Working Paper

Convex Optimization via Feedbacks

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Preface

The present paper finalizes the research carried out by the author at IIASA's Dynamic Systems project in June – September 1994.

The goal of the study was to solve a static optimization problem via a dynamical model. A static solution process was looked at as *a dynamical control problem.* For a dynamical model, a simple control system was chosen. The problem of guiding the system to an unknown static solution point was posed. Controls were formed as *feedbacks.* This pattern was motivated by the following. On the one hand, the pool of feedbacks is extremely broad, allowing one to design practically every possible system's dynamics. On the other hand, the mathematical control theory suggests a great variety of feedback-selection methods for synthesizing the desired system's dynamics. The control problem in question consisted in finding an appropriate feedback-selection method for building a needed solution-approaching dynamics. An adequate method was found for the case where the initial static problem is that of *convex optimization* (meaning in particular that the optimized index is convex).

A static solution turnes out to be approached by ratios $x(t)/t$ where $x(t)$ is a system's state at time t (the system proceeds on the time interval $0 \leq t < \infty$ starting from $x(0) = 0$). These ratios, normally, converge to a solution with time running to infinity. There are however such "reverse time" feedbacks that make ratios $x(t)/t$ meet a solution with time shrinking back to zero.

The results were announced at the Workshop on *Decomposition and Parallel Computing Techniques for Large-Scale Systems*, 13 - 23 June 1994, IIASA. The author is grateful to Andrzej Ruszczyński, the workshop organizer, for his kind invitation to participate in that meeting, and stimulating discussions, and thankful to Yurii Ermoliev for his comments on linkages to path following methods.

Contents

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Abstract

A method to approach a solution to a finite-dimensional convex optimization problem via trajectories of a control system is suggested. The feedbacks exploit the idea of extremal shifting control from the theory of closed-loop differential games (Krasovskii and Subbotin [1]). Under these feedbacks, system's velocities are formed through current relaxations of the initial problem. In relaxed problems, the initial equality constraint is replaced by a scalar equality or a scalar inequality showing, respectively, directions to keep or non-increase a current value of the discrepancy. The first (α-shifting) feedback minimizes Lagrangians for current relaxed problems, and results in a dynamical implementation of the penalty method. The second (half-space shifting) feedback solves relaxed problems directly. The first feedback is simpler but less accurate (accuracy bounds are pointed out). The sought solutions are approximated by state-over-time ratios. Discrete and continuous control patterns are considered. Asymptotical convergence with time growing to infinity is proved, and "immediate solution" trajectories having proper asymptotics with time shrinking to zero are designed.

Key words: convex programming, feedback control, differential inclusions.

AMS subject classification: 90C25, 93B52, 49M30, 34A60.

1 Approach

1.1 Problem: Setting, Assumptions, Examples

We are concerned with the optimization problem

$$\text{minimize } J(x), \quad x \in M, \quad Fx = b\,. \tag{1.1}$$

Here J is a convex function on $\mathbf{R}^n$ (note that J is automatically continuous), M is a closed, convex and bounded subset of $\mathbf{R}^n$, F is an $r \times n$ matrix, and $b \in \mathbf{R}^r$. As usual, $\mathbf{R}^k$ denotes the k-dimensional euclidean space of column vectors; $x^{(j)}$ stands for the j-th coordinate of an $x \in \mathbf{R}^n$; the superscript ^{T} marks transposition; $|\cdot|$ is used for the euclidean norm. By J^0 and X^0 are denoted, respectively, the optimal value and the solution set of the problem (1.1) (note that X^0 is nonempty). We describe a solution method for (1.1) and specify it for the problems of linear and quadratic programming.

Problem of Linear Programming. The initial formulation is

$$\text{minimize } c^T y, \quad y \in \mathbf{R}^k, \quad Ay \le b\,. \tag{1.2}$$

Here A is a $k \times r$ matrix whose rank equals k (implying $k \le r$). In (1.2) and in what follows the vector inequality is understood coordinate-wise. Introducing the extra variable $s \in \mathbf{R}^r$, reduce the inequality from (1.2) to

$$F(y, s) = b, \quad s \ge 0 \tag{1.3}$$

where

$$F(y,s) = Ay + s \ . \tag{1.4}$$

We suppose that the set X of all $(y,s) \in \mathbf{R}^{k+r}$ satisfying (1.3) is bounded (we identify $\mathbf{R}^k \times \mathbf{R}^r$ with $\mathbf{R}^{k+r}$). Every convex, closed and bounded set $M \subset \mathbf{R}^{k+r}$ containing X and such that $s \geq 0$ for every $(y,s) \in M$ will be called (A,b)-*admissible.* It is easily seen that any above M has the following property: $(y,s) \in M$ and $F(y,s) = b$ hold if and only if (1.3) is fulfilled. Thus for every (A,b)-admissible M the problem (1.2) is equivalent to (1.1) where $n = k + r$ ($x = (y,s)$, $y \in \mathbf{R}^k$, $s \in \mathbf{R}^r$),

$$J(y,s) = c^T y \tag{1.5}$$

and F is defined by (1.4). Equivalency is understood in the sence that, first, y^0 solves (1.2) if and only if there is a s_0 such that (y^0, s^0) solves (1.1), and the optimal values of these two problems coinside. The above problem (1.1) will be identified with (1.2) and referred to as the problem of linear programming (with the (A,b)-admissible set M).

Problem of Quadratic Programming. The initial formulation is

$$\text{minimize} \quad y^T D^T D y + c^T y, \quad y \in \mathbf{R}^k, \quad Ay \leq b \tag{1.6}$$

where D is a nondegenerate $k \times k$ matrix and A is as above. Passing to the new variable Dy, we reduce (1.6) to

$$\text{minimize} \quad | \, y \, |^2 + c^T y, \quad y \in \mathbf{R}^k, \quad Ay \leq b \tag{1.7}$$

(we do not change notations for the new vector c and the new matrix A having the same rank k). As above, (1.7) is equivalent to (1.1) where F is given by (1.4), M is an (A,b)-admissible set, and

$$J(y,s) = | \, y \, |^2 + c^T y \ . \tag{1.8}$$

We identify (1.7) with the so defined problem (1.1) and refer to it as the problem of quadratic programming (with the (A,b)-admissible set M).

1.2 Outline of the Method

In this paper, a solution method for (1.1) based on the extremal shifting control principle (Krasovskii and Subbotin [1]) is described. The idea is close to that of homotopy methods (path following of Zangwill and Garcia [2], or analytic centers of Sonnevend [3]), but, unlike them, results in non-gradient successive approximation procedures, thus linking to penalty and Lagrange myltipliers approaches (see, e.g., Bertsekas [4]). The paper develops Kryazhimskii and Osipov [5].

To approach a solution of (1.1) we use the dynamical control system

$$\dot{x}(t) = u(t), \quad x(0) = 0 \tag{1.9}$$

operating on the time interval $\mathbf{R}_+ = [0, \infty[$. System's states $x(t)$ and control values $u(t)$ belong to $\mathbf{R}^n$; it is assumed that

$$u(t) \in M \ .$$

Our goal is to build a control function $u(\cdot)$ such that for the corresponding trajectory $x(\cdot)$ of the system (1.9) the ratios $x(t)/t$ lie close to the solution set X^0 for sufficiently large t. Later, the case where, conversely, t tends to zero is studied.

1.3 Feedbacks

Controls $u(t)$ are formed as feedbacks: $u(t) = U(t, x(t))$. Formally, a *feedback* is identified with an arbitrary function $U : \mathbf{R}_+ \times \mathbf{R}^n \mapsto M$. The system (1.9) closed with a feedback U, has the form

$$\dot{x}(t) = U(t, x(t)), \quad x(0) = 0 .$$

The closed system may have no trajectories understood as solutions of the above differential equation (if say, U is discontinuous). We use the definition of trajectories coming from the theory of closed-loop differential games of Krasovskii and Subbotin [1]. For every $\delta > 0$, define the *δ-trajectory* $x_\delta(\cdot)$ under a feedback U by

$$x_\delta(0) = 0, \quad x_\delta(t) = x_\delta(t_i) + U(t_i, x_\delta(t_i))(t - t_i) \quad (t \in [t_i, t_{i+1}],\ i = 0, 1, ...) \tag{1.10}$$

where $t_i = i\delta$. A function $x(\cdot) : \mathbf{R}_+ \mapsto \mathbf{R}^n$ is called a *trajectory* under U if at every bounded subinterval of $\mathbf{R}_+$, $x(\cdot)$ is the uniform limit of a sequence of δ_j-trajectories with $\delta_j \to 0$.

Remark 1.1 The set of all δ-trajectories under U with arbitrary δ is equicontinuous and bounded (i.e. compact in the sup-norm) at every bounded interval; hence easily follows the existence of a trajectory under U.

Remark 1.2 Note that for a δ-trajectory $x_\delta(\cdot)$ under an arbitrary feedback U we have $x_\delta(t)/t \in M$ $(t > 0)$. This follows from the representation

$$\frac{x_\delta(t)}{t} = \frac{1}{t} \int_0^t \dot{x}_\delta(\tau) d\tau$$

and the inclusion $\dot{x}_\delta(\tau) \in M$ $(\tau \geq 0,\ \tau \neq t_i)$, due to convexity and closedness of M. Hence $x(t)/t \in M$ $(t > 0)$ for every trajectory $x(\cdot)$ under U.

In the next section, two feedbacks are described. The first guarantees that for every trajectory $x(\cdot)$ the distance from $x(t)/t$ to X^0 is small if t is sufficiently large. The second feedback guarantees $x(t)/t \in X^0$ for all trajectories $x(\cdot)$ and all $t > 0$ (producing therefore "ideal" optimization trajectories). The δ-trajectories (1.10) under these feedbacks result in two numerical optimization procedures. The one corresponding to the second feedback provides better quality of approximation being, however, more complicated.

2 Solutions

2.1 α-Shifting

Let

$$L^0 = \{u \in \mathbf{R}^n \ : \ Fu = b\} . \tag{2.1}$$

For every $t \geq 0$, $x \in \mathbf{R}^n$, fix a nonempty, closed and convex set $Q(t, x) \subset M$ such that

$$M \cap L^0 \subset Q(t, x) . \tag{2.2}$$

Define the *α-shifting* feedback U_α $(\alpha > 0)$ by

$$U_\alpha(t, x) \in \operatorname{argmin}\{2(Fx - tb)^T Fu + \alpha J(u) \ : \ u \in Q(t, x)\} . \tag{2.3}$$

Remark 2.1 Clearly, $U_\alpha(t,x)$ minimizes the Lagrangian for the problem

$$\text{minimize } J(u), \quad u \in Q(t,x), \quad (Fx - tb)^T(Fu - b) = 0 \tag{2.4}$$

with the Lagrange multiplyer $2/\alpha$. The problem (2.4) is a relaxation of the problem (1.1); namely, the set M is replaced by $Q(t,x)$, and the vector equality constraint is replaced by the scalar one; the latter indicates directions u to shift x in time so as to keep the discrepancy value $| Fx - tb |^2$.

