Working Paper

Evolution of Coalitions Governed by Mutational Equations

Jean Pierre Aubin

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IIASA

International Institute for Applied Systems Analysis □ A-2361 Laxenburg □ Austria
Telephone: +43 2236 715210 □ Telex: 079 137 iiasa a □ Telefax: +43 2236 71313

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FOREWORD

In cooperative game theory as well as in some domains of economic regulation by shortages (queues or unemployment), one is confronted to the problem of evolution of coalitions of players or economic agents. Since coalitions are subsets and cannot be represented by vectors — except if we embed subsets in the family of fuzzy sets, which are functions — the need to adapt the theory of differential equations and dynamical systems to govern the evolution of coalitions or subsets instead of vectors did emerge. Evolution of subsets (regarded as shapes or images) was also motivated by evolution equations of "tubes" in "visual servoing" on one hand, mathematical morphology on the other.

The "power spaces" in which coalitions, images, shapes, etc. evolve are metric spaces without a linear structure. However, one can extend the differential calculus to a mutational calculus *for maps from one metric space to another, as we shall explain in this paper. The simple idea is to replace half-lines allowing to define difference quotients of maps and their various limits in the case of vector space by "transitions" with which we can also define differential quotients of a map. Their various limits are called "mutations" of a map. Many results of differential calculus do not really rely on the linear structure and can be adapted to the nonlinear case of metric spaces and exploited. Furthermore, the concept of differential equation can be extended to* mutational equation *governing the evolution in metric spaces. Basic Theorems as the Nagumo Theorem, the Cauchy-Lipschitz Theorem, the Center Manifold Theorem and the second Lyapunov Method hold true for mutational equations.*

Evolution of Coalitions Governed by Mutational Equations

Jean-Pierre Aubin

1 Introduction: Mutational Equations for Tubes

The topic of this paper is to build a "differential calculus" in metric spaces in order to study and control "differential equations" in metric spaces.

This study was motivated by problems arising in "visual servoing", where one needs to find feedback controls feeding back on subsets (shapes) instead of vectors (see [23,27, Doyen] for further results, applications and references). Mathematical morphology, introduced in [44, Matheron] is also another field of motivations (see [50, Mattioli]).

These problems first required a study of "differential equations" governing the evolution of "tubes", which are compact-valued maps $t \rightsquigarrow P(t)$ from $[0, \infty[$ to a vector space E. We can also regard tubes as single-valued maps P from $[0, \infty[$ to the metric space $\mathcal{K}(E)$ of **nonempty compact subsets** of E supplied with the Hausdorff distance. While attempting to give a meaning to a differential equation governing the evolution of tubes, it was observed that no specific property of the Hausdorff distance was used, and that the theorems could be formulated and proved in any metric space.

Instead of surveying in the introduction the theorems of this paper, we chose to present some basic ideas and some corollaries within the framework of tubes, which will be proved later on in the framework of metric spaces. This choice was dictated by the fact that, for the time being at least, these are tools needed in visual servoing and mathematical morphology.

The reader who is more interested to the detailed and rigorous results should skip this short presentation and start with the first section.

Hence, one motivational topic of this paper is to study the evolution of **tubes**, which are set-valued maps $P : t \in [0, T] \rightsquigarrow E$, governed by a kind of "differential equation", called **mutational equation**

$$\overset{\circ}{P}(t) \ni f(t, P(t))$$

governing the evolution of tubes.

When $f : E \mapsto E$ is a single valued map or, more generally, when $F : E \rightsquigarrow E$ is a set-valued map, the evolution of tubes have been studied either as "viability tubes"[1], or as solutions to "funnel differential equations or inclusions" by Russian and Bielorussian mathematicians[2].

But mutational equations of the form

$$\overset{\circ}{P}(t) \ni f(V(P(t)))$$

(where V maps subsets $P \subset E$ of the state space E to state vectors and where $f : E \mapsto E$) do not fall in these formulations (see Steiner mutational equations below for an example).

To give a meaning to a mutational equation, the issue of defining what is meant by the time derivative $\overset{\circ}{P}(t)$ is raised again.

The suggestion we propose in this paper is inspired by the concept of shape derivatives of shape maps V, which are in some sense "set-defined maps", mapping subsets $K \subset E$ to vectors $V(K) \in Y$ in a finite dimensional vector space Y. (See [16, Céa], [19,20,21,22, Delfour & Zolésio], [24, Doyen], [73, Zolésio], etc.). Their idea was to replace the usual differential quotients $\frac{U(x+hv)-U(x)}{h}$ measuring the variation of a function U on half-lines $x + hv$ by differential quotients $\frac{V(\vartheta_\varphi(h,K))-V(K)}{h}$ where $\varphi : E \mapsto E$ is a Lipschitz map, $\vartheta_\varphi(h,x) := x(h)$ denotes the value at time h of the solution to the differential equation $x' = \varphi(x)$ starting at x at time 0 and $\vartheta_\varphi(h,K) := \{\vartheta_\varphi(h,x)\}_{x\in K}$ the reachable set from K at time h of φ.

In other words, the "curve" $h \mapsto \vartheta_\varphi(h,K)$ plays the role of the half lines $h \mapsto x + hv$ for defining differential quotients measuring the variations of the function V along it. Since the set $\mathcal{K}(E)$ of nonempty compact subsets of E is only a metric space, without linear structure, replacing half-lines by curves to measure variations is indeed a very reasonable strategy. For this special metric space, these "curves" ϑ_φ, which are examples of "transitions" defined below, are in one to one correspondence with the space $\text{Lip}(E,E)$ of Lipschitz maps φ. They play the role of directions when one defines

[1] as in VIABILITY THEORY, [5, Aubin], for instance, and in [32, Frankowska]. For the general case when $F : [0,T] \times E \rightsquigarrow E$ where F depends measurably on time, see [36, Frankowska, Plaskacz & Rzeżuchowski].

[2] See ([37,38,39, Kurzhanski & Filippova], [40, Kurzhanski & Nikonov], [41,42, Kurzhanski & Valyi],[57, Panasyuk], [69, Tolstogonov], etc.

directional derivatives of usual functions. Hence, if the limit

$$\overset{\circ}{V}(K)\varphi := \lim_{h\to 0+} \frac{V(\vartheta_\varphi(h,K)) - V(K)}{h}$$

exists, it is called the **directional shape derivative of V at K** in the "direction" φ. With such a concept, an inverse function theorem allowing to inverse locally a shape map V whenever its shape derivative $\text{Lip}(E,E) \mapsto Y$ is surjective is proved in [24, Doyen] and many applications to shape optimization under constraints are derived in Doyen's paper.

Since this strategy works well for shape maps, it should work as well for set-valued maps, and indeed, it does for solving certain classes of problems.

For this purpose, we introduce the **Hausdorff demi-distance** $\delta : \mathcal{K}(E) \times \mathcal{K}(E) \mapsto \mathbf{R}_+$ defined by

$$\forall\, K, L \in \mathcal{K}(E),\ \ \delta(K,L) := \sup_{x\in K} d(x,L) = \sup_{x\in K} \inf_{y\in L} d(x,y)$$

and the associated **Hausdorff distance**

$$\boldsymbol{d}(K,L) := \max(\delta(K,L), \delta(L,K))$$

Hence, going back to tubes $t \rightsquigarrow P(t)$ with **nonempty compact values**, we suggest to look for differential quotients of the form

$$\frac{\boldsymbol{d}(\vartheta_\varphi(h,P(t)), P(t+h))}{h}$$

which compare the variation $P(t+h)$ and the variation $\vartheta_\varphi(h,P(t))$ produced by a transition ϑ_φ applied to $P(t)$.

Let $B(K,\varepsilon)$ denote the closed ball of radius ε around K. If

$$\lim_{h\to 0+} \frac{\boldsymbol{d}(\vartheta_\varphi(h,P(t)), P(t+h))}{h} = 0 \tag{1.1}$$

or, equivalently, if there exists $\beta(h) \to 0$ with h such that, for all $h \in]0,1]$,

$$\vartheta_\varphi(h,P(t)) \subset B(P(t+h), \beta(h)h)\ \&\ P(t+h) \subset B(\vartheta_\varphi(h,P(t)), \beta(h)h)$$

it is tempting to say that the transition ϑ_φ, or, equivalently, that the associated Lipschitz map $\varphi \in \text{Lip}(E,E)$, plays the role of the directional derivative of the tube P at t in the forward direction 1.

This is what we shall do: we propose to call **mutation $\overset{\circ}{P}(t)$ of the tube P at t** the set of Lipschitz maps φ satisfying the property (1.1). We do have to coin a new name, because many concepts of derivatives of a set-valued map — **graphical derivatives**[3], such as contingent derivatives[4], circatangent derivatives[5] or adjacent derivatives[6], as well as other **pointwise concepts**[7] — have been used extensively.

We observe that any two Lipschitz maps $\varphi, \psi \in \overset{\circ}{P}(t)$ (or the associated transitions) are **equivalent at** $P(t)$ in the sense that

$$\lim_{h \to 0+} \frac{d(\vartheta_\varphi(h, P(t)), \vartheta_\psi(h, P(t)))}{h} = 0$$

If $\varphi \equiv v$ is a constant map $v \in E \subset \text{Lip}(E, E)$ satisfying the above property, we find a concept of derivative implicitly involved in **funnel equations**.

Now, if $f : [0, T] \times \mathcal{K}(E) \mapsto \text{Lip}(E, E)$ is a continuous map associating with a pair (t, K) a Lipschitz map $y \mapsto f(t, K; y)$, we can define a **mutational equation for tubes** of the form

$$\forall\, t \geq 0, \;\; \overset{\circ}{P}(t) \ni f(t, P(t); \cdot)$$

or, equivalently,

$$\left\{ \begin{array}{l} \vartheta_{f(t,P(t);\cdot)}(h, P(t)) \subset B(P(t+h), \beta(h)h) \\ \qquad\qquad\qquad \& \\ P(t+h) \subset B(\vartheta_{f(t,P(t);\cdot)}(h, P(t)), \beta(h)h) \end{array} \right.$$

(By identifying Lipschitz maps which are equivalent at $P(t)$ in a same equivalence class, the above mutational equation could be written in the more familiar form $\overset{\circ}{P}(t) = f(t, P(t); \cdot)$. But, as often when we try to avoid using factor spaces, we have the choice between potential confusion and ponderousness).

For another approach using set-valued derivatives in the case of convex valued tubes, see [29].

[3] according to a term coined by R.T. Rockafellar. See [61,62, Rockafellar], [66, Rockafellar & Wets], SET-VALUED ANALYSIS, [10, Aubin & Frankowska] and VIABILITY THEORY, [5, Aubin], among other authors for an exposition of their properties.

[4] introduced in [3, Aubin].

[5] introduced in [4, Aubin].

[6] introduced in [30,31,32, Frankowska].

[7] See [14, Banks & Jakobs], [15, De Blasi], [43, Martelli & Vignoli] among many other authors.

1.1 The Nagumo Theorem for Tubes

Let $M \subset E$ be a closed subset of a finite dimensional vector space E. We denote by $T_M(x)$ its **contingent cone**[8] **to** M **at** $x \in M$ and by $N_M(x) := (T_M(x))^-$ its polar cone, called the **subnormal or regular normal cone.**

Nagumo's Theorem for differential equations (see [51, Nagumo], VIABILITY THEORY, [5, Aubin]) states that M is invariant under $\varphi \in \text{Lip}(E, E)$ if and only if

$$\forall\, x \in M, \;\; \varphi(x) \in -T_M(x) \cap T_M(x)$$

and, actually[9], if and only if

$$\forall\, x \in M, \;\; \forall\, p \in N_M(x), \;\; \langle p, \varphi(x)\rangle \;=\; 0 \tag{1.2}$$

We shall set

$$\text{Lip}_0(M, E) \;:=\; \{\varphi \in \text{Lip}(E, E) \mid \text{satisfying (1.2)}\,\}$$

When φ is Lipschitz, we denote by

$$\|\varphi\|_\Lambda \;:=\; \sup_{x \neq y} \frac{\|\varphi(x) - \varphi(y)\|}{\|x - y\|}$$

the Lipschitz semi-norm of φ.

We supply it with the distance $\|\varphi_1 - \varphi_2\|_\infty \;:=\; \sup_{x \in M} \|\varphi_1(x) - \varphi_2(x)\|$ of uniform convergence.

Let $\mathcal{M} \subset \mathcal{K}(M)$ be a family of **nonempty compact subsets** of M and $\boldsymbol{d}_{\mathcal{M}}(K) := \inf_{L \in \mathcal{M}} \boldsymbol{d}(K, L)$ denote the distance to $\mathcal{M}$ in the Hausdorff space $\mathcal{K}(M)$. We recall the definition of **contingent cone** $T_{\mathcal{M}}(K) \subset \text{Lip}_0(M, E)$ introduced and studied in [24, Doyen] under the name of **velocity cones:** We shall say that a Lipschitz map $\varphi \in \text{Lip}_0(M, E)$ is **contingent to** $\mathcal{M}$ **at** $K \in \mathcal{M}$ if and only

$$\liminf_{h \to 0+} \frac{\boldsymbol{d}_{\mathcal{M}}(\vartheta_\varphi(h, K))}{h} = 0$$

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[8] which is the cone of directions $v \in E$ such that

$$\liminf_{h \to 0+} \frac{d(x + hv, M)}{h} \;=\; 0$$

[9] See VIABILITY THEORY, [5, Aubin], Theorem 3.2.4.

i.e., if and only if there exist sequences h_n and ε_n converging to 0 and a sequence of subsets $K_n \in \mathcal{M}$ such that

$$\vartheta_{\varphi}(h_n, K) \subset K_n + \varepsilon_n h_n B \ \& \ K_n \subset \vartheta_{\varphi}(h_n, K) + \varepsilon_n h_n B$$

Constrained Inverse Function Theorems, a calculus of contingent cones and Lagrange multipliers for shape optimization under constraints, which use such concepts of tangent cones, can be found in [24, Doyen].

The Nagumo theorem can be adapted to characterize the evolution of tubes under constraints.

