(\cdot)) = \sum_{i=1}^{I} u_k\, \omega_i(x), \quad I = 1, \ldots,$$

it is sufficient to set

$$\lambda(t) = \sum_{i=1}^{I} \frac{u_i}{\mathrm{G}\omega_i(\cdot)}\, q_i(t).$$

This concludes the proof.

The system (1.5), (1.6) with stationary observations admits the following general representation for test-functions,

$$\gamma_2(x, \lambda(\cdot)) = \sum_{i=1}^{\infty} \frac{1}{\beta_i} \int_0^\theta (1 - e^{-\beta_i t})\, \lambda'(t) dt \ \sum_{k=1}^{k^i} \mathrm{G}\omega_{ki}(\cdot)\, \omega_{ki}(x). \tag{5.6}$$

Lemma 5.2. Let all the eigenvalues $\{\lambda_i\}$ be simple and (5.5) be fulfilled. Then, for the system (1.5), (2.2) with the stationary observations of types (3.1) or (3.3) the assertion of Lemma 5.1 is valid for the set Γ_2.

The proof of Lemma 5.2 follows the lines of the proof of Lemma 5.1 with the only change: we construct a biorthogonal sequence for the following sequence of functions:

$$1,\ e^{-\lambda_1 t},\ \ldots\ e^{-\lambda_i t},\ \ldots\ .$$

Remark 5.1. The condition (5.5) is the well-known necessary and sufficient condition for *observability* of the system (1.2) with $u_0(\cdot) \in L^2(\Omega)$, (1.6) under stationary observations [13, 2, 3].

The following theorems give sufficient conditions for ε-identifiability and identifiability of the one dimensional parabolic system with stationary observations.

Theorem 5.1. Let the conditions of Lemma 5.1 be fulfilled. Then, both systems (1.2), (2.2), (1.3) with the stationary observations of type (3.1) and (1.5), (2.2), (1.3) with the stationary observations of types (3.1) or (3.3) are ε-identifiable in every interval $[0, x_1]$ or $[x_2, 1]$, $\forall x_1, x_2 \in (0,1)$.

Proof. Let $\Omega^* = [0, x_1]$ be an arbitrary subinterval of $[0,1)$. Let us take an arbitrary continuous function $v(x)$ vanishing at the ends of $[0,1]$ and such that

$$v(x) = c\,x \quad \text{when} \quad x \in [0, x_1], \tag{5.7}$$

where c is a positive scalar parameter which will be selected below.

Since the eigenfunctions of the system (1.1) form a basis in the space $H_0^1(\Omega)$ [9, 10], we conclude that both Γ_m, $m = 1, 2$ are dense in $H_0^1(\Omega)$ and (for $n = 1$) in the space of continuous functions vanishing at $x = 0, 1$.

Let, for example, $m = 1$. Then, for any $\nu > 0$ there exists a function $\gamma_1(x, \lambda_\nu(\cdot))$ such that

$$|\,\gamma_1(x, \lambda_\nu(\cdot)) - v(x)\,| \leq \nu, \quad \forall x \in [0, x_1]. \tag{5.8}$$

By virtue of (4.6) and (4.7), in order to identify the value of x^0 in $[0, x_1]$, one has to solve the system

$$\begin{cases} \gamma_1(x^0, \lambda_\nu(\cdot)) = \int_0^\theta \lambda_\nu'(t)\, y(t)dt, \\ x^0 \in [0, x_1] \end{cases} \tag{5.9}$$

among the others in (4.6)'. If (5.9) has the unique solution, we skip the next step of the argument. Otherwise, denote by x^{01} and x^{02} any two different solutions of (5.9), so as $\gamma_1(x^{01}, \lambda_\nu(\cdot)) = \gamma_1(x^{02}, \lambda_\nu(\cdot))$. Then, (5.8) yields

$$|\,v(x^{01}) - v(x^{02})\,| \leq |\,v(x^{01}) - \gamma_1(x^{01}, \lambda_\nu(\cdot)) + \gamma_1(x^{02}, \lambda_\nu(\cdot)) - v(x^{02})\,| \leq 2\nu.$$

Let us take now an arbitrary positive ε. Then, we obtain the necessary estimate (4.9)' in $[0, x_1]$, if taking

$$c = \frac{2\nu}{\varepsilon}.$$

The same argument, as in the above, we may apply for any interval $[x_2, 1] \subset (0, 1]$. This concludes the proof.

