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Regulation of Oscillatory Systems Subjected to Viability Constraints

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WP-94-96
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REGULATION OF OSCILLATORY SYSTEMS SUBJECTED TO VIABILITY CONSTRAINTS

Katharina Müllers *

Abstract : Regarding forcing terms of oscillatory systems of second–order differential equations as controls, we look for feedback (set–valued) maps governing the evolution of solutions satisfying viability or state constraints. We give a condition ensuring the existence of minimally forced solutions: the oscillation is unforced as long as viability is not at stake. Finally we compare "minimally forced oscillations" with free oscillations.

Key words: Controlling oscillations, differential inclusions, viability theory.

1 Motivation and introductory example

In our paper, we study forced oscillations

(1) $$x''(t) + \omega(x(t), x'(t)) = \gamma(t)$$

as control problems. We regard the forcing term γ as a control for keeping the state $x(\cdot)$ in a given constraint set K. The first restriction we impose is a bound on the forcing term

(2) $$\|\gamma(t)\| \leq \varphi(t, x(t), x'(t))$$

for all $t \geq 0$, where φ is a positive continuous function. In other words, the set of feasible controls at time t and state $(x(t), x'(t))$ is given by the set $\varphi(t, x(t), x'(t))B$, where B denotes the unit ball in the state space. This is a differential inclusion:

(3) $$x''(t) + \omega(x(t), x'(t)) \in \varphi(t, x(t), x'(t))B,$$

where $(x(t), x'(t)) \in K$ for all $t \geq 0$.

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To explain the main questions, we consider the following simple example of a two dimensional linear oscillatory differential inclusion:

$$(4) \qquad x''(t) - 2ax(t) + \omega^2 x(t) \in [-c, c],$$

where

$$(5) \qquad x(t) \in [-b, b] \qquad \forall t \geq 0.$$

The parameter $0 < a < \omega$ defines the oscillation, $b > 0$ the constraint set and $c > 0$ the bound of feasible forcing terms. If the forcing term is equal to 0, the solution $x(\cdot)$ has the form

$$(6) \qquad x(t) = e^{at}(\alpha \cos(\kappa t) + \beta \sin(\kappa t)),$$

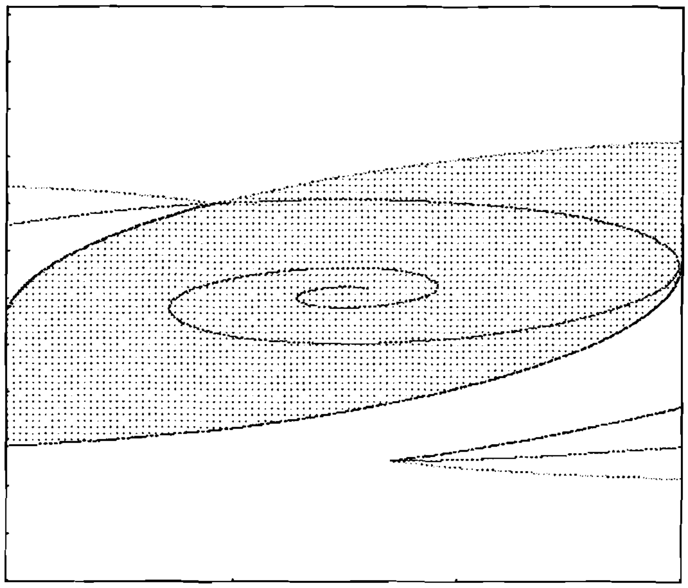
where $\kappa = \sqrt{\omega^2 - a^2}$ and where α and β depend on the initial condition. Therefore all unforced oscillations starting from $(x_0, x_0')) \neq (0,0)$ "explode", and the trajectories $x(\cdot)$ leave the intervall $[-b, b]$ in finite time. The only unforced solution remaining "viable" in the interval $[-b, b]$ is the equilibrium solution $x = 0$.

When we allow the forcing term to vary in the interval $[-c, c]$, the set $\mathcal{D}$ of all initial conditions from which starts a solution of (4) viable in $[-b, b]$ is much bigger (see figure 1). All solutions starting outside of $\mathcal{D}$ have to leave the constraint set in finite time. For all initial conditions in the interior of $\mathcal{D}$, we can find a solution regulated by a minimal forcing term, i. e., a forcing term equal to 0 as long as the viability of the solution is not at stake. Figure 1 shows such a solution: it oscillates freely until arriving at the boundary of the set $\mathcal{D}$, where it has to be forced with maximal force $|c|$ not to leave $\mathcal{D}$ and thus, not to leave the constraint set.