Theorem 1.1 *Let $M \subset E$ be a closed subset of a finite dimensional vector space E and $f : [0, \infty[\times \mathcal{K}(M) \mapsto \mathrm{Lip}_0(M, E)$ be a continuous map,* bounded *in the sense that*

$$\forall\, t \geq 0,\ \ \forall\, P \subset M,\ K \subset M,\ \ f(t, P; K) \text{ is bounded in } E$$

and that

$$\forall\, t \geq 0,\ \ \forall\, P,\ \ \|f(t, P; \cdot)\|_{\Lambda} \leq c$$

Assume that $\mathcal{M} \subset \mathcal{K}(M)$ is a viability domain *of f in the sense that*

$$\forall\, t \geq 0,\ \ \forall\, K \in \mathcal{M},\ \ f(t, K) \in T_{\mathcal{M}}(K)$$

Then, from any $K_0 \in \mathcal{M}$ starts a tube $t \mapsto P(t)$, solution to the mutational equation

$$\forall\, t \geq 0,\ \ \mathring{P}(t) \ni f(t, P(t); \cdot)$$

which is viable in *$\mathcal{M}$ in the sense that*

$$\forall\, t \geq 0,\ \ P(t) \in \mathcal{M}$$

This an easy corollary of Theorem 4.2 below.

1.2 The Cauchy-Lipschitz Theorem for Tubes

When f is Lipschitz, we obtain uniqueness of the solution to a differential mutation and estimates:

Theorem 1.2 *Let $f : \mathcal{K}(E) \mapsto \mathrm{Lip}(E, E)$ be a Lipschitz map, bounded in the sense that*

$$\forall\, P,\ K \subset E,\ \ f(P; K) \text{ is bounded in } E$$

and that

$$\mu := \max(0, \sup_{K \subset E} e^{\|f(K;\cdot)\|_\Lambda} - 1) \; < \; +\infty$$

Then, from any $K_0 \in \mathcal{K}(E)$ *starts a unique solution* $P(\cdot)$ *of the Cauchy problem to the mutational equation*

$$\forall\, t \geq 0, \;\; \overset{\circ}{P}(t) \; \ni \; f(P(t), \cdot)$$

If Q *is a mutable tube, we set*

$$\boldsymbol{d}(\sigma, \overset{\circ}{Q}(s))_\infty \; := \; \inf_{\tau \in \overset{\circ}{Q}(s)} \; \boldsymbol{d}(\sigma, \tau)$$

Then

$$\begin{cases} \forall\, t \in [0, T], \;\; \boldsymbol{d}(P(t), Q(t)) \; \leq \\ e^{(\mu + \|f\|_\Lambda)t} \boldsymbol{d}(K_0, Q(0)) + \int_0^t e^{(\mu + \|f\|_\Lambda)(t-s)} \boldsymbol{d}(f(Q(s)), \overset{\circ}{Q}(s))_\infty ds \end{cases}$$

which shows the Lipschitz dependence of the solution to the mutational equation with respect to the initial data and the right-hand side.

It follows from Theorem 4.5 below.

1.3 An Example: Steiner Mutational Equation

One class of mutational equation is provided by dynamics f which depend upon the subset P through a shape map.

Consider for instance the shape map $s_n : \mathcal{K}(E) \mapsto E$ associating with any $K \subset E$ its **Steiner point** $s_n(K)$ defined by

$$s_n(K) \; = \; n \int_{\Sigma^{n-1}} p\sigma(K, p)\omega(dp)$$

where Σ^{n-1} denotes the unit sphere in $E := \mathbf{R}^n$, $\sigma(K, \cdot)$ is the support function of K, ω is the measure on Σ^{n-1} proportional to the Lebesgue measure and satisfying $\omega(\Sigma^{n-1}) = 1$.

Since $\sigma(K, p) = \sigma(-K, -p)$, it follows that $s_n(K) = -s_n(-K)$[10]. The support function being also additive with respect to K, the map $s_n(\cdot)$ is

[10] so that if K is symmetric, i.e., $K = -K$, then $s_n(K) = 0$.

linear:

$$(1.3)\quad \begin{cases} \text{For all compact } K,\ L \subset \mathbf{R}^n \text{ and all } \lambda,\ \mu \in \mathbf{R}, \\ s_n(\lambda K + \mu L) = \lambda s_n(K) + \mu s_n(L) \end{cases}$$

One can prove that s_n is a selection in the sense that $s_n(K) \in \overline{co}(K)$ and that it is Lipschitz with respect to the Hausdorff distance (See for instance Theorem 9.4.1 of SET-VALUED ANALYSIS, [10, Aubin & Frankowska]).

Let us consider now a continuous map $g : E \mapsto \text{Lip}(E, E)$. Hence Steiner mutational equations are mutational equations of the form

$$\forall\, t \geq 0,\ \overset{\circ}{P}(t) \ni g(s_n(P(t)), \cdot)$$

In other words, the dynamics of the tube is governed by the dynamics of its Steiner point.

In particular, consider the case when $g : E \mapsto E$ where E is identified with a subspace of $\text{Lip}(E, E)$. Then we can compare the Steiner point $s_n(P(t))$ of a solution to the mutational equation

$$\forall\, t \geq 0,\ \overset{\circ}{P}(t) \ni g(s_n(P(t)))$$

starting from K_0 and the solution to the differential equation

$$\forall\, t \geq 0,\ y'(t) = g(y(t))$$

starting from $s_n(K_0)$. Then, it is easy to check that

$$\forall\, t \geq 0,\ s_n(P(t)) = y(t)$$

Indeed, from the definition

$$\begin{cases} P(t) + hg(s_n(P(t))) \subset B(P(t+h), \beta(h)h) \\ \& \\ P(t+h) \subset B(P(t) + hg(s_n(P(t))), \beta(h)h) \end{cases}$$

of a mutation and from property (1.3), we infer that

$$\begin{cases} s_n(P(t)) + hg(s_n(P(t))) \in B(s_n(P(t+h)), \beta(h)h) \\ \& \\ s_n(P(t+h)) \in B(s_n(P(t)) + hg(s_n(P(t))), \beta(h)h) \end{cases}$$

so that

$$\left| \frac{s_n(P(t+h)) - s_n(P(t))}{h} - g(s_n(P(t))) \right| \leq \beta(h)$$

and thus, $\dfrac{ds_n(P(t))}{dt} = g(s_n(P(t)))$.

1.4 The invariant manifold theorem for Tubes

More generally, many problems lead to the study of the evolution of observations $u(P(t))$ where $u : \mathcal{K}(E) \mapsto Y$, Y being a finite dimensional vector space.

For instance, one can regard a map u as a map associating with any set vector characteristics which are "adequate" in the sense that they "track" the evolution of the tubes.

Let $A \in \mathcal{L}(Y, Y)$ be a linear operator and $g : \mathcal{K}(E) \times Y \mapsto Y$ be given.

The problem arises to compare the evolution of $u(P(t))$ of the solution to a mutational equation

$$\forall\, t \geq 0, \;\; \overset{\circ}{P}(t) \ni f(P(t), y(t); \cdot)$$

starting from K_0 with the solution $y(t)$ to a differential equation

$$\forall\, t \geq 0, \;\; y'(t) = Ay(t) + g(P(t), y(t))$$

starting from $u(K_0)$, in the sense that $u(P(t)) = y(t)$. In other words, this means that Graph$(u) \subset \mathcal{K}(E) \times Y$ is viable (or invariant) under the "characteristic system": $\forall\, t \geq 0$,

$$\begin{cases} i) & \overset{\circ}{P}(t) \ni f(P(t), y(t); \cdot) \\ ii) & y'(t) = Ay(t) + g(P(t), y(t)) \end{cases} \tag{1.4}$$

We define the **contingent mutation** $\overset{\circ}{D}\, u(K)$ at K to be the set-valued map from Lip(E, E) to Y defined by $v \in \overset{\circ}{D}\, u(K)(\varphi)$ if and only if there exist sequences $h_n \to 0+$, $y_n \to u(K)$ and $K_n \to K$ such that

$$d(\vartheta_\varphi(h_n, K), K_n) \leq \alpha_n h_n \;\; \& \;\; \|u(K) + h_n v - y_n\| \leq \beta_n h_n$$

Naturally, $\overset{\circ}{D}\, u(K)(\varphi) = \left\{ \overset{\circ}{u}(K)\varphi \right\}$ coincides with the directional shape derivative of u at K whenever it exists.

We shall prove that the graph of u is a viability domain if and only if u is a solution to the **system of partial mutational equations**

$$\forall\, K \subset E, \;\; Au(K) \in \overset{\circ}{D}\, u(K)(f(K, u(K); \cdot)) - g(K, u(K)) \tag{1.5}$$

The existence theorem of [7,8, Aubin & Da Prato] can be extended to the case of partial mutational equations by using techniques of [11,12,13, Aubin & Frankowska]:

Theorem 1.3 *Let us assume that the maps* $f : \mathcal{K}(E) \times Y \mapsto \text{Lip}(E, E)$ *and* $g : \mathcal{K}(E) \times Y \mapsto Y$ *are Lipschitz and that there exists* $c > 0$ *such that*

$$\forall\, K \subset E,\ y \in Y,\ \ \|g(K; y)\| \le c(1 + \|y\|)\ \ \&\ \ \|f(K; \cdot)\|_\Lambda\ \le\ c$$

Then for $\inf_{x \neq 0} \frac{\langle Ax, x\rangle}{\|x\|^2}$ *large enough, there exists a unique bounded Lipschitz solution* $u : \mathcal{K}(E) \mapsto Y$ *to the system of partial mutational equations (1.5).*

This theorem follows from Theorem 5.2 below.

1.5 Mutational Calculus in Metric Spaces

The proofs of these theorems do not involve the explicit definition of the Hausdorff distance on the space $\mathcal{K}(E)$ of nonempty compact subsets of a vector space. Actually, these theorems are immediate corollaries of the analogous statements proved in any metric space. For instance, one can use other metric subspaces of the power set $\mathcal{P}(E)$, such as the space of closed convex subsets supplied with the **cosmic convergence** introduced in [65,66, Rockafellar & Wets], or a the σ-algebra $\mathcal{A}$ of a probability space $(\Omega, \mathcal{A}, \mu)$ supplied with the distance

$$\boldsymbol{d}_\mu(K, L)\ :=\ \frac{Var_\mu(K \ominus L)}{1 + Var_\mu(K \ominus L)}$$

where $K \ominus L := (K \cup L)\backslash(K \cap L)$ is the symmetric difference of K and L and $Var_\mu(M)$ the total variation of μ on M.

One can also use the distance

$$\boldsymbol{d}_2(K, L)\ :=\ \left(\int_K d_L(x)^2 dx + \int_L d_K(x)^2 dx\right)^{\frac{1}{2}}$$

introduced in [27, Doyen].

The need to extend concepts of derivatives in metric spaces is not new. As early as 1946, T. Ważewski introduced in [71,72, Ważewski] the concept of **allongements contingentiels supérieur et inférieur** (upper and lower contingent elongations) of a map $X \mapsto Y$[11] to prove implicit function theorems in metric

[11] Namely,

$$\overline{\text{all}}f(x)\ :=\ \limsup_{x' \to x} \frac{\boldsymbol{d}(f(x'), f(x))}{\boldsymbol{d}(x', x)}\ \ \&\ \ \underline{\text{all}}f(x)\ :=\ \liminf_{x' \to x} \frac{\boldsymbol{d}(f(x'), f(x))}{\boldsymbol{d}(x', x)}$$

spaces. More recently, H. Frankowska used first order and higher order "variations" in [34,35, Frankowska] to prove sophisticated inverse function theorems in metric spaces and L. Doyen to shape maps in [24, Doyen]. But we follow here another track motivated by the evolution of tubes, shape analysis and mathematical morphology.

We shall adapt to the case of metric spaces the Nagumo Theorem, the Cauchy-Lipschitz theorem and an existence theorem on invariant manifolds, as well as the Lyapunov method by introducing **epimutations** on metric spaces, the analogues of **epiderivatives** of functions defined on vector spaces, for studying asymptotic properties of mutational equations.

We shall adapt in another paper the inverse function theorems of Chapter 3 of SET-VALUED ANALYSIS, [10, Aubin & Frankowska] and of [24, Doyen] on local inversion to maps $X \mapsto Y$ from a complete metric space X to a normed space Y.

The main concepts of set-valued analysis shall be transferred to set-valued maps $F : X \rightsquigarrow Y$ from a metric space X to a metric space Y, by defining **contingent mutations** of a set-valued map at a point of its graph and other concepts of tangent mutations.

The main concepts of nonsmooth analysis shall also be extended to functions defined on metric spaces. By using **epimutations**, we will adapt to optimization of functions on metric spaces the Fermat and Ekeland rules.

2 Transitions on Metric spaces

Transitions adapt to metric spaces the concept of half line $x + hv$ starting from x in the direction v by replacing it by "curved" half-lines $\vartheta(h, x)$. Indeed, the "linear" structure of half lines in vector spaces is not really needed to build a differential calculus.

Definition 2.1 *Let X be a metric space for a distance $\boldsymbol{d}$. A map $\vartheta : [0,1] \times X \mapsto X$ satisfying*

$$\begin{cases} i) & \vartheta(0,x) = x \\ ii) & \|\vartheta(x)\| := \sup_{h \neq k} \dfrac{\boldsymbol{d}(\vartheta(h,x), \vartheta(k,x))}{|h-k|} < +\infty \\ iii) & \|\vartheta\|_\Lambda := \sup_{h \in [0,1],\, x \neq y} \dfrac{\boldsymbol{d}(\vartheta(h,x), \vartheta(h,y))}{\boldsymbol{d}(x,y)} < +\infty \\ iv) & \lim_{h \to 0+} \dfrac{\boldsymbol{d}(\vartheta(t+h,x), \vartheta(h, \vartheta(t,x)))}{h} = 0 \end{cases}$$

is called a **transition**. *When $\|\vartheta\|_\Lambda \leq 1$ in the above inequality, we say that ϑ is a* **nonexpansive transition**.

We denote by $\overline{\Theta}(X)$ the vector space of all transitions on X[12].

