ON "BADLY BEHAVED" DYNAMICS

Some Applications of Generalized Urn Schemes to Technological and Economic Change

Giovanni Dosi
University of Rome "La Sapienza", Rome, Italy
and
International Institute for Applied Systems Analysis
Laxenburg, Austria

Yuri Kaniovski
International Institute for Applied Systems Analysis
Laxenburg, Austria

RR-94-12
November 1994

Reprinted from the *Journal of Evolutionary Economics*, Volume 4, Number 2, 1994.

INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
Laxenburg, Austria

Reprinted with permission from the *Journal of Evolutionary Economics*, Volume 4, Number 2, 1994.

Printed by Novographic, Vienna, Austria.

Systems Analysis of Technological and Economic Dynamics

This new research project at IIASA is concerned with modeling technological and organizational change; the broader economic developments that are associated with technological change, both as cause and effect; the processes by which economic agents – first of all, business firms – acquire and develop the capabilities to generate, imitate, and adopt technological and organizational innovations; and the aggregate dynamics – at the levels of single industries and whole economies – engendered by the interactions among agents which are heterogeneous in their innovative abilities, behavioral rules and expectations. The central purpose is to develop stronger theory and better modeling techniques. However, the basic philosophy is that such theoretical and modeling work is most fruitful when attention is paid to the known empirical details of the phenomena the work aims to address: therefore, a considerable effort is put into a better understanding of the 'stylized facts' concerning corporate organization routines and strategy; industrial evolution and the 'demography' of firms; patterns of macroeconomic growth and trade.

From a modeling perspective, over the last decade considerable progress has been made on various techniques of dynamic modeling. Some of this work has employed ordinary differential and difference equations, and some of it stochastic equations. A number of efforts have taken advantage of the growing power of simulation techniques. Others have employed more traditional mathematics. As a result of this theoretical work, the toolkit for modeling technological and economic dynamics is significantly richer than it was a decade ago.

During the same period, there have been major advances in the empirical understanding. There are now many more detailed technological histories available. Much more is known about the similarities and differences of technical advance in different fields and industries and there is some understanding of the key variables that lie behind those differences. A number of studies have provided rich information about how industry structure co-evolves with technology. In addition to empirical work at the technology or sector level, the last decade has also seen a great deal of empirical research on productivity growth and measured technical advance at the level of whole economies. A considerable body of empirical research now exists on the facts that seem associated with different rates of productivity growth across the range of nations, with the dynamics of convergence and divergence in the levels and rates of growth of income in different countries, with the diverse national institutional arrangements in which technological change is embedded.

As a result of this recent empirical work, the questions that successful theory and useful modeling techniques ought to address now are much more clearly defined. The theoretical work described above often has been undertaken in appreciation of certain stylized facts that needed to be explained. The list of these 'facts' is indeed very long, ranging from the microeconomic evidence concerning for example dynamic increasing returns in learning activities or the persistence of particular sets of problem-solving routines within business firms; the industry-level evidence on entry, exit and size-distributions – approximately log-normal; all the way to the evidence regarding the time-series properties of major economic aggregates. However, the connection between the theoretical work and the empirical phenomena has so far not been very close. The philosophy of this project is that the chances of developing powerful new theory and useful new analytical techniques can be greatly enhanced by performing the work in an environment where scholars who understand the empirical phenomena provide questions and challenges for the theorists and their work.

In particular, the project is meant to pursue an 'evolutionary' interpretation of technological and economic dynamics modeling, first, the processes by which individual agents and organizations learn, search, and adapt; second, the economic analogs of 'natural selection' by which interactive environments – often markets – winnow out a population whose members have different attributes and behavioral traits; and, third, the collective emergence of statistical patterns, regularities, and higher-level structures as the aggregate outcomes of the two former processes.

Together with a group of researchers located permanently at IIASA, the project coordinates multiple research efforts undertaken in several institutions around the world, organizes workshops and provides a venue of scientific discussion among scholars working on evolutionary modeling, computer simulation and non-linear dynamical systems.

The research will focus upon the following three major areas:

1. Learning Processes and Organizational Competence.
2. Technological and Industrial Dynamics
3. Innovation, Competition, and Macrodynamics

Preface

Many novel techniques that have proved effective in multidisciplinary applied research were developed at IIASA. For example, in the early 1980s a group of economists from the West and mathematicians from the East made studies on generalized urn schemes and their applications to economics. This paper represents a review of the results that were obtained and outlines some new research topics, especially distributed economics and evolutionary games. The paper also provides an introduction to the application of this machinery in economics.

In January 1994, IIASA launched a new project on Systems Analysis of Technological and Economic Dynamics (TED). This paper represents the outcome of work carried out in one of the three major directions, namely, modeling of macroeconomic issues.

Peter E. de Jánosi
Director

J Evol Econ (1994) 4:93–123

Journal of
Evolutionary Economics

On "badly behaved" dynamics

Some applications of generalized urn schemes to technological and economic change

G. Dosi[1] and Y. Kaniovski[2]

[1] University of Rome "La Sapienza", Rome, Italy
[2] International Institute for Applied Systems Analysis, Laxenburg, Austria

Abstract. Adaptive (path dependent) processes of growth modeled by urn schemes are important for several fields of applications: biology, physics, chemistry, economics. In this paper we present a general introduction to urn schemes, together with some new results. We review the studies that have been done in the technological dynamics by means of such schemes. Also several other domains of economic dynamics are analysed by the same machinery and its new modifications allowing to tackle non-homogeneity of the phase space. We demonstrate the phenomena of multiple equilibria, different convergence rates for different limit patterns, locally positive and locally negative feedbacks, limit behavior associated with non-homogeneity of economic environment where producers (firms) are operating. It is also shown that the above urn processes represent a natural and convenient stochastic replicator dynamics which can be used in evolutionary games.

Key words: Urn scheme – Innovation – Ugly dynamics – Multiple limit states

JEL-classification: D83

1. Introduction

Microeconomic heterogeneity, non-equilibrium interactions and the co-existence of *negative* and *positive* feedbacks appear to be quite general characteristics of economic change. They are particularly evident in the case of technological innovation and diffusion – but by no means limited to them. Technical change typically involves diversity amongst the agents who generate or are effected by it; various forms of learning often based on trial-and-error procedures; and mechanisms of

Correspondence to: G. Dosi, Dipartimento di Science Economichie, Università degli Studi di Roma, Via Nomentana 41, I-00161 Rome, Italy

selection which reward particular types of technologies, agents or behaviors at the expenses of others.

These appear to be, indeed, general features of the competitive process driving economic dynamics. "Competition" entails the interaction among heterogeneous firms embodying different technologies, different expectations and, quite often, displaying different behaviors. Moreover, it is often the case that technological and organizational learning is associated with various types of externalities and increasing returns.

Over the last two decades, at last, such dynamic phenomena have drawn an increasing attention within the economic discipline – especially with reference to technological change. A number of conceptual approaches and mathematical tools have been applied, often benefiting from contemporary developments in the analysis of dynamical systems in natural sciences.

In this work, we shall discuss some of these approaches and, in particular, present the basic structure and the interpretative scope of one "formal machinery", namely *generalized urn schemes*. This paper can be considered as an introduction to generalized urn models, containing both known and new results together with a sketch of some directions for the future research. In section 2, we shall outline some phenomena which are central to technological and economic dynamics, and briefly review alternative formal representations of them. Section 3 introduces the basics of urn schemes. In the following sections we illustrate some applications to relatively simple competitive environments (section 4), and further refinements, contemplating local feedback processes (section 5); phenomena of increasing returns deriving from system compatibility (section 6); non-homogeneous environments (section 7) and "evolutionary games" (section 8). Finally, in the conclusion we shall point out some promising areas of application of this formal apparatus, including the economics of innovation, industrial dynamics, macroeconomics, finance.

2. Processes of economic evolution

In very general terms, the impulses driving economic change stem, first, from variations in the knowledge and physical resources upon which individual agents can draw in order to pursue their activities; second, from the process by which agents learn, adapt, invent – on the grounds of whatever they perceive to be the available knowledge and resources, and, third, from the interactions amongst the agents themselves. Of course, these sources of change are by no means independent: for example, learning activities obviously affect the available knowledge and the efficiency by which resources are used; interactions might trigger learning and entail externalities; learning itself may be associated with particular forms of economic activity, such as learning-by-doning. The variety of sources and mechanisms of economic change highlighted by economic history, most likely, in our view, precludes the identification of some unique or archetypical dynamic form which could apply across industries, phases of development, historical contexts. Still, it might be possible (and indeed is a challenging area of research) to identify few relatively invariant characteristics of the process of change and, with them, also the "formal machineries" most apt to represent them.

Some basic features of economic evolution are the following: (i) imperfect and time-consuming microeconomic learning; (ii) microheterogeneity; (iii) most often, various form of increasing returns – especially in the accumulation of knowledge – and non-linearities; (iv) aggregate dynamics driven by both individual learning and collective selection mechanisms; (v) "orderly" structural properties resulting from non-equilibrium fluctuations.

Correspondingly, let us examine the formal representations which can account for at least some of these features of evolutionary dynamics. As a general reference, let us start from "order-through-fluctuation" dynamics (cf. Nicolis and Prigogine 1971 and 1989; Prigogine and Stengers 1984): it is a quite broad paradigm for the interpretation of complex non-linear processes, initially developed with reference to physical chemistry and molecular biology, but more generally emphasizing the properties of self-reinforcing mechanisms and out-of-equilibrium self-organization. Such systems turn out to be sensitive to (however small) early perturbations and display multiplicity of patterns in their long-term behaviour. The cumulation of small early disturbances (or small disturbances around unstable states) "pushes" the system toward one of these patterns and thus "select" the structure towards which the system will eventually tend. These properties apply to a very wide class of dynamical systems, highlighting, loosely speaking, some general "evolutionary" features well beyond the domain of social sciences and biology.

Further specifications of evolutionary dynamics come from mathematical biology (see Eigen and Schuster 1979). Evolution in many of such models occurs in a way that some integral characteristics (mean fitness for biological systems or mean "competitiveness" in the economic analogy) "improves" along the trajectory. In the simplest case of Fisher's selection model, "improvements" straightforwardly imply that the mean fitness increases along the path. However, even in biology this equivalence does not hold in general (due, for example, to phenomena of hyper-selection, co-evolution, symmetry-breaking: see Allen (1988) and Silverberg (1988) for discussions directly linked to economic applications). Even more so, this *non*-equivalence between "evolution" and "increasing fitness", however defined, is likely to emerge whenever there is no identifiable "fundamental law of nature" or conservation principle. Putting it another way: evolutionary dynamics – in biology as well in economics – involves some kind of selection process grounded on the relevant distributions of agents' characteristics, on the one hand, and on some environmental criterion of "adaptiveness", on the other. (Until recently, most economic models have avoided the issue simply by *assuming* that all the agents were perfectly "adapted", either via some unspecified selection process that occured just before the economist started looking at the world, or via some optimization process that occured in the head of the agents themselves.) Replicator dynamics is a common formal tool to represent such selection-driven adaptation (for applications to economics, see Silverberg (1988) and Silverberg et al. (1988); adaptation processes of various types in "evolutionary games" are discussed by Banerjee and Weibull (1992), Cabrales (1992), Kandori et al. (1993), Samuelson and Zhang (1992) Young (1993)). However, at least the simplest replicator process imposes quite stringent conditions on the way selection occurs. In essence, these restrictions turn out to be negative feedbacks, i.e. diminishing returns, deriving from some underlying "conservation principle".[1] On the contrary, positive feedbacks lead to multiple limit states and generate a much richer variety of trajectories which the system may follow. For example, it is increasingly acknowledged that technological innovations are likely

[1] Conventionally, in economics, profit (or utility) maximization under a constraint of given and scarce resources clearly performs this role.

to involve some forms of dynamic increasing returns – hence, positive feedbacks – along their development and diffusion (cf. Freeman (1982), Dosi et al. (1988), Anderson et al. (1988), David (1988), and for an interpretation of the empirical evidence, Dosi (1988)). Relatedly, there is no guarantee that the particular economic outcome which happens to be historically selected amongst many notional alternatives will be the "best" one, irrespectively of the "fitness" or welfare yardsticks.[2]

Concerning the mathematical tools that have been proposed within and outside economics for the analysis of the competitive process, ordinary differential equations have a paramount importance (not surprisingly, since they are also the most common language of modern science and especially physics). They are applied to most analyses of economic and technological dynamics (for our purposes here, cf. Nelson and Winter (1982), Polterovich and Henkin (1988), Day (1992), and the works surveyed in Boldrin (1988); in general, cf. Brock and Malliaris (1989) and Rosser (1991)). In particular, ordinary differential equations with trajectories on the unit simplex – i.e. of the replicator type – borrow, as already mentioned, an idea of selection-driven evolution from biology (cf. Silverberg et al. 1988).[3] For stochastic (Markov) perturbations of these equations see Nicolis and Prigogine (1971) – for general equations –, and Foster and Young (1993), – for equations of the replicator type. However, while these continuous-time formulations work well, they involve a not so harmless approximation for events that are by nature discrete (the main example being a phase space which is discrete and changes by discrete increments). More intuitively, the continuous-time approximation is bound to take very literally the old saying that *natura non facet saltum.*

Moreover, from a technical point of view, the approximation carries unnecessary hypotheses of mathematical nature (a classical example is the Lipschitz condition on the coefficients of the differential equation describing the system) and specific difficulties (such as the requirement of rigorously defining the stochastic perturbations of replicator equations). In this respect, it might be worth mentioning here some recent results from so-called "evolutionary games" showing convergence to conventional Nash-type equilibria in the continuous approximation but not in the discrete formulation (Banerjee and Weibull 1992; Dekel and Scotchmer 1991). Moreover, formal representations of selection processes in economics often rely on replicator dynamics satisfying the weak monotonicity condition (Friedman 1991; Samuelson and Zhang 1992; Baherjee and Weibull 1992) (loosely speaking, the condition guarantees that, given an environment, there is no reversal in the "forces of selection" along the trajectory). However, even in simple cases the results on limit properties obtained under replicator dynamics might not hold under more general selection processes (see, for example, Cabrales 1992).