From the above results and Proposition 4.1 it follows

Theorem 5.2. Let the assumptions of Theorem 5.1 be fulfilled. Then for any possible output the localization problems (1.2), (2.2), (1.3) with the stationary observations of type (3.1) and (1.5), (2.2), (1.3) with the stationary observations of types (3.1) or (3.3) have unique solutions.

Assumption 5.1. Let $n = 1$ and

$$\mid \frac{\partial a_{11}}{\partial x} \mid \leq const.$$

Theorem 5.3. Let Assumption 5.1 and the conditions of Lemma 5.1 be fulfilled. Then, both systems (1.2), (2.2), (1.3) with the stationary observations of type (3.1) and (1.5), (2.2), (1.3) with the stationary observations of types (3.1) or (3.3) are identifiable in any closed subinterval of Ω.

Proof. Assumption 5.1 implies ([9, 10]) that all the eigenfunctions $\{\omega_i(\cdot)\}_{i=1}^{\infty}$ are elements of the space $H^2(\Omega) \bigcap H_0^1(\Omega)$ and an arbitrary function from $H^2(\Omega) \bigcap H_0^1(\Omega)$ may be represented by its Fourier-series expansion along the sequence of eigenvalues that converges in $H^2(\Omega)$. Hence, applying Lemmas 5.1 and 5.2, we obtain that both Γ_m, $m = 1, 2$ are dense in $H^2(\Omega) \bigcap H_0^1(\Omega)$.

Let, for example, $m = 1$ and $[x_1, x_2]$ be an arbitrary subinterval of $(0, 1)$. Take an arbitrary twice continuously differentiable and vanishing at $x = 0, 1$ function $v(x)$ that has a strictly positive first derivative in $[x_1, x_2]$. Since $v(\cdot) \in H^2(\Omega) \bigcap H_0^1(\Omega)$, from the above it follows that for any $\nu > 0$ there exists a function $\gamma_1(x, \lambda_\nu(\cdot))$ such that

$$\mid \gamma_1(x, \lambda_\nu(\cdot)) - v(x) \mid \leq \nu, \quad \forall x \in [x_1, x_2],$$

$$\mid \frac{\partial \gamma_1(x, \lambda_\nu(\cdot))}{\partial x} - \frac{\partial v(x)}{\partial x} \mid \leq \nu, \quad \forall x \in [x_1, x_2].$$

These two estimates imply the existence of $\nu^* > 0$ such that the associated test-function $\gamma_1(x, \lambda_{\nu^*}(\cdot))$ is also strictly monotone in $[x_1, x_2]$ and, hence, the corresponding system (5.9) does not have more than one solution in $[x_1, x_2]$. This concludes the proof.

Remark 5.2. We note that all the results of this section can also be extended for the case of multiple eigenvalues when all the multiplicities are uniformly bounded (see [13, 2, 3]).

6. The General Case: Sources of Type (1.4).

The main result of this section is the following

Theorem 6.1. Let Γ_m, $m = 1, 2$ be dense in $L^2(\Omega)$. Then both problems (1.2), (1.4), (1.6) and (1.5), (1.4), (1.6) are εh–identifiable in $\bar{\Omega}$.