We define an equivalence relation $\sim_x$ between transitions by

$$\vartheta_1 \sim_x \vartheta_2 \text{ if and only if } \lim_{h \to 0+} \frac{d(\vartheta_1(h,x), \vartheta_2(h,x))}{h} = 0$$

We say that $(X, \Theta(X))$ is a (complete) **mutational space** *if X is a (complete) metric space and $\Theta(X) \subset \overline{\Theta}(X)$ is a nontrivial vector subspace of transitions, closed in $\mathcal{C}([0,1] \times X, X)$ supplied with the pointwise convergence.*

Remark — We could have introduced the factor space of equivalence classes of transitions, by identifying at each point equivalent transitions. But this may be too cumbersome. □

[12] One may sometimes need more regular transitions: A transition is strict if

$$\limsup_{y \to x} \sup_{h \neq k} \frac{\boldsymbol{d}(\vartheta(h,y), \vartheta(k,y))}{|h-k|} < +\infty$$

and

$$\liminf_{h \to 0+, y \to x} \frac{\boldsymbol{d}(\vartheta(t+h,y), \vartheta(h, \vartheta(t,y)))}{h} = 0$$

We shall say that ϑ_1 and ϑ_2 are strictly equivalent if

$$\vartheta_1 \sim_x \vartheta_2 \text{ if and only if } \lim_{h \to 0+, y \to x} \frac{\boldsymbol{d}(\vartheta_1(h,x), \vartheta_2(h,y))}{h} = 0$$

One observes that the transitions $\vartheta(h,\cdot)$ are Lipschitz uniformly with respect to $h \in [0,1]$ and that for every $x \in X$, the maps $\vartheta(\cdot,x)$ are Lipschitz. The unit transition defined by $\mathbf{1}(h,x) = x$ is denoted by $\mathbf{1}$.

We shall supply a space $\Theta(X)$ of transitions with the distances $\boldsymbol{d}_\infty$ of uniform convergence[13] and Lipschitz semidistance defined respectively by

$$\boldsymbol{d}_\infty(\vartheta,\tau) := \sup_{h\in[0,1],\, z\in X} \boldsymbol{d}(\vartheta(h,z),\tau(h,z))$$

and

$$\boldsymbol{d}_\Lambda(\vartheta,\tau) := \sup_{h\in[0,1],\, z\in X} \frac{\boldsymbol{d}(\vartheta(h,z),\tau(h,z))}{h}$$

We shall need the following estimate on transitions:

Lemma 2.2 *Consider two transitions ϑ and τ. Then*

$$\forall\, t \in [0,1[,\ \ \boldsymbol{d}(\vartheta(t,x),\tau(t,y)) \le \boldsymbol{d}_\Lambda(\vartheta,\tau)\frac{e^{(\|\vartheta\|_\Lambda-1)t}-1}{\|\vartheta\|_\Lambda - 1} + \boldsymbol{d}(x,y)e^{(\|\vartheta\|_\Lambda-1)t} \tag{2.1}$$

and

$$\lim_{h\to 0+} \frac{\boldsymbol{d}(\vartheta_1(h,\vartheta_2(h,x)),\vartheta_2(h,\vartheta_1(h,x)))}{h} \le (1+\|\vartheta_1\|_\Lambda)\boldsymbol{d}_\Lambda(\vartheta_1,\vartheta_2)$$

Proof — Indeed, let us set $\varphi(t) := \boldsymbol{d}(\vartheta(t,x),\tau(t,y))$, which is a Lipschitz function, thus almost everywhere differentiable. Let us estimate its derivative:

$$\begin{cases} \dfrac{\boldsymbol{d}(\vartheta(t+h,x),\tau(t+h,y))}{h} \\[2mm] \le \dfrac{\boldsymbol{d}(\vartheta(t+h,x),\vartheta(h,\vartheta(t,x)))}{h} + \dfrac{\boldsymbol{d}(\vartheta(h,\vartheta(t,x)),\vartheta(h,\tau(t,y)))}{h} \\[2mm] +\dfrac{\boldsymbol{d}(\vartheta(h,\tau(t,y)),\tau(h,\tau(t,y))}{h} + \dfrac{\boldsymbol{d}(\tau(h,\tau(t,y)),\tau(t+h,y))}{h} \\[2mm] \le \dfrac{\boldsymbol{d}(\vartheta(t+h,x),\vartheta(h,\tau(t,x)))}{h} + \|\vartheta\|_\Lambda \boldsymbol{d}(\vartheta(t,x),\tau(t,y)) \\[2mm] +\boldsymbol{d}_\Lambda(\vartheta,\tau) + \dfrac{\boldsymbol{d}(\tau(h,\tau(t,y)),\tau(t+h,y))}{h} \end{cases}$$

[13] We can, if needed, use weaker topologies such as the compact topology (if the transitions are not bounded) or the pointwise topology with respect to $z \in X$.

imply that

$$\varphi'(t) \leq (\|\vartheta\|_\Lambda - 1)\varphi(t) + \boldsymbol{d}_\Lambda(\vartheta, \tau)$$

Since $\varphi(0) = \boldsymbol{d}(x, y)$, we infer that

$$\varphi(t) \leq \varphi(0) + \boldsymbol{d}_\Lambda(\vartheta, \tau)t + (\|\vartheta\|_\Lambda - 1)\int_0^t \varphi(s)ds$$

We then use the Gronwall Lemma.

Example: Transitions on Normed Spaces Let E be a finite dimensional vector space. We can associate with any $v \in E$ the transition $\vartheta_v \in \Theta(E)$ defined by

$$\vartheta_v(h, x) := x + hv$$

for which we have $\|\vartheta_v(x)\| = \|v\|$ and $\|v\|_\Lambda = 1$ (it is nonexpansive).

Therefore, we shall identify a normed space E with the mutational space (E, E) by taking for space of transitions the space $\Theta(E) = E$ of vectors regarded as "directions".

We can enlarge the space of transitions by using the Cauchy-Lipschitz Theorem. We associate with any Lipschitz map $\varphi : X \mapsto X$ the transition $\vartheta_\varphi \in \Theta(E)$ defined by

$$\vartheta_\varphi(h, x) := x(h)$$

where $x(h)$ is the unique solution to the differential equation $x'(t) = \varphi(x(t))$ starting from x.

Indeed, we deduce from the Cauchy-Lipschitz Theorem that

$$\|\vartheta_\varphi(x)\| \leq e^{\|\varphi\|_\Lambda}\frac{e^{\|\varphi\|_\Lambda} - 1}{\|\varphi\|_\Lambda}\|\varphi(x)\|$$

and that $\|\vartheta_\varphi\|_\Lambda \leq e^{\|\varphi\|_\Lambda}$ because

$$\boldsymbol{d}(\vartheta_\varphi(h, x), \vartheta_\varphi(h, y)) \leq e^{\|\varphi\|_\Lambda} d(x, y)$$

They satisfy $\vartheta_\varphi(h + t, x) = \vartheta_\varphi(h, \vartheta_\varphi(t, x))$.

We also deduce that

$$\boldsymbol{d}_\Lambda(\vartheta_\varphi, \vartheta_\psi) \leq \frac{e^{\|\varphi\|_\Lambda} - 1}{\|\varphi\|_\Lambda}\|\varphi - \psi\|_\infty$$

because

$$\boldsymbol{d}(\vartheta_\varphi(h,x),\vartheta_\psi(h,x)) \;\leq\; \frac{e^{\|\varphi\|_\Lambda h}-1}{\|\varphi\|_\Lambda h}\|\varphi-\psi\|_\infty$$

Then the space of Lipschitz maps $\varphi : E \mapsto E$ can be embedded in the space $\overline{\Theta}(E)$ of all transitions:

$$E \;\subset\; \mathrm{Lip}(E,E) \;\subset\; \overline{\Theta}(E)$$

We observe that for any $x \in E$, φ is equivalent to the vector $\varphi(x)$ at x: $\varphi \sim_x \varphi(x)$.

Example: Transitions on a subset of a vector space
Let $M \subset E$ be a closed subset of a finite dimensional vector space E. We denote by $T_M(x)$ its **contingent cone** and by $N_M(x) := (T_M(x))^-$ the **subnormal cone.**

We recall that M is invariant under $\varphi \in \mathrm{Lip}(E,E)$ if and only if

$$\forall\, x \in M,\;\; \forall\, p \in N_M(x),\;\; \langle p, \varphi(x)\rangle \;=\; 0 \tag{2.2}$$

We shall denote by

$$\mathrm{Lip}_0(M,E) \;:=\; \{\varphi \in \mathrm{Lip}(E,E) \,|\, \text{satisfying (2.2)}\,\}$$

We thus infer that

$$\mathrm{Lip}_0(M,E) \;\subset\; \overline{\Theta}(M)$$

is a space of transitions of the metric subset M.

Example: Transitions on Power Sets This is our main example. Let $M \subset E$ be a closed subset of a finite dimensional vector space E and $X := \mathcal{K}(M)$ be the family of nonempty compact subsets $K \subset M$.

We recall that the **Hausdorff demi-distance** $\delta : \mathcal{K}(E) \times \mathcal{K}(E) \mapsto \mathbf{R}_+$ is defined by

$$\forall\, K, L \in \mathcal{K}(E),\;\; \delta(K,L) \;:=\; \sup_{x\in K} d(x,L) \;=\; \sup_{x\in K}\,\inf_{y\in L} d(x,y)$$

and that the associated **Hausdorff distance**, as well as its restriction to $\mathcal{K}(M)$, is defined by

$$\boldsymbol{d}(K,L) \;:=\; \max(\delta(K,L),\delta(L,K))$$

We can also associate with any Lipschitz map $\varphi : E \mapsto E$ a transition $\vartheta_\varphi \in \Theta(X)$ defined by

$$\vartheta_\varphi(h,K) := \{\vartheta_\varphi(h,x)\}_{x\in K}$$

Indeed, we deduce that

$$\|\vartheta_\varphi(K)\| \le e^{\|\varphi\|_\Lambda}\frac{e^{\|\varphi\|_\Lambda}-1}{\|\varphi\|_\Lambda}\|\varphi(K)\|$$

and that

$$\|\vartheta_\varphi\|_\Lambda \le e^{\|\varphi\|_\Lambda}$$

because

$$\boldsymbol{d}(\vartheta_\varphi(h,K),\vartheta_\varphi(h,L)) \le e^{\|\varphi\|_\Lambda}\boldsymbol{d}(K,L)$$

We also observe that

$$\boldsymbol{d}_\Lambda(\vartheta_\varphi,\vartheta_\psi) \le \frac{e^{\|\varphi\|_\Lambda}-1}{\|\varphi\|_\Lambda}\|\varphi-\psi\|_\infty$$

Therefore,

$$\mathrm{Lip}_0(M,E) \subset \overline{\Theta}(\mathcal{K}(M))$$

is a space of transitions of $\mathcal{K}(M)$ and $(\mathcal{K}(M),\mathrm{Lip}_0(M,E))$ is a mutational space, the one we presented in the introduction.

Actually, there are other transitions on the metric space $\mathcal{K}(M)$.

Example: Morphological transitions. Indeed, more generally, we associate with any Lipschitz set-valued map $\Phi : X \rightsquigarrow X$ with compact values the set-valued map $\vartheta_\Phi \in \overline{\Theta}(E)$ defined by

$$\vartheta_\Phi(h,x) := \{x(h)\}_{x(\cdot)\in\mathcal{S}(x)}$$

where $x(\cdot)$ range over the set $\mathcal{S}(x)$ of solutions to the differential inclusion $x'(t) \in \Phi(x(t))$ starting from x.

We deduce from the Filippov Theorem[14], which extends the Cauchy-Lipschitz theorem to differential inclusions, that

$$\|\vartheta_\Phi(x)\| \le e^{\|\Phi\|_\Lambda}\frac{e^{\|\Phi\|_\Lambda}-1}{\|\Phi\|_\Lambda}\|\Phi(x)\|$$

[14] See [5, Theorem 5.3.1] and [35] for instance.

that $\|\vartheta_\Phi\|_\Lambda \leq e^{\|\Phi\|_\Lambda}$ because

$$d(\vartheta_\Phi(h,x), \vartheta_\Phi(h,y)) \leq e^{\|\Phi\|_\Lambda} d(x,y)$$

and that

$$\boldsymbol{d}_\Lambda(\vartheta_\Phi, \vartheta_\Psi) \leq \frac{e^{\|\Phi\|_\Lambda} - 1}{\|\Phi\|_\Lambda} \boldsymbol{d}(\Phi, \Psi)_\infty$$

where

$$\boldsymbol{d}(\Phi, \Psi)_\infty := \sup_{x \in E} \boldsymbol{d}(\Phi(x), \Psi(x))$$

The Invariance Theorem for differential inclusions (see Theorem 5.3.4 of VIABILITY THEORY, [5, Aubin]) states that a closed subset $M \subset E$ is invariant under $\Phi \in \mathrm{LIP}(E,E)$ if and only if

$$\forall\, x \in M, \quad \Phi(x) \subset -T_M(x) \cap T_M(x) \tag{2.3}$$

We shall denote by

$$\mathrm{LIP}_0(M,E) := \{\Phi \in \mathrm{LIP}(E,E) \mid \text{satisfying (2.3)}\}$$

Therefore, we can also associate with any Lipschitz set-valued map $\Phi : E \mapsto E$ with compact values the **morphological transition** $\vartheta_\Phi \in \overline{\Theta}(\mathcal{K}(M))$ defined by

$$\vartheta_\Phi(h,K) := \{\vartheta_\Phi(h,x)\}_{x \in K}$$

Indeed, we deduce that

$$\|\vartheta_\Phi(K)\| \leq e^{\|\Phi\|_\Lambda} \frac{e^{\|\Phi\|_\Lambda} - 1}{\|\Phi\|_\Lambda} \|\Phi(K)\|$$

and that

$$\|\vartheta_\Phi\|_\Lambda \leq e^{\|\Phi\|_\Lambda}$$

because

$$\boldsymbol{d}(\vartheta_\Phi(h,K), \vartheta_\Phi(h,L)) \leq e^{\|\Phi\|_\Lambda} \boldsymbol{d}(K,L)$$

We also observe that

$$\boldsymbol{d}_\Lambda(\vartheta_\Phi, \vartheta_\Psi) \leq \frac{e^{\|\Phi\|_\Lambda} - 1}{\|\Phi\|_\Lambda} \boldsymbol{d}(\Phi, \Psi)_\infty$$

Therefore,

$$\mathrm{LIP}_0(M,E) \subset \overline{\Theta}(\mathcal{K}(M))$$

is another space of transitions contained in $\mathcal{K}(M)$. □

Example: Morphological Dilatations They are defined by the constant set-valued maps $\Phi(x) := B$ where $B \subset E$ is a closed subset containing the origin, called the **structuring element** in mathematical morphology (See [45, Mattioli & Schmitt] for more details on this domain of image processing).