The α-shifting feedback is intended to realize the penalty method (see, e.g., Bertsekas [4, Chapter 4]). Introduce several notations. Let constants K_J and K_F be such that $| J(u) | \leq K_J$ and $| F - b | \leq K_F$ for all $u \in M$, $J^0[\gamma]$ be the optimal value of the γ-perturbed problem

$$\text{minimize} \quad J(x), \quad x \in M, \quad | Fx - b |^2 \leq \gamma \tag{2.5}$$

$X^0[\gamma,\sigma]$ $(\gamma, \sigma \geq 0)$ denote the the set of all x feasible for (2.5) satisfying $J(x) \leq J^0 + \sigma$, and finally,

$$O(\gamma,\sigma) = \sup\{\text{dist}(x, X^0) \ : \ x \in X^0[\gamma,\sigma]\} \ ; \tag{2.6}$$

here and in the sequel, $\text{dist}(\xi, X)$ stands for the distance of a point ξ to a set X in $\mathbf{R}^n$.

Remark 2.2 Clearly,

$$\sup\{\text{dist}(x, X^0[0,\sigma]) \ : x \in X^0[\gamma,\sigma]\} \to 0 \quad \text{as} \quad \gamma \to 0 \tag{2.7}$$

$$\sup\{\text{dist}(x, X^0[\gamma,0]) \ : x \in X^0[\gamma,\sigma]\} \to 0 \quad \text{as} \quad \sigma \to 0$$

yielding, respectively,

$$O(\gamma,\sigma) \to O(0,\sigma) \quad \text{as} \quad \gamma \to 0 \tag{2.8}$$

$$O(\gamma,\sigma) \to O(\gamma,0) \quad \text{as} \quad \sigma \to 0\,. \tag{2.9}$$

Similarly,

$$O(\gamma,\sigma) \to 0 \quad \text{as} \quad \gamma,\ \sigma \to 0 \tag{2.10}$$

$$J^0[\gamma] \to J^0 \quad \text{as} \quad \gamma \to 0\,.$$

Remark 2.3 For the problems of linear and quadratic programming (section 1), the deviation $J^0[\gamma] - J^0$ can be estimated explicitly provided there is a strictly feasible point (y_0, s_0) characterized by $Ay_0 + s_0 = b$ and $s_0^{(j)} > \epsilon_0 > 0$ $(j = 1, ..., r)$ (corresponding to the regular case in convex programming; see Eremin and Astafyev [6]). Indeed, let (y_γ, s_γ) be a solution to the perturbed problem (2.5). We have $Ay_\gamma + s_\gamma + d_\gamma = b$, $| d_\gamma | \leq \gamma^{1/2}$. Starting from this, one can easily prove that $y = y_0 + \mu(y_\gamma^{1/2} - y_0)$ satisfies $Ay \leq b$ for $\mu = \epsilon_0/(\gamma + \epsilon_0)$. Hence

$$| y - y_\gamma | = (1 - \mu) \mid y_0 - y_\gamma | \leq \gamma^{1/2} \mid y_0 - y_\gamma \mid / \epsilon_0 \leq K\gamma/\epsilon_0$$

where $K = 2\sup\{| \ y \ | : \ y \in M\}$. Thus, for the problem of linear programming (see (1.5)),

$$J^0[\gamma] = c^T y_\gamma = c^T y - c^T(y - y_\gamma) \geq J^0 - K \mid c \mid \gamma^{1/2}$$

and we obtain

$$-K \mid c \mid \gamma^{1/2} \leq J^0[\gamma] - J^0 \leq 0\,.$$

Similar estimates hold for the problem of quadratic programming.

Introduce

$$\lambda_a(x) = \mid Fx - b \mid^2 + aJ(x) - aJ^0 \tag{2.11}$$

($a > 0$) and formulate the penalty method as follows (see, e.g., Vasilyev [7, p. 182]).

Lemma 2.1 *For every $x \in M$ satisfying*

$$\lambda_a(x) \leq \epsilon \tag{2.12}$$

it holds that

$$x \in X^0[2K_J a, \epsilon/a] \tag{2.13}$$

$$\text{dist}(x, X^0) \leq O(2K_J a, \epsilon/a) \tag{2.14}$$

$$J^0[2K_J a] - J^0 \leq J(x) - J^0 \leq \epsilon/a \; . \tag{2.15}$$

Proof. From (2.12) and (2.11) follow (2.15) and $\mid Fx - b \mid^2 \leq 2K_J a$. These inequalities yield (2.13). The latter implies (2.14). □

For each $x(\cdot) : \mathbf{R}^+ \mapsto \mathbf{R}^n$ absolutely continuous on every bounded interval, and $\alpha > 0$, define

$$\Lambda_\alpha(t \mid x(\cdot)) = \mid Fx(t) - tb \mid^2 + \alpha \int_0^t J(\dot{x}(\tau)) d\tau - \alpha t J^0 \; . \tag{2.16}$$

Lemma 2.2 *It holds that*

$$\Lambda_\alpha(t \mid x(\cdot))/t^2 \geq \lambda_{\alpha/t}(x(t)/t) \quad (t > 0) \; .$$

Proof. We have

$$\Lambda_\alpha(t \mid x(\cdot))/t^2 - \lambda_{\alpha/t}(x(t)/t) = \frac{\alpha}{t}\frac{1}{t} \int_0^t J(\dot{x}(\tau)) d\tau - \frac{\alpha}{t} J \left(\frac{1}{t} \int_0^t \dot{x}(\tau) d\tau \right) \; .$$

The right hand side is nonnegative due to convexity of J. □

Lemma 2.3 *A δ-trajectory $x_\delta(\cdot)$ under the α-shifting feedback U_α satisfies*

$$\Lambda_\alpha(t \mid x_\delta(\cdot)) \leq K_F^2 t \delta \; . \tag{2.17}$$

Proof. Use induction. For $t = 0$, we have $\Lambda_\alpha(0 \mid x_\delta(\cdot)) = 0$, and (2.17) is true. Suppose that (2.17) is true for all $t \in [0, t_i]$ ($t_i = i\delta$). Take a $\tau \in [t_i, t_{i+1}]$ and prove (2.17) for $t = \tau$. Put $l(t_i) = Fx_\delta(t_i) - t_i b$, $u(t_i) = U_\alpha(t_i, x_\delta(t_i))$. Referring to (2.16), we get

$$\begin{aligned} \Lambda_\alpha(\tau \mid x_\delta(\cdot)) &= \Lambda_\alpha(t_i \mid x_\delta(\cdot)) + \\ &\quad 2l(t_i)^T (Fu(t_i) - b)(\tau - t_i) + \mid Fu(t_i) - b \mid^2 (\tau - t_i)^2 + \\ &\quad \alpha J(u(t_i)(\tau - t_i) - \alpha J^0 (\tau - t_i) \; . \end{aligned}$$

Let x^0 be a solution to (1.1). Noticing that $Fx^0 - b = 0$ and $J^0 = J(x^0)$, and using the definition of K_F, continue as follows:

$$\begin{aligned} \Lambda_\alpha(\tau \mid x_\delta(\cdot)) &\leq \Lambda_\alpha(t_i \mid x_\delta(\cdot)) + \\ &\quad [2l(t_i)^T (Fu(t_i) - b) + \alpha J(u(t_i)](\tau - t_i) - \\ &\quad [2l(t_i)^T (Fx^0 - b) + \alpha J(x^0)](\tau - t_i) + K_F^2 (\tau - t_i)\delta \; . \end{aligned}$$

By the definition (2.3) of U_α, and taking into account (2.2) and (2.1), we conclude that the second term on the right is no greater than the first; hence

$$\Lambda_\alpha(\tau \mid x_\delta(\cdot)) \leq \Lambda_\alpha(t_i \mid x_\delta(\cdot)) + K_F^2(\tau - t_i)\delta \,.$$

Estimate the first term on the right by (2.17) and obtain (2.17) for $t = \tau$. □

Theorem 2.1 *Let $\alpha > 0$.*

1) For a δ-trajectory $x_\delta(\cdot)$ under the α-shifting feedback U_α, it holds that

$$x_\delta(t)/t \in X^0[2K_J\alpha/t, K_F^2\delta\alpha] \quad (t > 0) \tag{2.18}$$

$$\text{dist}(x_\delta(t)/t, X^0) \leq O(2K_J\alpha/t, K_F^2\delta\alpha) \quad (t > 0) \tag{2.19}$$

$$J^0[2K_J\alpha/t] - J^0 \leq J(x_\delta(t)/t) - J^0 \leq K_F^2\delta\alpha \quad (t > 0) \,. \tag{2.20}$$

2) For every trajectory $x(\cdot)$ under U_α, it holds that

$$x(t)/t \in X[2K_J\alpha/t, 0] \quad (t > 0) \tag{2.21}$$

$$\text{dist}(x(t)/t, X^0) \leq O(2K_J\alpha/t, 0) \quad (t > 0) \tag{2.22}$$

$$J^0[2K_J\alpha/t] - J^0 \leq J(x(t)/t) - J^0 \leq 0 \quad (t > 0) \,. \tag{2.23}$$

Proof. Lemmas 2.2, 2.2, and 2.1 yield assertion 1. Assertion 1 and (2.9) imply assertion 2. □

Combining Theorem 2.1 and correlations (2.8), (2.10), (2.7), we obtain asymptotics for t going to infinity:

Corollary 2.1 *Let $\alpha > 0$. For a δ-trajectory $x_\delta(\cdot)$ and every trajectory $x(\cdot)$ under the α-shifting feedback U_α, it holds that*

$$\text{dist}(x_\delta(t)/t, X^0[0, K_F^2\delta\alpha) \to 0 \quad \textit{as} \quad t \to \infty \tag{2.24}$$

$$\limsup_{t\to\infty} \ \text{dist}(x_\delta(t)/t, X^0) \leq O(0, K_F^2\delta\alpha) \tag{2.25}$$

$$\text{dist}(x(t)/t, X^0) \to 0 \quad \textit{as} \quad t \to \infty \,. \tag{2.26}$$

The combination of Theorem 2.1 and (2.10) provides asymptotics for α tending to zero:

Corollary 2.2 *Let $t_0(\alpha) > 0$ and $\delta(\alpha) > 0$ satisfy $\alpha/t_0(\alpha)$, $\delta(\alpha)/\alpha \to 0$ as $\alpha \to 0$. Then for $\delta(\alpha)$-trajectories $x_{\delta(\alpha)}(\cdot)$ and arbitrary trajectories $x(\cdot)_\alpha$ under the α-shifting feedbacks U_α, it holds that*

$$\max\{\text{dist}(x_{\delta(\alpha)}(t)/t, X^0) \ : \ t \geq t_0(\alpha)\} \to 0 \quad \textit{as} \quad \alpha \to 0$$

$$\max\{\text{dist}(x(t)_\alpha/t, X^0) \ : \ t \geq t_0(\alpha)\} \to 0 \quad \textit{as} \quad \alpha \to 0 \,. \tag{2.27}$$

Example 2.1 Consider the problem of linear programming. Take the (A, b)-admissible set in the form

$$M = M_1 \times M_2 \tag{2.28}$$

where M_1 and M_2 are parallelopipeds:

$$M_1 = [K_-^{(1)}, K_+^{(1)}] \times ... \times [K_-^{(r)}, K_+^{(r)}] \tag{2.29}$$

$$M_2 = [0, P^{(1)}] \times ... \times [0, P^{(k)}] \, . \tag{2.30}$$

Such a choice is evidently possible. Put $Q(t, x) = M$. Referring to (1.4), (1.5), rewrite (2.3) as

$$U_\alpha(t, y, s) \in \operatorname{argmin}\{ \, 2(Ay + s - tb)^T(Av + w) + \alpha c^T v \ : \ v \in M_1, \ w \in M_2 \, \} \, .$$