In our paper, for the general oscillatory differential inclusion (3), we shall

- characterize those closed sets of initial conditions from which starts at least one solution of (3) viable in K,
- characterize the solutions remaining in K by a differential equation or inclusion,
- find a feedback regulation map defining the forcing terms providing a viable evolution,

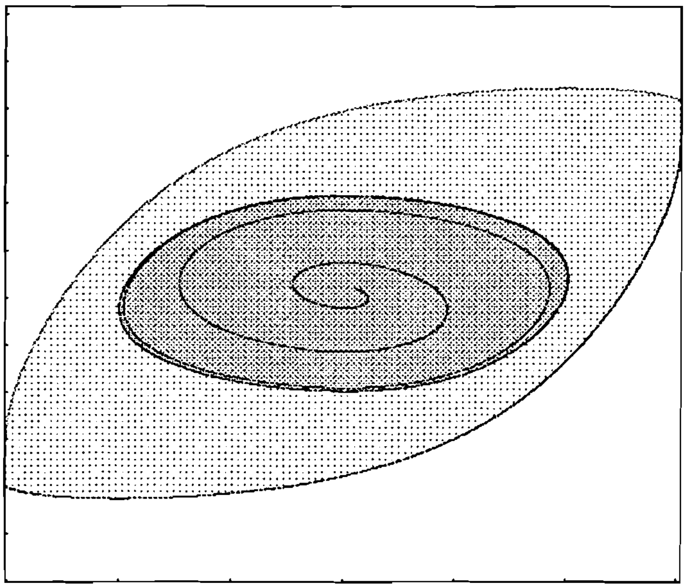
Figure 1:



- establish the existence of solutions regulated by a minimal forcing term, i. e., a forcing term equal to 0 as long as the viablity of the solution is *not* at stake,
- look for "free oscillatory cells", i. e., subsets of K from which a viable unforced oscillation is possible.

In the previous example, the free oscillatory cell contains only the equilibrium $(0, 0)$. In general, it can be bigger; figure 2 shows the set $\mathcal{D}$ together with its free oscillation cell for another oscillatory differential inclusion, which will be studied in section 6.

Figure 2: Free oscillation cell



2 Some basic definitions

We shall introduce some basic notations to set the problem in a general framework.

Definition 2.1 *A* constrainted oscillatory differential inclusion *is a second order differential inclusion*

(7) $$x''(t) + \omega(x(t), x'(t)) \in \varphi(x(t), x'(t))B,$$

where

(8) $$(x(t), x'(t)) \in K \qquad \forall t \geq 0.$$

The set K is a subset of the phase space $X \times X$, the function $\omega : K \longrightarrow X$ describes the dynamic of the system, B denotes the unit ball of X, and $\varphi : K \longrightarrow \mathbb{R}^+$ defines the set of feasible forcing terms.

We want to associate with a constrainted oscillatory differential inclusion a first order differential inclusion. This can be realized by introducing an auxiliary function $u(\cdot)$ depending on the state $x(\cdot)$ and its derivative $x'(\cdot)$ in the following manner:

Definition 2.2 *A* decomposition *of the oscillatory differential inclusion (7) is a differential inclusion*

$$(9) \qquad \begin{cases} i. \quad x'(t) + \vartheta(x(t)) = u(t) \\ ii. \quad u'(t) + \eta(x(t), u(t)) \in \psi(x(t), u(t))B \end{cases}$$

under the constraints

$$(10) \qquad u(t) \in U(x(t)) \quad \forall t \geq 0,$$

where the functions ϑ, η, ψ satisfy

$$(11) \qquad \begin{cases} \vartheta'(x)(u - \vartheta(x)) + \eta(x, u) = \omega(x, u - \vartheta(x)), \\ \psi(x, u) = \varphi(x, u - \vartheta(x)) \end{cases}$$

for all $x, u \in X$, and where the set-valued map U is defined by

$$(12) \qquad U(x) = \{u \in X, \ (x, u - \vartheta(x)) \in K\}.$$

The set-valued map U is closed whenever K is closed, because ϑ is continuous by assumption. Note that a decomposition always exists; for instance, take $\vartheta = 0$:

$$(13) \qquad \begin{cases} x'(t) = u(t) \\ u'(t) + \omega(x(t), u(t)) \in \varphi(x(t), u(t))B, \end{cases}$$

where the constraint map U is defined by $Graph(U) = K$. We see that a decomposition of the oscillatory differential inclusion (7) leads to a special case of