Proof. Let us take an arbitrary non-degenerate $[n \times n]$-matrix

$$C = \{c_{ij}\}, \quad i, j = 1, \ldots, n$$

and set

$$v_i(x) = \sum_{j=1}^{n} c_{ij} x_j, \quad j = 1, \ldots, n, \quad x = (x_1, \ldots, x_n)' \in \bar{\Omega}. \tag{6.1}$$

Assume first that all the functions $\{v_i(\cdot)\}$ belong to the set Γ_m, so as there exist $\{\lambda_{mi}(\cdot)\}_{i=1}^{n}$ for which

$$v_i(x) = \gamma_m(x, \lambda_{mi}(\cdot)), \; i = 1, \ldots, n, \; m = 1, 2, \quad x \in \bar{\Omega}.$$

Due to (4.8),

$$\int_{\Omega} \phi_2(z, x)\, \gamma_m(z, \lambda_{mi}(\cdot))\, dz \; = \; \sum_{j=1}^{n} c_{ij} x_j \; + \; b_i(x), \quad \forall x \in \bar{\Omega}, \; i = 1, \ldots, n, \tag{6.2}$$

where $b_i(x)$, $i = 1, \ldots$ depend upon the choice of C and h and are uniformly bounded in $\bar{\Omega}$, so as

$$| b_i(x) | \leq \mu = \mu(h, C), \quad \forall x \in \bar{\Omega}, \; i = 1, \ldots, n.$$

Indeed,

$$\int_{\Omega} \phi_2(z, x) \begin{pmatrix} \gamma_m(z, \lambda_{m1}(\cdot)) \\ \gamma_m(z, \lambda_{m2}(\cdot)) \\ \vdots \\ \gamma_m(z, \lambda_{mn}(\cdot)) \end{pmatrix} dz = \frac{1}{\text{meas}\,\{S_h(x) \bigcap \Omega\}} \int_{\{S_h(x) \bigcap \Omega - x\}} C(x+z)\, dz, \quad \forall\, x \in \bar{\Omega},$$

where $\bar{0}$ stands for the origin. Hence,

$$\| (b_1(x), \ldots, b_n(x))' \|_{R^n} \leq \sup_{z \in S_h(\bar{0})} \| Cz \|_{R^n} \leq h \, \| C \|, \quad \forall x \in \bar{\Omega}. \tag{6.3}$$

By virtue of (4.6) and (6.2), the unknown point x^0 is a solution of the following system

$$\begin{cases} x^0 = C^{-1} \begin{pmatrix} \int_0^\theta \lambda'_{m1}(t)\, y(t) dt - b_1(x^0) \\ \int_0^\theta \lambda'_{m2}(t)\, y(t) dt - b_2(x^0) \\ \vdots \\ \int_0^\theta \lambda'_{mn}(t)\, y(t) dt - b_n(x^0) \end{pmatrix}, \\ x^0 \in \bar{\Omega}. \end{cases} \tag{6.4}$$

Let x^{01}, x^{02} be two different solutions of the system (6.4) (if they exist, otherwise we skip the next step of the argument). Applying the estimate (6.3) yields

$$\| x^{01} - x^{02} \|_{R^n} \leq \underset{x^{01}, x^{02} \in \bar{\Omega}}{sup} \| C^{-1}((b_1(x^{01}), \ldots, b_n(x^{01}))' - (b_1(x^{02}), \ldots b_n(x^{02}))') \|_{R^n} \leq$$

$$\leq 2h \, \| C^{-1} \| \, \| C \| \,. \tag{6.5}$$

In order to obtain the required estimate (4.10) it is sufficient to select C be unit-matrix.

In the general case assumptions of Theorem 6.1 provide us with such sequences $\{\lambda^s_{mi}(\cdot)\}_{s=1}^{\infty}$, $i = 1, \ldots, n$, $m = 1, 2$ that the corresponding sequences of vector test-functions $\{(\gamma^s_m(\cdot, \lambda^s_{m1}(\cdot)), \ldots, \gamma^s_m(\cdot, \lambda^s_{mn}(\cdot)))'\}_{s=1}^{\infty}$, $m = 1, 2$ converge in the norm of $L^2_n(\Omega)$ to $(v_1(\cdot), \ldots, v_n(\cdot))'$, so as

$$\| \gamma_m^s(\cdot, \lambda_{mi}^s(\cdot)) - v_i(\cdot) \|_{L^2(\Omega)} \to 0 \quad \text{when} \quad s \to 0, \; i = 1, \ldots, n, \tag{6.6}$$

where, as in the above, $v(x) = Cx$.

