Working Paper

On the Lyapunov Second Method for Data Measurable in Time

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WP-93-54
September 1993

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Foreword

In this paper the authors study time dependent Lyapunov functions for nonatonomous systems described by differential inclusions. In particular it is shown that Lyapunov functions are viscosity supersolutions of a Hamilton-Jacobi equation. For this aim a new viability theorem for differential inclusions with time dependent state constraints is proved:

$$x'(t) \in F(t, x(t)), \quad x(t) \in P(t)$$

where $t \rightsquigarrow P(t)$ is absolutely continuous and $(t, x) \rightsquigarrow F(t, x)$ is a Lebesgue-Borel measurable set-valued map which is upper semicontinuous with respect to x and has closed convex images. The viability conditions are formulated both using contingent cones and in a dual way, using subnormal cones (negative polar of contingent cone).

1 Introduction

Consider two functions $V : \mathbf{R}_+ \times \mathbf{R}^d \mapsto \mathbf{R} \cup \{+\infty\}$, $W : \mathbf{R}_+ \times \mathbf{R}^d \mapsto \mathbf{R}$ and a set-valued map $F : \mathbf{R}_+ \times \mathbf{R}^d \rightsquigarrow \mathbf{R}^d$. Our aim is to study necessary and sufficient conditions for the existence of a solution $x(\cdot)$ to the differential inclusion

$$\begin{cases} x'(t) & \in & F(t,x(t)) \text{ for a.e. } t \geq t_0 \\ x(t_0) & = & x_0 \end{cases} \tag{1}$$

such that for every $t \geq t_0$

$$V(t,x(t)) + \int_{t_0}^{t} W(\tau, x(\tau))d\tau \ \leq \ V(t_0,x_0)$$

for every choice of $t_0 \geq 0$, $x_0 \in \mathbf{R}^d$. This problem is important for the investigation of stability in the sense of Lyapunov and the asymptotic stability. We refer to [4, Chapter 6] and [3, Chapter 9] for several applications and the bibliography concerning this problem. In the difference with these earlier works we allow F to be only measurable with respect to the time.

In particular we show that a continuous function $V : \mathbf{R}_+ \times \mathbf{R}^d \mapsto \mathbf{R}$ such that $t \rightsquigarrow \mathcal{E}p(V(t,\cdot))$ is absolutely continuous enjoys the following monotonicity property: for every $(t_0,x_0) \in \mathbf{R}_+ \times \mathbf{R}^d$ there exists a solution x of the differential inclusion (1) such that $t \mapsto V(t,x(t))$ is *nondecreasing* if and only if it is a (generalized) supersolution of the Hamilton-Jacobi-Bellman equation

$$-\frac{\partial V}{\partial t} + H\left(t,\ x,\ -\frac{\partial V}{\partial x}\right) = 0$$

Furthermore, V is a viscosity supersolution whenever either F is upper semicontinuous in both variables or when V is locally Lipschitz.

Our results are based on a new viability theorem for differential inclusions with dynamics measurable with respect to time and the state constraints, given by an *absolutely continuous* set-valued map $P : [0,T] \rightsquigarrow \mathbf{R}^d$ called *tube*.

We investigate the existence of solutions to the constrained problem

$$\begin{cases} x'(t) & \in & F(t,x(t)) \text{ a.e. in } [t_0,T] \\ x(t_0) & = & x_0 \\ x(t) & \in & P(t) \text{ for all } t \in [t_0,T] \end{cases} \tag{2}$$

for every $x_0 \in P(t_0)$ and all $t_0 \in [0,T[$.

The tube P is called *viable* if for every $t_0 \in [0,T[$, $x_0 \in P(t_0)$ there exists a solution to the above Cauchy problem. We refer to [3] for many results on the viability problem, applications of the viability theory and the historical comments. We prove here the following sufficient condition for viability:

$$\begin{array}{l} \exists\, A \subset [0,T] \text{ of full measure such that} \\ \forall\, t \in A,\ \forall\, x \in P(t) \quad (\{1\} \times F(t,x)) \cap \overline{co}\left(T_{\mathrm{Graph}(P)}(t,x)\right) \neq \emptyset \end{array}$$

where $\overline{co}$ stands for the closed convex hull and F has convex compact images, $F(t,\cdot)$ is upper semicontinuous, $F(\cdot,\cdot)$ is Lebesgue-Borel measurable and has a linear growth. For upper semicontinuous in both variables F similar conditions can be found in [3], [4] (without the convex hull $\overline{co}$) and in [11] (see also [3, Theorems 3.2.4, 3.3.4]) (with the convex hull $\overline{co}$). In the context of tubes and measurable in time dynamics the above condition was first proved in [9,10] under the additional hypothesis that $F(t,\cdot)$ is *locally Lipschitz*. In this way our result is a generalization of [10].

The outline of the paper is as follows. In section 2 we recall some basic definitions. The viability theorem is given in section 3. Section 4 is devoted to an application to the Lyapunov second method.