Definition 2.3 *An* affine oscillatory control system *is a system*

$$(14) \qquad \begin{cases} i. \quad x'(t) = g(x(t)) - h(x(t))u(t) \\ ii. \quad u'(t) + \omega(x(t), u(t)) \in \varphi(x(t), u(t))B, \end{cases}$$

where $u(t) \in U(x(t))$ for all $t \geq 0$. Here, the set-valued map $U : X \rightsquigarrow Z$ defined on the state X and going to the control space Z defines the constraints, B denotes the unit ball in Z, the set of feasible forces is defined by $\varphi : Graph(U) \longrightarrow I\!R^+$, and $g : X \longrightarrow X, h : X \longrightarrow Z$ and $f : Graph(U) \longrightarrow Z$ describe the dynamic of the system.

In this differential inclusion, the function $u(\cdot)$ is seen as a control, submitted to the constraints $u(t) \in U(x(t))$ for all $t \geq 0$. The forcing term in $\varphi(x,u)B$ can be seen as a "metacontrol" regulating the evolution of both state and control. The control system is affine in u in the first equation.

Remark — If the state space X and the control space Z coincide, and if the function h is positive, we can associate with a given affine oscillatory control system a constrainted oscillatory differential inclusion in the following manner. We differentiate the state equation, which leads to

$$(15) \qquad x''(t) - g'(x(t))x'(t) + h'(x(t))x'(t)u(t) \;=\; -h(x(t))u'(t),$$

and substitute $u(t) = (g(x(t)) - x(t))/h(x(t))$ and $u'(t) \in -\omega(x(t), u(t)) + \varphi(x(t), u(t))B$. We obtain the oscillatory differential inclusion

$$(16) \qquad x''(t) + \mu(x(t), x'(t)) \;\in\; \psi(x(t), x'(t))B,$$

under the constraints

$$(17) \qquad (x(t), x'(t)) \in K \qquad \forall t \geq 0,$$

where the functions μ and ψ are defined by

$$(18) \qquad \begin{array}{rcl} \mu(x,x') & = & -g'(x)x' + h'(x)x'\frac{g(x)-x'}{h(x)} - h(x)\omega(x, \frac{g(x)-x'}{h(x)}), \\ \psi(x,x') & = & h(x)\varphi(x, \frac{g(x)-x'}{h(x)}) \end{array}$$

for all $x, x' \in X$, and where the constraint set K is defined by

$$(19) \qquad (x,x') \in K \quad \Longleftrightarrow \quad \frac{g(x)-x'}{h(x)} \in U(x)$$

for all $x, x' \in X$.

Finally, we derive its most general form:

Definition 2.4 *A control system of the form*

$$(20) \qquad \left\{ \begin{array}{ll} i. & x'(t) \;=\; f(x(t), u(t)) \\ ii. & u'(t) + \omega(x(t), u(t)) \;\in\; \varphi(x(t), u(t))B, \end{array} \right.$$

where

$$(21) \qquad u(t) \in U(x(t)) \quad \textit{for all } t \geq 0,$$

is called general oscillatory control system.

In the following, we will treat the general oscillatory control system, and derive the results for oscillatory differential inclusions or affine oscillatory control systems only as examples.

Remark (Time dependent constraints and time dependent set of feasible forcing terms)

To treat oscillatory control systems where the dynamic, the constraint set and the set of feasible forcing terms are time dependent, i. e.,

$$(22)\qquad \begin{cases} i. & x'(t) = f(t, x(t), u(t)) \\ ii. & u'(t) + \omega(t, x(t), u(t)) \in \varphi(t, x(t), u(t))B, \end{cases}$$

under the constraints

$$(23)\qquad u(t) \in U(t, x(t)) \quad \text{for all } t \geq 0,$$

we introduce a second state s by

$$(24)\qquad \begin{cases} i. & s'(t) = 1 \\ ii. & x'(t) = f(s(t), x(t), u(t)) \\ iii. & u'(t) + \omega(s(t), x(t), u(t)) \in \varphi(s(t), x(t), u(t))B, \end{cases}$$

where $u(t) \in U(s(t), x(t))$ for all $t \geq 0$. This is again an oscillatory control system. If the nonautonomous system is affine, so is the new autonomous one.