The transitions produced by such differential inclusions are given by

$$\vartheta_\Phi(h,K) \ := \ K + hB$$

and called **morphological dilatations**. They play the role of the transitions $x + hv$ in vector spaces. □

We refer to [50, Mattioli] for further details.

We thus can associate with $\mathcal{K}(M)$ the two mutational subspaces $(\mathcal{K}(M), \mathrm{Lip}_0(M,E))$ and $(\mathcal{K}(M), \mathcal{K}(M))$ of the mutational space $(\mathcal{K}(M), \mathrm{LIP}_0(M,E))$. The mutational space $(\mathcal{K}(M), \mathrm{Lip}_0(M,E))$ is used in the framework of visual servoing whereas the mutational space $(\mathcal{K}(M), \mathcal{K}(M))$ is used in mathematical morphology.

3 Mutations of Smooth Single-Valued Maps

3.1 Definitions

We adapt first some classical definitions of differential calculus and notations to single-valued maps from a metric space to another.

Definition 3.1 *Consider two mutational spaces* $(X, \Theta(X))$, $(Y, \Theta(Y))$ *and a single-valued map* $f : X \mapsto Y$ *from* X *to* Y.

We shall say that the **mutation** $\overset{\circ}{f}(x)$ **of** f **at** x *is the set-valued map from* $\Theta(X)$ *to* $\Theta(Y)$ *defined by*

$$\tau \in \overset{\circ}{f}(x)\vartheta \ \textit{if and only if} \ \lim_{h \to 0+} \frac{d(f(\vartheta(h,x)), \tau(h, f(x)))}{h} \ = \ 0$$

We shall say that f is **mutable** *at x in the directions $\vartheta \in \Theta(X)$ if $\overset{\circ}{f}(x)\vartheta$ is nonempty for every $\vartheta \in \Theta(X)$ and that f is* **strictly mutable** *if*

$$\tau \in \overset{\circ}{f}(x)\vartheta \text{ if and only if } \lim_{h \to 0+,\, x' \to x} \frac{d(f(\vartheta(h,x')), \tau(h, f(x')))}{h} = 0$$

Proposition 3.2 *Consider two metric spaces X, Y and a single-valued map $f : X \mapsto Y$ from X to Y. If f is mutable at x, then two transitions $\tau_1 \in \overset{\circ}{f}(x)\vartheta$ and $\tau_2 \in \overset{\circ}{f}(x)\vartheta$ are equivalent at $f(x)$: $\tau_1 \sim_{f(x)} \tau_2$.*

If f is Lipschitz and if $\vartheta_1 \sim_x \vartheta_2$ are equivalent at x, then transitions $\tau_1 \in \overset{\circ}{f}(x)\vartheta_1$ and $\tau_2 \in \overset{\circ}{f}(x)\vartheta_2$ are also equivalent at $f(x)$.

Remark — When the context allows it, we may identify the transitions $\tau \in \overset{\circ}{f}(x)\vartheta$ since they are equivalent at $f(x)$ and make the mutation $\overset{\circ}{f}(x)$ single-valued by taking the factor space of $\Theta(Y)$. □

Remark: Composition of transitions —

Definition 3.3 *If $\vartheta_i \in \Theta(X)$ $(i = 1, 2)$, we denote by $\vartheta_1 \circ \vartheta_2$ the transition defined by*

$$(\vartheta_1 \circ \vartheta_2)(h, x) := \vartheta_1(h, \vartheta_2(h, x))$$

In the case of normed spaces, this composition boils down the addition since $\vartheta_{v_1} \circ \vartheta_{v_2} = \vartheta_{v_1+v_2}$.

We observe that if ϑ_1 is a strict transition, then

$$\begin{cases} i) & \|(\vartheta_1 \circ \vartheta_2)(x)\| \leq \|\vartheta_1(x)\| + \|\vartheta_1\|_\Lambda \|\vartheta_2(x)\| \\ ii) & \|\vartheta_1 \circ \vartheta_2\|_\Lambda \leq \|\vartheta_1\|_\Lambda + \|\vartheta_2\|_\Lambda \end{cases}$$

If f is strictly mutable at x, then $\overset{\circ}{f}(x)$ is an homomorphism from $\Theta(X)$ to $\Theta(Y)$ in the sense that for any $\tau_i \in \overset{\circ}{f}(x)\vartheta_i$ $(i = 1, 2)$:

$$\tau_1 \circ \tau_2 \in \overset{\circ}{f}(x)(\vartheta_1 \circ \vartheta_2)$$

We shall not use these algebraic properties in this paper. □

Consider the particular example of functions from an interval $I \subset \mathbf{R}$ to a mutational space $(X, \Theta(X))$.

Then its mutation $\overset{\circ}{x}(t)(1)$ in the direction $+1$ is the set of transitions $\vartheta \in \Theta(X)$ satisfying

$$\lim_{h \to 0+} \frac{d(\vartheta(h, x(t)), x(t+h))}{h} = 0$$

(which are all equivalent at $x(t)$). From now on, we shall set $\overset{\circ}{x}(t) := \overset{\circ}{x}(t)(1)$.

Since transitions are in particular maps $\vartheta_{t,x} : h \in [0, 1-t] \mapsto \vartheta(t+h, x) \in X$, we observe that the condition

$$\lim_{h \to 0+} \frac{d(\vartheta(t+h, x), \vartheta(t, \vartheta(t, x)))}{h} = 0$$

states that the transition ϑ belongs to the mutation of the map $h \mapsto \vartheta(t+h, x)$ at $t \in [0, 1[$:

$$\forall\, t \in [0, 1[, \ \ \forall\, x \in X, \ \ \overset{\circ}{\vartheta}_{t,x}(t)(1) \ni \vartheta$$

For maps defined from a normed space E to a mutational space $(Y, \Theta(Y))$, we usually restrict the transitions to be just vectors $u \in E$ by taking $\Theta(E) = E$, so that mutations $\overset{\circ}{f}(x)$ induce maps from E to $\Theta(Y)$ defined by

$$\tau \in \overset{\circ}{f}(x)u \ \text{ if and only if } \ \lim_{h \to 0+} \frac{d(f(x + hu), \tau(h, f(x)))}{h} = 0$$

For maps defined from a mutational space $(X, \Theta(X))$ to a vector space F, we restrict naturally the transitions to be vectors $u \in F$ by taking $\Theta(F) = F$, so that mutations $\overset{\circ}{f}(x)$ induce maps from $\Theta(X)$ to F defined by

$$\overset{\circ}{f}(x)\vartheta = \lim_{h \to 0+} \frac{f(\vartheta(h, x)) - f(x)}{h}$$

Remark — We can associate with a transition $\vartheta : [0, \infty[\times X \mapsto X$ satisfying

$$\forall\, h, l \geq 0, \ \ \vartheta(h + l, x) = \vartheta(h, \vartheta(l, x))$$

a semi-group of continuous linear operators $U_\vartheta(h)$ on the space $\mathcal{F}(X, F)$ (supplied with the pointwise convergence) defined by

$$\forall\, f \in \mathcal{F}(X, F), \ \ U_\vartheta(f)(x) := f(\vartheta(h, x))$$

Then the domain $\text{Dom}(L_\vartheta)$ of its infinitesimal generator is the space of mutable functions in the direction ϑ and its infinitesimal generator $L_\vartheta \in \mathcal{L}(\text{Dom}(L_\vartheta), \mathcal{F}(X, F))$ is defined by

$$\forall f \in \text{Dom}(L_\vartheta), \quad L_\vartheta f(x) = \lim_{h \mapsto 0+} \frac{f(\vartheta(h, x)) - f(x)}{h}$$

It can also be regarded as a Lie derivative of f. □

Let X and E be finite dimensional vector space s and $Y := \mathcal{K}(E)$. We regard a set-valued map $P : X \rightsquigarrow E$ with nonempty compact images as a single valued map $P : X \mapsto \mathcal{K}(E)$. We associate the mutational spaces (X, X) and $(\mathcal{K}(E), \text{Lip}(E, E))$.

We thus restrict the transitions $\vartheta \in \Theta(X)$ to be just vectors $u \in E$ and the transitions $\tau \in \Theta(\mathcal{K}(E))$ to be Lipschitz maps $\varphi \in \text{Lip}(E, E)$, so that mutations $\overset{\circ}{P}(x)$ are set-valued maps from X to $\text{Lip}(E, E)$ defined by

$$\varphi \in \overset{\circ}{P}(x)u \text{ if and only if } \lim_{h \to 0+} \frac{d(P(x + hu), \vartheta_\varphi(h, P(x)))}{h} = 0$$

In other words, the mutation $\overset{\circ}{P}(x)(u)$ is a set of Lipschitz maps $\varphi : E \mapsto E$ such that

$$\vartheta_\varphi(h, P(x)) \subset B(P(x + hu), \beta(h)h) \;\&\; P(x + hu) \subset B(\vartheta_\varphi(h, P(x)), \beta(h)h) \;\square$$

In particular, for mutable tubes $t \rightsquigarrow P(t)$, we shall set

$$\varphi \in \overset{\circ}{P}(t) \text{ if and only if } \lim_{h \to 0+} \frac{d(P(t + h), \vartheta_\varphi(h, P(t)))}{h} = 0$$

Remark — The contingent derivative of a set-valued map $P : X \rightsquigarrow E$ at a point (x, y) of its graph has no relations with the concept of mutation of this set-valued map regarded as a single-valued map from X to the power space $Y := \mathcal{K}(E)$.

In the first instance, the contingent derivative is a set-valued map $DP(x, y)$ from X to E depending upon a point $(x, y) \in \text{Graph}(P)$ whereas in the second point of view, the mutation $\overset{\circ}{P}(x)$ is a set-valued map from X to $\text{Lip}(E, E)$ depending only upon x and not on the choice of $y \in P(x)$.

This is the reason why we had to coin the word **mutation** instead of derivative to avoid this confusion. □

Let $M \subset E$ be a closed subset of a finite dimensional vector space, $X := \mathcal{K}(M)$ be the metric space of nonempty compact subsets of M and Y be a normed space. We associate with them the mutational spaces $(\mathcal{K}(M), \mathrm{Lip}_0(M, E))$ and (Y, Y).

A map $f : \mathcal{K}(M) \mapsto Y$ is often called a **shape map**, since they have been extensively used in shape design and shape optimization (see [16, Céa], [73, Zolésio], [19,20,21,22, Delfour & Zolésio], [24, Doyen], etc.).

Then, by restricting transitions on $\mathcal{K}(M)$ to $\mathrm{Lip}_0(M, E)$ and the transitions on Y to be directions $v \in Y$, we see that a mutation $\overset{\circ}{f}(K)$ is a set-valued map from the vector space $\mathrm{Lip}_0(M, E)$ to Y associating with a Lipschitz map φ the direction v defined by

$$v = \overset{\circ}{f}(K)\varphi := \lim_{h \to 0+} \frac{f(\vartheta_\varphi(h, K)) - f(K)}{h}$$

Assume that the interior Ω of M is not empty. Denote by $\mathcal{D}(\Omega, E)$ the space of indefinitely differentiable maps with compact support from Ω to Y. Let $f : \mathcal{K}(M) \mapsto \mathbf{R}$ be a shape function. If

$$\varphi \in \mathcal{D}(\Omega, E) \cap \mathrm{Lip}(E, E) \mapsto \overset{\circ}{f}(K)\varphi \text{ is linear and continuous}$$

then $\overset{\circ}{f}(K)$ is a **vector distribution** called the **shape gradient** of f at $K \subset M$.

Remark — Let us denote by $\mathcal{N}_K$ the subspace of vector distributions T satisfying

$$T\varphi = 0 \ \forall \varphi \in \mathcal{D}(\Omega, E) \text{ satisfying } \varphi(x) \in T_K(x) \ \forall x \in K$$

which is the subspace of **vector distributions normal** to K. This implies in particular that the support of a vector distribution normal to K is contained in the boundary ∂K of K.

Since $f(\vartheta_\varphi(h, K)) = f(K)$ for any Lipschitz map $\varphi \in \mathrm{Lip}_0(K, E)$, we see that the shape gradient $\overset{\circ}{f}(K)$ is a vector distribution which is normal to K, because[15]

$$\forall \varphi \in \mathrm{Lip}_0(K, E), \ \overset{\circ}{f}(K)\varphi = 0 \ \square$$

4 Mutational Equations

Let us consider a mutational space $(X, \Theta(X))$ and a single-valued map $f : X \times [0, \infty[\mapsto \Theta(X)$ from X to its space of transitions. We say that a function $x(\cdot)$ from $[0, T]$ to X is a **solution to the mutational equation** $\overset{\circ}{x} \ni f(t, x)$ if

[15]See [22, Delfour & Zolésio] for more details on this issue.

$$\forall\, t \in [0,T], \quad \overset{\circ}{x}(t) \ni f(t,x(t)) \tag{4.1}$$

or, equivalently, if

$$\forall\, t \geq 0, \quad \lim_{h\to 0+} \frac{\boldsymbol{d}(f(t,x(t);h,x(t)),x(t+h))}{h} = 0$$

We shall adapt both the Nagumo and the Cauchy-Lipschitz Theorems to the case of mutational equations. For the Nagumo Theorem, which states the existence of a solution $x(\cdot)$ viable in a subset $K \subset X$ (in the sense that for every $t \geq 0$, $x(t) \in K$), we need first to adapt the concept of contingent cone to the case of metric spaces.