2 Preliminaries

Let $K \subset \mathbf{R}^d$ be a nonempty subset and $x_0 \in K$. The *contingent cone* to K at x_0 is defined by

$$v \in T_K(x_0) \iff \liminf_{h \to 0+} \mathrm{dist}\left(v, \frac{K - x_0}{h}\right) = 0$$

where $\mathrm{dist}(a,A)$ denotes the distance from a point a to a set A. See [5, Chapter 4] for many properties of tangent cones.

The *subnormal cone* $N_K^0(x_0)$ to K at x_0 is the negative polar of the contingent cone:

$$N_K^0(x_0) \ := \ \left\{p \in \mathbf{R}^d \mid \forall\, v \in T_K(x_0),\ \ \langle p, v\rangle \le 0\right\}$$

Consider an extended function $\varphi : \mathbf{R}^d \mapsto \mathbf{R} \cup \{+\infty\}$. The domain of φ, $\mathrm{Dom}(\varphi)$, is the set of all x_0 such that $\varphi(x_0) \neq +\infty$. The *subdifferential* of φ at $x_0 \in \mathrm{Dom}(\varphi)$ is given by

$$\partial_-\varphi(x_0) \ = \ \left\{p \in \mathbf{R}^d \mid \ \liminf_{x \to x_0} \frac{\varphi(x) - \varphi(x_0) - \langle p, x - x_0\rangle}{\|x - x_0\|} \ \ge \ 0\right\}$$

The *contingent epiderivative* of φ at $x_0 \in \mathrm{Dom}(\varphi)$ in the direction $u \in \mathbf{R}^d$ is defined by

$$D_\uparrow \varphi(x_0)(u) \;=\; \liminf_{h\to 0+,\, u'\to u} \frac{\varphi(x_0+hu')-\varphi(x_0)}{h}$$

Let $\mathcal{T}$ be a metric space and $\{A_\tau\}_{\tau\in\mathcal{T}}$ be a family of subsets of a metric space X. The upper limit *Limsup* of A_τ at $\tau_0 \in T$ is the closed set

$$\mathrm{Limsup}_{\;\tau\to\tau_0} A_\tau = \{v \in X \mid \liminf_{\tau\to\tau_0} \mathrm{dist}\,(v, A_\tau) = 0\}$$

For a set-valued map $F : [0,T]\times\mathbf{R}^d \rightsquigarrow \mathbf{R}^d$ the graph $\mathrm{Graph}(F)$ is given by

$$\mathrm{Graph}(F) \;=\; \Big\{(t,x,y) \mid (t,x) \in [0,T]\times\mathbf{R}^d,\; y \in F(t,x)\Big\}$$

We associate to it the differential inclusion

$$x' \;\in\; F(t,x) \tag{3}$$

Recall that an absolutely continuous function $x : [t_0,T] \mapsto \mathbf{R}^d$ is a solution of (3) if $x'(t) \in F(t,x(t))$ almost everywhere in $[t_0,T]$.

Proposition 2.1 ([10]) *Assume that* $\mathrm{Graph}(F(t,\cdot))$ *are closed for almost all* $t \in [0,T]$ *and*

$$F(t,x) \text{ is closed and convex for almost all } t \in [0,T] \text{ and all } x \in \mathbf{R}^d \tag{4}$$

$$\exists\, \mu \in L^1(0,T),\; \|F(t,x)\| \le \mu(t) \text{ for a.e. } t \in [0,T] \text{ and all } x \in \mathbf{R}^d, \tag{5}$$

where $\|F(t,x)\| = \sup\{\|y\| \mid y \in F(t,x)\}$.

Then there exists a set $A \subset [0,T]$ *of full measure such that for every* $\tau \in A$ *and for every solution* x *to (3) defined on* $[0,T]$ *we have*

$$\emptyset \ne \mathrm{Limsup}_{\;h\to 0+} \left\{\frac{x(\tau+h)-x(\tau)}{h}\right\} \subset F(\tau, x(\tau))$$

Remark – The conclusion of Proposition 2.1 remains true if we replace (5) by the following linear growth assumption

$$\exists\, \mu \in L^1,\; \|F(t,x)\| \le \mu(t)(1+\|x\|) \text{ for a.e. } t \in [0,T] \text{ and all } x \in \mathbf{R}^d \tag{6}$$

Indeed, if F satisfies (6) and $x(\cdot)$ is a solution to (3) on $[0,T]$, then there exists an integer k such that $\|x(t)\| < k$ for $t \in [0,T]$. Let $A_k \subset [0,T]$ be a set of full measure given by Proposition 2.1 for the right hand side

$$F_k : [0,T] \times \mathbf{R}^d \rightsquigarrow \mathbf{R}^d$$

defined by

$$F_k(t,x) \; = \; \begin{cases} F(t,x) & \text{for} \quad \|x\| < k \\ \mu(t)(1+k)B & \text{for} \quad \|x\| \geq k \end{cases}$$

where B is the closed unit ball in $\mathbf{R}^d$. It is enough to take $A = \bigcap_{k=1}^{\infty} A_k$. □

