Working Paper

On the Modified Maximum Principle in Estimation Problems for Uncertain Systems

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Foreword

One of the principe features of modern modelling techniques for uncertain systems is to create a theory that would allow effective computation and graphic visualization. This requires a reconsideration of many previous schemes and the introduction of new insights. The present paper is written precisely in this context. It deals with estimation problems for uncertan dynamic models subjected to on line measurements. The estimation scheme gives effective rules for solving the problem to the end.

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Contents

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Abstract

The present report is devoted to the problems of estimating the state of a linear dynamic system on the basis of on-line observation. It is assumed that the disturbances in the system inputs and in the current measurements are uncertain, a set-membership description of their values being only given in advance.

A considerable number of problems concerning systems of the above type are covered by the theory of control and observation under uncertainty conditions [1–3]. The main problems of this paper deal with the description of certain informational domains that are consistent with the results of available measurements of the state space variables [3–5]. Here we consider the case when the disturbances in the system dynamics and in the observation equation are subjected to instantaneous (or "geometric") constraints.

One approach to the problem based on an imbedding procedure of the primary problem into an auxiliary one of linear-quadratic estimation theory is given in the paper. The proposed procedure involves certain quadratic forms to bound the uncertainties in the modified problem. This method allows one to derive an appropriate maximum principle that is satisfied by system trajectories leading to boundary points of the informational domain.

1 Introduction

The topic of this paper is motivated by problems of estimation and control of uncertain dynamic processes described by ordinary differential equations or differential inclusions [1–8]. Within the frame of linear models for system dynamics and observation mode

$$\dot{x} = A(t)x + C(t)v(t), \tag{1.1}$$

$$x(t_0) = x_0, \quad t_0 \le t \le \vartheta,$$

$$y(t) = G(t)x + F(t)w(t), \tag{1.2}$$

we study some problems of estimating the phase vectors $x(t)$ of a controlled process (1.1)–(1.2) that operates under imperfect information in the inputs or in the system parameters $\{x_0, v(\cdot), w(\cdot)\}$. The problems considered here are purely deterministic. It means that there is no statistical data for the unknown disturbances, the only information on these being the knowledge of some constraints on their admissible values.

One of the main steps to solve the estimation problem for uncertain system is the construction in the phase space of the so–called *informational domains* [3,5,6]. These domains include the unknown actual states of the system and consist of all the phase vectors that are compatible with the results of measurements. It is well known how to describe the informational domains for systems with squarely bounded uncertain parameters and input functions (in this case, the informational domains are ellipsoids [3]). But, when the unknown values of system disturbances are restricted instantaneously the solution to the problem under discussion is rather complicated. The method proposed here to handle the latter problem is based on the approximation techniques for the Lagrange functionals. This approach outlines the possibility to establish the "bridge" between the two main estimation problems with set-membership uncertainties for quadratic and non-quadratic constraints on their values. Necessary optimality conditions corresponding to the nonlinear duality Theorem 3.3 are among the results of the paper. The theoretical background of the present investigation has been laid in [6,9].

2 Problem Statement. Basic Definitions and Assumptions

Let R^k be the k-dimensional Euclidean space. For $x, y \in R^k$ let $x'y$ denote the usual inner product of x and y with the prime meaning transposition, $\|x\| = (x'x)^{1/2}$. Also denote by convR^k the set of convex compact subsets of R^k by $\rho(l|X)$ the support function for $X \in \text{conv}R^k$, $l \in R^k$ and by $R^{k\times m}$ – the set of all $k\times m$-matrixes.

Consider the following system

$$\dot{x} = A(t)x + C(t)v(t), \quad t_0 \le t \le \vartheta, \tag{2.1}$$

where $x \in R^n$, matrix functions $A(\cdot), C(\cdot)$ are continuous, $A : [t_0, \vartheta] \mapsto R^{n\times n}$; $C : [t_0, \vartheta] \mapsto R^{n\times q}$.