4.1 Contingent Transition Sets

Definition 4.1 (Contingent Transition Sets) *Let $(X, \Theta(X))$ be a mutational space, $K \subset X$ be a subset of X and $x \in K$ belong to K. The* contingent[16] transition set *$T_K(x)$ is defined by*

$$T_K(x) := \left\{ \vartheta \in \Theta(X) \mid \liminf_{h\to 0+} \frac{\boldsymbol{d}_K(\vartheta(h,x))}{h} = 0 \right\}$$

It is very convenient to have the following characterization of this transition set in terms of sequences:

$$\begin{cases} \vartheta \in T_K(x) \text{ if and only if } \exists\, h_n \to 0+,\ \exists\, \varepsilon_n \to 0+ \\ \text{and } \exists\, x_n \in K \to x \text{ such that } \forall\, n,\ \boldsymbol{d}(\vartheta(h_n,x),x_n) \leq \varepsilon_n h_n \end{cases}$$

Naturally, if $\vartheta_1 \sim_x \vartheta_2$ are equivalent at $x \in K$ and if ϑ_1 belongs to $T_K(x)$, then ϑ_2 is also a contingent transition to K at x.

Example: Normed Spaces Let E be a normed vector space. We can associate with any $v \in E$ the transition $\vartheta_v \in \overline{\Theta}(E)$ defined by

$$\vartheta_v(h,x) := x + hv$$

[16]This termed has been coined by G. Bouligand in the 30's. Since this is a concept consistent with the concept of contingent direction as we shall see below, we adopted the same terminology.

Then the vector $v \in E$ is contingent to K at $x \in K$ (in the usual sense of contingent cones to subsets in normed spaces) if and only if the associated transition ϑ_v is contingent to K at x.

Let us associate with any Lipschitz map $\varphi : X \mapsto X$ the transition $\vartheta_\varphi \in \overline{\Theta}(E)$ defined by

$$\vartheta_\varphi(h, x) := x(h)$$

where $x(\cdot)$ is the unique solution to the differential equation $x'(t) = \varphi(x(t))$ starting from x.

Then the associated transition is contingent to K at x if and only if the vector $\varphi(x)$ is contingent to K at x.

Example: Contingent Transition Sets on Power Sets

Let $M \subset E$ be a closed subset of a finite dimensional vector space and consider the mutational space $(\mathcal{K}(M), \mathrm{Lip}_0(M, E))$. Let $\mathcal{M} \subset \mathcal{K}(M)$ be the a family of nonempty compact subsets of M.

We shall say that a Lipschitz map $\varphi \in \mathrm{Lip}_0(M, E)$ is contingent to $\mathcal{M}$ at $K \in \mathcal{M}$ if and only if the associated transition ϑ_φ is contingent to $\mathcal{M}$ at K, i.e., if and only if there exist sequences h_n and ε_n converging to 0 and a sequence of subsets $K_n \in \mathcal{M}$ such that

$$\vartheta_\varphi(h_n, K) \subset K_n + \varepsilon_n h_n B \ \& \ K_n \subset \vartheta_\varphi(h_n, K) + \varepsilon_n h_n B$$

This contingent cone has been introduced and studied in [24, Doyen] under the name of velocity cone.

4.2 Nagumo's Theorem for Mutational Equations

Theorem 4.2 *Let $(X, \Theta(X))$ be a mutational space, $K \subset X$ be a closed subset and $f : [0, \infty[\times K \mapsto \Theta(X)$ be a uniformly continuous map* bounded *in the sense that:*

$$\forall\, t \geq 0,\ \forall\, x \in K,\ \|f(t, x)\|_\Lambda := \sup_{h \in [0,1],\ x \neq y} \frac{d(f(t, x; h, y), f(t, x; h, z))}{d(y, z)} \leq c_\Lambda$$

and that

$$\forall\, t \geq 0,\ \forall\, x \in K,\ \forall\, y \in X,\ \|f(t, x; y)\| := \sup_{k \neq h} \frac{d(f(t, x; h, y), f(t, x; k, y))}{|h - k|} \leq c$$

Assume that the closed bounded balls of X are compact.

If K is a viability domain *of f in the sense that*

$$\forall\, t \geq 0,\ \forall\, x \in K,\ \ f(t,x)\ \in\ T_K(x)$$

then, from any initial state $x_0 \in K$ starts one solution to the mutational equation $\mathring{x} \ni f(t,x)$ viable in K.

Proof

4.2.1 Construction of Approximate Solutions

We begin by proving that there exist approximate viable solutions to the mutational inclusion. We set

$$M\ :=\ \max(0, c_\Lambda - 1)$$

so that $M = 0$ when the mutations $f(t,x)$ are nonexpansive, and

$$\boldsymbol{d}_\infty(\tau, \mathring{x}\,(t))\ :=\ \inf_{\sigma \in \mathring{x}(t)} \boldsymbol{d}_\infty(\tau,\sigma)$$

Lemma 4.3 *We posit the assumptions of the Nagumo Theorem 4.2. Then, for any $\varepsilon > 0$, the set $\mathcal{S}_\varepsilon(x_0)$ of continuous functions $x(\cdot) \in \mathcal{C}(0,1;X)$ satisfying $x(0) = x_0$ and*

$$\begin{cases} i) & \forall\, t \in [0,1],\ \boldsymbol{d}(x(t),x_0)\ \leq\ ct \\ ii) & \forall\, t \in [0,1],\ \boldsymbol{d}(x(t),K)\ \leq\ \varepsilon\dfrac{e^M - 1}{M} \\ iii) & \forall\, t \in [0,1],\ \boldsymbol{d}_\infty(f(t,x(t)),\mathring{x}\,(t))\ \leq\ \varepsilon \\ iv) & \forall\, t \in [0,1],\ \boldsymbol{d}(x(t),x(t+h))\ \leq\ ch \end{cases}$$

is not empty.

Proof — Let us fix $\varepsilon > 0$. Since f is uniformly continuous, we can associate $\eta \in]0,\varepsilon]$ such that

$$\boldsymbol{d}_\infty(f(s,y), f(r,z))\ \leq\ \varepsilon \tag{4.2}$$

whenever $|s - r| \leq \eta$ and $\boldsymbol{d}(y,z) \leq \eta\dfrac{e^M - 1}{M}$.

We denote by $\mathcal{A}_\varepsilon(x_0)$ the set of pairs $(T_x, x(\cdot))$ where $T_x \in [0,1]$ and $x(\cdot) \in \mathcal{C}(0, T_x; X)$ is a continuous functions satisfying $x(0) = x_0$ and

$$\begin{cases} i) & \forall\, t \in [0,T_x], \;\; \boldsymbol{d}(x(t), x_0) \;\leq\; ct \\ ii) & \forall\, t \in [0,T_x], \;\; \boldsymbol{d}(x(T_x), K) \;\leq\; \eta \dfrac{e^{MT_x} - 1}{M} \\ iii) & \forall\, t \in [0,T_x], \;\; \boldsymbol{d}(x(t), K) \;\leq\; \eta \dfrac{e^{M} - 1}{M} \\ iv) & \forall\, t \in [0,T_x], \;\; \boldsymbol{d}_\infty(f(t, x(t)), \overset{\circ}{x}(t)) \;\leq\; \varepsilon \\ v) & \forall\, t \in [0,T_x], \;\; \boldsymbol{d}(x(t), x(t+h)) \;\leq\; ch \end{cases} \tag{4.3}$$

The set $\mathcal{A}_\varepsilon(x_0)$ is not empty: take $T_x = 0$ and $x(0) \equiv x_0$.
It is an inductive set for the order relation

$$(T_{x_1}, x_1(\cdot)) \;\preceq\; (T_{x_2}, x_2(\cdot))$$

if and only if

$$T_{x_1} \;\leq\; T_{x_2} \;\;\&\;\; x_2(\cdot)|_{[0,T_{x_1}]} \;=\; x_1(\cdot)$$

Zorn's Lemma implies that there exists a maximal element $(T_x, x(\cdot)) \in \mathcal{A}_\varepsilon(x_0)$. The Lemma follows from the claim that for such a maximal element, we have $T_x = 1$.

If not, we shall extend $x(\cdot)$ by a solution $\widehat{x}(\cdot)$ on an interval $[T_x, S_x]$ where $S_x > T_x$, contradicting the maximal character of $(T_x, x(\cdot))$.

Let us take $\widehat{x} \in K$ achieving the distance between $x(T_x)$ and K:

$$\boldsymbol{d}(x(T_x), \widehat{x}) \;=\; \boldsymbol{d}(x(T_x), K)$$

Let us set $\widehat{\vartheta} := f(t, \widehat{x}) \in T_K(\widehat{x})$ and

$$\forall\, t \in [T_x, 1], \;\; \widehat{x}(t) \;:=\; \widehat{\vartheta}(t - T_x, x(T_x))$$

Since the transition $\widehat{\vartheta}$ is mutable at $x(T_x)$, there exists $\beta \in]0, \varepsilon]$ such that

$$\boldsymbol{d}(\widehat{\vartheta}(h, \widehat{\vartheta}(t - T_x, x(T_x))), \widehat{\vartheta}(h + t - T_x, x(T_x))) \;\leq\; \varepsilon h \tag{4.4}$$

whenever $h \leq \beta$.

We then introduce

$$\alpha \;:=\; \min\left(\beta, \eta, \eta \frac{e^M - e^{MT_x}}{\|\widehat{\vartheta}(\widehat{x})\| M}, \frac{\eta}{\|\widehat{\vartheta}(\widehat{x})\|}\right) \;>\; 0$$

whenever $T_x < 1$.

By the definition of a contingent transition, there exists $h_x \in]0,\alpha]$ such that

$$\boldsymbol{d}(\widehat{\vartheta}(h_x,\widehat{x}),K) \leq \eta h_x \tag{4.5}$$

We then set $S_x := T_x + h_x > T_x$.

We obtain

$$\begin{cases} \boldsymbol{d}(\widehat{x}(S_x),K) = \boldsymbol{d}(\widehat{\vartheta}(S_x - T_x, x(T_x)),K) \\ \leq \boldsymbol{d}(\widehat{\vartheta}(h_x,\widehat{x}),\widehat{x}) + \boldsymbol{d}(\widehat{\vartheta}(h_x,x(T_x)),\widehat{\vartheta}(h_x,\widehat{x})) \\ \leq \eta h_x + e^{Mh_x}\boldsymbol{d}(x(T_x),\widehat{x}) \leq \eta h_x + e^{Mh_x}\eta\frac{e^{MT_x}-1}{M} \\ \leq \eta\frac{e^{M(T_x+h_x)}-1}{M} = \eta\frac{e^{MS_x}-1}{M} \end{cases}$$

by (4.5) and (4.3)ii) and Lemma 2.2, since

$$h + \frac{e^{Mh}(e^{MT}-1)}{M} \leq \frac{e^{M(T+h)}-1}{M}$$

Hence $\widehat{x}(\cdot)$ satisfies (4.3)ii) for S_x.

We observe that for any $t \in [T_x, S_x]$,

$$\boldsymbol{d}(\widehat{x}(t),x(T_x)) = \boldsymbol{d}(\widehat{\vartheta}((t-T_x,x(T_x)),x(T_x)) \leq (t-T_x)\|\widehat{\vartheta}(\widehat{x})\| \leq c(t-T_x)$$

so that $\widehat{x}(\cdot)(\cdot)$ satisfies (4.3)i).

Also, we note that

$$\begin{cases} \boldsymbol{d}(\widehat{x}(t),K) \leq \boldsymbol{d}(\widehat{x}(t),\widehat{x}) \leq \boldsymbol{d}(\widehat{x}(t),x(T_x)) + \boldsymbol{d}(x(T_x),\widehat{x}) \\ \leq (t-T_x)\|\widehat{\vartheta}(\widehat{x})\| + \eta\frac{e^{MT_x}-1}{M} \leq \alpha\|\widehat{\vartheta}(\widehat{x})\| + \eta\frac{e^{MT_x}-1}{M} \leq \eta\frac{e^{M}-1}{M} \end{cases}$$

from the very choice of α. Then $\widehat{x}(\cdot)$ satisfies (4.3)iii).

We note next that for any $t \in [T_x, S_x[$ and h small enough, $\widehat{x}(t+h) = \widehat{\vartheta}(h+t-T_x, x(T_x))$. Since $\widehat{\vartheta} := f(T_x,\widehat{x})$ is mutable and $\alpha \leq \beta$, inequality

$$\begin{cases} \boldsymbol{d}(\widehat{\vartheta}(h,\widehat{x}(t)),\widehat{x}(t+h)) \\ = \boldsymbol{d}(\widehat{\vartheta}(h,\widehat{\vartheta}(t-T_x,x(T_x)),\widehat{\vartheta}(h+t-T_x,x(T_x)))) \leq \varepsilon h \end{cases}$$

imply that for all $t \in [T_x, S_x]$, the constant transition $\widehat{\vartheta} := f(T_x, \widehat{x})$ belongs to the mutation $\overset{\circ}{\widehat{x}}(t)$. Therefore, for all $t \in [T_x, S_x]$,

$$\begin{cases} \boldsymbol{d}_\infty(f(t, \widehat{x}(t)), \overset{\circ}{\widehat{x}}(t)) \\ = \boldsymbol{d}_\infty(f(t, \widehat{x}(t)), f(T_x, \widehat{x})) \leq \varepsilon \end{cases}$$

since $|t - T_x| \leq \alpha \leq \eta$ and $\boldsymbol{d}(\widehat{x}(t), \widehat{x}) \leq \eta \frac{e^M - 1}{M}$. Therefore $\widehat{x}(\cdot)$ satisfies (4.3)iv).

Finally, we deduce that

$$\begin{cases} \boldsymbol{d}(\widehat{x}(t), \widehat{x}(t+h)) = \boldsymbol{d}(\widehat{\vartheta}(t - T_x, x(T_x)), \widehat{\vartheta}(h + t - T_x, x(T_x))) \\ \leq \|\widehat{\vartheta}(x(T_x))\| h \leq ch \end{cases}$$

so that $\widehat{x}$ satisfies (4.3)v).

Therefore, we have extended the maximal solution $(T_x, x(\cdot))$ on the interval $[0, S_x]$ and obtained the desired contradiction. Hence the proof of Lemma 4.3 is completed. □

4.2.2 Proof of the Nagumo Theorem for Mutational Equations

Consider now a sequence of ε-approximate solutions $x_\varepsilon(\cdot)$, which exist thanks to Lemma 4.3.

Since the closed bounded balls of X are compact and since the solutions remain in such closed balls X, we deduce that for every $t \in [0, 1]$, the images $x_\varepsilon(t)$ remain in a compact set of X.

Property (4.3)v) implies that the sequence of continuous functions $x_\varepsilon(\cdot))$ is equicontinuous.