To summarize this brief overview of the formalisms applied to economic dynamics and evolution: ideally, one would like some machinery able to capture as

---

[2] In fact, even environments that are stationary in their "fundamentals" (e.g. best practice technologies) selection-driven adaptation yields convergence to equilibria associated with Pareto-optimal properties only under further (and quite demanding) restrictions on the nature of the interactions, the related payoffs and the adaptation dynamics. This is certainly true in presence of "strategic" interactions, but it applies also under (quasi) pure competition: on the latter, see the pioneering investigation in Winter (1971).

[3] Of course, this does not bear any implication for the sources of "mutation" upon which environmental selection operates. For example, Silverberg et al. (1988) assume an exogenous drift in innovative opportunities with learning-by-using and diffusion-related externalities.

adequately as possible (a) increasing-returns phenomena, i.e. positive feedbacks; (b) "ugly" and badly behaved selection dynamics, involving also "jumps" and discontinuities, co-evolutionary effects, etc.; (c) a large variety of individual processes of adaptation and innovation (and, thus, being quite agnostic on the processes driving the perturbations); and (d) the process of accumulation of agents' individual behaviors into the regularities driving the dynamics of the whole population.

In the following, we shall assess to what extent an alternative class of models, namely *generalized urn schemes*, can fulfill these tasks. These schemes, sometimes called *non-linear Pólya processes* or *adaptive processes* of *growth*, generate stochastic discrete-time dynamic systems with trajectories on the set of points with rational coordinates from the unit simplex (cf. Arthur 1988, Arthur et al. 1983 and 1987c; Glaziev and Kaniovski 1991; Dosi et al. 1994; Arthur and Ruszczinski 1992). Formally, they represent non-stationary Markov chains with growing numbers of states. This allows to reach, under corresponding conditions, any state from the unit simplex (which is, by definition, not the case for finite Markov chains). The mathematical background comes from Hill et al. (1980) and Arthur et al. (1983), (1987a) and (1988). It does not rely on notions common for Markov processes such as "master equations": this essentially simplifies the argument and allows to produce deeper results. Moreover, this formal apparatus enables one to handle positive and/or negative feedbacks, possibly coexisting in the same process: see Arthur (1988) and Arthur et al. (1987c). In particular, these feedbacks may have a "local" nature – in the sense that they may occur only under particular states on the trajectories (Dosi et al. (1994)). This approach allows also to treat complementaries and network externalities in the adoption of competing technologies (Arthur et al. 1987b), whereby individual commodities – say, computers or telecommunication equipment – operate within networks requiring compatibility.[4] It must be also emphasized that in this work we generally suggest examples of application of this formalism drawn from the economics of innovation, but similar properties can easily be found in many other economic domains: rather than technologies, one could also consider e.g. organizational forms or strategies in business economics; cognitive models and decision rules in finance; etc. (see the final section). Using the generalized urn schemes one can analyse the emergenece of random market structure with more than one limit state occuring with positive probability (cf. Arthur et al. 1983 and Glaziev and Kaniovski 1991). Moreover, one may determine the different convergence rates to the various limit states attainable with positive probability (Arthur et al. 1988).

Generalized urn schemes are well suited to analyse increasing returns phenomena and, generally, the interaction of individual behaviors of agents who have incomplete information about the environment and its mechanisms of evolution. The two points are most often related: dynamic increasing returns tend to imply unpredictability of the particular limit state that will be attained. Conversely, as we shall see, the process of information acquisition entails dynamic consequences similar to purely "technological" increasing returns. The rules driving the collective

[4] Systems compatibility implies that one ought to consider combinations amongst individual technologies. In turn, this can hardly be done by adding to the "technological space", where choices are made, all possible combinations of technologies existing at any moment in time. At the very least, this procedure would lead to an enormous growth in the dimension of the phase space. For example, if N new technologies come to the market, considering all their possible combinations would imply the "explosion" of the dimension of the phase space up to $2^N - 1$.

dynamics are the cumulated effects of individual behaviors. For each agent, the impact of his own action is negligible, but the sequence of all of them shape the evolution of the system. Hence, one looks for the long run properties when the size of the population or, equivalently, time go to infinity. This does not restrict the applicability of the results for finite, but large enough, populations (although some caution is obviously required). This formal machinery is also a simulation tool as convenient and effective as ordinary differential equations (we shall tackle this type of application in a future publication).

In this work we shall analyse some of the patterns of system evolution which can be discovered by means of generalized urn schemes. In order to do this, we shall use some known models of technological dynamics and also introduce some novel modification highlighting the complex limit structures that these models generate.

Let us start with the simplest definition of a generalized urn scheme.

3. The basic elements of the theory of generalized urn schemes

In this section we give the basic version of the generalized urn scheme and outline the main patterns of the asymptotic behavior which it can demonstrate: multiplicity of the limit states, attainability and unattainability of them, and different convergence rates to the attainable ones.

To simplify the presentation, let us restrict ourselves to the case of two competing technologies which corresponds to urn schemes with balls of two colors (Hill et al. 1980 and Arthur et al. 1983). As illustrations, think for example of two technologies whose efficiency improves together with its diffusion, due e.g. to increasing returns in its production or to "network externalities" for the adopters.

Consider an urn of infinite capacity with black and white balls. Starting with $n_w \geqslant 1$ white balls and $n_b \geqslant 1$ black balls into the urn, a new ball is added into the urn at time instants $t = 1, 2, \ldots$. It will be white with probability $f_t(X_t)$ and black with probability $1 - f_t(X_t)$. By X_t we designate the proportion of white balls into the urn at time t. The general intuition is that, given the function $f_t(\cdot)$, one can build models of the stochastic evolution of X_t. The balls might be producers and white and black balls denote two technologies. The model is then one of adoption of competing innovations. Other interpretations might involve individuals selecting among products or even among "opinions". The path of X_t can take on a great variety of qualitative properties, depending on the specification of the function $f_t(\cdot)$: some of them will be explored in the following. Moreover, by allowing the addition of more than one ball, more than two colors, more than one urn, further urn models can be created. Here $f_t(\cdot)$ is a function,[5] which maps $R(0, 1)$ in $[0, 1]$ ($R(0, 1)$ stands for the set of rational numbers from $(0, 1)$). The dynamics of X_t is given by the relation

$$X_{t+1} = X_t + (t + n_w + n_b)^{-1}[\xi_t(X_t) - X_t], \quad t \geqslant 1, \quad X_1 = n_w(n_w + n_b)^{-1}.$$

Here $\xi_t(x)$, $t \geqslant 1$, are random variables independent in t, such that

$$\xi_t(x) = \begin{cases} 1 \text{ with probability } f_t(x), \\ 0 \text{ with probability } 1 - f_t(x). \end{cases}$$

[5] When it does not depend on t, it is called (Hill et al. (1980)) *urn function*.

Designate $\xi_t(x) - \mathbf{E}\xi_t(x) = \xi_t(x) - f_t(x)$ by $\zeta_t(x)$, where $\mathbf{E}$ stands for the mathematical expectation. Then we have

$$X_{t+1} = X_t + (t + n_w + n_b)^{-1}\{[f_t(X_t) - X_t] + \zeta_t(X_t)\}, \quad t \geq 1,$$
$$X_1 = n_w(n_w + n_b)^{-1}. \tag{1}$$

Due to $\mathbf{E}\zeta(x) = 0$, the system (1) shifts on average at time $t \geq 1$ from a point x on the value $(t + n_w + n_b)^{-1}[f_t(x) - x]$. Consequently, limit points of the sequence $\{X_t\}$ have to belong to the "set of zeros" of the function $f_t(x) - x$ (for $x \in [0, 1]$). It will really be the set of zeros if $f_t(\cdot)$ does not depend on t, i.e. $f_t(\cdot) = f(\cdot)$, $t \geq 1$, for $f(\cdot)$ being a continuous function.

In the general case one needs a specific mathematical machinery to describe this "set of zeros" (see Hill et al. (1980) for the case when the probabilities are discontinuous and do not depend on t; and Arthur et al. (1987b) for the case when the probabilities are discontinuous functions and depend on t).

To summarize the properties of the above urn scheme that are important for our purposes recall the following:

1. Representing a non-stationary Markov chain with growing number of states, the process X_t develops on the one-dimensional unit simplex $[0, 1]$ taking (discrete) values from the set $R(0, 1)$: at time $t + 1$, it can take the values $i(t + n_w + n_b)^{-1}$, where $n_w \leq i \leq n_w + t$;
2. Since in general we do not require any regularity of $f_t(\cdot)$, $t \geq 1$, the process can display a very complicated behavior; for example, its trajectories can produce "persistent fluctuations",[6] or even can "sweep off" an interval with probability 1 (see Arthur et al. 1987b);
3. If for a sequence $\{f_t(\cdot)\}$ there is a function $f(\cdot)$ such that $f_t(\cdot) = f(\cdot) + \delta_t(\cdot)$ and $\sup_{x \in R(0,1)} |\delta_t(x)| \to 0$ sufficiently fast as $t \to \infty$, then for an isolated root θ of $f(x) - x$, one can have convergence of X_t to θ with positive or zero probability (we call such points *attainable*[7] or *unattainable*, correspondingly) depending upon

$$(f(x) - x)(x - \theta) \leq 0 \tag{2}$$

or

$$(f(x) - x)(x - \theta) \geq 0 \tag{3}$$

in a neighborhood of θ (see Hill et al. 1980) and Dosi et al. (1994)); similar results

---

[6] By "persistent fluctuations" we mean the following. Assume that $f_t(\cdot)$ does not depend on t. Also let the set of zeros of $f(x) - x$ on $[0, 1]$ contain an interval (α, β) and X_t converge with probability 1 to a limit X_0 as $t \to \infty$. Then $\mathscr{P}\{X_0 \in (\alpha, \beta)\} > 0$ (see Hill et al. (1980)). For a fixed elementary outcome ω, X_t would converge to a certain limit. But we cannot observe the whole path for a fixed ω. At each time instant $t \geq 1$ we pick up a new elementary outcome and, consequently, the trajectory is unlikely to have a limit. This phenomenon of chaotic behavior of an (observed) trajectory we interpret as a "persistent fluctuation". More complicated almost "bubble-type" fluctuations appear if there is no convergence of X_t, $t \geq 1$, with probability 1, as in the above mentioned case when a trajectory "sweeps off" an interval.

[7] Note that in the case of a deterministic model described by an ordinary differential equation, in order to speak about attainability of certain limit state, one would have to operate with such notions as "domain of attraction" of this state, whose practical implementation is not often clear (especially for systems of nonlinear ordinary differential equations).

are known also for touchpoints, i.e. solutions of the equation $f(x) - x = 0$, where this function does not change its sign (see Pemantle 1991);[8]

4. Under the above representation for $\{f_t(\cdot)\}$, the convergence rate to those θ, which belong to the support of the limit variable (i.e. are attainable), depends upon the smoothness of $f(\cdot)$ at θ. In particular, if the smoothness decreases from differentiability, i.e.

$$f(x) = f'(\theta)(x - \theta) + o(|x - \theta|) \quad \text{as} \quad x \to \theta,$$

to the Hölder differentiability of the order $\gamma > 1/2$, i.e.

$$f(x) = f'_H(\theta)\operatorname{sgn}(x - \theta)|x - \theta|^\gamma + o(|x - \theta|^\gamma) \quad \text{as} \quad x \to \theta,$$

then the order of convergence of X_t to θ increases from $t^{-1/2}$ to $t^{-1/(1+\gamma)}$ (see Kaniovski and Pflug 1992).

The properties 1–3 listed above demonstrate the variety of possible long run behaviors of X_t, ranging from chaotic patterns to convergence to one of possibly multiple limit states. Therefore, it can describe an evolutionary process with many feasible outcomes. Developing in time, the process "selects" one of them. The different convergence rates mean that the rates of evolution are, in general, different for different limit states.