Due to (2.29), (2.30), specify as follows:

$$U_\alpha(t, y, s) = (V_\alpha(t, y, s), W(t, y, s)) \tag{2.31}$$

$$V_\alpha^{(j)}(t, y, s) = \begin{cases} K_-^{(j)}, & p^{(j)}(t, y, s) + \alpha c^{(j)} \geq 0 \\ K_+^{(j)}, & p^{(j)}(t, y, s) + \alpha c^{(j)} < 0 \end{cases} \tag{2.32}$$

$$W^{(j)}(t, y, s) = \begin{cases} 0, & q^{(j)}(t, y, s) \geq 0 \\ P^{(j)}, & q^{(j)}(t, y, s) < 0 \end{cases} \tag{2.33}$$

where

$$p(t, y, s) = A^T(Ay + s - tb) \tag{2.34}$$

$$q(t, y, s) = Ay + s - tb \, . \tag{2.35}$$

Example 2.2 Consider the problem of quadratic programming. As above, put $Q(t, x) = M$ where M is given by (2.28), (2.29), (2.30). The α-shifting feedback (2.3) has the form

$$U_\alpha(t, y, s) = \operatorname{argmin}\{ \, 2(Ay + s - tb)^T(Av + w) + | \, v \, |^2 + \alpha c^T v \ : \ v \in M_1, \ w \in M_2 \, \} \, .$$

Its explicit form is (2.31) where $W^{(j)}(t, y, s)$ are as above (see (2.33)), and

$$V_\alpha^{(j)}(t, y, s) = \begin{cases} K_-^{(j)}, & d_\alpha^{(j)}(t, y, s) < K_-^{(j)} \\ K_+^{(j)}, & d_\alpha^{(j)}(t, y, s) > K_+^{(j)} \\ d_\alpha^{(j)}(t, y, s), & d_\alpha^{(j)}(t, y, s) \in [K_-^{(j)}, K_-^{(j)}] \end{cases} \tag{2.36}$$

where

$$d_\alpha^{(j)}(t, y, s) = -[p(t, y, s)/\alpha + c/2]^{(j)} \, . \tag{2.37}$$

2.2 Half-Space Shifting

Denote

$$L^-(t,x) = \{u \in \mathbf{R}^n \ : \ (Fx - tb)^T(Fu - b) \leq 0\} \ . \tag{2.38}$$

Define the *half-space shifting* feedback U_0 by

$$U_0(t,x) \in \operatorname{argmin}\{J(u) \ : \ u \in Q(t,x) \cap L^-(t,x)\} \ . \tag{2.39}$$

Due to (2.2), the set on the right is nonempty.

Remark 2.4 Note that $U_0(t,x)$ is a solution to

$$\text{minimize} \ \ J(u), \quad u \in Q(t,x), \quad 2(Fx - tb)^T(Fu - b) \leq 0 \tag{2.40}$$

being a relaxation of the problem (1.1); namely, the set M is replaced by $Q(t,x)$, and the vector equality constrained is replaced by the scalar inequality; the latter describes all directions u to shift x locally in time so as not to increase the discrepancy $| \ Fx - tb \ |^2$. In the particular case where $Q(t,x) \subset \{u \in \mathbf{R}^n \ : \ (Fx - tb)^T(Fu - b) = 0\}$, the problem (2.40) turns into the relaxed problem (2.4) associated with the α-shifting feedback (Remark 2.1).

Lemma 2.4 *For a δ-trajectory $x_\delta(\cdot)$ under the half-space shifting feedback U_0, it holds that*

$$| \ F(x_\delta(t)/t) - b \ |^2 \leq K_F^2 \delta / t \quad (t > 0) \tag{2.41}$$

$$J(x_\delta(t)/t) \leq J^0 \quad (t > 0) \ . \tag{2.42}$$

Proof. Obviously, (2.41) is equivalent to

$$| \ Fx_\delta(t) - tb \ |^2 \leq K_F^2 \delta t \ . \tag{2.43}$$

Due to convexity of J, (2.42) follows from

$$\int_0^t J(\dot{x}(\tau) d\tau \leq tJ^0 \ . \tag{2.44}$$

Prove (2.43), (2.44). Denote $l(t_i) = Fx_\delta(t_i) - t_i b$, $u(t_i) = U_0(t_i, x_\delta(t_i))$ $(t_i = i\delta)$. Let x^0 solve (1.1). Note that $x^0 \in M \cap L^0 \subset Q(t_i, x_\delta(t_i))$ (see (2.1), (2.2)). Hence by the definition of U_0 (2.39) we have

$$J(u(t_i)) \leq J(x^0) = J^0 \ . \tag{2.45}$$

By (2.39) and (2.38) $u(t_i) \in L^-(t_i, x_\delta(t_i))$ yielding $l(t_i)^T(Fu(t_i) - b) \leq 0$. Therefore for $t \in]t_i, t_{i+1}]$, it holds that

$$\begin{aligned} | \ Fx_\delta(t) - tb) \ |^2 &= \ | \ l(t_i) \ |^2 + 2l(t_i)^T(Fu(t_i) - b)(\tau - t_i) + | \ F(u(t_i) - b) \ |^2 \ (\tau - t_i)^2 \\ &\leq \ | \ l(t_i) \ |^2 + K_F^2(\tau - t_i)^2 \end{aligned} \tag{2.46}$$

(we have used the definition of K_F), and

$$\int_0^t J(\dot{x}(\tau))d\tau = \int_0^{t_i} J(\dot{x}(\tau)d\tau + \alpha J(u(t_i))(t - t_i) \ . \tag{2.47}$$

Now use induction. Let $t \in]0, t_1]$. From $l(0) = l(t_0) = 0$ and (2.46) where $i = 0$, we get (2.43). From (2.45) and (2.47) follows (2.44). Suppose that (2.43) and (2.44) are true for all $t \in]0, t_i]$ where $i \geq 1$. Using (2.46) and (2.43) for $t = t_i$, we get (2.43) for $t \in]t_i, t_{i+1}]$. From (2.45), (2.47) and (2.44) where $t = t_i$, follows (2.44) for $t \in]t_i, t_{i+1}]$. $\square$.

Theorem 2.2 *1) For a δ-trajectory $x_\delta(\cdot)$ under the half-space shifting feedback U_0, it holds that*

$$x_\delta(t)/t \in X^0[0, K_J^2\delta/t] \quad (t > 0) \tag{2.48}$$

$$\text{dist}(x_\delta(t)/t, X^0) \le O(0, K_F^2\delta/t) \quad (t > 0) \tag{2.49}$$

$$J^0[2K_J\delta/t] - J^0 \le J(x_\delta(t)/t) - J^0 \le 0 \quad (t > 0)\,. \tag{2.50}$$

2) For every trajectory $x(\cdot)$ under U_α, it holds that

$$x(t)/t \in X^0 \quad (t > 0)\,. \tag{2.51}$$

Proof. Lemma 2.2 yields assertion 1. Assertion 1 and (2.10) imply assertion 2. □

Remark 2.5 Whenever $t > \alpha$, the above estimates (2.48), (2.49), (2.50), and (2.51) are more accurate than the estimates (2.18), (2.19) (2.20), and (2.22) provided in Theorem 2.1 for the α-shifting feedback.

Combining assertion 1 of Theorem 2.2 and correlation (2.10), we deduce the following two asymptotics:

Corollary 2.3 *For a δ-trajectory $x_\delta(\cdot)$ under the half-space shifting feedback U_0, it holds that*

$$\text{dist}(x_\delta(\cdot), X^0) \to 0 \quad \text{as} \quad t \to \infty\,. \tag{2.52}$$

Remark 2.6 The asymptotics (2.52) of δ-trajectories is accurate – unlike (2.24) guaranteed by the α-shifting feedback.

Corollary 2.4 *Let $t_0(\delta) > 0$, $\delta/t_0(\delta) \to 0$ as $\delta \to 0$. Then for δ-trajectories $x_\delta(\cdot)$ under the half-space shifting feedback U_0, it holds that*

$$\max\{\text{dist}(x_\delta(t)/t, X^0) \ : \ t \ge t_0(\delta)\} \to 0 \quad \text{as} \quad \delta \to 0\,.$$

Example 2.3 Consider the problem of linear programming with $c \ne 0$ and the (A, b)-admissible set $M = M_1 \times M_2$; the sets M_1, M_2 are specified later. Take Q in the form

$$Q(t, y, s) = Q_1(t, y, s) \times M_2\,. \tag{2.53}$$

Introduce the notations (2.34), (2.35). According to (2.38) and (1.4)

$$L^-(t, y, s) = \{(v, w) \in \mathbf{R}^{k+r} : p(t, y, s)^T v \le -q(t, y, s)^T w\}\,.$$

Combining with (2.53), we get

$$\begin{aligned} Q(t, y, s) \cap L^-(t, y, s) \ &= \ \{(v, w) \in \mathbf{R}^{k+r} : \ v \in Q_1(t, y, s), \\ &\quad p(t, y, s)^T v \le \beta(t, y, s), \ w \in Q_2(t, y, s, v)\} \end{aligned} \tag{2.54}$$

where

$$Q_2(t, y, s, v) = \{w \in M_2 \ : \ p(t, y, s)^T v \le -q(t, y, s)^T w\} \tag{2.55}$$

and $\beta(t,y,s)$ is the value of the problem

$$\text{maximize} \quad -q(t,y,s)^T w, \quad w \in M_2 . \tag{2.56}$$

The representation (2.54) together with the definitions (2.39), (1.5) of U_0 and J imply that

$$U_0(t,y,s) = (V_0(t,y,s), W(t,y,s)) \tag{2.57}$$

where $V_0(t,y,s)$ is a solution to the problem

$$\text{minimize} \quad c^T v, \quad v \in Q_1(t,y,s) \cap L_1^-(t,y,s) \tag{2.58}$$

with

$$L_1^-(t,y,s) = \{v \in \mathbf{R}^k \ : \ p(t,y,s)^T v \leq \beta(t,y,s)\} \tag{2.59}$$

and

$$W(t,y,s) \in Q_2(t,y,s,V_0(t,y,s)) . \tag{2.60}$$

In particular, we can let $W(t,y,s)$ be a solution to the problem (2.56).