Therefore, Ascoli's Theorem implies that a subsequence (again denoted by) $x_\varepsilon(\cdot)$ converges uniformly to $x(\cdot)$.

This limit is obviously a solution to the mutational equation, since for any $t \geq 0$,

$$\begin{cases} \boldsymbol{d}_\infty(f(t, x(t)), \overset{\circ}{x}_\varepsilon(t)) \\ \leq \boldsymbol{d}_\infty(f(t, x_\varepsilon(t)), \overset{\circ}{x}_\varepsilon(t)) + \boldsymbol{d}_\infty(f(t, x_\varepsilon(t)), f(t, x(t))) \end{cases}$$

This limit is viable in K since for all $t \in [0,1]$ and $\varepsilon > 0$, $d(x(t), K) \leq \varepsilon \frac{e^M - 1}{M}$.

Hence, there exists a solution to the mutational equation on the interval $[0,1]$, which can then be extended to $[0, \infty[$. □

4.3 Primitives of Mutations

Solutions to the mutational equation with state-independent right-hand side

$$\overset{\circ}{x}(t) \ni \vartheta(t)$$

are naturally regarded as a primitive of $\vartheta(t)$ starting at x_0. Then Gronwall's Lemma implies:

Proposition 4.4 *Let $(X, \Theta(X))$ be a mutational space. Consider two functions $t \mapsto \vartheta(t)$ and $t \mapsto \tau(t)$ from an interval $I \subset \mathbf{R}$ to $\Theta(X)$ and their primitives $x(\cdot)$ and $y(\cdot)$ starting at x_0 and y_0 respectively. Set $\mu(t) := \max\left(\int_0^t \|\vartheta(s)\|_\Lambda ds - t, 0\right)$. ($\mu(t) = 0$ whenever the transition ϑ is nonexpansive and bounded by Mt where $M := \sup_{t \in I}(\|\vartheta(t)\|_\Lambda - 1)$). Assume that the closed bounded balls of X are compact. Then*

$$d(x(t), y(t)) \leq d(x_0, y_0)e^{\mu(t)} + \int_0^t e^{\mu(t)-\mu(s)} d_\Lambda(\vartheta(s), \tau(s))ds \qquad (4.6)$$

In particular, from any initial state x_0 starts a unique primitive of $t \mapsto \vartheta(t) \in \Theta(X)$.

Remark — In [28, Doyen], one can find an existence theorem of primitives of "regulated transitions", which are uniform limits of piecewise constant transitions. Indeed, it is proved that if a sequence of transitions ϑ_n converges uniformly in $\Theta(X)$ to a transition ϑ, then the primitives $x_n(\cdot)$ of ϑ_n converge to a primitive $x(\cdot)$ of ϑ.

In particular, measurable mutational transitions with compact images do have primitives. □

4.4 Cauchy-Lipschitz's Theorem for Mutational Equations

For simplicity, we consider only the case when the dynamics of a mutational equation is described by a single-valued map f from X to $\Theta(X)$ independent of time. Consider the Cauchy problem associated with the mutational equation :

$$\forall\ t \in [0,T],\ \overset{\circ}{x}(t)\ \ni\ f(x(t)) \tag{4.7}$$

satisfying the initial condition $x(0) = x_0$.

In the case when the right-hand side of the mutational equation is Lipschitz, existence and uniqueness of the solution can be proven, but on top of it, estimates implying the Lipschitz dependence of the solution upon initial conditions and right-hand sides are provided.

We recall the following notations:

$$\|f(x;x)\|\ :=\ \sup_{h\neq k}\frac{\boldsymbol{d}(f(x;h,x),f(x;k,x))}{|h-k|}$$

and

$$\|f(x)\|_\Lambda\ :=\ \sup_{h\in[0,1],\,x\neq y}\frac{\boldsymbol{d}(f(x;h,x),f(x;h,y))}{\boldsymbol{d}(x,y)}$$

Theorem 4.5 *Let $(X,\Theta(X))$ be a complete mutational space and $f : X \mapsto \Theta(X)$ be a Lipschitz map with Lipschitz constant $\|f\|_\Lambda$. Assume also (for simplicity) that*

$$\sup_{x\in X}\|f(x)\|_\Lambda\ <\ +\infty$$

and set $M\ :=\ \max(0,\sup_{x\in X}\|f(x)\|_\Lambda - 1)$ (If the mutations $f(x)$ are nonexpansive, then $M = 0$.) Fix a mutable function $y(\cdot)\ :\ [0,\infty[\mapsto X$. Assume that the closed bounded balls of X are compact. Then there exists a unique solution $x(\cdot)$ to the Cauchy problem for the mutational equation (4.7) satisfying the inequality

$$\left\{\begin{array}{l}\forall\, t\in[0,\infty[,\ \ \boldsymbol{d}(x(t),y(t))\ \le\\[2ex] e^{(M+\|f\|_\Lambda)t}\boldsymbol{d}(x_0,y(0))+\displaystyle\int_0^t e^{(M+\|f\|_\Lambda)(t-s)}\boldsymbol{d}_\Lambda(f(y(s)),\overset{\circ}{y}(s))ds\end{array}\right.$$

By taking for function $y(\cdot)$ a solution of the Cauchy problem for the mutational equation $\overset{\circ}{y} \ni f(y)$ starting from y_0, we infer from this inequality that :

$$\sup_{t\in[0,T]}\ \boldsymbol{d}(x(t),y(t))\ \le\ e^{(M+\|f\|_\Lambda)T}\boldsymbol{d}(x_0,y_0)$$

which shows Lipschitz dependence with respect to initial states.

By taking for function $y(\cdot)$ a solution to the Cauchy problem for the mutational equation $\overset{\circ}{y} \ni g(y)$ starting from x_0, we obtain

$$\sup_{t\in[0,T]} \boldsymbol{d}(x(t),y(t)) \leq \frac{e^{(M+\|f\|_\Lambda)T}-1}{\|f\|_\Lambda}\boldsymbol{d}_\Lambda(g,f)$$

which shows Lipschitz dependence with respect to the right-hand sides.

Finally, we obtain

$$\boldsymbol{d}(x(t),x_0) \leq \frac{e^{(M+\|f\|_\Lambda)t}-1}{M+\|f\|_\Lambda}\|f(x_0;x_0)\|$$

We need the following Lemma[17]:

Lemma 4.6 *Let μ, $\gamma : \mathbf{R} \mapsto \mathbf{R}_+$ be differentiable functions. Then*

$$\left\{\begin{array}{l} \int_0^t e^{\mu(t)-\mu(s)}\gamma'(s)\int_0^s \psi(r)e^{\mu(s)-\mu(r)}\frac{(\gamma(s)-\gamma(r))^{n-1}}{(n-1)!}drds \\ = \int_0^t e^{\mu(t)-\mu(s)}\psi(s)\frac{(\gamma(t)-\gamma(s))^n}{n!}ds \end{array}\right. \tag{4.8}$$

and, in particular,

$$\left\{\begin{array}{l} \int_0^t e^{\mu(t)-\mu(s)}\gamma'(s)e^{\mu(s)-\mu(0)}\frac{(\gamma(s)-\gamma(0))^{n-1}}{(n-1)!}ds \\ = e^{\mu(t)-\mu(0)}\frac{(\gamma(t)-\gamma(0))^n}{n!} \end{array}\right. \tag{4.9}$$

[17] It follows from:

$$\left\{\begin{array}{l} \int_0^t e^{\mu(t)-\mu(s)}\gamma'(s)\int_0^s \psi(r)e^{\mu(s)-\mu(r)}\frac{(\gamma(s)-\gamma(r))^{n-1}}{(n-1)!}drds \\ = \int_0^t \psi(r)e^{\mu(t)-\mu(r)}dr\int_r^t \gamma'(s)\frac{(\gamma(s)-\gamma(r))^{n-1}}{(n-1)!}ds \\ = \int_0^t \psi(r)e^{\mu(t)-\mu(r)}dr\frac{(\gamma(t)-\gamma(r))^n}{n!} \quad \square \end{array}\right.$$

Proof

1. — **Construction of approximate solutions**

We introduce the map $G : \mathcal{C}([0,T],X) \mapsto \mathcal{C}([0,T],X)$ associating with $z(\cdot)$ the function $G(z)(t)$ is the (unique) primitive of $s \mapsto f(z(s))$ starting at x_0.

We denote by $e(\cdot)$ the error defined by

$$e(s) \;:=\; \boldsymbol{d}_\Lambda(f(y(s)), \overset{\circ}{y}(s))$$

We observe that

$$\boldsymbol{d}(G(y)(t), y(t)) \;\leq\; \boldsymbol{d}(x_0, y(0))e^{Mt} + \int_0^t e^{M(t-s)}e(s)ds \qquad (4.10)$$

by Proposition 4.4, since $y(\cdot)$ is the primitive of the function $s \mapsto \overset{\circ}{y}(s)$ starting at $y(0)$.

We introduce the sequence of approximate solutions $x_n(\cdot)$ defined by $x_1 := G(y)$ and, for every $n \geq 1$, par $x_{n+1} := G(x_n)$.

2. — **Convergence of approximate solutions**

We shall show that this is a Cauchy sequence in the complete metric space $\mathcal{C}([0,T],X)$, which thus, is convergent.

For simplicity, we set $\lambda := \|f\|_\Lambda$.

Indeed,

$$\left\{ \begin{array}{l} \boldsymbol{d}(G(x_n)(t), G(x_{n-1})(t)) \leq \lambda \int_0^t e^{M(t-s)}\boldsymbol{d}(x_n(s), x_{n-1}(s))ds \\ \leq \lambda \int_0^t e^{M(t-s)}\boldsymbol{d}(G(x_{n-1})(s), G(x_{n-2})(s))ds \end{array} \right.$$

so that, iterating these inequalities, we obtain

$$\begin{cases} \boldsymbol{d}(x_{n+1}(t), x_n(t)) \leq \\ \lambda^n \int_0^t dt \int_0^{t_1} e^{M(t-t_1)} dt_1 \dots \int_0^{t_{n-1}} e^{M(t_{n-2}-t_{n-1})} \boldsymbol{d}(x_1(t_{n-1}), y(t_{n-1})) dt_{n-1} \\ \leq \lambda^n \int_0^t dt \left(\int_0^{t_1} e^{M(t-t_1)} dt_1 \dots \int_0^{t_n} e^{M(t_{n-1}-t_n)} \boldsymbol{d}(x_0, y(0)) e^{Mt_n} dt_n \right) \\ +\lambda^n \int_0^t dt \left(\int_0^{t_1} dt_1 \dots \int_0^{t_n} e^{M(t_{n-1}-t_n)} e(t_n) dt_n \right) \\ = \frac{\lambda^n t^n}{n!} \boldsymbol{d}(x_0, y(0)) e^{Mt} + \int_0^t e^{M(t-s)} \frac{\lambda^n (t-s)^n}{n!} e(s) ds \end{cases}$$

thanks to Lemma 2.2.

Consequently,

$$\begin{cases} \boldsymbol{d}(x_p(t), x_q(t)) \leq \sum_{n=q}^{p-1} \boldsymbol{d}(x_{n+1}(t), x_n(t)) \\ \leq \sum_{n=q}^{p-1} \left(e^{Mt} \frac{\lambda^n t^n}{n!} \boldsymbol{d}(x_0, y(0)) + \int_0^t e^{M(t-s)} \frac{\lambda^n (t-s)^n}{n!} e(s) ds \right) \end{cases}$$

which shows that this is a Cauchy sequence, which converges uniformly on $[0, T]$ to a function $x(\cdot)$.

3. — The limit is a solution

This limit is a solution to the Cauchy problem since by taking the limit, equations $x_{n+1}(t) = G(x_n(t))$ imply that $x(t) = G(x(t))$, and thus, is a solution to the mutational equation.

By taking $q = 0$ in the preceding inequalities, we obtain

$$\begin{cases} \boldsymbol{d}(x_p(t), y(t)) \leq \left(\sum_{n=0}^{p-1} \frac{\lambda^n t^n}{n!} \right) e^{Mt} \boldsymbol{d}(x_0, y(0)) \\ + \int_0^t \left(\sum_{n=0}^{p-1} \frac{\lambda^n (t-s)^n}{n!} \right) e^{M(t-s)} e(s) ds \end{cases}$$

which imply the inequality we were looking for. □

Remark — This theorem has been extended in [28, Doyen] to the case of **mutational inclusions** with Lipschitz right-hand side by adapting the original proof of Filippov[18]. □

Example: Mutational Equations for Tubes

We have presented in the introduction the corollaries of the above theorem for mutational equations for tubes in the particular case when the right-hand sides are transitions associated with Lipschitz single-valued maps. Naturally, the same theorems hold true when the right-hand sides are transitions associated with Lipschitz set-valued maps Φ.

Therefore, we can extend the theorems dealing with mutational equations for tubes to the case of mutational equations of the form

$$\overset{\circ}{P}(t) \ni F(P(t); \cdot)$$

where $F(P; \cdot) \in \mathrm{LIP}_0(M, E)$.

This contains in particular the case in mathematical morphology when $F(P; x) := B(P)$ is a structuring element depending on P (called a **structuring function**).

The evolution of tubes P governed by such a mutational equation is given by

$$\forall\, t \geq 0, \quad \lim_{h \to 0+} \frac{\boldsymbol{d}(P(t+h), P(t) + hB(P(t)))}{h} = 0$$

For more details on mathematical morphology, see [45, Mattioli & Schmitt]. □

5 The Invariant Manifold Theorem

Let $(X, \Theta(X))$ be a mutational space and Y a finite dimensional vector space, where we take $\Theta(Y) = Y$. We supply the product $X \times Y$ with the space of transitions $\Theta(X) \times Y$.

[18]See also the extension of Filippov Theorem to operational differential inclusions in [33, Frankowska].

Let us consider a system of mutational-differential equations

$$\begin{cases} \overset{\circ}{x}(t) \ni f(x(t), y(t)) \\ y'(t) = Ay(t) + g(x(t), y(t)) \end{cases}$$

where $f : X \times Y \mapsto \Theta(X)$ and $g : X \times Y \mapsto Y$ are Lipschitz maps and $A \in \mathcal{L}(Y, Y)$ is a linear operator.