3 Application of the Viability Theorem in the framework of oscillatory control systems

We recall the Viability Theorem for differential inclusions.

Throughout the paper, X and Z denote finite dimensional vector spaces. Recall that the domain of a set–valued map $F : X \rightsquigarrow X$ is defined by

$$Dom(F) = \{x \in X;\ F(x) \neq \emptyset\}.$$

Theorem 3.1 (Viability Theorem) *[2, th. 3.3.5, th. 4.1.2] Let $F : X \rightsquigarrow X$ be a nontrivial, uppersemicontinuous set–valued map with compact convex images and linear growth, and let $K \subset Dom(F)$ be a closed set. The following properties are equivalent:*

i. For any $x_0 \in K$ there exists a viable solution on $[0, \infty[$ to the differential inclusion

$$\begin{cases} x'(t) & \in & F(x(t)) \text{ for almost all } t \geq 0 \\ x(0) & = & x_0, \end{cases} \tag{25}$$

i. e., a solution of system (25) remaining in K for all $t \geq 0$.

ii. The set K is a viability domain, *i. e., it satisfies the following tangential condition*

$$F(x) \cap T_K(x) \neq \emptyset \quad \text{for all } x \in K.$$

When K is not a viability domain, there exists a largest closed viability domain contained in K, called the viability kernel $Viab_F(K)$ *of K.*

We now consider the general oscillatory control system

$$\begin{cases} i. & x'(t) = f(x(t), u(t)) \\ ii. & u'(t) + \omega(x(t), u(t)) \in \varphi(x(t), u(t))B, \end{cases} \tag{26}$$

where $u(t) \in U(x(t))$ for all $t \geq 0$. To apply the Viability Theorem 3.1, we regard the state–dependent constraints on the controls as constraints on the state–control pairs:

$$(x(t), u(t)) \in K := Graph(U).$$

We posit the following assumptions

$$\begin{cases} \text{i. } Graph(U) \text{ is closed.} \\ \text{ii. } f \text{ and } \omega \text{ are continuous and have linear growth.} \\ \text{iii. } \varphi \text{ is continuous, positive and has linear growth.} \end{cases} \tag{27}$$

We deduce from the Viability Theorem 3.1 (see also [2, th. 7.2.5]) the following

Theorem 3.2 (Subregulation– and Metaregulation Map) *Let us assume that the oscillatory control system (26) satisfies the conditions (27). Let $R : X \rightsquigarrow Z$ be a closed set-valued map contained in U. Then the following two conditions are equivalent:*

i. For all initial state–control condition $(x_0, u_0) \in Graph(R)$*, there exists a state–control solution* $(x(\cdot), u(\cdot))$ *on* $[0, \infty[$ *to the oscillatory control system (26) starting at* (x_0, u_0) *and viable in* $Graph(R)$*.*

ii. R *is a solution to the partial differential inclusion*

$$0 \in DR(x,u)(f(x,u)) + \omega(x,u) - \varphi(x,u)B$$

for all $(x, u) \in Graph(R)$ *satisfying the constraints*

$$\forall x \in Dom(U), \quad R(x) \subset U(x).$$

In this case, the map R *is called a* φ–subregulation map *of* U*. The* metaregulation law *regulating the evolution of the state–control solutions viable in* $Graph(R)$ *takes the form of the system of differential inclusions*

$$(28) \qquad \begin{cases} i. & x'(t) = f(x(t), u(t)) \\ ii. & u'(t) + \omega(x(t), u(t)) \in G_R(x(t), u(t)), \end{cases}$$

where the set-valued map G_R *associated to the regulation map* R *is given by*

$$G_R(x,u) := (DR(x,u)(f(x,u)) + \omega(x,u)) \cap \varphi(x,u)B$$

and called the metaregulation map *associated with* R*.*
Furthermore, there exists a largest φ*–subregulation map* R^φ *contained in* U*.*

Let us now consider the trivial decomposition (13) of the constrainted oscillatory differential inclusion (7) . We obtain the following