Let $P : [0,T] \rightsquigarrow \mathbf{R}^d$ be a set-valued map with closed values. In this paper we call it a tube (of constraints). We say that P is *absolutely continuous* on $[0,T]$ if the following property holds true:

$$\begin{cases} \forall \varepsilon > 0 \,,\; \forall \text{compact } K \subset \mathbf{R}^d \,,\; \exists \delta > 0 \,,\; \forall 0 \leq t_1 < \tau_1 \leq \ldots \leq t_m < \tau_m \leq T, \\ \sum (\tau_i - t_i) \leq \delta \implies \sum \max\{e(P(t_i) \cap K,\, P(\tau_i)) \,,\, e(P(\tau_i) \cap K,\, P(t_i))\} \leq \varepsilon \end{cases}$$

where $e(U,V) = \inf\{\varepsilon > 0 \,|\, U \subset V + \varepsilon B\}$.

We get the definition of *left absolute continuity* by replacing

$$\max\{e(P(t_i) \cap K,\, P(\tau_i)) \,,\, e(P(\tau_i) \cap K,\, P(t_i))\}$$

by $e(P(t_i) \cap K,\, P(\tau_i))$. For a tube $P : [0,\infty[\rightsquigarrow \mathbf{R}^d$ with closed images we say that it is *locally absolutely continuous* (respectively *locally left absolutely continuous*) if the restriction of P to any finite time interval $[0,T]$ is absolutely continuous (respectively left absolutely continuous).

The Hamiltonian $H : [0,T] \times \mathbf{R}^d \times \mathbf{R}^d \mapsto \mathbf{R}$ associated to F is given by

$$H(t,x,p) \; = \; \sup_{v \in F(t,x)} \langle p, v \rangle$$

3 Viability Theorem

In this section we obtain a new viability theorem for unbounded set-valued maps. Consider the viability problem (2).

Theorem 3.1 *Let $\mu \in L^1(0,T)$ be a nonnegative function. Assume that a closed valued map $P : [0,T] \rightsquigarrow \mathbf{R}^d$ is absolutely continuous, that $F : [0,T] \times \mathbf{R}^d \rightsquigarrow \mathbf{R}^d$ has nonempty closed convex values and*

$$x \rightsquigarrow F(t,x) \text{ is upper semicontinuous for almost all } t \;; \tag{7}$$

$$F(\cdot,\cdot) \text{ is } \mathcal{L} \times \mathcal{B} \text{ (Lebesgue-Borel) measurable} \;; \tag{8}$$

Then the following statements are equivalent:

i) There exists $C \subset [0,T]$ of full measure such that for all $t \in C$, $x \in P(t)$

$$(\{1\} \times F(t,x) \cap \mu(t)(1+\|x\|)B) \cap \overline{co}\left(T_{\mathrm{Graph}(P)}(t,x)\right) \neq \emptyset$$

ii) For all $t_0 \in [0,T[$ and $x_0 \in P(t_0)$ there exists a solution $x(\cdot)$ of (2) satisfying $\|x'(t)\| \leq \mu(t)(1+\|x(t)\|)$ almost everywhere in $[t_0,T]$.

Corollary 3.2 *Let P, F be as in Theorem 3.1 and assume that (6) holds true. Then the following statements are equivalent:*

i) There exists $C \subset [0,T]$ of full measure such that for all $t \in C$, $x \in P(t)$

$$(\{1\} \times F(t,x)) \cap \overline{co}\left(T_{\mathrm{Graph}(P)}(t,x)\right) \neq \emptyset$$

ii) For all $t_0 \in [0,T[$ and $x_0 \in P(t_0)$ there exists a solution $x(\cdot)$ of (2).

iii) There exists $D \subset [0,T]$ of full measure such that for all $t \in D$, $x \in P(t)$

$$\forall\, (p_t,p_x) \in N^0_{\mathrm{Graph}(P)}(t,x), \quad -p_t \;+\; H(t,x,-p_x) \geq 0$$

Proof of Theorem 3.1 – By Proposition 2.1 applied to the map

$$(t,x) \rightsquigarrow F(t,x) \cap \mu(t)(1+\|x\|)B$$

and the Remark following it, *ii)* $\Rightarrow$ *i)*. To prove the converse, without any loss of generality, we may restrict our attention to the case $t_0 = 0$ and $x_0 \in P(0)$. By the Gronwall inequality, there exists $r > 0$ such that if an absolutely continuous function $x : [0,t_1] \rightarrow \mathbf{R}^d$ satisfies $\|x'(t)\| \leq$