We look for single-valued maps $u : X \mapsto Y$ **whose (closed) graphs are invariant under this system.**

Such a map u **tracks** the solutions $x(\cdot)$ to the mutational equation in the sense that if $x(\cdot)$ is a solution to

$$\overset{\circ}{x}(t) \ni f(x(t), u(x(t)))$$

starting from x_0, then $y(t) := u(x(t))$ is a solution to

$$y'(t) = Ay(t) + g(x(t), y(t))$$

starting from $u(x_0)$.

One then can characterize such maps $u : X \mapsto Y$ whose graphs are invariant under this system thanks to the Nagumo Theorem:

$$\forall x \in X, \ \ (f(x, u(x)), -Au(x) + g(x, u(x))) \in T_{\mathrm{Graph}(u)}(x, u(x)) \quad (5.1)$$

We shall say that the contingent set to the graph of a single-valued map u at $(x, u(x))$ is the graph of the **contingent mutation** $\overset{\circ}{D} u(x)$ at x. This is the set-valued map from $\Theta(X)$ to Y defined by $v \in \overset{\circ}{D} u(x)(\vartheta)$ if and only if there exist sequences $h_n \to 0+$, $y_n \to u(x)$ and $x_n \to x$ such that

$$d(\vartheta(h_n, x), x_n) \leq \alpha_n h_n \ \ \& \ \ \|u(x) + h_n v - y_n\| \leq \beta_n h_n$$

Naturally, $\overset{\circ}{D} u(x)(\vartheta) = \overset{\circ}{u}(x)\vartheta$ coincides with the usual mutation whenever u is mutable at x. It has nonempty values when u is Lipschitz.

Therefore, the graph of u is a viability domain if and only if

$$\forall x \in X, \ \ Au(x) \in \overset{\circ}{D} u(x)(f(x, u(x))) - g(x, u(x)) \qquad (5.2)$$

since it amounts to rewriting condition (5.1).

5.1 The Decomposable Case

We begin by proving the existence and uniqueness in the decomposable case when the real number λ defined by

$$\lambda := \inf_{x \neq 0} \frac{\langle Ax, x \rangle}{\|x\|^2}$$

is large enough. (We recall that $\forall\, y \in Y,\ \left\|e^{-At}y\right\| \leq e^{-\lambda t}\|y\|$).

Consider two maps $\varphi : X \mapsto \Theta(X)$ and $\psi : X \mapsto Y$ and the system

$$Au(x) \in \overset{\circ}{D}\, u(x)(\varphi(x)) - \psi(x) \tag{5.3}$$

We set

$$\|u\|_\infty := \sup_{x \in X} \|u(x)\| \in [0, \infty] \ \&\ \|u\|_\Lambda := \sup_{x \neq y} \frac{\|u(x) - u(y)\|}{\|x - y\|} \in [0, \infty]$$

When φ is Lipschitz, we denote by $S_\varphi(x, \cdot)$ the unique solution to the mutational equation

$$\overset{\circ}{x}(t) \ni \varphi(x(t))$$

starting from x at the initial time 0. We assume that φ is bounded in the sense that

$$M := \max(0, \sup_x \|\varphi(x)\|_\Lambda - 1) < +\infty$$

Theorem 5.1 *Suppose that the mutational space $(X, \Theta(X))$ is complete and that φ and ψ are Lipschitz and bounded. If $\lambda > 0$, then the single-valued map $u := \Gamma(\varphi, \psi)$ defined by*

$$\forall\, x \in X,\ u(x) = -\int_0^\infty e^{-At}\psi(S_\varphi(x,t))dt$$

is the unique solution to

$$Au(x) \in \overset{\circ}{D}\, u(x)(\varphi(x)) - \psi(x) \tag{5.4}$$

It is Lipschitz, bounded and satisfies

$$\|u\|_\infty \leq \frac{\|\psi\|_\infty}{(M + \lambda)} \ \&\ \forall\, \lambda > \|\varphi\|_\Lambda + M,\ \|u\|_\Lambda \leq \frac{\|\psi\|_\Lambda}{\lambda - M - \|\varphi\|_\Lambda} \tag{5.5}$$

The single-valued map $(\varphi, \psi) \mapsto \Gamma(\varphi, \psi)$ is continuous from $\mathcal{C}(X, \Theta(X)) \times \mathcal{C}(X, Y)$ to $\mathcal{C}(X, Y)$:

$$\boldsymbol{d}_\infty(\Gamma(\varphi_1, \psi_1), \Gamma(\varphi_2, \psi_2)) \leq \frac{\|\psi_1 - \psi_2\|_\infty}{\lambda} + \frac{\|\psi_2\|_\Lambda}{\lambda(\lambda - M - \|\varphi_2\|_\Lambda)} \boldsymbol{d}_\infty(\varphi_1, \varphi_2)$$

Proof

1. — We prove first that u is a solution to (5.4), by computing its contingent mutation: we have to check that there exist sequences $h_n \to 0+$, $x_n \to x$ such that

$$\begin{cases} \boldsymbol{d}(\vartheta_\varphi(h_n, x), x_n) \ \leq \ \alpha_n h_n \\ \|u(x) + h_n(Au(x) + \psi(x)) - u(x_n)\| \ \leq \ \beta_n h_n \end{cases}$$

Denote by $x(\cdot) = S_\varphi(x, \cdot)$ the solution to the mutational equation $\overset{\circ}{x}(t) \ni \varphi(x(t))$ starting from x. We know that $x(\cdot)$ being a solution to the mutational equation $\overset{\circ}{x}(t) \ni \varphi(x(t))$,

$$\boldsymbol{d}(\vartheta_\varphi(h_n, x), x(h_n)) \ \leq \ \alpha_n h_n$$

Setting $z(t) := \psi(x(t))$, we can write

$$u(x) \ := \ -\int_0^\infty e^{-At} z(t) dt$$

We check that for every $h_n > 0$

$$-\int_0^\infty e^{-At} z(t + h_n) dt \ = \ u(x(h_n))$$

Observing that

$$\begin{cases} \frac{1}{h_n} \int_0^\infty e^{-At} \left(z(t) - z(t + h_n)\right) dt \\ = \ -\frac{e^{Ah_n} - 1}{h_n} \int_0^\infty e^{-At} z(t) dt + \frac{e^{Ah_n}}{h_n} \int_0^{h_n} e^{-At} z(t) dt \end{cases}$$

we deduce that

$$\begin{cases} u(x) + h_n \left(-\frac{e^{Ah_n} - 1}{h_n} \int_0^\infty e^{-At} z(t) dt + \frac{e^{Ah_n}}{h_n} \int_0^{h_n} e^{-At} z(t) dt \right) \\ = \ u\left(x(h_n)\right) \end{cases}$$

We then remark that $\frac{1}{h_n}\int_0^{h_n} z(t)dt$ converges to $\psi(x)$ and thus, that

$$-\frac{e^{Ah_n}-1}{h_n}\int_0^{\infty} e^{-At}z(t)dt + \frac{e^{Ah_n}}{h_n}\int_0^{h_n} e^{-At}z(t)dt$$

converges to $Au(x)+\psi(x)$, so that, by the very definition of a contingent mutation, we obtain

$$Au(x)+\psi(x) \in \overset{\circ}{D}\, u(x)(\varphi(x))$$

2. — Estimate

$$\|u(x)\| \le \int_0^{\infty} \|\psi\|_{\infty} e^{-\lambda t} = \frac{\|\psi\|_{\infty}}{\lambda}$$

is obvious.

Let $v : X \rightsquigarrow Y$ be a bounded solution to (5.4).

We know that for every $x \in X$, there exists a solution $(x(\cdot), y(\cdot))$ to the system of mutational-differential equations

$$\begin{cases} i) & \overset{\circ}{x}(t) \ni \varphi(x(t)) \\ ii) & y'(t) - Ay(t) = \psi(x(t)) \end{cases}$$

starting from $(x, v(x))$ such that $y(t) = v(x(t))$ for every $t \ge 0$. We set $z(t) := \psi(x(t))$, which is bounded.

Therefore, if $\lambda > 0$, the function $e^{-At}z(t)$ is integrable. On the other hand, by integrating by parts $e^{-At}\psi(x(t)) := e^{-At}y'(t) - e^{-At}Ay(t)$, we obtain

$$e^{-AT}y(T) - v(x) = \int_0^T e^{-At}\psi(x(t))dt$$

which implies

$$v(x) = -\int_0^{\infty} e^{-At}\psi(x(t))dt$$

by letting $T \to \infty$. We have thus proved that $v(x) = u(x)$.

3. — We now fix a pair of elements x_1 and x_2 and we set for $i = 1, 2$: $u(x_i) = -\int_0^{\infty} e^{-At}z_i(t)dt$, where

$$x_i(\cdot) = S_{\varphi}(x_i, \cdot) \;\&\; z_i(t) = \psi(x_i(t))$$

Since φ is Lipschitz, Theorem 4.5 implies, setting $\alpha := M + \|\varphi\|_\Lambda$, that

$$\forall\, t \geq 0,\;\; \boldsymbol{d}(x_1(t), x_2(t)) \;\leq\; e^{\alpha t}\boldsymbol{d}(x_1, x_2)$$

and thus,

$$\forall\, t \geq 0,\; \|z_1(t) - z_2(t)\| \leq \|\psi\|_\Lambda \boldsymbol{d}(x_1(t), x_2(t)) \leq \|\psi\|_\Lambda e^{\alpha t}\boldsymbol{d}(x_1, x_2)$$

Consequently, if $\lambda > \alpha$, then $u(x_2) = -\int_0^\infty e^{-At} z_2(t)dt$ satisfies

$$\|u(x_1) - u(x_2)\| \leq \int_0^\infty \|\psi\|_\Lambda e^{-t(\lambda-\alpha)}\boldsymbol{d}(x_1, x_2)dt \leq \frac{\|\psi\|_\Lambda}{\lambda - \alpha}\boldsymbol{d}(x_1, x_2)$$

Let us consider now two pairs (φ_1, ψ_1) and (φ_2, ψ_2) and the solutions

$$\forall\, x \in X,\; u_i(x) \;:=\; -\int_0^\infty e^{-At}\psi_i(S_{\varphi_i}(x, t))dt \;\; (i = 1, 2)$$

Set $z_i(t) \;:=\; \psi_i(x_i(t))$ Since the functions are Lipschitz, Theorem 4.5 with $e(s) := \boldsymbol{d}(\overset{\circ}{x}_1(t), \varphi_2(x_1(t)))$ implies that

$$\forall\, t \geq 0,\;\; \boldsymbol{d}(x_1(t), x_2(t)) \;\leq\; \boldsymbol{d}_\infty(\varphi_1, \varphi_2)\frac{e^{t\|(\varphi_2\|_\Lambda + M)} - 1}{\|\varphi_2\|_\Lambda + M}$$

Hence

$$\begin{cases} \forall\, t \geq 0,\;\; \|z_1(t) - z_2(t)\| \;\leq\; \|\psi_1 - \psi_2\|_\infty + \|\psi_2\|_\Lambda \boldsymbol{d}(x_1(t), x_2(t)) \\ \leq\; \|\psi_1 - \psi_2\|_\infty + \|\psi_2\|_\Lambda \boldsymbol{d}_\infty(\varphi_1, \varphi_2)\dfrac{e^{t\|(\varphi_2\|_\Lambda + M)} - 1}{\|\varphi_2\|_\Lambda + M} \end{cases}$$

Therefore,

$$\begin{cases} \|u_1(x) - u_2(x)\| \;\leq \\ \displaystyle\int_0^\infty e^{-\lambda t}\|\psi_1 - \psi_2\|_\infty dt + \|\psi_2\|_\Lambda \boldsymbol{d}_\infty(\varphi_1, \varphi_2)\int_0^\infty \frac{e^{t\|(\varphi_2\|_\Lambda + M)} - 1}{\|\varphi_2\|_\Lambda + M}e^{-\lambda t}dt \\ \leq\; \dfrac{\|\psi_1 - \psi_2\|_\infty}{\lambda} + \dfrac{\|\psi_2\|_\Lambda}{\lambda(\lambda - \alpha_2)}\boldsymbol{d}_\infty(\varphi_1, \varphi_2) \;\; \square \end{cases}$$

5.2 The General Case

Let us consider now the system of first order partial mutational equations (5.2).

Theorem 5.2 *Suppose that the mutational space $(X, \Theta(X))$ is complete, that the maps $f : X \times Y \mapsto \Theta(X)$ and $g : X \times Y \mapsto Y$ are Lipschitz, that f is bounded and that*

$$\forall\, x, y,\ \ \|g(x,y)\| \leq c(1 + \|y\|)$$

Then for λ large enough, there exists a unique bounded Lipschitz solution to the system of mutational equations (5.2).