In order to show the analytical power of this formal apparatus, let us begin by considering some examples of technological dynamics in homogeneous economic environments, where competing firms, producing either one of the technologies, are operating.

4. Some examples of competition under global feedbacks in an homogeneous economic environment

We start with the simplest model which displays (global) positive feedback and, as a consequence, multiple patterns of limits behavior (two in this case). In this section we demonstrate in particular how the global forces ruling the dynamics of whole populations can be derived from the individual behavior of economic agents.

Suppose that we have two competing technologies, say, A and B, and a market with imperfectly informed and risk-averse adopters.[9] The two technologies have already been introduced in the market, say $n_A \geqslant 1$ units of A and $n_B \geqslant 1$ units of B. Let us study their diffusion on the market. At time instants $t = 1, 2, \ldots$ one new adopter enters the market. Since he is imperfectly informed and risk-averse, he uses some "boundedly rational" decision rule to make his choice.[10] For example,

---

[8] Depending upon whether $f(\cdot)$ attains or not the values 0 and 1, these properties can hold for X_1 belonging to a certain domain in $R(0, 1)$ or for any X_1 from $R(0, 1)$ (for details see Dosi et al. 1994).

[9] Note that some general system properties – such as the multiplicity of limit states under positive feedbacks – are independent from the exact characterization of microeconomic decision rules, although the latter influence both the processes and the nature of limit structures themselves.

[10] In any case, fascinating issues, which cannot be pursued here, regard the meaning of "rationality" in environments driven by positive feedbacks and showing multiple limit states. For example, even if the agents knew the "true" urn model, what use could they make of this cognitive representation? How could they be more than "boundedly rational"?

in Arthur et al. (1983) and Glaziev and Kaniovski (1991) the following rule was considered:

$R1$. Ask an odd number $r > 1$ of users which technology they adopt. If the majority of them use A, choose A. Otherwise choose B.

According to this rule, technologies are symmetric. Alternatively, suppose that they are not. For example, A comes from a well-known firm with a lot of "goodwill" and B from a new and unknown one. Hence, potential users perceive a different risk in this choice and require different evidence. Assume that this corresponds to the following rule:

$R2$. Fix $\alpha \in (0, 1)$. Ask $q \geqslant 3$ users of the technologies. If more than αq of them use A, choose A. Otherwise choose B.

Here α measures the relative uncertainty of the adopters concerning the two technologies. Clearly if $\alpha = 1/2$ and q is an odd number, then $R2$ converts into $R1$.

An alternative interpretation of the choice process described by $R1$ and $R2$ is in terms of increasing returns to the technologies, rather than risk-aversion of the adopters: the latter know that the greater the number of past adopters, the bigger are also the improvements which a technology has undergone (although the improvements themselves are not directly observable). Hence, in this case, sampling provides an indirect measure of unobservable technological characteristics.

Rule $R1$ generates the probability to choose A as a function of its current proportion on the market. Such probability is given by:

$$f_t(x) = p_{R1}(x) + \delta_t(x), \tag{4}$$

where

$$p_{R1}(x) = \sum_{i=(r+1)/2}^{r} C_r^i x^i (1-x)^{r-i},$$

$$\sup_{x \in R(0,1)} |\delta_t(x)| \leqslant \text{const} \min(x, 1-x) t^{-1},$$

and C_r^i stands for the number of combinations from r to i.

The function $p_{R1}(x) - x$ has three roots 0, 1/2 and 1 on [0, 1]. The root 1/2, satisfying (3), proves to be unattainable, i.e. there is no feasible asymptotic market structure corresponding to it or, speaking in mathematical terms, X_t converges to this root with zero probability as $t \to \infty$ (see Glaziev and Kaniovski (1991)), while the roots 0 and 1, satisfying (2), are indeed attainable, i.e. X_t converges to each of them with positive probability for any ratio between $n_A \geqslant 1$ and $n_B \geqslant 1$: in other words, they both identify a feasible asymptotic market structure. Moreover, the probability for $A(B)$ to dominate in the limit (i.e. that $X_t \to 1 (X_t \to 0)$ as $t \to \infty$) will be greater than 1/2 if the initial number of units $n_A(n_B)$ of the technology is greater than the initial number of units of the alternative technology (for details see Glaziev and Kaniovski 1991).

Consequently, we observe here a mechanism of "selection" which is "history-dependent": the past shapes, in probability, the future, and this effect self-reinforces along the diffusion trajectory.

Quite similarly, rule $R2$ generates

$$f_t(x) = p_{R2}(x) + \delta_t(x), \tag{5}$$

where

$$p_{R2}(x) = \sum_{i=[\alpha q]+1}^{q} C_q^i x^i (1-x)^{q-i},$$

$$\sup_{x \in R(0,1)} |\delta_t(x)| \leqslant \text{const} \min(x, 1-x) t^{-1}.$$

Here we designate by $[a]$ the integer part of a. The function $p_{R2}(x) - x$ has three roots $0, \theta$ and 1 on $[0, 1]$, where θ shifts to the right as α increases.[11] It can be shown, that similarly to the previous case, also this rule generates a mechanism for establishing the dominance of one of the competing technologies (and both have a positive probability to dominate). However, one cannot explicitly trace here the influence of the initial frequencies of the technologies on the probabilities to dominate.

In general, it is not true that a representation similar to (4) can be derived, with a function which does not depend on t ($p_{R1}(\cdot)$ in the case of (4)). To demonstrate this, consider the following example.

*R*3. At time $t \geqslant 1$ ask an odd number $r_t > 1$ of the users of alternative technologies. If the majority of them use A, choose A. Otherwise choose B.

Here each of the new adopters uses his own sample size to make his decision. Requiring that $r_t \leqslant N < \infty$ (i.e. that one can not infinitely increase the size of the sample used for decision making), we see that

$$f_t(x) = p_{R3}^t(x) + \delta_t(x),$$

where

$$p_{R3}^t(x) = \sum_{i=(r_t+1)/2}^{r_t} C_{r_t}^i x^i (1-x)^{r_t - i},$$

$$\sup_{x \in R(0,1)} |\delta_t(x)| \leqslant \text{const} \min(x, 1-x) t^{-1}.$$

Generally speaking, $p_{R3}^t(\cdot)$ does not display any regularity as $t \to \infty$. Consequently, the representation from the previous section does not hold. At the same time, all the functions $p_{R3}^t(x) - x, t \geqslant 1$, have the same roots $0, 1/2$ and 1 on $[0, 1]$. Also, since $3 \leqslant r_t \leqslant N$, derivatives of these functions at $1/2$ are uniformly bounded from zero and from above. These properties imply that 0 and 1 are attainable, while 1/2 turns out to be unattainable.[12] In contrast, assuming that the choice of the sample size is random according to a fixed distribution, i.e. that $r_t, t \geqslant 1$, are random variables and have the same distribution

$$\mathscr{P}\{r_t = 2i + 1\} = p_i > 0, \quad i = 1, 2, \ldots, n, \quad \sum_{i=1}^{n} p_i = 1,$$

---

[11] The above mentioned facts concerning the estimate for $\delta_t(\cdot)$ and the root θ hold true only for large enough q (depending on α). This becomes clear if notice that for $\alpha q < 1$ one gets $[\alpha q] = 0$ and $p_{R2}(x) - x$ has only two roots 0 and 1. The same is true for all other cases when asymmetric decision rules are involved (i.e. rules different from the simple majority/minority ones).

[12] Examples of essentially nonstationary functional sequences, i.e. $f_t(\cdot), t \geqslant 1$, that do not exhibit any regularity as $t \to \infty$, can be studied by means of a theory especially developed to tackle such issues (see Arthur et al. 1987a and 1988).

we have

$$f_t(x) = \tilde{p}_{R3}(x) + \delta_t(x),$$

where

$$\tilde{p}_{R3}(x) = \sum_{i=1}^{n} p_i \sum_{j=(i+1)/2}^{2i+1} C^j_{2i+1} x^j (1-x)^{2i+1-j},$$

$$\sup_{x \in R(0,1)} |\delta_t(x)| \leqslant \text{const} \min(x, 1-x) t^{-1}.$$

The function $\tilde{p}_{R3}(\cdot)$, satisfying the representation from the previous section, also has three roots 0, 1/2 and 1, among which 0 and 1 are attainable and 1/2 is unattainable. Note that here stochasticity simplifies the problem by removing the intrinsic nonstationarity of the process.

The three foregoing examples display (global) positive feedbacks.[13] Examples of (global) negative feedbacks can be similarly derived.

Consider the following rules:

*R*4. Ask an odd number r of users which technology they adopt. If the majority of them use A, choose B. Otherwise choose A.

*R*5. Fix $\alpha \in (0, 1)$. Ask $q \geqslant 3$ users of the technologies. If more than αq of them use A, choose B. Otherwise choose A.

If $\alpha = 1/2$ and q is an odd number, then *R*5 converts into *R*4.

These rules may accommodate behaviors such as the search for diversity in consumption or implicitly capture the outcomes of strategic behaviors on the side of the producers of the technologies aimed at the exploitation of "market power" (cf. Dosi et al. 1994 and Glaziev and Kaniovski 1991). We have relations here similar to (4) and (5) with

$$p_{R4}(x) = \sum_{i=0}^{(r-1)/2} C^i_r x^i (1-x)^{r-1},$$

and

$$p_{R5}(x) = \sum_{i=0}^{[\alpha q]} C^i_q x^i (1-x)^{q-1}.$$

In both cases there is a unique solution of the corresponding equations $p_{R4}(x) - x = 0$ and $p_{R5}(x) - x = 0$. For *R*4 it is 1/2, and for *R*5 the root θ shifts to the right as α increases.[14] The negative feedback determines a limit market structure, whereby both technologies are represented in the market with equal share in *R*4, or they share the market in the proportion $\theta{:}(1-\theta)$ (the limit for the ratio of the number of units of A to the number of units of B) in the case of *R*5.

[13] Actually this statement is not completely correct. Consider for example the rule *R*2. In the feasible domain $R(0, 1)$ (i.e. the set of points which can be attained with positive probability through a finite number of steps from the initial state) there is actually a global positive feedback with respect to θ. In other words, this root is a global repeller in $R(0, 1)$. But adding to this domain 0 and 1, two asymptotically attainable points, we see that there is a local negative feedback in $R(0, \theta)$ with respect to 0 and a local negative feedback in $R(\theta, 1)$ with respect to 1. Or, in other words, 0 is a local attractor in $R(0, \theta)$ and 1 is a local attractor in $R(\theta, 1)$.

[14] For a fixed α one can show that θ converges to α as $q \to \infty$.

For both rules, we know the rates of convergence of X_t to the root, i.e. $\sqrt{t}(X_t - 1/2)$ for $R4$ or $\sqrt{t}(X_t - \theta)$ for $R5$, are asymptotically normal as $t \to \infty$. The means of the limit normal distributions equal zero for both cases and one can also specify the corresponding variances (see Arthur et al. (1983) for the case of $R4$). Consequently, we can characterize the rate of emergence of the limit market structures.[15]

More complicated $f(\cdot)$ functions appear if we introduce additional hypotheses concerning the characteristics and/or dynamics of the pool of adopters. If we assume that adopters who use some decision rule R_i occur with frequency (probability) $\alpha_i > 0$, $i = 1, 2, \ldots, k$, $(\sum_{i=1}^{k} \alpha_i = 1)$, then the function $f_t(\cdot)$, corresponding to the behavior of the whole pool, is a randomization with weights α_i of functions $f_t^i(\cdot)$ generated by the rules R_i, i.e.

$$f_t(x) = \sum_{i=1}^{k} \alpha_i f_t^i(x), \quad x \in R(0, 1), t \geq 1.$$

The simplest example, where adopters who use $R1$ come up with probability $\alpha > 0$, while those who use $R4$ come up with probability $1 - \alpha > 0$, has been considered in Dosi et al. (1994).

More generally, meaningful applications of generalized urn schemes to particular problems of technological and economic dynamics imply an "inductive" specification of the $f(\cdot)$ function, which, loosely speaking, "summarizes" the "intrinsic" or behavioral features of the agents and the nature of their interactions.

Beyond these properties of general positive and negative feedbacks, let us now consider those more complicated situations with *locally* positive and/or *locally* negative feedbacks.

5. Examples of technological dynamics under local feedbacks in homogeneous economic environments

In this section we deal with the situation when there are more than one interior attainable limit state (or root of the corresponding function). Conceptually it might mean for example that there are several patterns of the long run behavior which do not imply monopoly of either technology. Moreover, the second type of models considered in this section, suggests another interpretation of urn schemes. In contrast to the previous examples, where we assume that the uncertainty is due to imperfect information of individual adopters about the choices of the whole population, we shall introduce uncertainty generated by "imperfectness" of the adopters themselves (somewhat analogous to fluctuations of their preferences).

One of the simplest examples of technological dynamics under "local" feedbacks is the following.[16] Think of adopters of competing technologies who are risk-averse enough not to follow the choice of a minuscule minority of the pool of users, but,

---

[15] For this particular rules one can determine an even sharper asymptotic characterization – the law of iterated logarithm (see Arthur et al. 1983).