Specify the above construction. (Note that if, as in Example 2.1, we define M_1 by (2.29) and put $Q_1(t,y,s) = M_1$, then the feasible set of the problem (2.58) is a perallelopiped intersected with a half-space; to obtain a solution of this problem, a logical analysis of the vertexes of the above intersection is needed, and the final form for $V_0(t,y,s)$ is not so simple.) Let

$$Q_1(t,y,s) = M_1 = B(0,R) \tag{2.61}$$

where $B(0,R)$ is the closed ball in $\mathbf{R}^k$ centered at zero with radius R. Define M_2, as in Example 2.1, by (2.30). Clearly, $W(t,y,s)$ can be given by (2.33), and

$$\beta(t,y,s) = \sum_{j+1}^{r} P^{(j)} \max\{0, -q^{(j)}(t,y,s)\} . \tag{2.62}$$

Solve the problem (2.58). If $p(t,y,s) = 0$, then (see (2.59)) $L_1^-(t,y,s) = \mathbf{R}^n$, and

$$V_0(t,y,s) = -R\frac{c}{|\, c \,|} . \tag{2.63}$$

Let $p(t,y,s) \neq 0$. Represent

$$c = \mu(t,y,s)p(t,y,s) + c_1(t,y,s)$$

where $c_1(t,y,s)$ is orthogonal to $p(t,y,s)$; we have therefore

$$\mu(t,y,s) = -\frac{c^T p(t,y,s)}{|\, p(t,y,s) \,|^2}, \quad c_1 = c - \mu(t,y,s)p(t,y,s) . \tag{2.64}$$

Similarly, represent the free variable in the problem (2.58) as

$$v = \nu p(t,y,s) + v_1 \tag{2.65}$$

where v_1 is orthogonal to $p(t,y,s)$. The problem (2.58) is reduced then to

$$\text{minimize} \quad \mu(t,y,s)\nu \,|\, p(t,y,s) \,|^2 + c_1^T(t,y,s)v_1 \tag{2.66}$$

$$v_1^T p(t,y,s) = 0$$

$$\nu^2 \mid p(t,y,s) \mid^2 + \mid v_1 \mid^2 \le R^2$$
$$\nu \le \frac{\beta(t,y,s)}{\mid p(t,y,s) \mid^2} .$$

The minimizing $v_1 = v_1(t,y,s)$ for a fixed ν is obviously

$$v_1(t,y,s) = -\kappa(t,y,s) c_1(t,y,s) \tag{2.67}$$

$$\kappa(t,y,s) = \left(\frac{R^2 - \nu^2 \mid p(t,y,s) \mid^2}{\mid c_1(t,y,s) \mid^2} \right)^{1/2}$$

if $c_1(t,y,s) \neq 0$; otherwise $v_1(t,y,s)$ is arbitrary feasible, i.e. satisfying

$$(v_1(t,y,s))^T p(t,y,s) = 0, \quad \nu^2 \mid p(t,y,s) \mid^2 + \mid v_1(t,y,s) \mid^2 \le R^2 . \tag{2.68}$$

Substituting $v_1 = v_1(t,y,s)$ reduce (2.66) to

$$\text{minimize} \quad \mu(t,y,s)\nu \mid p(t,y,s) \mid^2 - \left(\frac{R^2 - \nu^2 \mid p(t,y,s) \mid^2}{\mid c_1(t,y,s) \mid^2} \right)^{1/2} \tag{2.69}$$

$$\nu^2 \mid p(t,y,s) \mid^2 \le R^2$$
$$\nu \le \frac{\beta(t,y,s)}{\mid p(t,y,s) \mid^2}$$

if $c_1(t,y,s) \neq 0$, and

$$\text{minimize} \quad \mu(t,y,s)\nu \mid p(t,y,s) \mid^2 \tag{2.70}$$
$$\nu^2 \mid p(t,y,s) \mid^2 \le R^2$$
$$\nu \le \frac{\beta(t,y,s)}{\mid p(t,y,s) \mid^2}$$

if $c_1(t,y,s) = 0$. Consider the first case. Rewrite (2.69) in short notations:

$$\text{minimize} \quad a_0(t,y,s)\nu - (a_1(t,y,s) - a_2(t,y,s)\nu^2)^{1/2} \tag{2.71}$$
$$\nu \in [g_1(t,y,s), g_2(t,y,s)]$$

where

$$a_0(t,y,s) = \mu(t,y,s) \mid p(t,y,s) \mid^2 \tag{2.72}$$
$$a_1(t,y,s) = \frac{R^2}{\mid c_1(t,y,s) \mid^2} \tag{2.73}$$
$$a_2(t,y,s) = \frac{\mid p(t,y,s) \mid^2}{\mid c_1(t,y,s) \mid^2} \tag{2.74}$$
$$g_1(t,y,s) = -\frac{R}{\mid p(t,y,s) \mid} = -\frac{a_1(t,y,s)^{1/2}}{a_2(t,y,s)^{1/2}} \tag{2.75}$$
$$g_2(t,y,s) = \min \left\{ \frac{a_1(t,y,s)^{1/2}}{a_2(t,y,s)^{1/2}}, \frac{\beta(t,y,s)}{\mid p(t,y,s) \mid^2} \right\} . \tag{2.76}$$

The function minimized in (2.71) is strictly decreasing and strictly increasing respectively on the left and on the right from

$$\nu_0(t,y,s) = -\text{sign } a_0(t,y,s) \frac{a_0(t,y,s) a_1(t,y,s)^{1/2}}{(a_0(t,y,s)^2 + a_2(t,y,s))^{1/2} a_2(t,y,s)^{1/2}} . \tag{2.77}$$

Hence the solution to (2.71) is

$$\nu_1(t,y,s) = \min\{\nu_0(t,y,s), g_2(t,y,s)\} . \tag{2.78}$$

Coming back to the representation (2.65), write out the solution to (2.58):

$$V_0(t,y,s) = \nu_1(t,y,s)p(t,y,s) + v_1(t,y,s) . \tag{2.79}$$

The problem (2.70) corresponding to $c_1(t,y,s) = 0$ is obviously solved by

$$\nu_1(t,y,s) = \begin{cases} g_2(t,y,s), & a_0(t,y,s) > 0 \\ g_1(t,y,s), & a_0(t,y,s) < 0 . \\ \text{any } \nu \in [g_1(t,y,s), g_2(t,y,s)], & a_0(t,y,s) = 0 \end{cases} \tag{2.80}$$

Let us sum up. The half-space shifting feedback U_0 is defined by (2.57); its components depend on the vectors (2.34) and (2.35). The component $W(t,y,s)$ is given by (2.60); in particular the form (2.33) is admissible. The component $V_0(t,y,s)$ is defined through the parameters (2.62), (2.64), and (2.72) through (2.76) If $p(t,y,s) = 0$, then $V_0(t,y,s)$ is determined by (2.63). Otherwise $V_0(t,y,s)$ has the form (2.79). If $c_1(t,y,s) \neq 0$, the right hand side of (2.79) is defined by (2.78), (2.77), (2.67); if $c_1(t,y,s) = 0$, it is defined by (2.80),

$$v_1(t,y,s)^T p(t,y,s) = 0, \quad \nu_1(t,y,s)^2 \mid p(t,y,s) \mid^2 + \mid v_1(t,y,s) \mid^2 \leq R^2 \tag{2.81}$$

(i.e. (2.68) specified for $\nu = \nu_1(t,y,s)$). The construction corresponds to M, and $Q(t,y,s)$ given by (2.28), (2.53), (2.61), (2.30).

Example 2.4 Consider the problem of quadratic programming under the assumptions of Example 2.3; namely, we define M and $Q(t,y,s)$ by (2.28), (2.53), (2.61), (2.30); recall that the notations (2.34), (2.35) are assumed, and $\beta(t,y,s)$ is the optimal value of the problem (2.56). Like in Example 2.3, (2.54), (2.39), and (1.8) lead to U_0 of the form (2.57) where $V_0(t,y,s)$ is the solution to the problem

$$\text{minimize} \quad \mid v \mid^2 + c^T v, \quad v \in Q_1(t,y,s) \cap L_1^-(t,y,s) \tag{2.82}$$

($L_1^-(t,y,s)$ is determined by (2.59)), and $W(t,y,s)$ satisfies (2.60); in particular, it can be given by (2.33); note that for $\beta(t,y,s)$ we have (2.62). Let the radius R of the ball M_1 (2.61) be so large that M_1 contains the global minimizer $-c/2$ of the objective function in (2.82). If

$$-\frac{c}{2} \in Q_1(t,y,s) \cap L_1^-(t,y,s)$$

or, equivalently,

$$-c^T p(t,y,s) \leq 2\beta(t,y,s) \tag{2.83}$$

then obviously

$$V_0(t,y,s) = -\frac{c}{2} . \tag{2.84}$$

Otherwise $V_0(t,y,s)$ is the projection of the global minimizer $-c/2$ onto the hypeplane bordering $L_1^-(t,y,s)$; note that so far as $\beta(t,y,s) \geq 0$ (see (2.62)) this projection does not escape the ball $Q_1(t,y,s)$. Writing out the explicit expression for the projection, we get

$$V_0(t,y,s) = p_0(t,y,s) + \frac{c}{2} - \frac{c^T p_0(t,y,s)}{2 \mid p_0(t,y,s) \mid^2} p_0(t,y,s) \tag{2.85}$$

where

$$p_0(t,y,s) = \beta(t,y,s)\frac{p(t,y,s)}{|\,p(t,y,s)\,|^2} \tag{2.86}$$

(the above formula is correct, since the assumption that (2.83) is untrue yields $p(t,y,s) \neq 0$). Sum up. The half-space shifting feedback U_0 has the form (2.57). The component $W(t,y,s)$ is given by (2.60); in particular the form (2.33) is admissible. The component $V_0(t,y,s)$ is defined by (2.84) if (2.83) holds, and by (2.85), (2.86) otherwise.

3 Continuous Time Control Pattern

3.1 Contingent Feedbacks

Above, a discrete control pattern implying that controls are being worked out at discrete times t_i was analysed. In this and the next sections, we consider a continuous control pattern assuming controls to react at system's states at every current time. Following the standard approach, formalize it through differential inclusions. Namely, identify a *contingent feedback* with an arbitrary mapping $\mathcal{U}$ from $\mathbf{R}_+ \times \mathbf{R}^n$ into the set of all nonempty subsets of M, and define a *trajectory under* $\mathcal{U}$ to be a solution to the differential inclusion

$$\dot{x}(t) \in \mathcal{U}(t, x(t))\ , \tag{3.1}$$

i.e. a function $x(\cdot) : \mathbf{R}_+ \mapsto \mathbf{R}^n$ absolutely continuous at every bounded interval, and satisfying $x(0) = 0$ and (3.1) for almost all t (with respect to the Lebesgue measure). In this section we study contingent analogues of the α-shifting and half-space shifting feedbacks.

A set-valued map $\mathcal{F}$ on a $E \subset \mathbf{R}_+ \times \mathbf{R}^n$ is as usual identified with a function associating to every $(t,x) \in E$ a nonempty set $\mathcal{F}(t,x) \subset \mathbf{R}^n$; in case $E = \mathbf{R}_+ \times \mathbf{R}^n$, we call $\mathcal{F}$ a set-valued map (without mentioning its set of definition). Continuity and upper semicontinuity of a set-valued map $\mathcal{F}$ on E at a point are understood in a standard way (see Aubin and Cellina [8, p. 41, 43]). A set-valued map $\mathcal{F}$ whose restriction to a set E is continous or upper semicontinuous at every point $(t,x) \in E$ is said to be, respectively, continuous and upper semicontinous on E; if $E = \mathbf{R}_+ \times \mathbf{R}^n$, then $\mathcal{F}$ is called, respectively, continuous and upper semicontinuous. We will use the two lemmas easily implied by standard results of set-valued analysis (see Aubin and Frankowska [9, Section 1.4]).