$\mu(t)(1+\|x(t)\|)$ a.e. in $[0,t_1]$ and $x(0)=x_0$, then $\|x\|_\infty < r$. Hence it is sufficient to prove the existence of a solution to the problem

$$\begin{cases} x'(t) \in \tilde{F}(t,x(t)) \text{ a.e. in } [0,T] \\ x(0) = x_0 \\ x(t) \in \tilde{P}(t) \text{ for every } t \in [0,T] \end{cases}$$

where $\tilde{P}(t) = P(t) \cup \{x \in \mathbf{R}^d : \|x\| \geq r\}$ and

$$\tilde{F}(t,x) = \begin{cases} F(t,x) \cap \mu(t)(1+\|x\|)B & \text{if } \|x\| < r \\ \\ \mu(t)(1+r)B & \text{if } \|x\| \geq r \end{cases}$$

In the same time, $x(\cdot)$ is a solution to the differential inclusion (2). Furthermore, $\tilde{F}$ is integrably bounded, because for almost all $t \geq 0$

$$\forall\, x \in \mathbf{R}^d,\ \ \|\tilde{F}(t,x)\| \leq \mu(t)(1+r) := \tilde{\mu}(t)$$

and the viability condition holds true for almost all $t \geq 0$ and all $x \in \mathbf{R}^d$:

$$\left(\{1\} \times \tilde{F}(t,x)\right) \cap \overline{co}\left(T_{\mathrm{Graph}(\tilde{P})}(t,x)\right) \neq \emptyset$$

To simplify the presentation of the proof we shall rather use the initial notations, i.e. F for $\tilde{F}$ and P for $\tilde{P}$, and μ for $\tilde{\mu}$.

Step 1. – Using [12, Theorem 2.4], we construct an increasing sequence $\{K_k\}$ of closed subsets of $[0,T]$ such that $\bigcup_1^\infty K_k$ is of full measure, for every k, the restriction $F\,|_{K_k \times \mathbf{R}^d}$ is upper semicontinuous and the function

$$\nu := \sum_{k=1}^{\infty} \sup\{\mu(t) |\, t \in K_k\}\, \chi_{(K_k \setminus K_{k-1})}$$

is integrable, where $\chi_{(K)}$ denotes the characteristic function of $K \subset [0,T]$.

Step 2. – Fix k. By [4, Theorem 1.13.1] there exists a sequence $\{F_m^k\}_{m=1}^\infty$ of convex compact valued maps from $K_k \times \mathbf{R}^d$ into $\mathbf{R}^d$ such that

a) $\forall\, t \in K_k$, $\forall\, x \in \mathbf{R}^d$, $\forall\, m$, $\ F_{m+1}^k(t,x) \subset F_m^k(t,x)$,

b) $\forall\, t \in K_k$, $\forall\, x \in \mathbf{R}^d$, $\ F(t,x) = \bigcap_{m=1}^\infty F_m^k(t,x)$,

c) $\forall\, m$, $\ F_m^k$ is locally Lipschitz,

d) $\forall t \in K_k,\ \forall x \in \mathbf{R}^d,\ \forall m,\ \ F_m^k(t,x) \subset \overline{co}\, F(K_k \times \mathbf{R}^d) \subset \sup_{t \in K_k} \mu(t)\, B$

We define the set-valued map $F_k : [0,T] \times \mathbf{R}^d \rightsquigarrow \mathbf{R}^d$ by:

$$F_k(t,x) = \begin{cases} \nu(t)B & \text{if } \ t \notin K_k \\ F_k^m(t,x) & \text{if } \ t \in K_m \setminus K_{m-1} \text{ and } m \in \{1,2,...,k\} \end{cases}$$

and denote by $\mathcal{S}_k$ the set of all solutions to the following viability problem

$$\begin{cases} x'(t) \in F_k(t,x(t)) \ \text{ a.e. in } \ [0,T] \\ x(0) = x_0 \\ x(t) \in P(t) \ \text{ for all } \ t \in [0,T] \end{cases}$$

It is easy to check that F_k satisfies all the assumptions of Theorem 4.7 from [10]. Thus the set $\mathcal{S}_k$ is nonempty and compact.

It follows directly from the construction that

$$\begin{cases} F_{k+1}(t,x) \subset F_k(t,x), \ \ \forall t \in [0,T],\ \forall x \in \mathbf{R}^d \\ \\ F(t,x) = \bigcap_{k=1}^{\infty} F_k(t,x), \ \ \forall t \in \bigcup_{k=1}^{\infty} K_k,\ \forall x \in \mathbf{R}^d \end{cases}$$

Thus $\mathcal{S}_{k+1} \subset \mathcal{S}_k$, for every k, which in turn implies that $\mathcal{S} = \bigcap_{k=1}^{\infty} \mathcal{S}_k$ is nonempty, where $\mathcal{S}$ denote the set of solutions to (2) with $t_0 = 0$ defined on $[0,T]$. □

Using the same construction as in the above proof we obtain the following generalization of [10, Theorem 4.2]:

Theorem 3.3 *Assume that a closed valued map $P : [0,T] \rightsquigarrow \mathbf{R}^d$ is left absolutely continuous, that $F : [0,T] \times \mathbf{R}^d \rightsquigarrow \mathbf{R}^d$ has nonempty closed convex values and satisfies (7), (8). Let $\mu \in L^1(0,T)$ be a nonnegative function.*

Then the following statements are equivalent:

i) There exists $C \subset [0,T]$ of full measure such that for all $t \in C$, $x \in P(t)$

$$(\{1\} \times F(t,x) \cap \mu(t)(1 + \|x\|)B) \cap DP(t,x)(1) \neq \emptyset$$

ii) For all $t_0 \in [0,T[$ and $x_0 \in P(t_0)$ there exists a solution $x(\cdot)$ to (2) satisfying $\|x'(t)\| \leq \mu(t)(1 + \|x(t)\|)$ almost everywhere in $[t_0,T]$.

When P does not satisfy the viability condition $i)$ of Theorem 3.3, then we may look for the largest subtube of P, which is viable under F. In the stationary case such subset was introduced and studied by Aubin in [2].

Definition 3.4 *Consider a tube* $P : [0,T] \rightsquigarrow \mathbf{R}^d$, $t_1 \geq 0$ *and a left absolutely continuous set-valued map* $\mathcal{P} : [t_1,T] \rightsquigarrow \mathbf{R}^d$ *with closed images such that for every* $t \in [t_1,T]$, $\mathcal{P}(t) \subset P(t)$. $\mathcal{P}$ *is called a viability subtube of* P *with respect to* F *if there exists* $A \subset [t_1,T]$ *with* $m([t_1,T] \setminus A) = 0$ *such that*

$$\forall\, t \in A,\ \forall\, x \in \mathcal{P}(t),\ \ F(t,x) \cap D\mathcal{P}(t,x)(1) \neq \emptyset$$

The largest viability subtube of P *with respect to* F *is called the viability kernel of* P *with respect to* F.

Theorem 3.5 *Let* $P : [0,T] \rightsquigarrow \mathbf{R}^d$ *be closed valued and* $F : [0,T] \times \mathbf{R}^d \rightsquigarrow \mathbf{R}^d$ *satisfy (4), (6), (7) and (8). Then the set of all initial conditions* $(t_0, x_0) \in \mathrm{Graph}(P)$ *such that the constrained Cauchy problem (2) has a solution is the* closed viability kernel *of* P *with respect to* F.

Proof — For every $t_0 \in [0,T]$, consider the set $K(t_0)$ of all initial conditions $x_0 \in P(t_0)$ such that the constrained Cauchy problem (2) has a solution. From our assumptions, using the same arguments as in the convergence theorem [5, p.271], we deduce that the set $\{(t,x) \mid x \in K(t)\}$ is closed. From (6) we deduce that K is left absolutely continuous. Theorem 3.3 implies that every viability subtube $\mathcal{P}$ of P is smaller than K.

4 Lyapunov Functions

Consider a lower semicontinuous function $V : \mathbf{R}_+ \times \mathbf{R}^d \mapsto \mathbf{R} \cup \{+\infty\}$, an $\mathcal{L} \times \mathcal{B}$ measurable function $W : \mathbf{R}_+ \times \mathbf{R}^d \mapsto \mathbf{R}$ and a set-valued map $F : \mathbf{R}_+ \times \mathbf{R}^d \leadsto \mathbf{R}^d$. Let $t_0 \geq 0$, $x_0 \in \mathbf{R}^d$. A function $x : [t_0, \infty[\mapsto \mathbf{R}^d$ is called locally absolutely continuous if its restriction to any finite time interval is absolutely continuous. A locally absolutely continuous function $x : [t_0, \infty[\mapsto \mathbf{R}^d$ is a solution to (1) if $x'(t) \in F(t, x(t))$ almost everywhere in $[t_0, \infty[$ and $x(t_0) = x_0$.

Throughout the whole section we impose the following assumptions:

- For almost all $t \geq 0$, $W(t, \cdot)$ is lower semicontinuous and for some $k \in L^1_{loc}(\mathbf{R}_+, \mathbf{R}_+)$ we have

$$|W(t,x)| \leq k(t)(1 + \|x\|) \text{ for a.e. } t \geq 0 \text{ and all } x \in \mathbf{R}^d$$

- F has nonempty convex compact values, satisfies (7), (8), (6), where $\mu \geq 0$ is a locally integrable function.