[16] In this section, by "local" we mean specific to particular states of the process, without however any "spatial" connotation. Feedbacks that the "local" in terms of some "topology" of the environment will be considered in section 7.

for some reasons, are not inclined to conform to the absolute majority of them (say, due to a preference for variety, "inwardedly" generated judgements, eccentricity, etc.). To trivialize, imagine the example of someone who might not want to buy a touch-tone phone instead of a classic rotary one when less than 10% of his friends do so, but may as well desire an old-fashioned rotary phone when more than 90% of his friends have touch-tone ones.

Somewhat similar dynamics are present also outside the domains of technology adoption and consumption patterns: for example, the "bullish" and "bearish" phases on financial markets retain some of these characteristics (although admittedly one should be cautious in applying without appropriate modifications the formal machinery presented here to speculative phenomena, since in the latter the "weight of history" might well be lower than that implied by these urn schemes).

This type of behavior gives rise to the following rule.

*R*6. Fix $\alpha \in (0, 1/2)$. Ask $q > 1/\alpha$ users of the technologies. If the number of those of them who use *A* is greater than αq and smaller than $(1-\alpha)q$, choose *A*. Otherwise choose *B*.

Arguments similar to the ones given in the previous section show that an analog of the relation (4) in this case holds true and

$$p_{R6}(x) = \sum_{i=[\alpha q]+1}^{[(1-\alpha)q]} C_q^i x^i (1-x)^{q-i}.$$

If α is small enough, then $p_{R6}(x) - x$ has three roots 0, θ_1 and θ_2 on [0, 1]. Here $0 < \theta_1 < 1/2 < \theta_2 < 1$.[17] Satisfying (3), the root θ_1 turns out to be unattainable, while 0 and θ_2 are attainable roots. Consequently, in the limit we can have either monopoly of *B*, or the situation when with positive probability the market is shared by *A* and *B* in the proportion $\theta_2:(1-\theta_2) > 1$. For large α close to 1/2 there could be only one root 0, i.e. the corresponding limit market pattern is monopoly of *B*. For an intermediate value of α one can imagine a situation when $p_{R6}(x) - x$ has two roots on [0, 1] – a crosspoint 0 and a touchpoint $\theta \in (0, 1/2)$. Then both are attainable with positive probability (for θ this follows from the results of Pemantle (1991)). Consequently, in the limit we have either monopoly of *B*, or the ratio between *A* and *B* equals to $\theta:(1-\theta) < 1$.

We now turn to a different class of models.

Let us introduce a price dynamics for the two technologies. As in Dosi et al. (1994), assume that two firms (producers of *A* and *B*, respectively) use the following strategy: up to a certain market share, defined by the proportion of the product of the firm among all products which have been sold until the current time (usually greater than 1/2), the firm reduces the price and above that level increases it. Let us consider the simplest (linear) case of this policy which is graphically represented in Fig. 1. Here $Pr_A(x_A)$ designates the dependence of the price of technology *A* as a function of its proportion x_A among adopters who are using either technology. $Pr_B(x_A)$ designates the dependence of the price of the technology *B* as a function of x_A. (Note that the proportions of the technologies *A* and *B* are related by: $x_A + x_B = 1$.) Define x_A^* and x_B^* as the "critical" market shares which switch from falling- to rising-price rules. Hence, the dependence of the price of the *A* (*B*)

[17] For a fixed α one can show that $\theta_1 \to \alpha$ and $\theta_2 \to 1-\alpha$ as $r \to \infty$.

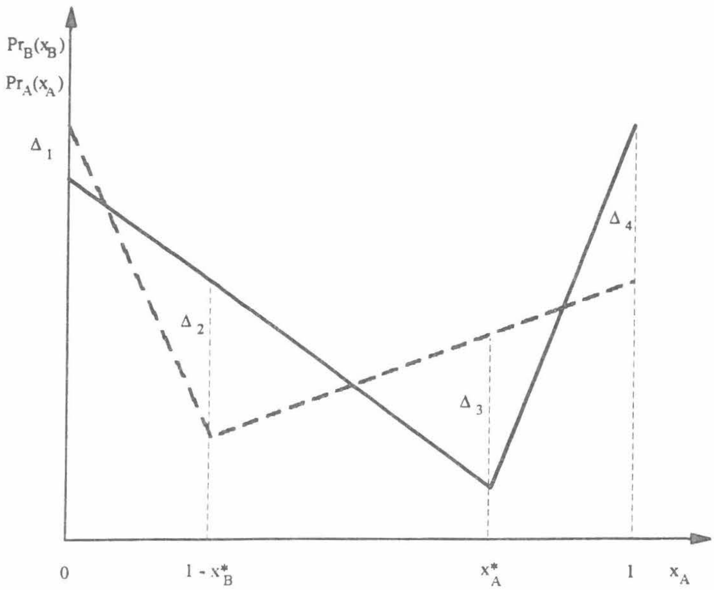

Fig. 1. Dependence of prices of *A* and *B* on the market share of *A*

technology on its proportion on the market $x_A(x_B)$ is given by four parameters: $Pr_A(0)\ x_A^*$; $Pr_A(x_A^*)$; $Pr_A(1)(Pr_B(1)$; x_B^*; $Pr_B(1-x_B^*)$; $Pr_B(0))$.[18]

This price dynamics embodies both positive and negative feedback mechanisms of diffusion. Within the domain of positive feedback the price falls with increasing market shares possibly due to learning economies, dynamic increasing returns, etc., and/or, on the behavioral side, to market-penetration strategies. Then, above a certain market share, the price starts to increase (hence entailing negative feedbacks), possibly due to monopolistic behaviors of the firm or to the progressive exhaustion of technological opportunities to lower production costs. Note that the model accounts also for those particular cases when firms follow different "non-symmetric" policies -- e.g. one increases the price and another lowers it, or both increase (lower) them,[19] or one increases (lowers) price and the other follows the above general strategy. These special cases can be obtained from the general one by simply changing the relations between $Pr_A(0)$, $Pr_A(x_A^*)$, $Pr_A(1)(Pr_B(1)$, $Pr_B(1-x_B^*)$, $Pr_B(0))$.

It is natural to suppose that in the case when the "value" of the technologies for the users is approximately the same and potential adopters know about it, the technology which is cheaper has more chances to be sold, i.e. the *A* technology is bought if $Pr_A(x_A) - Pr_B(x_A) < 0$. However, if the prices only slightly differ or consumers have some specific preferences (which can be characterized only statistically or on average), that may sometimes lead to the adoption of the more expensive technology. Mathematically this case can be formalized in the following way (see

[18] Note that one accounts also for the circumstances when $Pr_A(1) \leqslant Pr_A(x_A^*)$ $(Pr_B(0) \leqslant Pr_B(1-x_B^*))$, such as when $x_A^* = 1 (x_B^* = 1)$: in this case, firm $A(B)$ still reduces the price on its product as its proportion on the market goes to one.

[19] For the case when both lower prices, see Glasiev and Kaniovski (1991) where formally the same situation is interpreted somewhat differently.

also Hanson 1985). The A technology is bought if $Pr_A(x_A) - Pr_B(x_A) + \xi < 0$, where ξ is a random variable. (Consequently, the B technology is bought if $Pr_A(x_A) - Pr_B(x_A) + \xi > 0$.) To preserve the symmetry of the decision rule we should avoid the situation when the event "$Pr_B(x_A) - Pr_A(x_A) = \xi$" has nonzero probability. This is definitely not the case when the distribution of ξ possesses a density with respect to the Lebesgue measure on the set of real numbers. Consequently, we will assume that the distribution ξ has a density in R^1. The probability $f(x_A)$ to choose the A technology, as a function of x_A, equals to $P\{\zeta < Pr_B(x_A) - Pr_A(x_A)\}$. To avoid unnecessary sophistications of the model, we shall assume that ξ has a uniform distribution on $[-\alpha, \alpha]$. The probability to choose A as a function of x_A in this case has the form

$$f(x_A) = \begin{cases} 1 & \text{if} \quad Pr_B(x_A) - Pr_A(x_A) \geqslant \alpha, \\ 0 & \text{if} \quad Pr_B(x_A) - Pr_A(x_A) \leqslant -\alpha, \\ \dfrac{Pr_B(x_A) - Pr_A(x_A) + \alpha}{2\alpha} & \text{if} \quad -\alpha < Pr_B(x_A) - Pr_A(x_A) < \alpha. \end{cases}$$

For $\alpha > \max_{i=1,2,3,4} \Delta_i$ this is graphically represented in Fig 2. Here we have three roots – θ_1, θ_2 and θ_3 – of the function $f(x) - x$ on $[0, 1]$. Satisfying (3), the root θ_2 proves to be unattainable, while θ_1 and θ_3, satisfying (2), are attainable, i.e. the process X_t converges to each of them with positive probability for any initial proportions of the technologies on the market. Using results of Arthur et al. (1988),

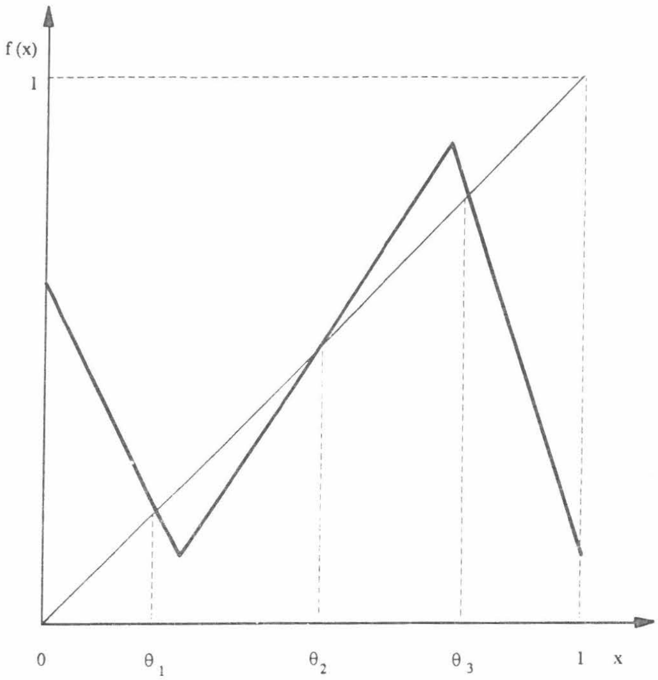

Fig. 2. Probability to choose A depending on its market share

we find the rates of convergence to the attainable roots

$$\theta_1 = \frac{(\alpha + \Delta_1)(1 - x_B^*)}{2\alpha(1 - x_B^*) + \Delta_1 + \Delta_2},$$

$$\theta_3 = 1 - \frac{(\alpha + \Delta_4)(1 - x_A^*)}{2\alpha(1 - x_A^*) + \Delta_3 + \Delta_4}.$$

In particular,

$$\lim_{t \to \infty} \mathscr{P}\{\sqrt{t}(X_t - \theta_i) < y, X_s \to \theta_i\} = \mathscr{P}\{X_s \to \theta_i\}\mathscr{P}\{\mathscr{N}(0, \sigma_i^2) < y\}. \quad (6)$$

Here $\mathscr{N}(0, \sigma_i^2)$ stands for a Gaussian distribution with zero mean and variance

$$\sigma_i^2 = \frac{\theta_i(1 - \theta_i)}{1 - 2f'(\theta_i)}, \quad (7)$$

where $f'(\cdot)$ designates the derivative of $f(\cdot)$. It can be shown that

$$f'(\theta_1) = -\frac{\Delta_1 + \Delta_2}{2\alpha(1 - x_B^*)} \quad (8)$$

and

$$f'(\theta_3) = -\frac{\Delta_3 + \Delta_4}{2\alpha(1 - x_A^*)}. \quad (9)$$

One sees from (6)–(9) that convergence to both θ_1 and θ_3 occurs with the rate $t^{-1/2}$ but the random fluctuations around this, which are determined by the variances of the corresponding limit distributions, can be different.

In this example, the above dynamics of prices together with the described behavior of adopters generate multiple limit patterns with slightly different rates of emergence. Under the same price dynamics and marginally more sophisticated assumptions concerning the behavior of adopters, one can have even more complicated limit market structures where the initial proportions of the technologies on the market influence those structures (see Dosi et al. 1994). Similar considerations concerning convergence rates also apply (with corresponding modifications).

The analytical procedure is to introduce further specifications on the statistical frequences (probabilities) of the producers of $A(B)$ to follow a particular shape of the above price dynamics and/or hypotheses concerning statistical frequences of the adopters who use variants of the above decision rules: thus, one can construct much more complicated functions $f_t(\cdot)$.

Next, let us discuss one important generalization of the urn scheme presented so far.

6. Urn schemes with multiple additions – a tool for analysis of system compatibilities

As mentioned in section 2, quite a few modern high-technology products require compatibility. Think for example of compatibility requirements of software packages and hardware in computers. Moreover, it is reasonable to expect that competing technologies might arrive in different "lumpy" quantities and in different combinations with each other. In this section we present the modification of the basic scheme apt to handle such phenomena. We have hinted earlier that considering all notional combinations of new technologies as a sort of "higher level" new technologies, although formally possible, does not look too attractive. An alternative method for handling inter-technological compatibilities has been introduced by Arthur et al. (1987a). For the case of two (A and B) competing technologies it looks like the following.