Lemma 3.1 *Let $\mathcal{F}$ be a closed- and convex-valued set-valued map continuous on a set E, and f be a convex function on $\mathbf{R}^n$. Then the set-valued map $U : (t,x) \mapsto U(t,x) =$ argmin$\{f(u)\ :\ u \in \mathcal{F}(t,x)\}$ is closed- and convex-valued and upper semicontinuous on E.*

Lemma 3.2 *Let $E_1, ..., E_m$ be closed subsets of $\mathbf{R}_+ \times \mathbf{R}^n$, and a set-valued map $\mathcal{F}$ be bounded, closed- and convex-valued, and upper semicontinuous on each of them. Then $\mathcal{F}$ is bounded, closed- and convex-valued and upper semicontinuous on $\cup\{E_j\ :\ j = 1, ..., m\}$.*

3.2 Contingent α-Shifting

In this subsection we assume the following.

Condition 3.1 The set-valued map Q is continuous.

Define the *α-shifting contingent feedback* $\mathcal{U}_\alpha$ ($\alpha > 0$) by the right hand side of the inclusion (2.3) determining the α-shifting feedback U_α:

$$\mathcal{U}_\alpha(t,x) = \text{argmin}\{2(Fx - tb)^T Fu + \alpha J(u) \ : \ u \in Q(t,x)\} \ . \tag{3.2}$$

Remark 3.1 By Condition 3.1 and Lemma 3.1, $\mathcal{U}_\alpha$ is closed- and convex-valued and upper semicontinuous; this yields existence of a trajectory under $\mathcal{U}_\alpha$ (see, e.g., Aubin and Cellina [8, Theorem 4, p. 101]).

Theorem 3.1 *Let $\alpha > 0$.*

1) There exists a trajectory under the α-shifting contingent feedback $\mathcal{U}_\alpha$.

2) For every trajectory $x(\cdot)$ under $\mathcal{U}_\alpha$, the inclusion (2.21) and the inequalities (2.22) (2.23) hold.

3) The convergence (2.26) holds uniformly with respect to all trajectories $x(\cdot)$ under $\mathcal{U}_\alpha$.

Proof. Assertion 1 is justified in Remark 3.1. Assertion 3 follows from assertion 2 and (2.10). Prove assertion 2. Due to Lemmas 2.2 and 2.1, it is sufficient to show that

$$\Lambda_\alpha(t \mid x(\cdot)) \leq 0 \quad (t \geq 0) \ . \tag{3.3}$$

For almost all t, we have

$$\dot{\Lambda}_\alpha(t \mid x(\cdot)) = 2l(t)^T(F(\dot{x}(t) - b) + \alpha J(\dot{x}(t)) - \alpha J^0 \ ;$$

here $l(t) = Fx(t) - tb$, and $\dot{\Lambda}_\alpha(t \mid x(\cdot)$ stands for the derivative at t of the function $\tau \mapsto \Lambda(\tau \mid x(\cdot))$. Let x^0 be a solution to (1.1). Observing that $Fx^0 - b = 0$ and $J^0 = J(x^0)$, continue as follows:

$$\dot{\Lambda}_\alpha(t \mid x_\alpha(\cdot)) = 2[l(t)^T(F(\dot{x}(t) - b) + \alpha J(\dot{x}(t))] - 2[l(t)^T(F(x^0 - b) + \alpha J(x^0)] \ .$$

The inclusion $\dot{x}(t) \in U_\alpha(t, x(t))$, the definition (3.2) and the fact that $x^0 \in M$, imply that for almost all t,

$$\dot{\Lambda}_\alpha(t \mid x(\cdot)) \leq 0 \ .$$

Noticing that $\Lambda_\alpha(\cdot \mid x(\cdot))$ is absolutely continuous, and $\Lambda_\alpha(0 \mid x(\cdot)) = 0$, obtain (3.3). □

Theorem 3.1 shows that, as $\alpha \to 0$, α-shifting contingent feedbacks have the same asymptotics as α-shifting ones (see Corollaries 2.1 and 2.2).

Corollary 3.1 *Let $t_0(\alpha)$ satisfy $\alpha/t_0(\alpha) \to 0$ as $\alpha \to 0$. Then for arbitrary trajectories $x(\cdot)_\alpha$ under α-shifting contingent feedbacks $\mathcal{U}_\alpha$, the convergence (2.27) takes place.*

Example 3.1 Consider the problem of linear programming. Following Example 2.1, take the (A,b)-admissible set M in the form (2.28), (2.29), (2.30), and put $Q(t,x) = M$. Assume the notations (2.34) and (2.35). The α-shifting contingent feedback $\mathcal{U}_\alpha$ is determined by modified formulas (2.31), (2.32), (2.33):

$$\mathcal{U}_\alpha(t,y,s) = \mathcal{V}_\alpha(t,y,s) \times \mathcal{W}(t,y,s) \tag{3.4}$$

$$\mathcal{V}_\alpha(t,y,s) = \mathcal{V}_\alpha^{(1)}(t,y,s) \times ... \times \mathcal{V}_\alpha^{(k)}(t,y,s) \tag{3.5}$$

$$\mathcal{W}(t,y,s) = \mathcal{W}^{(1)}(t,y,s) \times ... \times \mathcal{W}^{(r)}(t,y,s) \tag{3.6}$$

$$\mathcal{V}_\alpha^{(j)}(t,y,s) = \begin{cases} K_-^{(j)}, & p^{(j)}(t,y,s)+\alpha c^{(j)} > 0 \\ K_+^{(j)}, & p^{(j)}(t,y,s)+\alpha c^{(j)} < 0 \\ [K_-^{(j)}, K_+^{(j)}], & p^{(j)}(t,y,s)+\alpha c^{(j)} = 0 \end{cases}$$

$$\mathcal{W}^{(j)}(t,y,s) = \begin{cases} 0, & q^{(j)}(t,y,s) > 0 \\ L^{(j)}, & q^{(j)}(t,y,s) < 0\ . \\ [0, L^{(j)}], & q(j)(t,y,s) = 0 \end{cases} \tag{3.7}$$

Example 3.2 Under the assumptions of Example 3.1, consider the problem of quadratic programming. The α-shifting feedback $\mathcal{U}_\alpha$ has the form (3.4), (3.5), (3.6) with $\mathcal{W}^{(j)}(t,y,s)$ given as above by (3.7), and $\mathcal{V}_\alpha^{(j)}(t,y,s) = \{V_\alpha^{(j)}(t,y,s)\}$ where $V_\alpha^{(j)}(t,y,s)$ is defined by (2.36), (2.37).

3.3 Contingent Half-Space Shifting

Define the *half-space shifting contingent feedback* $\mathcal{U}_0$ by the right hand side of the inclusion (2.39) determining the half-space shifting feedback U_0:

$$\mathcal{U}_0(t,x) = \text{argmin}\{J(u)\ :\ u \in Q(t,x) \cap L^-(t,x)\}\ . \tag{3.8}$$

Note that $\mathcal{U}_0$ is closed- and convex-valued. However, in nontrivial cases, a trajectory under $\mathcal{U}_0$ does not exist. Namely, the following theorem is true. Denote by $Q_0(t,x)$ the set of all solutions to

$$\text{minimize } J(u), \quad u \in Q(t,x)\ . \tag{3.9}$$

Theorem 3.2 *For an arbitrary $x(\cdot) : \mathbf{R}_+ \mapsto \mathbf{R}^n$ absolutely continuous at every bounded interval and satisfying $x(0) = 0$, the following statements are equivalent:*

(i) for almost all $t \geq 0$,

$$\dot{x}(t) \in Q_0(t, x(t)) \cap X^0\ ; \tag{3.10}$$

(ii) $x(\cdot)$ is a trajectory under the half-space shifting contingent feedback $\mathcal{U}_0$.

Proof. Let (i) hold. Then for almost all t, we have that $\dot{x}(t)$ solves both (3.9) with $x = x(t)$, and (1.1). Since $Q(t,x(t)) \cap L^-(t,x(t))$ lies between the feasible sets of these two problems, $\dot{x}(t) \in \mathcal{U}_0(t,x(t))$ for almost all t implying (ii). Let (ii) be satisfied. Due to (3.8), we have $\dot{x}(t) \in L^-(t,x(t))$ for almost all t; this implies that $d(\cdot) : t \mapsto \mid Fx(t) - tb \mid^2$ is nonincreacing. Hence, so far as $d(0) = 0$, we get $Fx(t) = tb$ for all $t \geq 0$. The latter yields $L^-(t,x(t)) = \mathbf{R}^n$ (see (2.38)). Consequently $\mathcal{U}_0(t,x(t))$ coincides with the solution set $Q_0(t,x(t))$ to the problem (3.9) where $x = x(t)$. Thus we obtain (i). □

Corollary 3.2 *Let there exist a measurable set $E \subset \mathbf{R}_+$ with positive measure such that $Q_0(t,x) \cap X^0$ is empty for every $t \in E$ and $x \in tX^0$. Then there does not exist a trajectory under $\mathcal{U}_0$.*

Proof. Suppose that there is a trajectory $x(\cdot)$ under $\mathcal{U}_0$. By Theorem 3.2 (i) is satisfied. Hence $x(t) \in tX^0$. By assumption $Q_0(t,x(t)) \cap X^0$ is empty for $t \in E$. Consequently, (3.10) is untrue for these t, contradicting (i). □

Remark 3.2 Assume that the problems (1.1) and (3.9) have the single solutions; denote them, respectively, x^0 and $u^0(t,x)$. The fact that the assumption of Corollary 3.2 is violated means $u^0(t,tx^0) = x^0$ for almost all $t \in [0,1]$. Ensuring this by selection of $Q(t,x)$ is practically equivalent to knowing x^0. This case is trivial.

Example 3.3 Consider the problem of linear programming under the conditions of Example 2.3. (Recall that these conditions lead to an explicit description of the half-space shifting feedback.) Namely, M has the form (2.28) where M_1, M_2, and $Q(t,y,s)$ are defined by (2.53), (2.61), (2.30). Obviously $Q_0(t,y,s) = \{-Rc/ \mid c \mid\} \times M_2$. The requirement that the intersection $Q_0(t,y,s) \cap X^0$ is nonempty implies that $-Rc/ \mid c \mid$ is a solution to the initial linear programming problem (1.2). If this is not so, then the above intersection is empty for all $t > 0$ and $(y,s) \in tX^0$, and by Corollary 3.2 there does not exist a trajectory under $\mathcal{U}_0$.

Remark 3.3 Theorem 3.2 yields (see also the proof of Corollary 3.2) that if a trajectory $x(\cdot)$ under $\mathcal{U}_0$ exists, then it satisfies $x(t)/t \in X^0$ for all $t > 0$ (see (2.51).

4 Immediate Solution Trajectories

4.1 Problem

Come back to Corollary 3.1 which says that for trajectories $x(\cdot)_\alpha$ under the α-shifting contingent feedback $\mathcal{U}_\alpha$, the convergence (2.27) holds; recall that $\alpha/t_0(\alpha) \to 0$ as $\alpha \to 0$. In particular, we have $\text{dist}(x(t_0(\alpha))_\alpha/t_0(\alpha), X^0) \to 0 \quad \text{as } \alpha \to 0$. Introducing the function $\alpha(\cdot) : t \mapsto \alpha(t)$ inverse to $t_0(\cdot) : \alpha \mapsto T_0(\alpha)$, rewrite:

$$\text{dist}(x(t)_{\alpha(t)}/t, X^0) \to 0 \quad \text{as} \quad t \to 0 \ . \tag{4.1}$$

The convergence (4.1) demonstrates the effect of approaching a solution with time shrinking to zero (the effect of "immediate solution"). Here the approximating points $x(t)_{\alpha(t)}$ finalize at times t trajectories under the different contingent feedbacks $\mathcal{U}_{\alpha(t)}$. We now ask, whether there exists a contingent feedback $\mathcal{U}$ such that the "immediate solution" property

$$\text{dist}(x(t)/t, X^0) \to 0 \quad \text{as} \quad t \to 0 \tag{4.2}$$

holds for all trajectories under $\mathcal{U}$. Below, we answer positively.