Let us select any $\rho > 0$. Then, by (6.6) and (4.8), we can find such $s = s^*$ that

$$\left| \int_\Omega \phi_2(z,x) \gamma_m^{s^*}(z, \lambda_{mi}^{s^*}(\cdot))\, dz - \int_\Omega \phi_2(z,x) v_i(z)\, dz \right| \le \rho, \quad i = 1, \ldots, n, \quad \forall x \in \bar{\Omega}.$$

The last set of estimates leads, along the lines (6.2)-(6.4), to the system

$$\begin{cases} x^0 = C^{-1} \begin{pmatrix} \int_0^\theta \lambda_{m1}^{s^*\prime}(t)\, y(t) dt - b_1(x^0) + \rho_1(x^0) \\ \int_0^\theta \lambda_{m2}^{s^*\prime}(t)\, y(t) dt - b_2(x^0) + \rho_2(x^0) \\ \vdots \\ \int_0^\theta \lambda_{mn}^{s^*\prime}(t)\, y(t) dt - b_n(x^0) + \rho_n(x^0) \end{pmatrix}, \\ x^0 \in \bar{\Omega} \end{cases}$$

for the unknown location point with

$$| \rho_i(x) | \le \rho, \quad \forall x \in \bar{\Omega}, \; i = 1, \ldots, n.$$

Thus, we obtain, via a slight modification of (6.5), the needed estimate (4.10) with

$$\varepsilon = 2\sqrt{n}\rho.$$

This concludes the proof of Theorem 6.1.

Remark 6.1. In the next section we give examples of systems for which the assumptions of Theorem 6.1 are fulfilled.

7. Associated Control Problems. Duality Relations.

In this section we establish the linkage between the nonlinear localization problems (1.2) or (1.5), (1.6) and some classes of linear open loop control problems. We show that the latter can serve as a tool in the construction of desirable set of test-functions that enables us to solve the localization problem in question.

Let us consider first the following control problem associated with the system (1.2), (1.6),

$$\frac{\partial \varphi}{\partial t} = -A\varphi - \mathbf{G}^*(\cdot)\lambda(\cdot), \quad t \in T, \ x \in \Omega, \tag{7.1}$$

$$\varphi\,|_\Sigma = 0, \quad \varphi(x,\theta) = 0.$$

Problem 7.1. Find a *control* $\lambda(\cdot) \in L^2_r(T)$ that drives the system (7.1) at $\varphi(\cdot,0) = v(\cdot)$.

From (2.4) it immediately follows

Proposition 7.1. Let $\lambda(\cdot) = \lambda^*(\cdot)$ solves Problem 7.1 for $v(\cdot) = v^*(\cdot)$. Then for the localization problem (1.2), (1.6) we obtain

$$\int_\Omega \phi_l(x,x^0)v^*(x,0)dx = \int_0^\theta \lambda^{*\prime}(t)\, y(t)dt. \tag{7.2}$$

Problem 7.2. Find a *control* $\lambda(\cdot) \in L^2_r(T)$ that drives the system

$$\frac{\partial \varphi}{\partial t} = -A\varphi - \mathbf{G}^*(\cdot)\lambda(\cdot), \quad t \in T, \ x \in \Omega, \tag{7.3}$$

$$\varphi\,|_\Sigma = 0, \quad \varphi(x,\theta) = 0$$

at $\varphi(\cdot,\cdot) = w(\cdot,\cdot)$.

The relation (2.5) leads to

Proposition 7.2. Let $\lambda(\cdot) = \lambda^*(\cdot)$ solves Problem 7.2 for $w(\cdot,\cdot) = w^*(\cdot,\cdot)$. Then for the localization problem (1.5), (1.6) we obtain

$$\int_\Omega \phi_l(x,x^0)\Big(\int_0^\theta w^*(x,t)dt\Big)dx = \int_0^\theta \lambda^{*\prime}(t)\, y(t)dt. \tag{7.4}$$

Remark 7.1. Both systems (7.1) and (7.3) are well-posed in backward time under assumptions discussed in Sections 2, 3.