Consider Z^2_+, the set of two dimensional vectors with non-negative integer coordinates. Introduce $\vec{\xi}^t(x)$, $t \geqslant 1$, $x \in R(0,1)$, random vectors with values in Z^2_+ independent in t. If $\vec{\xi}^t(x) = (\xi^t_i(x), \xi^t_2(x))$ takes the value $\vec{i} = (i_1, i_2)$ we can interpret this both as additions of $i_1 \geqslant 0$ white and $i_2 \geqslant 0$ black balls into an urn of infinite capacity with black and white balls or, equivalently, as adoption in a market of infinite capacity of i_1 units of A and i_2 units of B.

Mathematical results similar to those presented in section 3 are obtained (see Arthur et al. 1987a, 1987b and 1988). An important property of this generalization is that $\vec{\xi}^t(x)$ can take the value $\vec{0} = (0,0)$ with nonzero probability. Consequently, no adoption might happen at time t. Taking into account that the scheme allows multiple adoptions, one sees that sequential instances of adoption do not coincide with physical time "periods". Hence, loosely speaking, history may "accelerate" by discrete jumps of variable length.

Designate by X_t the proportion of white balls in the urn at time $t \geqslant 1$. Then the number w_t of white balls and γ_t, the total number of balls in the urn at time t, follow the dynamics

$$w_{t+1} = w_t + \xi^t_1(X_t), \qquad t \geqslant 1, \tag{10}$$

$$\gamma_{t+1} = \gamma_t + \xi^t_1(X_t) + \xi^t_2(X_t), \quad t \geqslant 1. \tag{11}$$

Here $w_1 \geqslant 1$ and $b_1 \geqslant 1$ stand for the initial numbers of white and black balls in the urn. Also $\gamma_1 = w_1 + b_1$. Dividing (10) by (11), one has

$$X_{t+1} = X_t + \gamma_t^{-1} \frac{\xi^t_1(X_t) - X_t[\xi^t_1(X_t) + \xi^t_2(X_t)]}{1 + \gamma_t^{-1}[\xi^t_1(X_t) + \xi^t_2(X_t)]}, \quad t \geqslant 1, \quad X_1 = \frac{w_1}{\gamma_1}. \tag{12}$$

Let $p(\vec{i}, x)$, $\vec{i} \in Z^2_+, x \in R(0,1)$, be the distribution of $\vec{\xi}(x)$, i.e.[20]

$$\mathscr{P}\{\vec{\xi}^t(x) = \vec{i}\} = p(\vec{i}, x).$$

Since

$$\mathbf{E}[\xi^t_1(X_t) + \xi^t_2(X_t)] = \sum_{\vec{i} \in Z^2_+} (i_1 + i_2) p(\vec{i}, x), \tag{13}$$

then, requiring that for all $x \in R(0,1)$

$$p(\vec{0}, x) \leqslant \alpha < 1 \quad \text{and} \quad \sum_{\vec{i} \in Z^2_+} (i_1 + i_2)^2 p(\vec{i}, x) \leqslant c_1, \tag{14}$$

[20] In general this distribution can also depend on t and/or the total current number of balls in the urn γ_t, but, to simplify the formulae, here we restrict ourselves to the simplest situation.

one has with probability 1

$$1 - \alpha \leqslant \liminf_{t \to \infty} \frac{\gamma_t}{t-1} \leqslant \limsup_{t \to \infty} \frac{\gamma_t}{t-1} \leqslant c_2. \tag{15}$$

Here $c_i, i = 1, 2$, stand for some constants. Notice that

$$\mathbf{E}\left\{\gamma_t^{-1} \frac{\xi_1^t(X_t) - X_t[\xi_1^t(X_t) + \xi_2^t(X_t)]}{1 + \gamma_t^{-1}[\xi_1^t(X_t) + \xi_2^t(X_t)]} | X_t = x, \gamma_t = \gamma\right\} =$$

$$= \gamma^{-1} \sum_{\vec{i} \in Z_+^2} \frac{i_1 - x(i_1 + i_2)}{1 + \gamma^{-1}(i_1 + i_2)} p(\vec{i}, x). \tag{16}$$

Relations (12), (15) and (16) allow to show that X_t converges with probabilitiy 1 as $t \to \infty$ to the properly defined (since the function involved may be discontinuous) set of zeros on $[0, 1]$ of the function

$$g(x) = \sum_{\vec{i} \in Z_+^2} [i_1 - x(i_1 + i_2)] p(\vec{i}, x). \tag{17}$$

Furthermore, if the function $G(\cdot)$ in the right hand side of (13) turns out to be continuous, and X_t a.s. converges to X^0, then, from (11) and (13), one has that $(t-1)^{-1}\gamma_t$ converges with probability 1, and the limit γ^0 has the form $\gamma^0 = G(X^0)$.

For this case we can derive the same set of asymptotic statements as for the basic scheme.

As an example of how this modification of the basic scheme can work, let us consider the following generalizations of the decision rules $R2$ and $R5$ given in section 4.

R7. Fix $\alpha \in (1/2, 1)$. Ask $q \geqslant 3$ users of technologies. If more than αq of them use A, choose A. If not more than $(1 - \alpha)q$ of them use A, choose B. Otherwise do not choose any technology. This rule generates the following probability to choose A:

$$p((1, 0), x) = p_{R7}^{(1)}(x) + \delta_\gamma^{(1)}(x),$$

and, analogously, B:

$$p((0, 1), x) = p_{R7}^{(2)}(x) + \delta_\gamma^{(2)}(x).$$

Finally, the probability not to choose anything, i.e. $p((0, 0), x)$, is

$$p((0, 0), x) = p_{R7}^{(3)}(x) + \delta_\gamma^{(3)}(x).$$

Here γ stands for the total current number of balls in the urn,

$$p_{R7}^{(1)}(x) = \sum_{i = [\alpha q] + 1}^{q} C_q^i x^i (1 - x)^{q-i}, \quad p_{R7}^{(2)}(x) = \sum_{i=0}^{[(1-\alpha)q]} C_q^i x^i (1 - x)^{q-i},$$

$$p_{R7}^{(3)}(x) = 1 - p_{R7}^{(1)}(x) - p_{R7}^{(2)}(x) = \sum_{i = [(1-\alpha)q] + 1}^{[\alpha q]} C_q^i x^i (1 - x)^{q-i},$$

$$\sup_{x \in R(0,1)} |\delta_\gamma^{(i)}(x)| \leqslant \text{const} \min(x, 1 - x)\gamma^{-1}, i = 1, 2, 3.$$

Also $g(\cdot)$ in this case is

$$g(x) = p_{R7}^{(1)}(x) - x[p_{R7}^{(1)}(x) + p_{R7}^{(2)}(x)]$$

and the only root on (0, 1) of $g(\cdot)$ is given by the equation

$$x = \frac{p_{R7}^{(1)}(x)}{p_{R7}^{(1)}(x) + p_{R7}^{(2)}(x)}.$$

Since $p_{R7}^{(1)}(x) + p_{R7}^{(2)}(x) < 1$ for $x \in R(0, 1)$, this root is smaller than the corresponding root of $R2$. But still it is unattainable. Two other roots, 0 and 1, are attainable.

Similarly to $R7$ we can introduce $R8$ – a counterpart of $R5$. In this case we have, as for $R5$, that there is only one attainable root. For reasons similar to those mentioned in the previous section, this root is larger than the corresponding one of $R5$ and for a fixed α it approaches α as $q \to \infty$.

As highlighted in all the foregoing examples, applications of urn schemes allow an analytical investigation of questions such as the possibility of "lock-in" into one of alternative technological systems or, conversely, the feasibility of their long-term co-existence. Clearly, in several of the examples, lock-in and history-dependent selection of a particular system does occur. But these models show also that the notional multiplicity of limit states and the evolutionary importance of early historical events depend upon the precise mechanics by which agents acquire information and change their preferences. (In the simple examples here these mechanics are captured by the different decision rules.) Similar considerations are likely to apply to more complex models allowing for learning processes also among suppliers of the technologies themselves. However, as already mentioned, it is not the purpose of this work to discuss the specific characteristics of market interactions and learning processes: rather, one of our major points here is that urn schemes, with the appropriate modifications, can be applied to a wide variety of them.

Further, let us introduce the urn model corresponding to the case when competition occurs in non-homogeneous economic environments, and thus interactions have "local" characteristics in some "spatial" sense.

7. Generalized urn schemes with non-homogeneous environments and their economic applications

In this section we introduce a new modification of the basic scheme allowing for a "distributed" economic system, composed of interacting parts which can be metaphorically understood as "regions". The parameters of the whole system are dynamically formed by all the regions, involving different kinds of non-linear interacting. Interestingly, the dynamics of the system proves to be much more complex than the behaviors of its components. In the following we shall mainly consider the technical aspect of this modification; however it is intuitive that obvious candidates for economic application are growth processes involving "local" learning.

Think of m urns of infinite capacity with black and white balls. Starting with $n_i^w \geqslant 1$ white balls and $n_i^b \geqslant 1$ black balls into the i-th urn, a ball is added in one of the urns at time instants $t = 1, 2, \ldots$.[21] With probability $f_i(\vec{X}(t))$ it will be added into the i-th urn. It will be white with probability $f_i^w(\vec{X}(t))$ and black with probability

[21] In general one does not require that $n_i^w \geqslant 1$ and $n_i^b \geqslant 1$. The only thing one really needs is positiveness of $n_i^w + n_i^b$. Consequently, the process can start from zero number of balls of one of the colors into the urns. The same is true for all urn processes considered in the paper.

$f_i^b(\vec{X}(t))$. Here $\vec{f}(\cdot), \vec{f}^{\,w}(\cdot), \vec{f}^{\,b}(\cdot)$, are vector functions which map $R(\vec{0}, \vec{1})$ in S_m, and $\vec{f}^{\,w}(\cdot) + \vec{f}^{\,b}(\cdot) = \vec{f}(\cdot)$. By $R(\vec{0}, \vec{1})$ we designate the Cartesian product of m copies of $R(0, 1)$ and

$$S_m = \left\{ \vec{x} \in R^m : x_i \geqslant 0, \sum_{i=1}^{m} x_i = 1 \right\}.$$

$\vec{X}(t)$ stands for the vector whose i-th coordinate $X_i(t)$ represents the proportion of white balls in the i-th urn at time t. To introduce the dynamics of $\vec{X}(t)$ consider $\xi^t(\vec{x})$, $t \geqslant 1$, $\vec{x} \in R(\vec{0}, \vec{1})$, random $m \times 2$ matrices independent in t with the elements $\xi^t_{i,j}(\vec{x})$, $i = 1, 2, \ldots, m, j = 1, 2$, such that $\mathscr{P}\{\xi^t_{i,1}(\vec{x}) = 1\} = f_i^w(\vec{x})$ and $\mathscr{P}\{\xi^t_{i,2}(\vec{x}) = 1\} = f_i^b(\vec{x})$. This means that a white (black) ball is added into the i-th urn at time t if $\xi_{i,1}(\vec{X}(t)) = 1 (\xi_{i,2}(\vec{X}(t)) = 1)$. Then the total number γ_i^t of balls in i-th urn at time $t \geqslant 1$ follows the dynamics

$$\gamma_i^{t+1} = \gamma_i^t + \xi^t_{i,1}(\vec{X}(t)) + \xi^t_{i,2}(\vec{X}(t)), \quad t \geqslant 1, \quad \gamma_i^1 = n_i^w + n_i^b. \tag{18}$$

Since

$$\mathbf{E}[\xi^t_{i,1}(\vec{x}) + \xi^t_{i,2}(\vec{x})] = f_i(\vec{x}), \tag{19}$$

then, requiring that

$$f_i(\vec{x}) \geqslant f_i^0 > 0, \tag{20}$$

one has

$$f_i^0 \leqslant \liminf_{t \to \infty} \frac{\gamma_i^t}{t} \leqslant \limsup_{t \to \infty} \frac{\gamma_i^t}{t} \leqslant 1. \tag{21}$$

The number w_i^t of white balls and the number b_i^t of black balls in the urn follow the dynamics

$$w_i^{t+1} = w_i^t + \xi^t_{i,1}(\vec{X}(t)), \quad t \geqslant 1, \quad w_i^1 = n_i^w, \tag{22}$$
$$b_i^{t+1} = b_i^t + \xi^t_{i,2}(\vec{X}(t)), \quad t \geqslant 1, \quad b_i^1 = n_i^b.$$

Dividing (22) by (18) one has the following dynamics for the proportion of white balls in the i-th urn

$$X_i(t+1) = X_i(t) + \frac{1}{\gamma_i^t} \frac{\xi^t_{i,1}(\vec{X}(t)) - X_i(t)[\xi^t_{i,1}(\vec{X}(t)) + \xi^t_{i,2}(\vec{X}(t))]}{1 + (\gamma_i^t)^{-1}[\xi^t_{i,1}(\vec{X}(t)) + \xi^t_{i,2}(\vec{X}(t))]},$$
$$t \geqslant 1, \quad X_i(1) = \frac{n_i^w}{\gamma_i^1}. \tag{23}$$