4.2 Regularized Contingent Half-Space Shifting

In this section, we assume the following.

Condition 4.1 The multi-valued map Q is continuous on $]0,\infty[\times \mathbf{R}^n$ and upper semicontinuous at every point of $\{0\} \times \mathbf{R}^n$.

Remark 4.1 The feedbacks considered below have values no broader than those of Q. Trajectories under these feedbacks do not depend on values of Q at points of $\{0\} \times \mathbf{R}^n$, and therefore the requirement of upper semicontinuity of Q at these points can be omitted. Inversely, being imposed, it does not reduce generality.

In this subsection we assume the following.

Condition 4.2 The set-valued map $(t,x) \mapsto Q(t,x) \cap L^-(t,x)$ is continuous on

$$L^+ = \{(t,x) \ : \ t > 0, \ x \in \mathbf{R}^n, \ \mid Fx - tb \mid^2 > 0\} \ . \tag{4.3}$$

Remark 4.2 One cannot guarantee continuity of the above set-valued map unless $Q(t,x) \subset L^0$. For example, for a one-dimensional problem (1.1) with the constraints $x \in [-1,1]$, $x = 0$, and $Q(t,x) = [-1,1]$, the set $Q(t,x) \cap L^-(t,x)$ equals $[-1,0]$ if $x > 0$, and $[0,1]$ if $x < 0$; there is no continuity at points $(t,0)$. Note that lack of continuity of $(t,x) \mapsto Q(t,x) \cap L^-(t,x)$ is a reason for nonexistence (in nontrivial cases) of trajectories under the half-space shifting contingent feedback $\mathcal{U}_0$ (Corollary 3.2): if the above map were continuous, then by Lemma 3.1 the set-valued map $\mathcal{U}_0$ would be semicontinuous implying existence of a trajectory.

Let us give a condition sufficient for Condition 4.2.

Lemma 4.1 *Let Condition 4.1 be satisfied, and for every $(t,x) \in L^+$ there exist an inner point of $Q(t,x)$ belonging to L^0. Then Condition 4.2 is satisfied.*

Proof. Continuity of Q on L^+ following from Condition 4.1 and obvious upper semicontinuity of the map L^- imply that the map $(t,x) \mapsto Q(t,x) \cap L^-(t,x)$ is upper semicontinuous on L^+. Let us show its lower semicontinuity on L^+. Represent $Q(t,x) \cap L^-(t,x) = \{u \in L^-(t,x) \ : u \in Q(t,x)\}$, and use Aubin and Frankowska [9, Proposition 1.5.2]. This states desired lower semicontinuity under the following conditions: (i) the set-valued map L^- is lower semicontinuous on L^+, (ii) for every $(t,x) \in L^+$, $Q(t,x)$ is convex and has the nonempty interiour $\mathrm{int}Q(t,x)$, (iii) the graph of the map $(t,x) \mapsto \mathrm{int}Q(t,x)$ on L^+ is open, and (iv) for every $(t,x) \in L^+$ there is an $u \in L^-(t,x) \cap \mathrm{int}Q(t,x)$. Condition (i) is easily verified (see (2.38)). Conditions (ii) and (iv) follow from the assumptions. It is sufficient to prove condition (iii). Take an $(t_*,x_*) \in L^+$ and an $u_* \in \mathrm{int}Q(t,x)$. Let $\epsilon > 0$ be the radius of a neighborhood of u_* contained in $Q(t_*,x_*)$. Denoting by $\Delta(\cdot \mid D)$ the support function of a set $D \subset \mathbf{R}^n$ (see [Rockafellar [10, Section 13]), we get

$$\Delta(\psi \mid Q(t_*,x_*)) - \epsilon \mid \psi \mid \geq \psi^T u_* \quad (\psi \in \mathbf{R}^n) .$$

Continuity of the map Q implies that for all (τ,ξ) sufficiently close to (t_*,x_*), it holds that

$$\Delta(\psi \mid Q(\tau,\xi)) \geq \Delta(\psi \mid Q(t_*,x_*)) - \epsilon/2 \quad (\psi \in \mathbf{R}^n) .$$

These two inequalities yield that that for all (τ,ξ) sufficiently close to (t_*,x_*), the $(\epsilon/2)$-neighborhood of u lies in $Q(\tau,\xi))$. Thus, (t_*,x_*,u_*) belongs to the interior of the graph of the map $(t,x) \mapsto \mathrm{int}Q(t,x)$. This graph is therefore open. □.

Example 4.1 Consider the problem of linear or quadratic programming. The problem in equivalent setting (1.2) or (1.6) will be referred to as the *initial* one. Let, as in Examples 2.1 and 2.2, $Q(t,x) = M$ and M be given by (2.28), (2.29), (2.30). Let, besides, M_2 (2.30) be such that $L^j > s^j$ $(j = 1,...,r)$ for all nonnegative $s \in \mathbf{R}^r$ satisfying $Ay+s = b$ with a y feasible for the initial problem. Let, finally, be a strictly feasible point (y_0,s_0) (see Remark 2.3). Then the assumption of Lemma 4.1 is satisfied, namely $(y_0,s_0) \in Q(t,x) \cap L^0$; consequently, by Lemma 4.1 Condition 4.2 is true. Indeed, feasibility of (y_0,s_0) implies $(y_0,s_0) \in L^0$. Its strict feasibility implies that, first, all y from a neighborhood of y_0 are feasible for the initial problem, and thus belong to M_1, and, second, $s_0^{(j)} > 0$ $(j = 1,...,r)$. Due to the above property M_2, a neighborhood of s_0 lies in M_2. Thus, (y_0,s_0) is an inner point of $M_1 \times M_2 = M = Q(t,x)$.

We build a contingent feedback guaranteeing the "immediate solution" effect (4.2) as a modified (regularized) half-space shifting contingent feedback $\mathcal{U}_0$. Regularization allows us, simultaneously, to overcome the fundamental drawback of $\mathcal{U}_0$, i.e. nonexistence of trajectories in nontrivial cases (see subsection 3.4); note that the stated below "immediate solution" effect (4.2) is weaker than $x(t)/t \in X^0$ $(t > 0)$ guaranteed for the trajectories $x(\cdot)$ under $\mathcal{U}_0$, provided they exist.

Fix a scalar continuous monotonically increasing function $\rho(\cdot)$ on $\mathbf{R}_+$ such that $\rho(t) > 0$ for $t > 0$, and

$$\rho(t)/t^2 \to 0 \quad \text{as} \quad t \to 0 \,. \tag{4.4}$$

Define the *regularized half-space shifting feedback* $\mathcal{U}_*$ by

$$\mathcal{U}_*(t,x) = \begin{cases} \mathcal{U}_0(t,x), & \mid Fx - tb \mid^2 \geq \rho(t),\ t > 0 \\ Q_0(t,x), & \mid Fx - tb \mid^2 \leq \rho(t)/2,\ t > 0 \\ (1-\omega(t,x))\mathcal{U}_0(t,x) & \mid Fx - tb \mid^2 = (1-\omega(t,x))\rho(t) \\ \quad + \omega(t,x)Q_0(t,x), & \quad + \omega(t,x)\rho(t)/2,\ \omega(t,x) \in]0,1[,\ t > 0 \\ Q(0,x), & t = 0 \,. \end{cases} \tag{4.5}$$

Recall that $Q_0(t,x)$ is the solution set to the problem (3.9). Observing the third line on the right of (4.5), note that whenever $\mid Fx - tb \mid^2 \in]\rho(t)/2, \rho(t)[$ $(t > 0)$,

$$\omega(t,x) = 2\frac{\rho(t) - \mid Fx - tb \mid^2}{\rho(t)} \,. \tag{4.6}$$

Lemma 4.2 *The regularized half-space shifting feedback $\mathcal{U}_*$ is closed- and convex-valued and upper semicontinuous.*

Proof. The last line on the right of (4.5) shows that at $\{0\} \times \mathbf{R}^n$ the map $\mathcal{U}_*$ is closed- and convex-valued; in view of Condition 4.1 $\mathcal{U}_*$ is upper semicontinuous at every point of this set. Let E_1 E_2, E_3 be the sets of all $(t,x) \in]0,\infty[\times \mathbf{R}^n$ such that $\mid Fx - tb \mid^2$ belongs, respectively, to $[\rho(t),\infty[$, $]0,\rho(t)/2]$, $[\rho(t)/2,\rho(t)]$. Define E_1^0, E_2^0, E_3^0 as, respectively, E_1 E_2, E_3 united with $\{0\} \times \mathbf{R}^n$. Clearly, E_1^0 E_2^0, E_3^0 are closed and their union is $\mathbf{R}_+ \times \mathbf{R}^n$. Therefore thanks to Lemma 3.2 it is sufficient to prove that $\mathcal{U}_*$ is closed- and convex-valued and upper semicontinuous at each of these sets. Closed- and convex-validity and upper semicontinuity of $\mathcal{U}_*$ on $\{0\} \times \mathbf{R}^n\}$ were stated above. Hence, in view of the definition of E_1^0 E_2^0, E_3^0, it remaines to verify these properties on E_1 E_2, E_3. Condition 4.2 implies that the set-valued map $(t,x) \mapsto Q(t,x) \cap L^-(t,x)$ is continuous on E_1. Consequently, by Lemma 3.1 the half-space shifting feedback $\mathcal{U}_0$ (2.39) is closed- and convex-valued and upper semicontinuous on E_1. According to (4.5) $\mathcal{U}_*$ and $\mathcal{U}_0$ have the same restriction to E_1. Hence $\mathcal{U}_*$ is closed- and convex-valued and upper semicontinuous on E_1. Continuity of Q (Condition 4.1) and Lemma 3.1 imply that the set-valued map Q_0 is closed- and convex-valued and upper semicontinuous. By (4.5) $\mathcal{U}_*$ and Q_0 have the same restriction to E_2. Hence $\mathcal{U}_*$ is closed- and convex-valued and upper semicontinuous on E_2 . Consider finally the restriction of $\mathcal{U}_*$ to E_3. Take a $(t,x) \in E_3$. By (4.5)

$$\mathcal{U}_*(t,x)(1-\omega(t,x))\mathcal{U}_0(t,x) + \omega(t,x)Q_0(t,x) \,.$$

Convexity and closedness of $\mathcal{U}_0(t,x)$ and $Q_0(t,x)$ imply those of $\mathcal{U}_*(t,x)$. Show semicontinuity of the restriction of $\mathcal{U}_*$ to E_3 at point (t,x). Take an arbitrary $\epsilon > 0$. It must be shown that there is a neighborhood B of (t,x) such that for every $(\tau,\xi) \in B \cap E_3$, we have

$$\sup\{\text{dist}(u,\mathcal{U}_*(t,x)) \ : \ u \in \mathcal{U}_*(\tau,\xi))\} < \epsilon \,. \tag{4.7}$$

Note that $(t,x) \in L^+$ (see (4.3)). Hence, in view of Condition 4.2 and Lemma 3.1, $\mathcal{U}_0$ is upper semicontinuous at (t,x). Then, due to continuity of $\omega(\cdot)$ in a neighborhood of (t,x)

(see (4.6)) the map $(\tau, \xi) \mapsto (1 - \omega(\tau, \xi))\mathcal{U}_0(\tau, \xi)$ is also upper semicontinuous at (t, x). This means that

$$\sup\{\text{dist}(u, (1 - \omega(t, x))\mathcal{U}_0(t, x)) \ : \ u \in (1 - \omega(\tau, \xi))\mathcal{U}_0(\tau, \xi))\} < \epsilon/2 \ .$$

for all (τ, ξ) from a certain neighborhood B_1 of (t, x). As noted above, Q_0 is upper semicontinuous. Hence, in view of continuity of $\omega(\cdot)$ (in a neighborhood of (t, x)) we conclude that

$$\sup\{\text{dist}(u, \omega(t, x)Q_0(t, x)) \ : \ u \in \omega(\tau, \xi)Q_0(\tau, \xi))\} < \epsilon/2$$

for all (τ, ξ) from a certain neighborhood B_2 of (t, x). Observing the third line of (4.5), we see that for all $(\tau, \xi) \in B_1 \cap B_2$, the inequality (4.7) is satisfied. □

Remark 4.3 The proof of Lemma 4.2 does not use monotonicity of $\rho(\cdot)$ and the assumption (4.4).