The input measurable function $v(\cdot)$ and the initial state $x(t_0) = x_0$ are assumed to be unknown being restricted in advance by instantaneous "geometric" constraints

$$v(t) \in Q(t), \quad t_0 \le t \le \vartheta, \tag{2.2}$$

$$x(t_0) = x_0 \in X_0 \tag{2.3}$$

where the set $X_0 \in \mathrm{conv}R^n$ and the continuous map $Q(\cdot)$ are given ($Q : [t_0, \vartheta] \mapsto \mathrm{conv}R^q$).

It is assumed further that direct observations of the current phase states $x(t)$ are impossible, the available information of the system dynamics being generated by the equation

$$y(\tau) = G(\tau)x(\tau) + F(\tau)w(\tau), \quad t_0 \leq \tau \leq t \tag{2.4}$$

with continuous matrix functions $G(\cdot), F(\cdot)(G : [t_0, \vartheta] \mapsto R^{m\times n}; F : [t_0, \vartheta] \mapsto R^{m\times r};\ m \leq n)$.

The disturbances $w(\cdot)$ are also unknown and restricted by

$$w(\tau) \in W(\tau), \quad t_0 \leq \tau \leq t \tag{2.5}$$

with the continuous multivalued function $W(\cdot)(W : [t_0, \vartheta] \mapsto \mathrm{conv}R^r)$ given a priori.

The solution of system (2.1) that starts from a point x_0 at the instant t_0 and is generated by an admissible input $v(\cdot)$ will be denoted as $x(\cdot\,; t_0, x_0, v(\cdot))$.

According to the known formula we have

$$x(t; t_0, x_0, v(\cdot)) = S(t_0, t)x_0 +$$

$$+ \int_{t_0}^{t} S(\tau, t)C(\tau)v(\tau)d\tau, \quad t_0 \leq t \leq \vartheta, \tag{2.6}$$

where $S(\tau, s)$ is the matrix solution of the system

$$dS(\tau, s)/ds = -S(\tau, s)A(\tau), \quad S(\tau, \tau) = E$$

(here E is the identity matrix in $R^{n\times n}$).

Consider the problem of determining the current state $x(t)$ of dynamic process (2.1) via on-line measurements $y(\tau)$ ($t_0 \leq \tau \leq t$). In order to indicate the interval $[t_0, t]$ of observation time we shall use further the symbol $y_t(\tau)$ instead of $y(\tau)$.

Definition 2.1 [3]. The informational domain $X(t, y_t(\cdot))$ of the system states compatible with measured signal $y_t(\cdot)$ is the set of all the points $\{x_*\}$ in R^n through each of which at the instant t there passes at least one of the trajectories $x(\cdot\ ; t_0, x_0, v(\cdot))$ of system (2.1)–(2.3) ($x_* = x(t; t_0, x_0, v(\cdot))$) that generates (together with certain $w(\cdot)$) due to equation (2.4) the same measurement $y_t(\cdot)$.

Note that the set $X(t, y_t(\cdot))$ is compact and convex in R^n with the support function $\rho(l|X(t, y_t(\cdot)))$ being determined by the following formula [3]:

$$\rho(l|X(t, y_t(\cdot))) = \inf\Big\{ \Psi_t(l, \lambda(\cdot)) \mid \lambda(\cdot) \in L_2^m[t_0, t] \Big\} \tag{2.7}$$

where

$$\begin{aligned} \Psi_t(l, \lambda(\cdot)) \quad = \quad & \rho(S'(t_0, t)l - \int_{t_0}^{t} S'(t_0, \tau)G'(\tau)\lambda(\tau)d\tau | X_0) + \\ & + \int_{t_0}^{t} \rho(S'(\tau, t)l - \end{aligned}$$

$$\begin{aligned} - \quad & \int_{\rho}^{t} S'(\tau, s)G'(s)\lambda(s)ds | C(\tau)Q(\tau))d\tau + \\ + \quad & \int_{t_0}^{t} \lambda'(\tau)y_t(\tau)d\tau + \int_{t_0}^{t} \rho(-\lambda(\tau)|F(\tau)W(\tau))d\tau \end{aligned} \tag{2.8}$$

Here $L_2^m[t_0, t]$ denotes the space of m-vector functions squarely integrable on the interval $[t_0, t]$.