Definition 4.1 $V : \mathbf{R}_+ \times \mathbf{R}^d \mapsto \mathbf{R} \cup \{+\infty\}$ *is called a* Lyapunov function *for F with respect to W if there exists a set $D \subset \mathbf{R}_+$ of full measure such that*

$$\forall\, (t,x) \in \mathrm{Dom}(V) \cap D \times \mathbf{R}^d, \quad \inf_{v \in F(t,x)} D_\uparrow V(t,x)(1,v) \leq -W(t,x)$$

Theorem 4.2 *If the set-valued map $t \leadsto \mathcal{E}p(V(t,\cdot))$ is locally absolutely continuous, then the following three statements are equivalent:*

i) $\forall\, (t_0, x_0) \in \mathbf{R}_+ \times \mathbf{R}^d$ there exists a solution $x(\cdot)$ to (1) such that

$$\forall\, t \geq t_0, \quad V(t, x(t)) + \int_{t_0}^{t} W(\tau, x(\tau))d\tau \;\leq\; V(t_0, x_0)$$

ii) V is a Lyapunov function for F with respect to W

iii) $\exists\, C \subset \mathbf{R}_+$ of full measure such that for all $(t,x) \in \mathrm{Dom}(V) \cap C \times \mathbf{R}^d$

$$\forall\, (p_t, p_x, q) \in N^0_{\mathcal{E}p(V)}(t, x, V(t,x)), \; -p_t + H(t, x, -p_x) \geq q\, W(t,x)$$

If in addition V is locally Lipschitz, then iii) is equivalent to

iii)' $\exists\, C \subset \mathbf{R}_+$ of full measure such that for all $(t,x) \in C \times \mathbf{R}^d$

$$\forall\, (p_t, p_x) \in \partial_- V(t,x), \quad -p_t + H(t, x, -p_x) \;\geq\; -W(t,x)$$

Theorem 4.3 *Assume that F is upper semicontinuous in both variables and that at least one of the following two assumptions holds true:*

H_1) $k \equiv const$ and W is lower semicontinuous in both variables

H_2) W is continuous.

Then the following three statements are equivalent:

i) $\forall\,(t_0, x_0) \in \mathbf{R}_+ \times \mathbf{R}^d$ *there exists a solution $x(\cdot)$ to (1) such that*

$$\forall\, t \geq t_0, \quad V(t, x(t)) + \int_{t_0}^{t} W(\tau, x(\tau))d\tau \;\leq\; V(t_0, x_0)$$

ii) $\forall\,(t,x) \in \mathrm{Dom}(V),\ \ \inf_{v\in F(t,x)} D_\uparrow V(t,x)(1,v) \leq -W(t,x)$

iii) $\forall\,(t,x) \in \mathrm{Dom}(V),\ \forall\,(p_t,p_x) \in \partial_- V(t,x),\ -p_t + H(t,x,-p_x) \geq -W(t,x)$, *i.e. V is a viscosity supersolution to the Hamilton-Jacobi equation* $-\frac{\partial V}{\partial t} + H(t,x,-\frac{\partial V}{\partial x}) + W(t,x) = 0$.

Proof of Theorem 4.2 — We first show that *ii)* $\Longrightarrow$ *iii)*. Let D be as in the Definition 4.1. By [5, p.226,228] for every $(t,x,z) \in \mathcal{E}p(V)$ such that $t \in D$

$$(\{1\} \times F(t,x) \times \{-W(t,x)\}) \cap \overline{co}\, T_{\mathcal{E}p(V)}(t,x,z) \neq \emptyset \qquad (9)$$

Applying the separation theorem we deduce *iii)*.

To prove that *iii)* $\Longrightarrow$ *i)* it is enough to consider $(t_0,x_0) \in \mathrm{Dom}(V)$. Using the time translation, we may restrict our attention to the case $t_0 = 0$. Consider the set-valued map $\widetilde{F} : \mathbf{R}_+ \times \mathbf{R}^d \times \mathbf{R} \rightsquigarrow \mathbf{R}^{d+1}$ with nonempty convex compact images defined by

$$\widetilde{F}(t,x,y) \;=\; F(t,x) \times [-k(t)(1+\|x\|), -W(t,x)]$$

Then

a) for almost every $t \geq 0$, $\widetilde{F}(t,\cdot)$ is upper semicontinuous

b) $\widetilde{F}$ is $\mathcal{L} \times \mathcal{B}$ measurable ;

c) For a.e. t and all $(x,y) \in \mathbf{R}^{n+1}$, $\|\widetilde{F}(t,x,y)\| \leq (\mu(t)+2k(t))(1+\|x\|)$

By *iii)*, (9) holds true for all $(t,x,z) \in \mathcal{E}p(V) \cap (C \times \mathbf{R}^d \times \mathbf{R})$. According to Theorem 3.1 there exist $(x_n, y_n) : [0,n] \mapsto \mathbf{R}^d$ solving the problem

$$\begin{cases} x' \;\in\; F(t,x) \quad , \quad x(0) = x_0 \\ y' \;\in\; [-k(t)(1+\|x\|), -W(t,x)] \quad , \quad y(0) = V(0,x_0) \end{cases}$$

such that $y_n(t) \geq V(t, x_n(t))$ for all $t \in [0, n]$. Hence for every $t \in [0, n]$,