Theorem 4.1 *1) There exists a trajectory $x(\cdot)$ under the regularized half-space shifting feedback $\mathcal{U}_*$.*

2) For every trajectory $x(\cdot)$ under $\mathcal{U}_*$, it holds that

$$\text{dist}(x(t)/t, X^0) \leq O(\rho(t)/t^2, 0) \quad (t > 0) \tag{4.8}$$

$$J^0[\rho(t)/t^2] - J^0 \leq J(x(t)/t) - J^0 \leq 0 \quad (t > 0) \ . \tag{4.9}$$

3) The convergence (4.2) holds uniformly with respect to all trajectories $x(\cdot)$ under $\mathcal{U}_*$.

Proof. Assertion 1 is justified by Lemma 4.2. Assertion 3 follows straightforwardly from assertion 2, and the convergences (4.4) and (2.10). Prove assertion 2. Take a trajectory $x(\cdot)$ under $\mathcal{U}_*$. We have

$$| \ Fx(t) - bt \ |^2 \leq \rho(t) \tag{4.10}$$

for all $t \geq 0$. Indeed, suppose that this is untrue. Then, as long as for $t = 0$ (4.10) is satisfied, there exist $\tau_1 > 0$ and $\tau_2 > 0$ such that

$$d(t) = | \ Fx(t) - tb \ |^2 > \rho(t) \quad (t \in [\tau_1, \tau_2]) \tag{4.11}$$

$$d(\tau_2) > d(\tau_1) \ . \tag{4.12}$$

From (4.11) and the definition of $\mathcal{U}_*$ (4.5) it follows that

$$\dot{x}(t) \in \mathcal{U}_0(t, x(t)) \tag{4.13}$$

for almost all $t \in [\tau_1, \tau_2]$. Hence, referring to the definition of $\mathcal{U}_0$ (2.39), we deduce that for the above t, we have $\dot{x}(t) \in L^-(t, x(t))$ yielding $\dot{d}(t) \leq 0$ (see the definition of $L^-(t, x)$ (2.38)). This contradicts (4.12), and (4.10) is proved. Let us show that

$$J(\dot{x}(t)) \leq J^0 \tag{4.14}$$

for almost all t, namely all $t > 0$ such that $\dot{x}(t) \in \mathcal{U}_*(t, x(t))$. According to (4.5), this inclusion takes either the form (4.13), or one of the forms

$$\dot{x}(t) \in Q_0(t, x(t)) \tag{4.15}$$

$$\dot{x}(t) \in (1 - \omega(t, x(t))\mathcal{U}_0(t, x(t)) + \omega(t, x(t))Q_0(t, x(t)) \tag{4.16}$$

where

$$\omega(t, x(t)) \in [0, 1] \tag{4.17}$$

(see (4.6)). Let (4.13) hold. Observe the definition of $\mathcal{U}_0$ (3.8) and note that due to (2.2), (2.1), and (2.38), the set $Q(t, x(t)) \cap L^-(t, x(t))$ contains the feasible set $M \cap L^0$ of the problem (1.1). This and (4.13) straightforwardly imply (4.14). If (4.15) holds, then we have (4.14) as the feasible set $Q(t, x)$ of the problem (3.9) contains that of (1.1). Consider finally the case (4.16). We have

$$\dot{x}(t) = (1 - \omega(t, x(t)))u_1 + \omega(t, x(t))u_2 \tag{4.18}$$

where $u_1 \in \mathcal{U}_0(t, x(t))$, $u_2 \in Q_0(t, x(t))$. Repeating the speculations used for the cases (4.13) and (4.15), we obtain

$$J(u_1) \leq J^0, \quad J(u_2) \leq J^0$$

which together with (4.18) and (4.17) yield (4.14) in view of convexity of J. Consider the obtained inequalities (4.10) and (4.14). From (4.10) follows

$$| Fx(t)/t - b |^2 \leq \rho(t)/t^2 \quad (t > 0) . \tag{4.19}$$

From (4.14) and convexity of J we deduce

$$J(x(t)/t) \leq \frac{1}{t} \int_0^t J(\dot{x}(\tau))d\tau \leq J^0 \quad (t > 0) . \tag{4.20}$$

Recalling the definitions of $O(\cdot)$ (2.6) and $J^0[\cdot]$, we see that the inequalities (4.19) and (4.20) imply (4.8), (4.9) □

Example 4.2 Consider the problem of linear programming under the conditions of Example 2.3. Recall that in Example 3.3 it was shown that under the conditions of Example 2.3 there does not exist a trajectory under the half-space shifting contingent feedback $\mathcal{U}_0$ unless a very unnatural condition is fulfilled.

Assume that there exists a strictly feasible point (see Remark 2.3). Remind that M and Q are given by (2.28), (2.53), (2.61), (2.30). The notations (2.34), (2.35) are assumed, and $\beta(t, y, s)$ stands for the value of the problem (2.56) given by (2.62). Condition 4.1 is clearly satisfied. Following Example 4.1, assume that M_2 is wide enough, namely, M_2 contains a neighborhood of the set of all s with nonnegative coordinates such that $Ay + s = b$ for a certain y feasible for the initial problem (1.2). Then, like in Example 4.1, we state that the assumption of Lemma 4.1 is satisfied; therefore Condition 4.2 is fulfilled. Consequently, Theorem 4.1 is true.

Describe the regularized half-space shifting contingent feedback $\mathcal{U}_*$. First build $\mathcal{U}_0$. All elements $U_0(t, y, s)$ of $\mathcal{U}_0(t, y, s)$ are described in the last paragragh of Example 2.3. Rewrite it in a more explicit manner. For $p(t, y, s) \neq 0$, assume the notations (2.64). For the case $p(t, y, s) \neq 0$, $c_1(t, y, s) \neq 0$, introduce the notations (2.72) through (2.78). For the case $p(t, y, s) \neq 0$, $c_1(t, y, s) = 0$, rearrange (2.80):

$$\mathcal{V}_1(t, y, s) = \begin{cases} \{g_2(t, y, s)\}, & a_0(t, y, s) > 0 \\ \{g_1(t, y, s)\}, & a_0(t, y, s) < 0 \\ [g_1(t, y, s), g_2(t, y, s)], & a_0(t, y, s) = 0 . \end{cases} \tag{4.21}$$

The set of all $V_0(t,y,s)$ described in Example 2.3 is then

$$\mathcal{V}_0(t,y,s) = \begin{cases} \{-R\frac{c}{|c|}\}, & p(t,y,s)=0 \\ \{\nu_1(t,y,s)p(t,y,s)+v_1(t,y,s)\}, & p(t,y,s)\neq 0,\ c_1(t,y,s)\neq 0 \\ \nu_1(t,y,s)p(t,y,s)+\mathcal{V}_1(t,y,s), & p(t,y,s)\neq 0,\ c_1(t,y,s)=0 \end{cases} . \tag{4.22}$$

Let now $V_0(\cdot)$ be an arbitrary selector of $\mathcal{V}_0$, i.e.

$$V_0(t,y,s) \in \mathcal{V}_0(t,y,s) \tag{4.23}$$

and

$$\mathcal{W}(t,y,s) = Q_2(t,y,s,V_0(t,y,s)) . \tag{4.24}$$

Recall that Q_2 is defined by (2.55). The half-space shifting contingent feedback $\mathcal{U}_0$ is given by

$$\mathcal{U}_0(t,y,s) = \mathcal{V}_0(t,y,s) \times \mathcal{W}(t,y,s) . \tag{4.25}$$

It is easily seen that

$$Q_0(t,y,s) = \left\{-R\frac{c}{|c|^2}\right\} \times M_2 .$$

Thus the formula (4.5) for the regularized half-space shifting contingent feedback takes the form

$$\mathcal{U}_*(t,y,s) = \mathcal{V}_*(t,y,s) \times \mathcal{W}_*(t,y,s) \tag{4.26}$$

where

$$\mathcal{V}_*(t,y,s) = \begin{cases} \mathcal{V}_0(t,y,s), & |q(t,y,s)|^2 \geq \rho(t),\ t>0 \\ \left\{-R\frac{c}{|c|^2}\right\}, & |q(t,y,s)|^2 \leq \rho(t)/2,\ t>0 \\ (1-\omega(t,y,s))\mathcal{V}_0(t,y,s)) & \\ \quad +\omega(t,y,s) - R\frac{c}{|c|^2}, & \rho(t)/2 < |q(t,y,s)|^2 < \rho(t),\ t>0 \\ B(0,R), & t=0 \end{cases} \tag{4.27}$$

$$\mathcal{W}_*(t,y,s) = \begin{cases} \mathcal{W}(t,y,s), & |q(t,y,s)|^2 \geq \rho(t),\ t>0 \\ M_2, & |q(t,y,s)|^2 \leq \rho(t)/2,\ t>0 \\ (1-\omega(t,y,s))\mathcal{W}(t,y,s)) & \\ \quad +\omega(t,y,s)M_2, & \rho(t)/2 < |q(t,y,s)|^2 < \rho(t),\ t>0 \\ M_2, & t=0 . \end{cases} \tag{4.28}$$

Here $\omega(\cdot)$ is defined in accordance with (4.6):

$$\omega(t,y,s) = 2\frac{\rho(t) - |q(t,y,s)|^2}{\rho(t)} . \tag{4.29}$$

By Theorem 4.1 every trajectory $x(\cdot) = (y(\cdot),s(\cdot))$ under $\mathcal{U}_*$ posesses the "immediate solution" property (4.2).

Example 4.3 Consider the problem of quadratic programming under the assumptions of Example 4.2. As it is shown in this Example, Conditions 4.1 and 4.2 are satisfied, and therefore Theorem 4.1 is true. We use notations of Example 4.2. Like in Example 2.4, assume $-c/2 \in M_1$. Combining constructions of Examples 4.2 and 2.4, we obtain that the regularized half-space shifting contingent feedback $\mathcal{U}_*$ is determined by (4.26), (4.27), (4.28), (4.29) where $\mathcal{W}$ is given by (4.24), and

$$\mathcal{V}_0 = \{V_0(t,y,s)\} . \tag{4.30}$$

with $V_0(t,y,s)$ defined, as in Example 2.4, through (2.84) if (2.83) holds, and (2.85), (2.86) otherwise.