It should be pointed out that a direct computation due to formulae (2.7)–(2.8) of the values of support function $\rho(l|X(t, y_t(\cdot)))$ is rather cumbersome procedure. Therefore, it would appear to be of significant interest to characterize $X(t, y_t(\cdot))$ in a different manner. The main result of the present paper does this. The augmented Lagrangian method introduced in [6,9] for uncertain dynamic processes is developed here.

Another scheme to prove the main approximation theorem than that in [6,9] is also suggested. The well-known results of linear-quadratic estimation theory constitute the basis for the further consideration.

Concluding this paragraph we mention the close relations between the problem studied here and the viability one in the differential inclusions theory [10,13].

3 Approximation of the Lagrangians

Let us set a few notations for standard function spaces. Denote by $C^n_k[t_0,\vartheta]$ the space of all k times continuously differentiable functions $f(\cdot)$ ($f:[t_0,\vartheta]\mapsto R^n$) and $C^{n\times q}_k[t_0,\vartheta]$ to be the set of all k times continuously differentiable matrix functions $Z(\cdot)$ ($Z:[t_0,\vartheta]\Rightarrow R^{n\times q}$). Let $C^n[t_0,\vartheta]$, $C^{n\times q}[t_0,\vartheta]$ be respectively the spaces $C^n_k[t_0,\vartheta]$, $C^{n\times q}_k[t_0,\vartheta]$ for $k=0$. The symbol R^q_+ stands for the cone in R^q that consists of symmetric positively definite $q\times q$-matrices and the symbol $C^q_{+k}[t_0,\vartheta]$ denotes the cone in $C^{q\times q}_k[t_0,\vartheta]$ formed by matrix functions $Z(\cdot)$ with values $Z(t)$ in R^q_+.

Let l be an arbitrary vector in R^n. Consider the following set of functions displaced the $\{p(\cdot\,;l,\Lambda)|\Lambda=\{M,R(\cdot),H(\cdot)\}\in R^n_+\times C^n_+[t_0,\vartheta]\times C^m_+[t_0,\vartheta]\} in L^m_2[t_0,\vartheta]$:

$$p(\cdot\,;l,\Lambda)=(\mathcal{L}^{-l}_\Lambda\circ\mathcal{D}_\Lambda)l \tag{3.1}$$

where the linear operators $\mathcal{D}_\Lambda:R^n\mapsto L^m_2[t_0,\vartheta]$ and $\mathcal{L}_\Lambda:L^m_2[t_0,\vartheta]\mapsto L^m_2[t_0,\vartheta]$ are defined by

$$(\mathcal{D}_\Lambda b)(t)=G(t)(S(t_0,t)M^{-1}S'(t_0,\vartheta)+$$

$$+\int_{t_0}^{t}S(\tau,t)R^{-1}(\tau)S'(\tau,\vartheta)d\tau)b,\ b\in R^n,\ t_0\le t\le\vartheta,$$

$$\mathcal{L}_\Lambda\lambda(\cdot)=(K_1+K_2)\lambda(\cdot),\lambda(\cdot)\in L^m_2[t_0,\vartheta] \tag{3.2}$$

$$\begin{aligned}(K_1\lambda(\cdot))(t) &= \int_{t_0}^{\vartheta}K(t,\tau)\lambda(\tau)d\tau,\ K(t,\tau)=G(t)(S(t_0,t)M^{-1}S'(t_0,\tau)+\\ &+ \int_{t_0}^{t\wedge\tau}S(\sigma,t)R^{-1}(\sigma)S'(\sigma,\tau)d\sigma)G'(\tau),\quad t\wedge\tau=min\{t,\tau\},\\ &(K_2\lambda(\cdot))(t)=H^{-1}(t)\lambda(t),t_0\le t\le\vartheta,\end{aligned}$$

Note that the functions $\{p(\cdot;l,\Lambda)\}$ are well defined for all $l\in R^n$ and $\Lambda=\{M,R(\cdot),H(\cdot)\}\in R^n_+\times C^n_+[t_0,\vartheta]\times C^m_+[t_0,\vartheta]$ [14]. We shall use also the notation coΦ for a convex hull ([15]) of a function $\Phi:R^n\mapsto R^1$ and the symbol $r(A)$ will signify the rank of matrix $A\in R^{m\times n}$.