Proof — We introduce the map H defined by

$$u := H(v) \text{ is the solution to } Au(x)\ \in \overset{\circ}{D}\, u(x) f(x, v(x)) - g(x, v(x))$$

We observe that the functions $\varphi(x) := f(x, v(x))$ and $\psi(x) := g(x, v(x))$ satisfy

$$\|\varphi\|_\Lambda \ \leq\ \|f\|_\Lambda(1 + \|v\|_\Lambda),\ \ \|\psi\|_\Lambda \ \leq\ \|g\|_\Lambda(1 + \|v\|_\Lambda)$$

and

$$\|\psi\|_\infty \ \leq\ \frac{c(1 + \|v\|_\infty)}{\lambda}$$

Theorem 5.1 implies the inequalities

$$\|H(v)\|_\infty \ \leq\ \frac{c}{\lambda}(1 + \|v\|_\infty)\ \ \&\ \ \|H(v)\|_\Lambda \ \leq\ \frac{\|g\|_\Lambda(1 + \|v\|_\Lambda)}{\lambda - \|f\|_\Lambda(1 + \|v\|_\Lambda) - M}$$

We observe first that when $\lambda > c$,

$$\forall\, \|v\|_\infty \ \leq\ \frac{c}{\lambda - c},\ \ \|H(v)\|_\infty \ \leq \frac{c}{\lambda - c}$$

When $\lambda > 4\|f\|_\Lambda\, \|g\|_\Lambda$, we denote by

$$\rho(\lambda)\ :=\ \frac{\lambda - \|f\|_\Lambda - \|g\|_\Lambda - \sqrt{\lambda^2 - 2\lambda(\|f\|_\Lambda + \|g\|_\Lambda) + (\|f\|_\Lambda - \|g\|_\Lambda)^2}}{2\|f\|_\Lambda} > 0$$

a root $\rho(\lambda)$ of the equation

$$\lambda\rho\ =\ \|f\|_\Lambda\rho^2 + (\|f\|_\Lambda + \|g\|_\Lambda)\rho + \|g\|_\Lambda$$

We observe that for λ large enough,

$$\lim_{\lambda \to +\infty} \lambda\rho(\lambda) = \|g\|_\Lambda$$

Therefore, for $\lambda > 4\|f\|_\Lambda \|g\|_\Lambda + M$,

$$\forall \|v\|_\Lambda \leq \rho(\lambda - M), \ \|H(v)\|_\Lambda \leq \rho(\lambda - M)$$

We denote by $\mathcal{B}$ the subset defined by

$$\mathcal{B} := \left\{ u \in \mathcal{C}(X,Y) \mid \|u\|_\infty \leq \frac{c}{\lambda - c} \ \& \ \|u\|_\Lambda \leq \rho(\lambda - M) \right\}$$

When $\lambda > \max(c, 4\|f\|_\Lambda \|g\|_\Lambda + M)$, the preceding inequalities imply that H maps the closed subset $\mathcal{B}$ to itself. On the other hand, the preceding Proposition implies that H is Lipschitz :

$$\boldsymbol{d}_\infty(H(v_1), H(v_2)) \leq \frac{\|g\|_\Lambda + \rho(\lambda - M)\|f\|_\Lambda}{\lambda - M} \boldsymbol{d}_\infty(v_1, v_2)$$

Therefore, by taking λ satisfying

$$\|g\|_\Lambda + \rho(\lambda - M)\|f\|_\Lambda < \lambda - M$$

which is possible because $\lim_{\lambda \to +\infty} \lambda\rho(\lambda) = \|g\|_\Lambda$.

Then the single-valued map H is a strict contraction, so that there exists a unique fixed point $u = H(u)$ of H, which is a solution to (5.2). □

6 Lyapunov Functions

6.1 Lower-Semicontinuous Lyapunov Functions

Let $(X, \Theta(X))$ be a mutational space. Consider a mutational equation

$$\mathring{x}(t) \ni f(x(t)) \tag{6.1}$$

a function $V : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ and a real-valued function $w(\cdot)$.

The function V is said *to enjoy the Lyapunov property* if and only if from any initial state x_0 starts a solution to the mutational equation satisfying

$$\forall t \geq 0, \ V(x(t)) \leq w(t), \ w(0) = V(x(0)) \tag{6.2}$$

Such inequalities imply many properties on the asymptotic behavior of V along the solutions to the mutational equation (in numerous instances, $w(t)$ goes to 0 when $t \to +\infty$, so that $V(x(t))$ converges also to 0).

Recall that the ***epigraph*** of V is defined by

$$\mathcal{E}p(V) \ := \ \{(x, \lambda) \in X \times \mathbf{R} \mid V(x) \leq \lambda\}$$

We see right away that when $w(\cdot)$ is a solution to a differential equation $w' = -\varphi(w)$, we have actually a viability problem in the epigraph of V because the Lyapunov property can be written: For any initial state x_0, there exists a solution to the mutational equation satisfying

$$\forall\, t \geq 0, \ \ (x(t), w(t)) \ \in \ \mathcal{E}p(V)$$

This function φ is used as a parameter in what follows. (The main instance of such a function φ is the affine function $\varphi(w) := aw - b$, the solutions of which are $w(t) = (w(0) - \frac{b}{a})e^{-at} + \frac{b}{a})$.

So that we can apply the Nagumo theorem whenever the epigraph of V is closed, i.e., whenever V is lower semicontinuous: V enjoys the Lyapunov property if and only if its epigraph is a viability domain of the map $(x, w) \rightsquigarrow f(x) \times \{-\varphi(w)\}$.

Therefore, our first task is to study the contingent transition set to the epigraph of an extended function V at some point $(x, V(x))$: it is the epigraph of a function denoted $\overset{\circ}{D}_\uparrow V(x)$ and called the *contingent epimutation* of V at x. Let $V : X \mapsto \mathbf{R} \cup \{+\infty\}$ be a nontrivial extended function and x belong to its domain. Then, for any transition $\vartheta \in \Theta(X)$,

$$\overset{\circ}{D}_\uparrow V(x)(\vartheta) \ := \ \sup_{\varepsilon > 0} \inf_{h \in]0,\varepsilon],\ y \in B(\vartheta(h,x), \varepsilon h)} \frac{V(y) - V(x)}{h}$$

is called the **contingent epimutation** of V at x in the direction $\vartheta \in \Theta(X)$.

The function V is said to be **contingently epimutable** at x if its contingent epimutation never takes the value $-\infty$.

If V is Lipschitz around x, this is a "Dini directional mutation" in the sense that

$$\overset{\circ}{D}_\uparrow V(x)(\vartheta) \ := \ \liminf_{h \to 0+} \frac{V(\vartheta(h, x)) - V(x)}{h}$$

It is an extension of the concept of directional mutation: If V is mutable at x, then

$$\forall\, \vartheta \in \Theta(X), \ \overset{\circ}{D}_\uparrow V(x)(\vartheta) \ = \overset{\circ}{V}(x)(\vartheta)$$

We shall prove that

$$\lambda \geq \overset{\circ}{D}_{\uparrow} V(x)(\vartheta) \text{ if and only if } (\vartheta,\lambda) \in T_{\mathcal{E}p(V)}(x,V(x))$$

Proposition 6.1 *Let $V : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ be a nontrivial extended function and x belong to its domain. For all $w \geq V(x)$,*

$$T_{\mathcal{E}p(V)}(x,w) \subset T_{\mathrm{Dom}(V)}(x) \times \mathbf{R}$$

For $w = V(x)$, we have

$$T_{\mathcal{E}p(V)}(x,V(x)) = \mathcal{E}p(\overset{\circ}{D}_{\uparrow} V(x))$$

and for all $w > V(x)$,

$$\mathrm{Dom}(\overset{\circ}{D}_{\uparrow} V(x)) \times \mathbf{R} \subset T_{\mathcal{E}p(V)}(x,w)$$

Proof

1. — Fix $w \geq V(x)$. Let us assume that (ϑ,λ) belongs to $T_{\mathcal{E}p(V)}(x,w)$. We infer that there exist sequences x_n, $w_n \geq V(x_n)$ and $h_n > 0$ converging to x, w and 0 such that

$$\begin{cases} i) & d(\vartheta(h_n,x),x_n) \leq \alpha_n h_n \\ ii) & |w + h_n\lambda - w_n| \leq \beta_n h_n \end{cases} \tag{6.3}$$

Since $V(x_n)$ are finite, we thus deduce that ϑ belongs to the contingent transition set to the domain of V at x, and thus, that $T_{\mathcal{E}p(V)}(x,w) \subset T_{\mathrm{Dom}(V)}(x) \times \mathbf{R}$.

When $w = V(x)$, this implies that

$$\lambda \geq \sup_{\varepsilon>0} \inf_{h\in]0,\varepsilon],\, y\in B(\vartheta(h,x),\varepsilon h)} \frac{V(y)-V(x)}{h}$$

i.e., that

$$(\vartheta,\lambda) \in \mathcal{E}p(\overset{\circ}{V}(x))$$

2. — Let $\vartheta \in \Theta(X)$ belong to the domain of the contingent epimutation of V at x. Then $\lambda_0 = \overset{\circ}{D}_{\uparrow} V(x)(\vartheta)$ is finite, so that there exist sequences

of elements $h_n > 0$, x_n and $w_n \geq V(x_n)$ converging to 0, x and $V(x)$ respectively such that

$$\begin{cases} i) & \boldsymbol{d}(\vartheta(h_n, x), x_n) \leq \alpha_n h_n \\ ii) & |V(x) + h_n\lambda_0 - w_n| \leq \beta_n h_n \end{cases} \tag{6.4}$$

If $w > V(x)$ and if λ is any real number, we see that (ϑ, λ) belongs to $T_{\mathcal{E}p(V)}(x, w)$ because we can write

$$(x_n, w + h_n\lambda) = (x_n, w_n) + (0, w - w_n + h_n\lambda)$$

Since $w - w_n + h_n\lambda$ is strictly positive when h_n is small enough, we infer that $(x_n, w + h_n\lambda)$ belongs to the epigraph of V, i.e., that (ϑ, λ) belongs to the transition set $T_{\mathcal{E}p(V)}(x, w)$. □

6.2 The Characterization Theorem

Let $(X, \Theta(X))$ be a mutational space and $f : X \mapsto \Theta(X)$ describe the dynamics of a mutational equation $\overset{\circ}{x} \ni f(x)$ We consider a time-dependent function $w(\cdot)$ defined as a solution to the mutational equation

$$w'(t) = -\varphi(w(t)) \tag{6.5}$$

where $\varphi : \mathbf{R}_+ \to \mathbf{R}$ is a given continuous function with linear growth.

Our problem is to characterize functions enjoying the Lyapunov property.

Definition 6.2 (Lyapunov Functions) *Let $(X, \Theta(X))$ be a mutational space and $f : X \mapsto \Theta(X)$. We shall say that a nonnegative contingently epimutable extended function V is a* Lyapunov function *of f associated with a function $\varphi(\cdot) : \mathbf{R}_+ \mapsto \mathbf{R}$ if and only if V*

$$\forall\, x \in \mathrm{Dom}(V), \ \overset{\circ}{D}_\uparrow V(x)(f(x)) + \varphi(V(x)) \leq 0 \tag{6.6}$$

Theorem 6.3 *Let $(X, \Theta(X))$ be a mutational space. Assume that the closed bounded balls of X are compact. Let V be a nonnegative contingently epimutable lower semicontinuous extended function and $f : X \rightsquigarrow \Theta(x)$ be a continuous and bounded map. Then V is a Lyapunov function of f associated with $\varphi(\cdot)$ if and only if for any initial state $x_0 \in$ Dom(V),* there exist solutions *$x(\cdot)$ to the mutational equation $\overset{\circ}{x} \ni f(x)$ and $w(\cdot)$ to (6.5) satisfying differential inequality (6.2).*

Proof — We set $G(x,w) := (f(x), -\varphi(w))$. Obviously, the system (6.1), (6.5) has a solution satisfying (6.2) if and only if the system of mutational equations

$$(\overset{\circ}{x}(t), w'(t)) \ni G(x(t), w(t)) \quad (6.7)$$

has a solution starting at $(x_0, V(x_0))$ viable in $\mathcal{K} := \mathcal{E}p(V)$. We first observe that $\mathcal{K}$ is a viability domain for G if and only if V is a Lyapunov function for f with respect to φ: If $\mathcal{K}$ is a viability domain of G, by taking $z = (x, V(x))$, we infer that

$$(f(x), -\varphi(V(x))) \in T_{\mathcal{K}}(x, V(x)) = \mathcal{E}p(\overset{\circ}{D}_{\uparrow} V(x))$$

hence (6.6).

Conversely, (6.6) implies that the pair

$$(f(x), -\varphi(V(x))) \in T_{\mathcal{E}p(V)}(x, V(x))$$

In particular, $f(x)$ belongs to the domain of $\overset{\circ}{D}_{\uparrow} V(x)$, so that Proposition 6.1 implies that $(f(x), -\varphi(w)) \in T_{\mathcal{K}}(x, w)$ whenever $w > V(x)$.

6.3 Attractors

Using distance functions as Lyapunov functions, we can study attractors:

Definition 6.4 *We shall say that a closed subset K is an* attractor *of order $\alpha \geq 0$ if and only if for any $x_0 \in \mathrm{Dom}(f)$, there exists at least one solution $x(\cdot)$ to mutational equation (6.1) such that*

$$\forall\, t \geq 0, \;\; d_K(x(t)) \;\leq\; d_K(x_0)e^{-\alpha t}$$

We can recognize attractors by checking whether the distance function to K is a Lyapunov function:

We define the directional Dini mutation

$$D_{\uparrow}\boldsymbol{d}_K(x)(\vartheta) \;:=\; \liminf_{h \to 0+} \frac{\boldsymbol{d}_K(\vartheta(h,x)) - \boldsymbol{d}_K(x)}{h}$$

(We observe that when $x \in K$, a transition ϑ is contingent to K at x if and only if $D_{\uparrow}\boldsymbol{d}_K(x)(\vartheta) \leq 0$.)

Corollary 6.5 *Let X be a metric space whose closed balls are compact and $f : X \rightsquigarrow \Theta(x)$ be a continuous and bounded map.*

Then a closed subset $K \subset \mathrm{Dom}(f)$ is an attractor if and only if the function $d_K(\cdot)$ is a solution to the contingent inequalities:

$$\forall\, x \in X, \;\; \overset{\circ}{D}_{\uparrow}\, d_K(x)(f(x)) + \alpha d_K(x) \;\leq\; 0$$

6.4 Dissipative Systems

Let X be a finite dimensional vector space and $f : X \mapsto X$ be a C^1 map. A differential equation $x' = f(x)$ is said to be **dissipative** if the measure $V(\vartheta_f(t,K))$ of $\vartheta_f(t,K))$ decreases along the reachable sets. These reachable sets are solutions to the mutational equation $\overset{\circ}{K}(t) \ni f$ with constant right-hand side. Therefore, a system is dissipative if the shape map V defined by

$$V(K) := \int_K dx$$

is a Lyapunov function for this mutational equation. More generally, shape functions W defined by

$$W(K) := \int_K h(x)dx$$

where h is C^1 are shape differentiable and thus, epimutable:

$$\overset{\circ}{W}(f) = \int_K \mathrm{div}(h(x)f(x))dx$$

Such a function is thus a Lyapunov function of f if and only if

$$\forall\, K \in \mathcal{K}(X),\ \int_K \mathrm{div}(h(x)f(x)) + \int_K h(x)dx \le 0$$

i.e., if and only if $\mathrm{div}(h(x)f(x)) + ah(x) \le 0$ for every $x \in X$. If this is the case, then

$$\forall\, K \in \mathcal{K}(X),\ \forall \ge 0, \int_{\vartheta_f(t,K)} h(x)dx \le e^{-at} \int_K h(x)dx$$

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