Since

$$\mathbf{E}\left\{ \frac{1}{\gamma_i^t} \frac{\xi^t_{i,1}(\vec{X}(t)) - X_i(t)[\xi^t_{i,1}(\vec{X}(t)) + \xi^t_{i,2}(\vec{X}(t))]}{1 + (\gamma_i^t)^{-1}[\xi^t_{i,1}(\vec{X}(t)) + \xi^t_{i,2}(\vec{X}(t))]} \,\Big|\, \vec{X}(t) = \vec{x}, \vec{\gamma}^t = \vec{\gamma} \right\}$$
$$= \frac{1}{\gamma_i} \frac{f_i^w(\vec{x}) - x_i f_i(\vec{x})}{1 + (\gamma_i)^{-1} f_i(\vec{x})},$$

relations (21) and (23) allow to show that $\vec{X}(t)$ converges wth probability 1 as $t \to \infty$ to the set of zeros (properly defined) on $[\vec{0}, \vec{1}]$ of the m-dimensional vector-function $\vec{F}(\cdot)$ whose i-th coordinate is $f_i^w(\vec{x}) - x_i f_i(\vec{x})$. Assume that both $\vec{f}^{\,w}(\cdot)$ and $\vec{f}^{\,b}(\cdot)$ are continuous and there is a limit $\vec{X}^0$ *for* $\vec{X}(t)$. Then from equality (19) one can conclude that $t^{-1}\vec{\gamma}^t$ converges with probability 1 as $t \to \infty$ and the limit $\vec{\gamma}^0$ has the form

$$\gamma_i^0 = f_i(\vec{X}^0), \quad i = 1, 2, \ldots, m. \tag{24}$$

Using the above relations we can obtain analogs of the results listed in section 3 for the basic generalized urn scheme.

In principle, multiple urns models are capable of capturing positive (and, possibly, negative) feedback processes which are "local" on some appropriately defined space (it could be "regions" or "countries", but also groups of agents with particular features).

Consider a particular model of technological dynamics in a non-homogeneous economic environment which can be treated by means of the modification given here.

Suppose that we have two possible locations (which can be thought as urns), 1 and 2, for the producers of two competing technologies – A and B. At each of the locations (they could be understood as "economic regions" or "countries") there are one firm producing A and one firm producing B. Producers use the strategy described in section 5 (with their own sets of parameters). Suppose for example that there are (bounded) increasing returns and market-share dependent pricing strategies. Then for each of the locations there exists a minimal price of the technologies as a function of the current concentration of, say, A, i.e. $M(x_A) = \min(Pr_A(x_A), Pr_B(x_A))$. For the case represented by Fig. 1, the function is given in Fig. 3. Note that at points λ_i technologies reverse their order as the cheaper ones. Designate the proportion of A for the first and the second locations by x_1 and x_2 correspondingly. Also let λ_j^i, $j = 1, 2, 3, i = 1, 2$, be the points where the minimal prices switch from one technology to another. (Consequently, we consider the case when the minimal

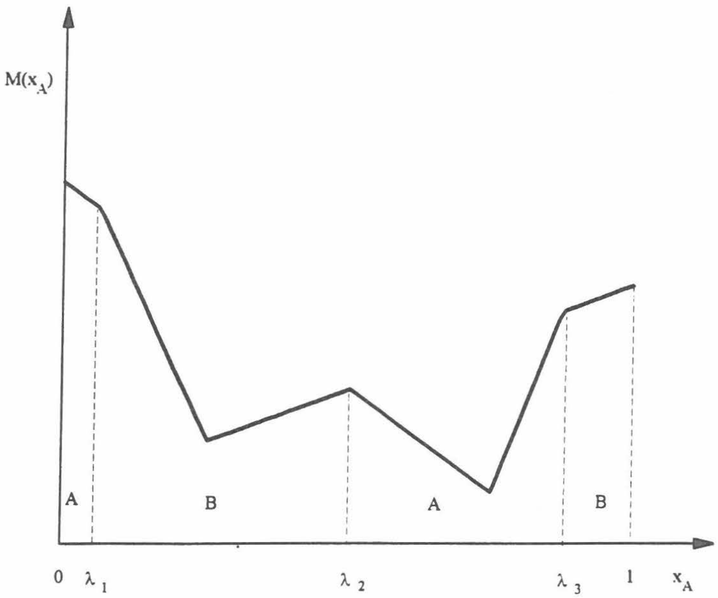

Fig. 3. The price of the cheapest among A and B technologies as a function of x_A

prices for both locations have a shape similar to that presented in figure 3.) Suppose that at time instants $t = 1, 2, \ldots$, a consumer buys a unit of either technology. (A "consumer", irrespectively of where he is located, can demand a technology produced in either region.) He adopts the cheapest among the technologies, but, as before (section 5), because of some specific preferences or other reasons which can be taken into account statistically, he measures the difference between $M_1(x_1)$ and $M_2(x_2)$ with a random error. Here $M_i(\cdot)$ stands for the minimal price for the i-th location as a function of the market-share of A at this location. A unit of the technologies from the first location is bought if $M_1(x_1) - M_2(x_2) + \zeta < 0$; otherwise, i.e. when $M_1(x_1) - M_2(x_2) + \zeta > 0$, a unit from the second location is bought. As before (section 5) to preserve the symmetry of the decision rule we should avoid the situation when the event "$M_2(x_2) - M_1(x_1) = \zeta$" has nonzero probability. Consequently, we should again assume that the distribution of ζ possesses a density with respect to the Lebesgue measure on the set of real numbers. The probability to choose the first location is $f_1(x_1, x_2) = \mathscr{P}\{\zeta < M_2(x_2) - M_1(x_1)\}$. To simplify our considerations let us suppose that ζ has a uniform distribution on $[-\beta, \beta]$. Then the probability to choose the first location is

$$f_1(\vec{x}) = \begin{cases} 1 & \text{if} \quad M_2(x_2) - M_1(x_1) \geqslant \beta, \\ 0 & \text{if} \quad M_2(x_2) - M_1(x_1) \leqslant -\beta, \\ [M_2(x_2) - M_1(x_1) + \beta]/2\beta & \text{if} \quad -\beta < M_2(x_2) - M_1(x_1) < \beta. \end{cases}$$

Suppose that $\beta > \max_{0 \leqslant x_i \leqslant 1, i=1,2} |M_2(x_2) - M_1(x_1)|$. Then (20) holds with

$$f_1^0 = \left\{ \min_{0 \leqslant x_i \leqslant 1, i=1,2} [M_2(x_2) - M_1(x_1)] + \beta \right\} \Big/ 2\beta$$

and

$$f_2^0 = 1/2 - \left\{ \max_{0 \leqslant x_i \leqslant 1, i=1,2} [M_2(x_2) - M_1(x_1)] \right\} \Big/ 2\beta.$$

The simplest decision rule for choosing a specific technology when a location has been chosen is the following: a unit of $A(B)$ is adopted at the i-th location if $x_i \in I_i^A (x_i \in I_i^B)$. Here $I_i^A = (0, \lambda_1^i) \cup (\lambda_2^i, \lambda_3^i)$ and $I_i^B = [\lambda_1^i, \lambda_2^i] \cup [\lambda_3^i, 1)$. The corresponding vector-function $\vec{F}(\cdot)$ has the form

$$F_i(\vec{x}) = \begin{cases} (1 - x_i) f_i(\vec{x}) & \text{for} \quad x_i \in I_i^A, \\ -x_i f_i(\vec{x}) & \text{for} \quad x_i \in I_i^B. \end{cases}$$

We can show that $\vec{X}(t)$ converges (for any initial number of A and B at the both locations) with probability 1 as $t \to \infty$ to a random vector $\vec{X}$. The limit takes with positive probability four values: $(\lambda_1^1, \lambda_1^2)$, $(\lambda_1^1, \lambda_3^2)$, $(\lambda_3^1, \lambda_1^2)$, $(\lambda_3^1, \lambda_3^2)$. Of course, one may easily refine these examples by introducing more complicated decision rules (e.g. mixed strategies randomizing the choice among technologies after having chosen the location, etc.), or by specifying the technological and behavioral relationships between location-specific prices and market-shares. In any case what is important to notice here is that this class of models is apt to analyse the dynamics of what can be interpreted as the inter-regional (or international) location of "production" of diverse technologies under non-constant returns and heterogeneous behaviors among producers. "Spatial" asymmetries in the limit distributions are the general outcome, while – as mentioned earlier – more precise long-term properties can be analysed by adding the appropriate "inductive" specifications on technological dynamics and behavioral responses.

Further along these lines, consider the application of this formal apparatus to a sort of "reduced form" model of international growth driven by investment in two alternative technologies, A and B. The "world economy" consists of m interacting parts (for example, economic regions) of infinite capacity. Technologies are capital-embodied so that each instance of adoption is associated with an investment decision. The time sequence $t = 1, 2, \ldots,$ is defined by the sequence of investments (i.e. adoption decisions). This will not interfere with our considerations since we are interested in the proportions of the total capital stock embodying each of the technologies. Further, let us make the following assumptions:

1. Assume (in a quite non-Schumpeterian fashion) that there is no "creative destruction", and units of capital pile up on each other without depreciation.
2. Each technology yields a different productivity, which changes along the diffusion trajectory. The productivity of each technology in each region, and their shares by region, ultimately determines per capita incomes (along the process and in the limit). However, private agents do not select technologies on the grounds of productivities but in terms of their relative profitabilities.

For the i-th region we consider the following indicators: $x_i(1 - x_i)$ – the fraction of the capitals embodied by $A(B)$; $P_i^A(x_i)(P_i^B(x_i)$ – the cost of a unit of investments embodying $A(B)$ and dependent on x_i; $R_i^A(x_i)(R_i^B(x_i))$ – the overhead costs per unit of investment (which may include the cost of capital services) as a function of x_i; $W_i^A(x_i)(W_i^B(x_i))$ – the unit labour costs for a given x_i. (Note that the dynamics of $W_i^A(\cdot)(W_i^B(\cdot))$ may well depend on technology-specific changes in labour productivity associated with e.g. dynamic increasing returns, and region-specific changes in wage rates denominated in some constant unit of measure. However, in the most general formulation of the model we do not need to specify the exact form of these relationships.) For convenience, assume that each "investment" adds one unit of output. Thus, the profit $\pi_i^A(x_i)(\pi_i^B(x_i))$ from using $A(B)$ is

$$\pi_i^A(x_i) = P_i^A(x_i) - R_i^A(x_i) - W_i^A(x_i),$$
$$\pi_i^B(x_i) = P_i^B(x_i) - R_i^B(x_i) - W_i^B(x_i).$$

Also the total profit for the i-th region is

$$\pi_i(x_i) = \pi_i^A(x_i) + \pi_i^B(x_i).$$

Now we can consider the following mechanism of investment/adaptation. Suppose that $\pi_i^A(x_i) > 0$ and $\pi_i^B(x_i) > 0$ for $x_i \in [0, 1]$, $i = 1, 2, \ldots, m$ (i.e. all the technologies have non-negative profitabilities). Assuming that the current fractions of A in the regions are given by the corresponding coordinates of an m-dimensional vector $\vec{x} \in [\vec{0}, \vec{1}]$, a unit of $A(B)$ is adapted in the i-th region with probability

$$\frac{\pi_i^A(x_i)}{\sum_{k=1}^m \pi_k(x_k)} \left(\frac{\pi_i^B(x_i)}{\sum_{k=1}^m \pi_k(x_k)} \right). \tag{25}$$

The economic interpretation is a sort of "Ricardian" mechanism – whereby investment depends, in probability, on the net surplus generated by each technology –

jointly with less than perfect mobility of investment across technologies (say, due to technology-specific learning-by-doing and -by-using).[22]

By thinking of a unit of A as a white ball and a unit of B as a black ball, we obtain the above scheme with

$$f_i^w(\cdot) = \frac{\pi_i^A(\cdot)}{\sum_{k=1}^m \pi_k(\cdot)} \quad \text{and} \quad f_i^b(\cdot) = \frac{\pi_i^B(\cdot)}{\sum_{k=1}^m \pi_k(\cdot)}.$$

Further conceptual results can then be derived, for example, by imposing particular restrictions on the dynamics of labour productivity, wages, overheads and mark-ups, therefore determining the shape of the functions $\pi_i^A(\cdot)$ and $\pi_i^B(\cdot)$. Other refinements might involve the introduction of a stochastic element in the decisions governing the allocation of investment to regions and technologies. Assuming (small enough) random errors ε_i^A and ε_i^B, $i = 1,2,\ldots,m$, instead of (25) we shall have

$$\frac{\pi_i^A(x_i) + \varepsilon_i^A}{\sum_{k=1}^m [\pi_k(x_k) + \varepsilon_k^A + \varepsilon_k^B]} \left(\frac{\pi_i^B(x_i) + \varepsilon_i^B}{\sum_{k=1}^m [\pi_k(x_k) + \varepsilon_k^A + \varepsilon_k^B]} \right).$$

In any case, whenever our assumption holds on the different (and endogenous) productivities associated with different technologies, our multiple urn scheme allows the analysis of the process of long-term differentiation in per capita incomes driven by "local" learning and other forms of "virtuous" and "vicious" circles – as Nicholas Kaldor would put it.[23] The limit shares of white and black balls in each urn, under a rather innocent hypothesis of monotonicity of productivities and incomes, determine also the limit distribution of the latter among countries or regions[24] (One may visualize this distribution as the shares into growing pie representing "world income" normalized with world population.)