Theorem 3.1 *Suppose $r(G(t))=m$ for all $t\in[t_0,\vartheta]$. Then for every $l\in R^n$ the following equality is true*

$$\inf\Big\{\Psi_\vartheta(l,\lambda(\cdot)) \mid \lambda(\cdot) \in L_2^m[t_0,\vartheta]\Big\} = \tag{3.3}$$

$$= \operatorname{co}\inf \quad \Big\{\Phi(l,\Lambda) \mid \Lambda \in \mathbf{R}_+^n \times C_+^n[t_0,\vartheta] \times C_+^m[t_0,\vartheta]\Big\}$$

where $\Phi(l,\Lambda) = \Psi_\vartheta(l, p(\cdot;l,\Lambda))$ *and functions* Ψ_ϑ, p *are define by (2.8), (3.1) (3.2).*

To sketch the proof of this main theorem we first describe in a different ("geometric") fashion the set $\{p(\cdot;l,\Lambda)\}$ introduced by relations (3.3). This new description will follow from a sequence of lemmas.

Lemma 3.1 *For any vectors* $l, b \in R^n$ $(l \neq 0, b \neq 0)$ *with positive scalar product* $l'b > 0$ *one can present* $l = Nb$ *where* $N \in R_+^n$, *and vice versa.*

Lemma 3.2 *Suppose* $g(\cdot), \phi(\cdot) \in C^n[t_0,\vartheta]$ *and for all* $t \in [t_0,\vartheta]$ $g(t) \neq 0, \phi(t) \neq 0$. *Then the following conditions are equivalent:*

i. $g(t)'\phi(t) > 0$ *for every* $t \in [t_0,\vartheta]$,

ii. $g(t) = N(t)\phi(t)$ *for some matrix function* $N(\cdot) \in C_+^n[t_0,\vartheta]$.

Lemma 3.3 *Assume* $g(\cdot) \in C_1^n[t_0,\vartheta]$ *is such that for all* $t \in [t_0,\vartheta]$ $g(t) \neq 0, \dot{g}(t) \neq 0$ *and vectors* $(-g(t)$ *and* $\dot{g}(t)$ *do not have the same directions. Then there exists a function* $\phi(\cdot) \in C_1^n[t_0,\vartheta]$ *with the properties*

i. $g(t_0)'\phi(t_0) > 0$,

ii. $\dot{g}(t)'\phi(t) > 0$ *for every* $t \in [t_0,\vartheta]$,

iii. $g(t)'\dot{\phi}(t) > 0$ *for every* $t \in [t_0,\vartheta]$.

Conversely if conditions (i)–(iii) are fulfilled for some functions $g(\cdot), \phi(\cdot) \in C_1^n[t_0,\vartheta]$ *then vectors* $g(t)$, $\dot{g}(t)$ *are not of opposite directions and* $g(t) \neq 0, \dot{g}(t) \neq 0$ *for every* $t \in [t_0,\vartheta]$ *is.*

Lemma 3.4 *Assume* $g(\cdot) \in C_1^n[t_0,\vartheta]$ *and* $g(t) \neq 0, \dot{g}(t) \neq 0$ *for every* $t \in [t_0,\vartheta]$. *Then, the following two conditions are equivalent* *i.* *vectors* $g(t)$, $\dot{g}(t)$ *are not directed oppositely in* R^n *for every* $t \in [t_0, \vartheta]$,

ii. *there exists a triplet* $\Lambda = \{M, R(\cdot), H(\cdot)\} \in R^n_+ \times C^n_+[t_0, \vartheta] \times C^m_+[t_0, \vartheta]$ *so that the following equation holds*

$$M^{-1}g(t_0) + \int_{t_0}^{t} R^{-1}(\tau)g(\tau)d\tau = H^{-1}(t)\dot{g}(t), \quad t_0 \leq t \leq \vartheta. \tag{3.4}$$

Note that for

$$g(t) = l - \int_{t_0}^{\vartheta} p(s)ds, \quad t_0 \leq t \leq \vartheta$$

the latter equation coincides with the operator relation $p(\cdot) = (\mathcal{L}_\Lambda^{-1} \circ \mathcal{D}_\Lambda)l$ defining the function $p(\cdot) = p(\cdot; l, \Lambda)$ in the case $A(t) = 0, G(t) = E$ for all $t \in [t_0, \vartheta]$ in the system (2.1) and in the measurement equation (2.4).