Corollary 3.1 *Let* R *be a subregulation map of the constraint map* U *for the oscillatory differential inclusion (7). The metaregulation law regulating the evolution of viable solutions takes the form of a second order differential inclusion*

$$(29) \qquad x''(t) + \omega(x(t), x'(t)) \in G_R(x(t), x'(t)),$$

where the metaregulation map G_R *associated to the regulation map* R *defines the admissible (feedback) forces providing viable evolutions and is given by*

$$G_R(x,u) := (DR(x,u)(u) + \omega(x,u)) \cap \varphi(x,u)B$$

for all $(x, u) \in Graph(R)$*.*

Remark (Determination of φ–subregulation maps) We have several possibilities for finding φ–subregulation maps.

i. The graph of the largest φ–subregulation map, which is the viability kernel of the graph of U for the oscillatory control system (26), can be determinated numerically by the viability kernel algorithm [9] [7].

ii. In section 5 we shall determinate the maximal regulation map for a large class of two–dimensional systems.

iii. In the case of inequality constraints, we can use Maderner's results [5] [6] to find regulation maps.

Remark (Closed metaregulation map) In some special cases, we need the metaregulation map G_R to be closed, which is not always true. Once again, we can use Maderner's results [5] [6]. A second possibility is the following one:

For the oscillatory control system (26), we fix a regulation map R with associated metaregulation map G_R corresponding to Theorem 3.2. We consider for fixed $\rho > 0$ the extended control system

$$(30) \qquad \begin{cases} i. & x'(t) \ = \ f(x(t), u(t)) \\ ii. & u'(t) + \omega(x(t), u(t)) \ = \ v(t) \\ iii. & v'(t) \ \in \ \rho B, \end{cases}$$

under the constraints

$$(31) \qquad v(t) \in \overline{G}_R(x(t), u(t)) \subset \varphi(x(t), u(t))B \quad \forall t \geq 0,$$

where the set–valued map $\overline{G}_R$ is defined by

$$(32) \qquad Graph(\overline{G}_R) \ := \ \overline{Graph(G_R)}.$$

Since $Graph(R)$ is closed, we have $Dom(\overline{G}_R) = Dom(G_R) = Graph(R)$. Theorem 3.2 ensures the existence of a largest closed "submetaregulation" map $\tilde{G}_R$ contained in $\overline{G}_R$, such that for all $(x_0, u_0, v_0) \in Graph(\tilde{G}_R)$, there exists a solution $(x(\cdot), u(\cdot), v(\cdot))$ of the system (30) starting at (x_0, u_0, v_0), defined on $[0, \infty]$ and viable in $Graph(\overline{G}_R)$. In particular, we have $(x(t), u(t)) \in Dom(\overline{G}_R) = Graph(R) \subset Graph(U)$ for all $t \geq 0$, so $(x(\cdot), u(\cdot))$ is a viable solution of system (26). Hence all solutions regulated by the closed sub-metaregulation map $\tilde{G}_R$, i. e., the solutions of

$$(33) \qquad \begin{cases} i. & x'(t) = f(x(t), u(t)) \\ ii. & u'(t) + \omega(x(t), u(t)) \in \tilde{G}_R(x(t), u(t)) \end{cases}$$

are viable solutions of the oscillatory control system (26).

4 Selection Procedures

We now want to look for some explicit dynamical closed loop systems using selection procedures of the metaregulation map G_R. A detailed presentation of selection procedures can be found in [2, ch. 6]. Let us recall the definition and the main theorems.

Definition 4.1 *Let $G : X \rightsquigarrow Z$ be a set-valued map. A set-valued map $S : X \rightsquigarrow Z$ is a* selection procedure *of G if $Graph(S)$ is closed, and if*

$$S_G(x) := S(x) \cap G(x) \neq \emptyset \qquad \forall x \in Dom(G).$$

The main example of a selection procedure for a compact-valued map $G : X \rightsquigarrow Z$ is given by

$$(34) \qquad \text{minselec}(x) := \{u \in Z;\ \|u\| \leq \inf_{\tilde{u} \in G(x)} \|\tilde{u}\|\} \qquad \forall x \in Dom(G),$$

which provides the minimal selection

$$\begin{aligned} \text{minselec}_G(x) &= \text{minselec}(x) \cap G(x) \\ &= \{u \in Z;\ \|u\| = \min_{\tilde{u} \in G(x)} \|\tilde{u}\|\} \qquad \forall x \in Dom(G). \end{aligned}$$

For minselec to be a selection procedure of G, we have to guarantee that it has closed graph. The following proposition is an immediate corollary of [2, Prop. 6. 5. 3.].