8. Generalized urn process and evolutionary games

Generalized urn schemes generate stochastic replicator dynamics. In particular, all of the above urn process represent a kind of discrete time stochastic replicator equations and, consequently, can be used in the general setting of "evolutionary games" instead of deterministic ones (see Friedman (1991) for the corresponding construction with deterministic dynamics). Let us give a simple sketch of how this can be done, reserving to further works a more detailed analysis of this problem.

Following Friedman (1991), consider a set of interacting populations, indexed $k = 1,2,\ldots,m$. If $m = 2$ they could be thought, for example, as "sellers" and "buyers". A member of each population has a finite number of available actions (or "behaviors" or "strategies"). Let us restrict ourselves for simplicity to the case of two possible actions, indexed $i = 1,2$. Then, any point of the one dimensional

---

[22] This mechanism is somewhat analogous to the diffusion patterns of capital-embodied innovations modeled by Soete and Turner (1984). See also Metcalfe (1988).

[23] The formal apparatus presented here clearly allows generalizations in an explicit dynamic setting on the models of "local" learning put forward by Atkinson and Stiglitz (1969) and David (1975). Note that by "local" in this section we mean *both region- and technology-specific.*

[24] In this there is an intuitive link also with endogenously generated absolute advantages/disadvantages which shape the possibilities of growth in open economies as in Dosi et al. (1990).

simplex $[0, 1]$ represents a possible mixed strategy for an individual member of a population. Any point in the same simplex also represents the fraction of a population employing the first strategy. Hence $[\vec{0}, \vec{1}]$, the Cartesian product of m copies of the simplex $[0, 1]$, is the set of strategies profiles and also the state space under the maintained interpretation that interactions are anonymous.

Interactions are summarized in a fitness function which specifies the relevant evolutionary payoff for the individuals in each population as a function of their own strategy and the current state. Formally a fitness function consists of maps: $[0, 1] \times [\vec{0}, \vec{1}] \to R^1$, $k = 1, 2, \ldots, m$, which are assumed linear in the first (own strategy) argument and continuously differentiable in the second (population state) argument. If $m = 2$, the payoffs of a bi-matrix game give a simplest example of a fitness function.

The final basic element of the model, which radically departs from other models currently available (see Friedman 1991), is a stochastic replicator-type dynamic structure specifying how a state evolves over time. The urn machinery allows a quite general and powerful formalization. We postulate that

$$\vec{X}(t+1) = \vec{Q}(t, \vec{X}(t), \vec{\xi}_t), \quad t \geqslant 1, \quad \vec{X}(1) \in [\vec{0}, \vec{1}]. \tag{26}$$

Here $\vec{X}(t)$ stands for the vector whose i-th coordinate $X_i(t)$ equals to the proportion of players in the i-th population who are using at time t the first strategy (then the proportion of players in the i-population who keep the second strategy is $1 - X_i(t)$). Moreover, $\vec{\xi}_t, t \geqslant 1$, m-dimensional random vectors are independent it t and $\vec{Q}(.,.,.)$ stands for a deterministic function, which:

(a) keeps $[\vec{0}, \vec{1}]$ invariant;

(b) is measurable with respect to the product of two σ-fields of Borel sets on R^m.

To illustrate this concept, let us consider the following example.

Assume a dynamics which satisfies the above requirements and consider two populations, say "buyers" and "sellers". Suppose the interaction concerns the exchange of some object under imperfect and incomplete information such that the two populations can undertake two (pure) strategies: "be honest" or "cheat" for the sellers and "inspect" or "trust" for the buyers (so $m = 2$). Assume that, starting from $b_1 \geqslant 1$ who inspect and $b_2 \geqslant 1$ who don't, at time instants $t = 1, 2, \ldots$, a new buyer joins that population. He can be of the inspecting or non-inspecting kinds and this depends upon the current frequencies of inspecting and non-inspecting buyers and of honest and cheating sellers. This dependency does not act deterministically, but randomly. In particular, there is a function $f_1(\cdot,\cdot){:}[\vec{0}, \vec{1}] \to [0, 1]$ and random variables $\xi_1(t, \cdot, \cdot)$, independent in $t \geqslant 1$, such that

$$\xi_1(t, x, y) = \begin{cases} 1 \text{ with probability } f_1(x, y), \\ 0 \text{ with probability } 1 - f_1(x, y), \end{cases}$$

where $(x, y) \in R(\vec{0}, \vec{1})$. Then $X(t)$, the proportion of buyers who inspects, evolves in the following way

$$X(t+1) = X(t) + \frac{1}{t+b}[\xi_1(t, X(t), Y(t)) - X(t)], \quad t \geqslant 1,$$

$$X(1) = \frac{b_1}{b}, \quad (b = b_1 + b_2). \tag{27}$$

Consequently, at time t a new inspecting (non-inspecting) buyer joins the corresponding population if $\xi_1(t, X(t), Y(t)) = 1(0)$. Here $Y(t)$ stands for the current proportion of honest sellers. Similarly, the dynamics of the latter is

$$Y(t+1) = Y(t) + \frac{1}{t+s}[\xi_2(t, X(t), Y(t)) - Y(t)], \quad t \geqslant 1,$$

$$Y(1) = \frac{s_1}{s}, \quad (s = s_1 + s_2). \tag{28}$$

We set $s_1 \geqslant 1$ for the initial number of honest sellers and $s_2 \geqslant 1$ for the initial number of cheating ones. Also $\xi_2(t, \cdot, \cdot)$ are independent in $t \geqslant 1$ and such that

$$\xi_2(t,x,y) = \begin{cases} 1 & \text{with probability} \quad f_2(x,y), \\ 0 & \text{with probability} \quad 1 - f_2(x,y), \end{cases}$$

for $(x, y) \in R(\vec{0}, \vec{1})$. Therefore, at time t a new honest (cheating) seller joins the corresponding population if $\xi_2(t, X(t), Y(t)) = 1(0)$. The function $f_2(\cdot, \cdot)$ maps $[\vec{0}, \vec{1}]$ on $[0, 1]$. It is assumed that $\{\xi_1(t, \cdot, \cdot)\}$ and $\{\xi_2(t, \cdot, \cdot)\}$ are independent.

If we set $\vec{X}(t) = (X(t), Y(t))$, then (27) and (28) represents a dynamics of the form (26). Indeed, condition (a) here holds automatically and the measurability condition (b) is also met since $\vec{X}(\cdot)$ takes in this case at most a countable number of values.[25]

Taking the conditional expectations in (27) and (28), one gets

$$X(t+1) = X(t) + \frac{1}{t+b}[f_1(X(t), Y(t)) - X(t)]$$

$$+ \frac{1}{t+b}\zeta_1(t, X(t), Y(t)), \quad t \geqslant 1, \quad X(1) = \frac{b_1}{b} \tag{29}$$

and

$$Y(t+1) = Y(t) + \frac{1}{t+s}[f_2(X(t), Y(t)) - Y(t)]$$

$$+ \frac{1}{t+s}\zeta_2(t, X(t), Y(t)), \quad t \geqslant 1, \quad Y(1) = \frac{s_1}{s}. \tag{30}$$

Here $\zeta_i(t, x, y) = \xi_i(t, x, y) - \mathbf{E}\xi_i(t, x, y)$, i.e. $\mathbf{E}\zeta_i(t, x, y) = 0$. Hence, at time t the system shifts on average from a point (x, y) on

$$\left(\frac{1}{t+b}[f_1(x,y) - x], \frac{1}{t+s}[f_2(x,y) - y]\right).$$

This gives us two hints. First, that, under certain assumptions (see, for example, Ljung and Söderström 1983), the system of finite difference equations (29) and (30) asymptotically (as $t \to \infty$) behaves like the following system of ordinary differential

[25] Here $\vec{X}(\cdot)$ is a nonstationary Markov process with growing number of states. In particular, $\vec{X}(t)$ can attain only the following values: $\left(\frac{b_1 + i}{b + t - 1}, \frac{s_1 + j}{s + t - 1}\right)$, $0 \leqslant i \leqslant t - 1, 0 \leqslant j \leqslant t - 1$. If, for the purpose of the analysis, one would prefer populations which do not grow, then a number of conceptually interesting maps $\vec{Q}(\cdot, ., \cdot)$ can be produced by means of finite state Markov chains (see, for example, Kandori et al. 1993 and Samuelson and Zhang 1992). An important feature of any dynamics like (26) developing in a discrete space is that condition (b) holds automatically.

equations

$$\dot{x} = f_1(x, y) - x, \quad \dot{y} = f_2(x, y) - y. \tag{31}$$

Second, possible limits of $\vec{X}(\cdot)$ are given as the solutions of the following system of nonlinear equations

$$f_1(x, y) - x = 0, \quad f_2(x, y) - y = 0, \tag{32}$$

where $(x, y) \in [\vec{0}, \vec{1}]$. Since, in general, we do not assume continuity of the functions $f_i(\cdot, \cdot)$, the solutions should be defined in the appropriate sense (see Arthur et al. 1987a).

So far, one has been totally agnostic in the form of the functions $f_i(\cdot, \cdot)$, $i = 1, 2$,:[26] in our earlier example they depend on how buyers and sellers adjust their behaviors in the course of their interactions and, thus, on the fitness functions of the populations, $q_i(\cdot, \cdot)$, $i = 1, 2$, but several other processes come easily to mind.[27] Note that, at one extreme, one can give a totally "ecological" interpretation of the link between the $f_i(\cdot, \cdot)$ and $q_i(\cdot, \cdot)$ functions: newly arriving agents do not "learn" anything by the observation of frequencies and payoffs, but relative fitness directly affects the probabilities of arrival of the cheating/non-cheating, inspecting/trusting types. In a crude biological analogy, relative fitness affects the rates of reproduction of the various "types". (In the economic domain, an analogy is the expansion/contraction of organizations characterized by fixed behavioral routines.) Alternatively, one may think also of various processes of adaptive learning. Models of this type are examined by Fudenberg and Kreps (1993) and Kaniovski and Young (1994). Hence, the dynamics of the form (27) and (28) will depend, of course, on the shape of the fitness functions[28] and also on the assumptions that one makes about the information agents are able to access – e.g. on the "true" fitness of their own population and the other ones, on the current combination of different types of agents (the "strategy profile") and also on the "cognitive" processes at work in adaptation. In this respect, the notion of "compatibility" (Friedman 1991) can be interpreted as a special restriction of the relationship between frequency dynamics and fitness functions, built on the deterministic analog (31) of the system (27) and (28).

9. Conclusions

Innovation and technology diffusion and more generally economic change involve competition among different technologies, and, most often, endogenous changes

[26] In general, they can also depend on t.

[27] Postponing a detailed analysis of the theoretical applications covered by this formalism to a separate publication, we only mention here the following examples. First, the agents, having no (explicit) fitness function, use a majority rule similar to the ones discussed in section 4. Second, playing a bi-matrix game, they use the corresponding payoffs as a fitness function. Since the proportions of players following a certain strategy from the opposite team are given by statistical estimates, opponents define the best reply strategy with a random error. Third, a combination of the previous two when the pool of players is not homogeneous, i.e. with positive probabilities new-comers can use both of the above decision rules.

[28] Incidentally, notice that for the purposes of this work, our "agnosticism" extends also to the precise form of the $q_i(\cdot, \cdot)$ functions.

in the costs/prices of technologies themselves and in adopters' choices. In the economic domain (as well as in other disciplines) the formal representation of such processes involves some dynamics of competing "populations" (i.e., technologies, firms, or even behavioral traits and "models" of expectation formation). A growing literature on such dynamics has begun studying the properties of those (generally non-linear) processes that innovation and diffusion entails. As by now robustly established, multiple equilibria are normally to be expected and "history matters", also in the sence that out-of-equilibrium fluctuations may bear system-level consequences on notional asymptotic outcomes. Developing on previous results showing – under dynamic increasing returns – the likely "lock-in" of diffusion trajectories onto particular technologies, we have presented a formal modeling apparatus aimed at handling the interaction between diffusion patterns, on the one hand, and technology learning or endogenous preferences formation or endogenous price formation, on the other. As examples, we presented three classes of stochastic models of shares dynamics on a market of infinite capacity by two competing new technologies. In the first of them, we assumed that the adoption dynamics is essentially driven by endogenous changes in the choices of risk-averse, imperfectly informed adopters (or, in a formally equivalent analogy, by some positive or negative externality imperfectly estimated by would-be users of alternative technologies). In the second example, we considered an endogenous price dynamics of two alternative technologies, driven by e.g., changes in their costs of production and/or by the intertemporal behaviors of their producers. In the third example we dealt with the same economic set-up as in the second one, but with an explicit "spatial" representation of the location of producers, and with location-specific selection of capital-embodied technologies (this latter case has interesting implications in terms of macroeconomic "lock-ins" into diverse patterns of growth). Finally, we sketched some possible applications of generalized urn schemes to the dynamics of selection and adaptation by interacting populations (including "evolutionary games").