From this remark and lemmas 3.1–3.4, one can conclude that for validity of the Theorem 3.1 it is sufficient to prove the following property:

$$\inf\Big\{\Psi_\vartheta(l, \lambda(\cdot)) \mid \lambda(\cdot) \in L_2^m[t_0, \vartheta]\Big\} =$$
$$\inf\Big\{\Psi_\vartheta(l, \lambda(\cdot)) \mid \lambda(\cdot) \in \mathcal{M}_l\Big\}$$

where

$$\begin{aligned} \mathcal{M}_l \quad = \quad & \Big\{\lambda(\cdot) \in C_0^m[t_0, \vartheta] \mid S'(t, \vartheta)l - \int_t^{\vartheta} S'(t, s)G'(s)\lambda(s)ds \neq 0, \\ & G'(t)\lambda(t) \neq 0, S'(t, \vartheta)l - \int_t^{\vartheta} S'(t, s)G'(s)\lambda(s)ds \neq \\ & -\mu G'(t)\lambda(t), \quad \mu > 0, \quad t_0 \leq t \leq \vartheta\Big\}. \end{aligned}$$

This last step of the proof is verified using the smoothing technique and approximation ideas as in [11].

The next result clarifies the meaning of the complicated constructions of the previous theorem.

Having fixed a triplet $\{x_o^*, v^*(\cdot), w^*(\cdot)\}$ of uncertain variables in (2.1)–(2.5) consider again the linear system (2.1), but with additional disturbances:

$$\dot{z} = A(t)z + C(t)v^*(t) + \eta(t), \tag{3.5}$$

$$z(t_0) = z_0 = x_o^* + \zeta, \quad t_0 \le t \le \vartheta,$$

$$y_\vartheta(t) = G(t)z + F(t)w^*(t) + \xi(t), \tag{3.6}$$

where the unknown disturbances $\{\zeta, \eta(\cdot), \xi(\cdot)\}$ are square-bounded jointly by

$$\zeta' M \zeta + \int_{t_0}^{\vartheta} \eta'(t)R(t)\eta(t)dt + \int_{t_0}^{\vartheta} \xi'(t)H(t)\xi(t)dt \le \mu^2 \tag{3.7}$$

with $\{M, R(\cdot), H(\cdot)\} \in R_+^n \times C_+^n[t_0, \vartheta] \times C_+^m[t_0, \vartheta], \mu > 0$.

Denote $Z(\vartheta, y_\vartheta(\cdot); \omega, \Lambda, \mu)$ to be the set of all the states $z(\vartheta)$ of the system (3.5) compatible with measured signal $y_\vartheta(\cdot)$ due to (3.6); $\omega = \{x_o^*, v^*(\cdot), w^*(\cdot)\}$, $\Lambda = \{M, R(\cdot), H(\cdot)\}$. It is known that the set $Z(\vartheta, y_\vartheta(\cdot); \omega, \Lambda, \mu)$ is an ellipsoid and its center $z_o(\vartheta, y_\vartheta(\cdot); \omega, \Lambda)$ does not depend on μ [3]. Let us set one more notation

$$Z_o(\vartheta, y_\vartheta(\cdot); \Lambda) = \bigcup \Big\{ z_o(\vartheta, y_\vartheta(\cdot); \omega, \Lambda) \mid \omega \in X_o \times Q(\cdot) \times W(\cdot) \Big\} \tag{3.8}$$

Theorem 3.2 *[3,6]. The following equality is true*

$$\rho(l \mid Z_o(\vartheta, y_\vartheta(\cdot); \Lambda)) = \Phi(l, \Lambda)$$

for all $l \in R^n$ and $\Lambda = \{M, R(\cdot), H(\cdot)\} \in R_+^n \times C_+^n[t_0, \vartheta] \times C_+^m[t_0, \vartheta]$ where $\Phi(l, \Lambda)$ is defined in Theorem 3.1.