4.3 Weakly Regularized Contingent Half-Space Shifting

Let us provide another regularization of the half-space shifting contingent feedback having the same property as $\mathcal{U}_*$. The function $\rho(\cdot)$ introduced in the previous subsection is assumed here to be strictly monotonically increasing and continuously differentiable on $]0,\infty[$. We also assume Condition 4.1. Denote

$$L^-(t,x \mid \gamma) = \{u \in \mathbf{R}^n \ : \ 2(Fx-tb)^T(Fu-b) \le \gamma\} . \tag{4.31}$$

Assume the following.

Condition 4.3 The set-valued map $(t,x) \mapsto Q(t,x) \cap L^-(t,x \mid \dot\rho(t))$ is continuous on L^+.

The condition of Lemma 4.1 sufficient for Condition 4.2 is sufficient for Condition 4.3 too:

Lemma 4.3 *Let Condition 4.1 be satisfied, and for every $(t,x) \in L^+$ there exist an inner point of $Q(t,x)$ belonging to L^0. Then Condition 4.3 is satisfied.*

The proof is similar to that of Lemma 4.1.

Let

$$\mathcal{U}_{0w}(t,x) \in \operatorname{argmin}\{J(u) \ : \ u \in Q(t,x) \cap L^-(t,x \mid \dot\rho(t))\} \quad (t>0) \tag{4.32}$$

and K_M be a constant such that $K_M \ge \mid u \mid$ for all $u \in M$. Define the *weakly regularized half-space shifting feedback* $\mathcal{U}_{*w}$ by

$$\mathcal{U}_{*w}(t,x) = \begin{cases} \mathcal{U}_{0w}(t,x), & 4K_M \mid F^T(Fx-tb) \mid \ge \dot\rho(t), \ t>0 \\ Q_0(t,x), & 4K_M \mid F^T(Fx-tb) \mid \le \dot\rho(t)/2, \ t>0 \\ (1-\omega(t,x))\mathcal{U}_{0w}(t,x) & 4K_M \mid F^T(Fx-tb) \mid = \\ \quad +\,\omega(t,x)Q_0(t,x), & \quad (1-\omega(t,x))\dot\rho(t) + \omega(t,x)\dot\rho(t)/2 . \\ & \quad \omega(t,x) \in]0,1[, \ t>0 \\ Q(0,x), & t=0 \end{cases} \tag{4.33}$$

Note that

$$\omega(t,x) = 2\frac{\dot\rho(t) - 4K_M \mid Fx-tb \mid}{\dot\rho(t)} . \tag{4.34}$$

Lemma 4.4 *The weakly regularized half-space shifting feedback $\mathcal{U}_{*w}$ is closed- and convex-valued and upper semicontinuous.*

We omit the proof; it copies that of Lemma 4.2 with using Condition 4.3 instead of Condition 4.2.

Theorem 4.2 *1) There exists a trajectory $x(\cdot)$ under the weakly regularized half-space shifting feedback $\mathcal{U}_{*w}$.*

2) For every trajectory $x(\cdot)$ under $\mathcal{U}_{*w}$, the estimates (4.8) (4.9) take place.

3) The convergence (4.2) holds uniformly with respect to all trajectories $x(\cdot)$ under $\mathcal{U}_{*w}$.

Proof. Assertion 1 is justified by Lemma 4.4. Assertion 3 follows straightforwardly from assertion 2, and the convergences (4.4) and (2.10). Prove assertion 2. Take a trajectory

$x(\cdot)$ under $\mathcal{U}_{*w}$. Let us show that for all $t \geq 0$ the inequality (4.10) holds. It is sufficient to establish that for all $t > 0$ such that $\dot{x}(t) \in \mathcal{U}_{*w}(t, x(t))$,

$$\dot{d}(t) \leq \dot{\rho}(t) \tag{4.35}$$

where $d(t) = \mid Fx(t) - bt \mid^2$. Let for a t and $x = x(t)$ the condition of the first line in the definition of $\mathcal{U}_{*w}$ (4.33) be satisfied. Then $\mathcal{U}_{*w}(t, x(t)) = \mathcal{U}_{0w}(t, x(t))$; hence by (4.32) $\dot{x}(t) \in L^-(t, x(t) \mid \dot{\rho}(t))\}$ implying $\dot{d}(t) = 2(Fx(t) - tb)^T(F\dot{x}(t) - b) \leq \dot{\rho}(t)$. Thus (4.35) is verified. Let for t and $x = x(t)$ the condition of the second or third line in the definition of $\mathcal{U}_{*w}$ (4.33) be fulfilled, i.e. the vector

$$p = 2F^T(Fx(t) - tb)$$

satisfies

$$2K_M \mid p \mid \leq \dot{\rho}(t)\ . \tag{4.36}$$

Take an arbitrary $u_0 \in L^0 \cap M$; using $Fu_0 - b = 0$, proceed as follows:

$$\begin{aligned} \dot{d}(t) &= 2(Fx(t) - tb)^T(F\dot{x}(t) - b) \\ &= 2(Fx(t) - tb)^T(F\dot{x}(t) - b) - 2(Fx(t) - tb)^T(Fu_0 - b) \\ &= [p^T\dot{x}(t) - 2(Fx(t) - tb)^T b] - [p^T u_0 - 2(Fx(t) - tb)^T b] \\ &\leq 2K_M \mid p \mid \leq \dot{\rho}(t) \end{aligned}$$

the last inequality is provided by (4.36). Therefore (4.35) is established. Consequently, the estimate (4.10) is true. Now copying the proof of Lemma 4.2 (with replacing $\mathcal{U}_*$ by $\mathcal{U}_{*w}$), we obtain that for almost all $t \geq 0$, the inequality (4.14) holds; (4.10) and (4.14) lead to (4.19) and (4.20) which by the definition of $O(\cdot)$ (2.6) and $J^0[\cdot]$ give (4.8), (4.9). □

Example 4.4 Consider the problem of linear programming under the assumptions of Example 4.2. Condition 4.1 is obviously satisfied. As it is shown in the above example, the assumption of Lemma 4.3 is fulfilled; thus by this Lemma, Condition 4.3 is satisfied. Hence Theorem 4.2 is true. Modifying constructions of Examples 4.2 and 2.3, obtain a representation for the contingent feedback $\mathcal{U}_{0w}$ (4.32). For $p(t, y, s) \neq 0$, assume the notations (2.64). For the case $p(t, y, s) \neq 0$, $c_1(t, y, s) \neq 0$, introduce the notations (2.72) through (2.75), (2.77), and (2.78); the notation (2.76) is modified by replacing $\beta(t, y, s)$ with $\beta(t, y, s) + \dot{\rho}(t)$:

$$g_2(t, y, s) = \min\left\{\frac{a_1(t, y, s)^{1/2}}{a_2(t, y, s)^{1/2}}, \frac{\beta(t, y, s) + \dot{\rho}(t)}{\mid p(t, y, s) \mid^2}\right\}\ . \tag{4.37}$$

Finally, $Q_2(t, y, s, v)$ is modified as follows:

$$Q_2(t, y, s, v) = \{w \in M_2\ :\ p(t, y, s)^T v \leq -q(t, y, s)^T w + \dot{\rho}(t)\}\ . \tag{4.38}$$

The formula for $\mathcal{U}_{0w}$ is analogous to (4.25):

$$\mathcal{U}_{0w}(t, y, s) = \mathcal{V}_0(t, y, s) \times \mathcal{W}(t, y, s)\ ; \tag{4.39}$$

here $\mathcal{V}_0$ is defined by (4.22), (4.21), and $\mathcal{W}$ by (4.24), (4.23). Let

$$h(t, y, s) = (p(t, y, s), q(t, y, s)) \tag{4.40}$$

$$K_M = \left(R^2 + \sum_{j=1}^{r} (P^{(j)})^2\right)^{1/2} \qquad (4.41)$$

(see (2.29), (2.30)). Note that in view of (1.4) $F^T(F(y,s) - tb) = h(t,y,s)$, and K_M satisfies the definition given above in this subsection. Then $\omega(t,y,s)$ (4.34) takes the form

$$\omega(t,x) = 2\frac{\dot{\rho}(t) - 4K_M \mid h(t,y,s) \mid}{\dot{\rho}(t)}. \qquad (4.42)$$

For the weakly regularized half-space shifting feedback $\mathcal{U}_{*w}$ (4.33), we have the representation

$$\mathcal{U}_{*w}(t,y,s) = \mathcal{V}_*(t,y,s) \times \mathcal{W}_*(t,y,s) \qquad (4.43)$$

where

$$\mathcal{V}_*(t,y,s) = \begin{cases} \mathcal{V}_0(t,y,s), & 2K_N \mid h(t,y,s) \mid \geq \dot{\rho}(t),\ t > 0 \\ \left\{-R\frac{c}{|c|^2}\right\}, & 2K_N \mid h(t,y,s) \mid \leq \dot{\rho}(t)/2,\ t > 0 \\ (1 - \omega(t,y,s))\mathcal{V}_0(t,y,s)) & \\ \quad + \omega(t,y,s) - R\frac{c}{|c|^2}, & \dot{\rho}(t)/2 < \mid h(t,y,s) \mid < \dot{\rho}(t),\ t > 0 \\ B(0,R), & t = 0 \end{cases} \qquad (4.44)$$

$$\mathcal{W}_*(t,y,s) = \begin{cases} \mathcal{W}(t,y,s), & 2K_N \mid h(t,y,s) \mid \geq \dot{\rho}(t),\ t > 0 \\ M_2, & 2K_N \mid h(t,y,s) \mid \leq \dot{\rho}(t)/2,\ t > 0 \\ (1 - \omega(t,y,s))\mathcal{W}(t,y,s)) & \\ \quad + \omega(t,y,s)M_2, & \dot{\rho}(t)/2 < \mid h(t,y,s) \mid < \dot{\rho}(t),\ t > 0 \\ M_2, & t = 0 \end{cases} \qquad (4.45)$$

By Theorem 4.2 every trajectory $x(\cdot) = (y(\cdot), s(\cdot))$ under $\mathcal{U}_{*w}$ posesses the "immediate solution" property (4.2).

Example 4.5 Consider the problem of quadratic programming under the assumptions of Example 4.4 (4.2). Conditions 4.1 and 4.3 are satisfied, and therefore Theorem 4.2 is true. Use notations of Example 4.4. Assume $-c/2 \in M_1$. Combining constructions of Examples 4.4 and 2.4, we deduce that the weakly regularized half-space shifting contingent feedback $\mathcal{U}_*$ is determined by (4.43), (4.44), (4.45) (4.42); here $\mathcal{W}$ is given by (4.24), (4.38) and $\mathcal{V}_0$ has the form (4.30) with $V_0(t,y,s)$ defined through (2.84) if

$$-c^T p(t,y,s) \leq 2(\beta(t,y,s) + \dot{\rho}(t))$$

(replacing (2.83) from Example 4.3), and (2.85), (2.86) otherwise.

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