$$V(t, x_n(t)) + \int_0^t W(\tau, x_n(\tau)) d\tau \ \leq \ V(0, x_0)$$

We extend x_n on $\mathbf{R}_+$ by setting $\forall\, t \geq n, \ \ x_n(t) = x_n(n)$. We can find a subsequence $x_{n_k} : [0, n_k] \mapsto \mathbf{R}^d$ and a locally absolutely continuous $x : \mathbf{R}_+ \mapsto \mathbf{R}^d$ such that $x_{n_k} \to x$ uniformly on compact sets and for every $r > 0$, x'_{n_k} restricted to $[0, r]$ converge weakly in $L^1(0, r; \mathbf{R}^d)$ to x'. Exactly in the same way as in [5, p.271] we check that x is a solution to (1) with $t_0 = 0$. Finally $i)$ yields $ii)$ in view of Proposition 2.1 applied to $\tilde{F}$. When in addition V is locally Lipschitz, we deduce the equivalence of $iii)$ and $iii)'$ using [7]. □

Proof of Theorem 4.3 — By the Mean Value Theorem [4, p.21], if F is upper semicontinuous and W is lower semicontinuous, then $i)$ yields $ii)$. By [7], $ii) \Longrightarrow iii)$. Conversely, if $iii)$ is satisfied, then, exactly as in [8], the upper semicontinuity of the Hamiltonian H imply

$$\forall\, (p_t, p_x, q) \in N^0_{\mathcal{E}p(V)}(t, x, V(t, x)), \ \ -p_t + H(t, x, -p_x) \geq q\, W(t, x)$$

for all $(t, x) \in \mathrm{Dom}(V)$. From the separation theorem we deduce (9) at every $(t, x, z) \in \mathcal{E}p(V)$. If the assumption $H_1)$ is verified, then consider the upper semicontinuous set-valued map $\tilde{F}$ as in the proof of Theorem 4.2. By [3, Theorem 3.3.6] the viability problem

$$\left\{ \begin{array}{rcl} t'(s) & = & 1 \ \ , \ \ t(0) = 0 \\ x'(s) & \in & F(s, x(s)) \ \ , \ \ x(0) = x_0 \\ y'(s) & \in & [-k(1 + \|x(s)\|), -W(s, x(s))] \ \ , \ \ y(0) = V(0, x_0) \\ (t, x(t), y(t)) & \in & \mathcal{E}p(V) \end{array} \right.$$

has a solution defined on $[0, \infty[$. This yields $i)$ and completes the proof in this case. If $H_2)$ is verified then the viability problem

$$\left\{ \begin{array}{rcl} t'(s) & = & 1 \ \ , \ \ t(0) = 0 \\ x'(s) & \in & F(s, x(s)) \ \ , \ \ x(0) = x_0 \\ y'(s) & = & -W(s, x(s)) \ \ , \ \ y(0) = V(0, x_0) \\ (t, x(t), y(t)) & \in & \mathcal{E}p(V) \end{array} \right.$$

has a solution defined on $[0, \infty[$, which again implies $i)$. □

Theorem 4.2 allows, using an approximation procedure, to prove

Theorem 4.4 *In Theorem 4.2 assume in addition that V is continuous. Then the equivalent statements i) – iii) are equivalent to*

iv) For all $(t_0, x_0) \in \mathbf{R}_+ \times \mathbf{R}^d$ there exists a solution $x(\cdot)$ to (1) such that for all $t \geq s \geq t_0$,

$$V(t, x(t)) + \int_s^t W(\tau, x(\tau))d\tau \;\leq\; V(s, x(s)) \tag{10}$$

In particular, if $W > 0$, then $V(t, x(t)) < V(s, x(s))$ for all $t \geq s \geq t_0$, i.e., V is strictly decreasing along the trajectory x.

Corollary 4.5 *In Theorem 4.2 assume that V is nonnegative, locally Lipschitz and that for some $\alpha \geq 0$, $W \geq \alpha V$. Then for every $(t_0, x_0) \in \mathbf{R}_+ \times \mathbf{R}^d$ there exists a solution x of (1) such that*

$$\forall\, t_2 \geq t_1 \geq t_0, \;\; V(t_2, x(t_2)) \;\leq\; e^{-\alpha(t_2 - t_1)} V(t_1, x(t_1))$$

If in addition V does not depend on time, i.e., $V(t, x) = V(x)$,

$$\forall\, x \neq 0, \;\; V(x) > 0 \;\;\&\;\; V(0) = 0$$

and for some $r > 0$ the connected component Ω_r of the level set $\{x \,|\, V(x) \leq r\}$ containing zero is compact, then $x(t) \to 0$ whenever $x_0 \in \Omega_r$.

We next prove an existence theorem for the lower semicontinuous Lyapunov functions.

Theorem 4.6 *Consider a lower semicontinuous extended function $V_1 : \mathbf{R}_+ \times \mathbf{R}^d \mapsto \mathbf{R} \cup \{+\infty\}$ and assume that for a.e. $t > 0$, $W(t, \cdot)$ is continuous and the set-valued map $x \rightsquigarrow F(t, x)$ is continuous.*

Then there exists the smallest lower semicontinuous Lyapunov function V of F with respect to W satisfying $t \rightsquigarrow \mathcal{E}p(V(t, \cdot))$ is locally left absolutely continuous such that $V \geq V_1$.