In all of the cases, the process is allowed to embody some stochasticity, due to e.g., "imperfect" learning from other people's choices, marginal and formally undetectable differences in users' preferences, or some inertia in adjusting between different prices but identical-return technologies.

The formal apparatus presented here, based on the idea of the generalized urn scheme, allows, in the domain of its applicability, quite general analytical accounts of the relationships between some system-parameters (e.g., proxies for information "imperfection" by adopters; dynamic increasing returns and monopolistic exploitation of new technologies by their producers) and limit market shares. While path-dependency (i.e., "history matters") applies throughout, the foregoing analytical techniques appear to be able, at the very least, to discriminate those which turn out to be feasible limit equilibria (i.e., those which are attainable with positive probabilities) and, also, to discover the different rates of emergence of the limit patterns.

The apparatus can also be used for numerical simulation. In this case it proves to be as general as ordinary differential equations and as easy to implement. By means of numerical simulation one can also study much more complicated and "inductively rich" models. Still, the developed mathematical machinery serves in such numerical studies as a means of prediction and verification, showing the general kind of behavior one ought to expect. Yet another complementarity between the analytical exploration of these models and their numerical simulation concerns the study of their non-limit properties, e.g. the "transient" structures that might emerge along the trajectories and their degrees of persistence.

As the foregoing modeling illustrations show, "market imperfections" and "informational imperfections" often tend to foster technological variety, i.e., the equilibrium co-existence of different technologies and firms. Moreover, stochasticity in the choice process may well bifurcate limit market-shares outcomes. Finally, it is shown, corporate pricing strategies – possibly based on rationally-bounded procedures, imperfect information and systematically "wrong" expectation-formation mechanisms – are generally bound to influence long-term outcomes. Under all these circumstances, the foregoing modeling techniques allow, at the very least, a "qualitative" analytical assessment of diffusion/competition processes by no means restricted to those circumstances whereby microeconomic expectations, on average, represent unbiased estimations of the future.

If all this analytical representation is empirically adequate, then there seem to be no *a priori* reasons to restrict it to technological dynamics. In fact, under suitable modifications, it may apply as well to interdependent expectations, decisions and returns in many other economic domains. Just to give a few examples: the evolution of strategies and organizational forms in industrial dynamics; the dynamics of location in economic geography (Arthur 1990); adaptive processes and the emergence of social norms; "mimetic" effects on financial markets; macroeconomic coordination.[29] The list is likely to be indeed very long. Ultimately, what we have tried to implement is a relatively general analytical apparatus able to handle at least some qualitative properties of dynamic stochastic processes characterized by both positive, and, possibly negative, feedbacks of a functional form as "badly-behaved" as possible.

In principle, domains of applicability of generalized urn schemes correspond to the set of phenomena where not only "history matters" but the burden of the past increasingly shapes the present. Of course, we are far from claiming that this is always the case. However, we do indeed suggest that quite a few of the processes of economic change fall into this category.

Acknowledgments. We thank Sid Winter and two anonymous referees for their precious comments on earlier drafts. This research has benefitted at various stages from the support of the Italian Ministry of Research (MURST 40%); the Italian National Research Council (CNR); the Center for Research in Management, U.C. Berkeley; and throughout by the International Institute for Applied Systems Analysis (IIASA), Laxenburg, Austria.

References

Allen PM (1988) Evolution, Innovation and Economics. In: Dosi G, Freeman C, Nelson R, Silverberg G, Soete L. (eds) Technical Change and Economic Theory. Pinter Publishers, London and New York, pp 95–119

Anderson PW, Arrow KF, Pines R (eds) (1988) The Economy as an Evolving Complex System. SFI Studies in the Science of Complexity. Addison-Wesley, New York

Arthur WB, Ruszczinski A (1992) Strategic Pricing in Markets with Conformity Effects. Arch Control Sciences, 1: 7–31

Arthur WB (1983) On Competing Technologies and Historical Small Events: the Dynamics of Choice under Increasing Returns. International Institute for Applied Systems Analysis, Laxenburg, Austria WP-83-90, (Working paper)

[29] For some works in these different domains that link at least in spirit with the approach to economic dynamics suggested here, see among others, Kirman (1991), Kuran (1991), Boyer and Orléan (1992), Durlauf (1991).

Arthur WB (1990) "Silicon Valley" Location Clusters: When do Increasing Returns Imply Monopoly? Math Social Sci 19: 235–251
Arthur WB (1988) Self-Reinforcing Mechanisms in Economics. In: Anderson PW, Arrow KF, Pines R The Economy as an Evolving Complex System. SFI Studies in the Science of complexity. Addison-Wesley, New York, pp 9–31
Arthur WB, Ermoliev YM, Kaniovski YM (1983) The Generalized Urn Problem and Its Application. Kibernetika 1: 49–56 (in Russian)
Arthur WB, Ermoliev YM, Kaniovski YM (1987a) Adaptive Process of Growth Being Modeled by Urn Schemes. Kibernetika 6: 49–57 (in Russian)
Arthur WB, Ermoliev YM, Kaniovski YM (1987b) Nonlinear Urn Processes: Asymptotic Behavior and Applications. Interational Institute for Applied Systems Analysis, Laxenburg, Austria WP-87-85, (working paper)
Arthur WB, Ermoliev YM, Kaniovski YM (1987c) Path Dependent Processes and the Emergence of Macro-Structure. Eur J Operational Res 30: 294–303
Arthur WB, Ermoliev YM, Kaniovski YM (1988) Nonlinear adaptive Processes of Growth with General Increments: Attainable and Unattainable Components of Terminal Set. International Institute for Applied Systems Analysis, Laxenburg, Austria WP-88-86 (working paper)
Atkinson A, Stiglitz J (1969) A New View of Technological Change. Econ J 79: 573–578
Banerjee A, Weibull JW (1992) Evolution and Rationality: Some Recent Game-Theoretic Results. Presented at the Tenth World Congress of the International Economic Association, Moscow, August 1992
Boldrin M (1988) Persistent Oscillations and Chaos in Dynamic Economic Models: Notes for a Survey. In: Anderson PW, Arrow KF, Pines R (eds) The Economy as an Evolving Complex System. SFI Studies in the Science of Complexity. Addison-Wesley, New York, 49–75
Boyer R, Orléan A (1992) How do Conventions Evolve? J Evol Econ 2: 165–177
Brock WA, Malliaris AG (1989) Differential Equations, Stability and Chaos in Dynamic Economics. North Holland, Amsterdam
Cabrales A (1992) Stochastic Replicator Dynamics. Dept. of Economics, University of San Diego, California, mimeo
David PA (1985) Clio and the Economics of QWERTY. Am Econ Rev (Papers and Proceedings) 75: 332–337
David PA (1988) Path-Dependance: Putting the Past into the Future in Economics. Technical Report no. 533. Institute for Mathematical Studies in the Social Sciences, Stanford University
David PA (1975) Technical Choice, Innovation and Economic Growth. Cambridge University Press, Cambridge, Mass
Day R (1992) Irregular Growth Cycles. Am Econ Rev 72: 406–414
Dekel E, Scotchmer S (1991) On the evolution of Optimizing Behavior. J Econ Theory 59: 637–666
Dosi G (1988) Sources, Procedures and Microeconomic Effects of Innovation. J Econ Literature 26: 1120–1171
Dosi G, Freeman C, Nelson R, Silverberg G, Soete L (eds) (1988) Technical Change and Economic Theory. Francis Pinter, London and Columbia University Press, New York
Dosi G, Pavitt K, Soete L (eds) (1990) The Economics of Technological Change and International Trade. New York University Press, New York and Wheatsheaf, Brighton
Dosi G, Ermoliev Y, Kaniovski Y (1994) Generalized Urn Schemes and Technological Dynamics. J Math Econ 23: 1–19
Durlauf SN (1991) Non-Ergodic Economic Growth. Dept. of Economics, Stanford University, mimeo
Eigen M, Schuster P (1979) The Hypercycle: Principle of Natural Selforganization Springer, Berlin Heidelberg New York
Foster D, Young HP (1990) Stochastic Evolutionary Game Dynamics. Theoretical Population Biol 38: 219–232
Freeman C (1982) The Economics of Industrial Innovation, 2nd edn. Francis Pinter, London
Friedman D (1991) Evolutionary Games and Economics. Econometrica 59: 637–666
Frydman R (1982) Towards an Understanding of Market Processes: Individual Expectations, Learning and Convergence to Rational Expectations Equilibrium. Am Econ Rev 72: 652–668
Frydman R, Phelps ES (eds) (1983) Individual Forecasting and Aggregate Outcomes. Oxford University Press, Oxford

Fudenberg D, Kreps DM (1993) Learning Mixed Equilibria. Games Econ Behavior 4: 320–367

Glaziev SY, Kaniovski YM (1991) Diffusion of Innovations Under Conditions of Uncertainty: A Stochastic Approach. In: Nakicenovic N, Grubler A (eds), Diffusion of Technologies and Social Behavior. Springer, Berlin Heidelberg New York pp 231–246

Hanson WA (1985) Bandwagons and Orphans: Dynamic Pricing of Competing Systems Subject to Decreasing Costs. Ph.D. Dissertation, Standard University, Stanford

Hill BM, Lane D, Sudderth W (1980) A Strong Law for Some Generalized Urn Processes. Ann Probability 8: 214–226

Jordan J (1985) Learning Rational Expectation: The Finite State Case. J Econ Theory 36: pp 257–276

Kandori M, Mailath GJ, Rob R (1993) Learning, Mutation, and Long Run Equilibria in Games. Econometrica 61: 29–56

Kaniovski Y, Pflug G (1992) Non-standard Limit Theorems for Stochastic Approximation and Their Applications for Urn Schemes. International Institute for Applied Systems Analysis, Laxenburg, Austria WP-92-25 (working paper)

Kaniovski Y, Young HP (1994) Dynamic Equilibria Selection Under Incomplete Information. International Institute for applied Systems analysis, Laxenburg, Austria WP-94-30 (working paper)

Kirman A (1991) Epidemics of Opinion and Speculative Bubbles in Financial Markets. In: Taylor M (ed) Money and Financial Markets. Macmillan, London

Kuran T (1991) Congnitive Limitations and Preference Evolution. J Institutional Theoretical Econ 147: 241–273

Ljung L, Söderström T (1983) Theory and Practice of Recursive Identification. M.I.T. Press, Cambridge, Mass

Metcalfe JS (1988) The Diffusion of Innovations: an Interpretative Survey. In: Dosi G, Freeman C, Nelson R, Silverberg G, Soete L (eds) Technical Change and Economic Theory. Francis Pinter, London and Columbia University Press, New York, pp 560–589

Nelson R, Winter S (1982) An Evolutionary Theory of Economic Change. The Belknap Press of Harward University Press, Cambridge, Mass

Nicolis G, Prigogine I (1971) Self-Organization in Nonequilibrium Systems: From Dissipative Structures to Order through Fluctuations. Wiley, New York

Nicolis G, Prigogine I (1989) Exploring Complexity. An Introduction. Freeman, New York

Pemantle R (1991) When Are Touchpoints Limits for Generalized Pólya Urns? Proc Am Math Soc 113: 235–243

Polterovich VM, Henkin GM (1988) An Economic Model of Interaction of Processes of Creation and Leasing of Technologies. Ekonomika i Matematicheskie Metody 24: 1071–1083 (in Russian)

Prigogine I, Stengers I (1984) Order out of Chaos: Man's New Dialogue with Nature. Heinemann, London

Rosser JB, Jr (1991) From Catastrophe to Chaos: A General Theory of Economic Discontinuities. Kluwer, Boston Dordrecht London

Samuelson L, Zhang J (1992) Evolutionary Stability in Asymmetric Games. J Econ Theory 57: 363–391

Silverberg G, Dosi G, Orsenigo L (1988) Innovation, Diversity and Diffusion: a Self Organization Model. Econ J 98: 1032–1054

Silverberg G (1988) Modelling Economic Dynamics and Technical Change: Mathematical Models of Self-Organisation and Evolution. In: Dosi G, Freeman C, Nelson R, Silverberg G, Soete L (eds) Technical Change and Economic Theory. Francis Pinter, London and Columbia University Press, New York, pp 531–559

Soete L, Turner R (1984) Technological Diffusion and the Rate of Technical Change. Econ J 94: 612–623

Winter S (1971) Satisficing, Selection and the Innovating Remnant. Quarterly J Econ 85: pp 237–261

Young P (1993) The Evolution of Conventions. Econometrica 61: 57–84