Combining Theorems 3.1, 3.2 we obtain

Theorem 3.3 *Let $r(G(t)) = m$ for every $t \in [t_0, \vartheta]$. Then the following equality holds*

$$\begin{aligned} X(\vartheta, y_\vartheta(\cdot)) \;\; = \;\; & \bigcap \Big\{ Z_o(\vartheta, y_\vartheta(\cdot); \Lambda) \mid \Lambda = \{M, R(\cdot), H(\cdot)\} \\ & \in R_+^n \times C_+^n[t_0, \vartheta] \times C_+^m[t_0, \vartheta] \Big\} \end{aligned} \tag{3.9}$$

Theorem 3.3 gives a precise description of informational domains $X(\vartheta, y_\vartheta(\cdot))$ by means of solutions $Z_o(\vartheta, y_\vartheta(\cdot); \Lambda)$ to the linear-quadratic problem (3.5)–(3.7) allowing to vary matrix parameters in joint integral constraint (3.7) on uncertain disturbances.

4 The Maximum Principle

Introduce some more notations. Let $r^{(i)}(t;\Lambda)$ be the i-th column of matrix $R(t,\Lambda) \in R^{m\times n}$ $i(i=l,....,n)$,

$$R(t,\Lambda) = G(t)(S(t_0,t)M^{-1}S'(t_0,\vartheta) + \int_{t_0}^{t} S(\tau,t)R^{-1}(\tau)S'(\tau,\vartheta)d\tau,$$

$$\Lambda = \{M, R(\cdot), H(\cdot)\} \in R_+^n \times C_+^n[t_0,\vartheta] \times C_+^m[t_0,\vartheta].$$

Denote $\lambda^{(i)}(t;\Lambda)$ to be a solution to the integral equation

$$\mathcal{L}_\Lambda \lambda^{(i)}(\cdot;\Lambda) = r^{(i)}(\cdot;\Lambda), i = l,....,n,$$

where the operator $\mathcal{L}_\Lambda$ is defined by (3.2). Let $P(t;\Lambda)$ be the matrix $\{\lambda^{(1)}(t;\Lambda),\cdots,\lambda^{(n)}(t;\Lambda)\} \in R^{m\times n}$. The symbol $\Gamma(t,\vartheta;\Lambda)$ signifies a solution to the following system in $R^{n\times n}$

$$\dot{\Gamma} = -\Gamma A(t) + P'(t,\Lambda)G(t), \quad t_0 \le t \le \vartheta \tag{4.1}$$

with the end condition $\Gamma(\vartheta,\vartheta;\Lambda) = E$. For every $l \in R^n$ denote $\phi(t,l;\Lambda) = l'\Gamma(t,\vartheta;\Lambda)$.

Having fixed a direction $l \in R^n$ consider the problem of computing the value of the support function $\rho(l|X(\vartheta, y_\vartheta(\cdot)))$ to the informational domain $X(\vartheta, y_\vartheta(\cdot))$. Let x^* be a support point of $X(\vartheta, y_\vartheta(\cdot))$ corresponding to the given l:

$$l'x^* = \rho(l \mid X(\vartheta, y_\vartheta(\cdot)))$$

and $x^*(\cdot)$ be a solution of the system (2.1)–(2.5) so that $x^*(\vartheta) = x^*$.