In particular there exists the smallest nonnegative lower semicontinuous Lyapunov function V of F with respect to W satisfying $t \rightsquigarrow \mathcal{E}p(V(t, \cdot))$ is locally left absolutely continuous.

Remark — If there is no lower semicontinuous Lyapunov function V of F with respect to W larger than V_1 satisfying $t \rightsquigarrow \mathcal{E}p(V(t,\cdot))$ is left absolutely continuous, then $V \equiv +\infty$. □

Proof — We consider the set-valued map $t \rightsquigarrow P(t) := \mathcal{E}p(V_1(t,\cdot))$. For every $t_0 \geq 0$, let $K(t_0)$ be the set of all $x_0 \in P(t_0)$ such that the constrained Cauchy problem

$$\begin{cases} x'(t) & \in & F(t,x(t)) \text{ for a.e. } t \geq t_0 \\ x(t_0) & = & x_0 \\ x(t) & \in & P(t) \text{ for all } t \geq t_0 \end{cases}$$

has a solution (defined on $[t_0,\infty[$). The graph of the set-valued map K is closed and K is locally left absolutely continuous. Define V by

$$V(t,x) = \inf\{r \in \mathbf{R} \mid (x,r) \in K(t)\}$$

Then V is lower semicontinuous and $t \rightsquigarrow \mathcal{E}p(V(t,\cdot))$ is left absolutely continuous. From Theorem 4.2 we deduce that V is the smallest Lyapunov function of F with respect to W such that $V \geq V_1$ and: $t \rightsquigarrow \mathcal{E}p(V(t,\cdot))$ is locally left absolutely continuous.

5 Stabilizing Selections

We extend here the sufficiency part of [1, Theorem 3.1] to the time dependent case.

Consider a continuously differentiable $V : \mathbf{R}_+ \times \mathbf{R}^d \mapsto \mathbf{R}_+$ such that

$$V(t,0) = 0 \quad \& \quad \forall\, x \neq 0, \;\; V(t,x) > 0$$

Theorem 5.1 *Assume that F measurable with respect to t and satisfies (4), (6) with $T = +\infty$ and $\mu \in L^1(0,+\infty)$, that for almost all $t \geq 0$ the set-valued map $x \rightsquigarrow F(t,x)$ is continuous, $0 \in F(t,0)$ and*

$$\forall\, x \in \mathbf{R}^d \setminus \{0\}, \quad \alpha(t,x) \;:=\; -\frac{\partial V}{\partial t}(t,x) + H\left(t,x,-\frac{\partial V}{\partial x}(t,x)\right) \;>\; 0$$

For every $r > 0$ set $\gamma_r(t) \;:=\; \inf_{\|x\|\geq r} \alpha(t,x)$. If for all $r > 0$,

$$\forall\, t \geq 0, \quad \int_t^{\infty} \gamma_r(s)ds \;=\; \infty$$

then there exists a selection $f(t,x) \in F(t,x)$ *which is Carathéodory on* $\mathbf{R}_+ \times \mathbf{R}^d \backslash \{0\}$ *such that* $\forall\, t \geq 0,\ f(t,0) = 0$ *and every solution* $x(\cdot)$ *to*

$$x'(t) \;=\; f(t,x(t)) \text{ for a.e. } t \tag{11}$$

converges to zero as $t \to +\infty$.

Proof — Define a new set-valued map

$$G(t,x) \;=\; \left\{ y \in F(t,x) \mid \frac{\partial V}{\partial t}(t,x) + \left\langle \frac{\partial V}{\partial x}(t,x), y \right\rangle \leq -\frac{1}{2}\alpha(t,x) \right\}$$

Then G has convex compact images and is measurable with respect to t. Furthermore, it is not difficult to realize (see for instance [1, Lemma 2.1]) that for almost all $t \geq 0$, $G(t,\cdot)$ is continuous on $\mathbf{R}^d \backslash \{0\}$. By [5, p.374] there exists a Carathéodory selection

$$\mathbf{R}_+ \times (\mathbf{R}^d \setminus \{0\}) \ni (t,x) \mapsto f(t,x) \in G(t,x)$$

We set $f(t,0) = 0,\ \alpha(t,0) = 0$. Clearly the growth of f is at most linear. Consider any solution $x(\cdot)$ of (11) on $[0,\infty[$. Then, differentiating $V(t,x(t))$, we prove that for all $t \geq s \geq 0$

$$V(t,x(t)) + \frac{1}{2}\int_s^t \alpha(\tau, x(\tau))d\tau \leq V(s,x(s))$$

From assumptions of theorem we deduce that for some $t_n \to +\infty$ $x(t_n) \to 0$. Since $\mu \in L^1$ and $\|f(t,x)\| \leq \mu(t)(1 + \|x\|)$, using the Gronwall inequality, we end the proof.

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