Remark 7.2. We note that Propositions 7.1, 7.2 point out a way to numerical realization of solutions of localization problems on the basis of methods developed in the theory of optimal control.

Let $V_m(\cdot)$, $m = 1,2$ stand for the sets of all the solutions of the systems (7.1) and (7.3). Denote the attainable sets of these systems at $t = 0$ by V_m, $m = 1,2$.

From Propositions 7.1, 7.2 we obtain

Theorem 7.1. The following relations hold:

$$\Gamma_1 = V_1, \tag{7.5}$$

$$\Gamma_2 = \{v(x) | \int_0^\theta \varphi(x,t)\,dt, \quad \varphi(\cdot,\cdot) \in V_2(\cdot)\}. \tag{7.6}$$

The equality (7.5) establishes *the duality relations* between the localization problem (1.2), (1.6) and Problem 7.1. In turn, (7.6) connects the localization problem (1.5), (1.6) with Problem 7.2.

Problem 7.1 is well-studied [12, 2, 3]. Let us recall for the definition of *weak null-controllability* [4] (which we adjust here for the system (7.1)): the system (7.1) is said to be weakly null-controllable in T if its attainable set V_1 is dense in $L^2(\Omega)$.

From Theorem 6.1 and the equality (7.5) it follows

Proposition 7.3. Let the system (7.1) be weakly null-controllable in T. Then the localization problem (1.2), (1.4), (1.6) is εh-identifiable in Ω.

Recall for several results on null-controllability. Remark first that in the case of spatially averaged observations,

$$(\mathbf{G}^*(\cdot)\lambda(\cdot))(x,t) \;=\; \lambda'(t)\,\chi(x,t), \tag{7.7}$$

whereas for zone observations,

$$(\mathbf{G}^*(\cdot)\lambda(\cdot))(x,t) \;=\; \sum_{i=1}^{r} \lambda_{(i)}(t) \times \begin{cases} 1, & \text{if } x \in S_i(t), \\ 0, & \text{if } x \bar{\in} S_i(t), \end{cases} \qquad \lambda(\cdot) = (\lambda_{(1)}(\cdot), \ldots, \lambda_{(r)}(\cdot))', \tag{7.8}$$

and for pointwise observations,

$$(\mathbf{G}^*(\cdot)\lambda(\cdot))(x,t) \;=\; \lambda'(t)(\delta(x - \bar{x}^1(t)), \ldots, \delta(x - \bar{x}^n(t)))'. \tag{7.9}$$

For the case when observations are stationary it is known [12, 3] that if the operator A

has finite multiplicity $M = \max_i\{k^i\}$ (for notations Section 5), then the system (7.1), (7.7) or (7.9) is weakly null-controllable in any finite time if and only if the dimensionality of output $r \geq M$ and

$$rank\ \{\mathbf{G}\omega_{i1}(\cdot)\ \mathbf{G}\omega_{i2}(\cdot)\ \dots\ \mathbf{G}\omega_{ik^i}(\cdot)\} = k^i, \quad \forall i = 1, \dots .$$

Existence of dynamic (scanning) controls of type (7.8) with

$$S_j(t) = S_{h_j(t)}(\bar{x}(t)), \quad j = 1, \dots, m$$

and (7.9) that ensure weak null-controllability has been established in [8, 6].

Remark 7.3. In the present paper we consider the case of single unknown source. However, the approach discussed here may point out a way for investigation of the localization problem with several unknown sources on the basis of proper selection of test-functions that allow to separate individual sources, namely, of the following type

$$\gamma_m(x, \lambda(\cdot)) = \begin{cases} 1, & x \in \Omega^*, \\ 0, & x \in \bar{\Omega} \backslash \Omega^*. \end{cases}$$

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