Theorem 4.1 *Let the assumption of Theorem 3.1 be fulfilled and ε be an arbitrary positive number. Then, there exists a triplet $\Lambda = \{M, R(\cdot), H(\cdot)\} \in R_+^n \times C_+^n[t_0,\vartheta] \times C_+^m[t_0,\vartheta]$ such that the following relations are true:*

$$\phi(t_o,l;\Lambda)x^*(t_o) \ge \rho(\phi(t_o,l;\Lambda)|X_o) - \varepsilon_1, \tag{4.2}$$

$$\phi(t,l;\Lambda)C(t)v^*(t) \ge \rho(\phi(t,l;\Lambda)|C(t)Q(t)) - \varepsilon_2(t), t_0 \le t \le \vartheta, \tag{4.3}$$

$$l'P'(t;\Lambda)G(t)x^*(t) \geq \rho(P(t;\Lambda)l \mid y_\vartheta(t) - F(t)W(t))-$$

$$-\varepsilon_2(t), \quad t_0 \leq t \leq \vartheta, \tag{4.4}$$

$$\rho(\phi(t,l;\Lambda)\,|X(t,y_t(\cdot))) \geq \phi(t,l;\Lambda)x^*(t) \geq \rho(\phi(t,l;\Lambda)|$$

$$Z_o(t,y_t(\cdot);\Lambda)) - \varepsilon(t) \geq \rho(\phi(t,l;\Lambda)|X(t,y_t(\cdot))) - \varepsilon(t), t_0 \leq t \leq \vartheta, \tag{4.5}$$

where $y_t(\tau) = y_\vartheta(\tau) \quad (t_0 \leq \tau \leq t \leq \vartheta)$, $\varepsilon_1 > 0$, $\varepsilon_i(t) > 0$, $\varepsilon_i(\cdot) \in L_1^1[t_0,\vartheta] \quad (i = 2,3)$,

$$\varepsilon(t) = \varepsilon_1 + \int_{t_0}^{t} (\varepsilon_2(\tau) + \varepsilon_3(\tau))d\tau \leq \varepsilon,$$

function $v^*(t)$ *is the input defining* $x^*(\cdot)$ *via (2.1) and the sets* $Z_o(t,y_t(\cdot);\Lambda)$ *are constructed due to formulae (3.5)-(3.8).*

Let us comment these necessary optimality conditions. The first group of inequalities (4.2)-(4.4) corresponds with the classical maximum principle written here in an approximate form with modification (4.1) of the conjugate system. The latter assertion (4.5) reflects *the duality property* of the convex compact set $X(t,y_t(\cdot))$ given by the Theorem 3.3: a point $x \in R^n$ lies outside the set $X(t,y_t(\cdot))$ if there exists an aggregate $Z_o(t,y_t(\cdot);\Lambda)$ so that x does not belong to $Z_o(t,y_t(\cdot);\Lambda)$. It means that the sets $Z_o(t,y_t(\cdot);\Lambda)$ play the same role in the description of informational domains $X(t,y_t(\cdot))$ as the usual linear hyperplanes of separability theorem in convex analysis.

5 Dynamic Relations

The final section of the paper deals with the evolution problem arising in the control and estimation theory for uncertain systems with set-membership data [4,6,10,11]. The next theorems present the equations that describe the dynamics of informational domains $X(t,Y_t(\cdot))$ under variation of the observation time t.

Theorem 5.1 *[3]. The set $Z_o(t, y_t(\cdot); \Lambda)$ is a t-cross-section of the integral funnel to the following differential inclusion*

$$\dot{z}(A(s) - \Sigma(s)G'(s)H(s)G(s))z + \Sigma(s)G'(s)H(s)(y_t(s)$$

$$-F(s)W(s)) + C(s)Q(s), \quad z(t_o) = z_o, \quad t_o \leq s \leq t, \tag{5.1}$$

where $\Sigma : [t_0, \vartheta] \Rightarrow R^{n \times n}$ is the solution of the Riccati equation

$$\dot{\Sigma} = A(s)\Sigma + \Sigma A'(s) - \Sigma G'(s)H(s)G(s)\Sigma + R^{-1}(s),$$

$$\Sigma(t_o) = M^{-1}, \quad t_o \leq s \leq t. \tag{5.2}$$

Theorem 5.2 *Under the assumption of Theorem 3.1 the set $X(t, y_t(\cdot))$ equals the intersection over all matrix triplets $\Lambda = \{M, R(\cdot), H(\cdot)\} \in R^n_+ \times C^n_+[t_0, \vartheta] \times C^M_+[t_0, \vartheta]$ of the t-cross-sections of the trajectory assemblies of system (5.1)-(5.2).*

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