Proposition 4.1 *Let us assume that the set-valued map $G : X \rightsquigarrow Z$ is lower semicontinuous with compact convex values. Then* minselec *is a selection procedure of G with single-valued minimal selection* minselec_G.

We turn back to the general oscillatory control system (26), fixing a φ–subregulation map $R : X \rightsquigarrow Z$ of U. For the following theorem, see also [2, th. 7. 6. 5].

Theorem 4.1 (Selection) *Let us assume that the conditions (27) hold true. Let S be a selection procedure of the metaregulation map G_R with convex values. Then for all initial conditions $(x_0, u_0) \in Graph(R)$ there exists a solution $(x(\cdot), u(\cdot))$ of the closed loop oscillatory system*

$$\text{(35)} \qquad \begin{cases} i. & x'(t) \ = \ f(x(t), u(t)) \\ ii. & u'(t) + \omega(x(t), u(t)) \ \in \ S_{G_R}(x(t), u(t)) \end{cases}$$

defined on $[0, \infty[$ with $x(0) = x_0$, $u(0) = u_0$, and viable in $Graph(R)$.

Proof — We consider the set–valued map F defined by

$$F(x, u) \ := \ \{f(x, u)\} \times (-\omega(x, u) + S(x, u)) \cap \varphi(x, u)B$$

for all $(x, u) \in Graph(R)$, and the following differential inclusion

$$\text{(36)} \qquad \begin{cases} (x'(t), u'(t)) \in F(x(t), u(t)), \\ \text{where } (x(t), u(t)) \in Graph(R) \text{ for all } t \geq 0. \end{cases}$$

Since α, β and φ are continuous and have linear growth, and since S has closed graph and convex values and satisfies

$$S(x, u) \cap \varphi(x, u)B \supset S(x, u) \cap G_R(x, u) \neq \emptyset$$

for all $(x, u) \in Graph(R)$, F is upper semicontinuous with nonempty convex compact images and with linear growth. To apply the Viability Theorem 3.1, we have to verify that $Graph(R)$ is a viability domain of F, i. e.,

$$T_{Graph(R)}(x, u) \cap F(x, u) \neq \emptyset \qquad \forall (x, u) \in Graph(R).$$

But this is implied by the fact that the set $G_R(x, u) \cap S(x, u)$ is always nonempty. Therefore, for all initial state $(x_0, u_0) \in Graph(R)$ there exists a solution $(x(\cdot), u(\cdot))$ to (36) viable in $Graph(R)$. But this is also a solution of (35), because it satisfies

$$(x'(t), u'(t)) \in T_{Graph(R)}(x(t), u(t)) = Graph(DR(x(t), u(t))$$

almost everywhere, and hence

$$u'(t) \in DR(x(t), u(t))(f(x(t), u(t)) \cap (-\omega(x(t), u(t)) + \varphi(x(t), u(t))B)$$

almost everywhere, which is equivalent to

$$\begin{aligned} & u'(t) + \omega(x(t), u(t)) \\ & \quad \in (DR(x(t), u(t))(f(x(t), u(t)) + \omega(x(t), u(t))) \cap \varphi(x(t), u(t))B \\ & \quad = G_R(x(t), u(t)) \end{aligned}$$

almost everywhere □

To state the previous theorem for the minimal selection, we have to guarantee that the metaregulation map G_R is lower semicontinuous. We obtain as a consequence of [2, Prop. 7. 1. 3]

Lemma 4.1 *If the conditions*

$$(37) \qquad \begin{cases} i. & Graph(R) \ni (x', u') \rightsquigarrow T_{Graph(R)}(x', u') \text{ is lower semicontinuous.} \\ ii. & \sup_{(x,u) \in Graph(R)} \|DR(x, u)\| < +\infty. \end{cases}$$

are satisfied, then G_R is lower semicontinuous.

Corollary 4.1 *Let the conditions (27) and (37) hold true. Then for all initial conditions $(x_0, u_0) \in Graph(R)$ there exists a solution $(x(\cdot), u(\cdot))$ of the closed loop oscillatory control system*

$$(38) \qquad \begin{cases} i. & x'(t) = f(x(t), u(t)) \\ ii. & u'(t) + \omega(x(t), u(t)) = \mathrm{minselec}_{G_R}(x(t), u(t)) \end{cases}$$

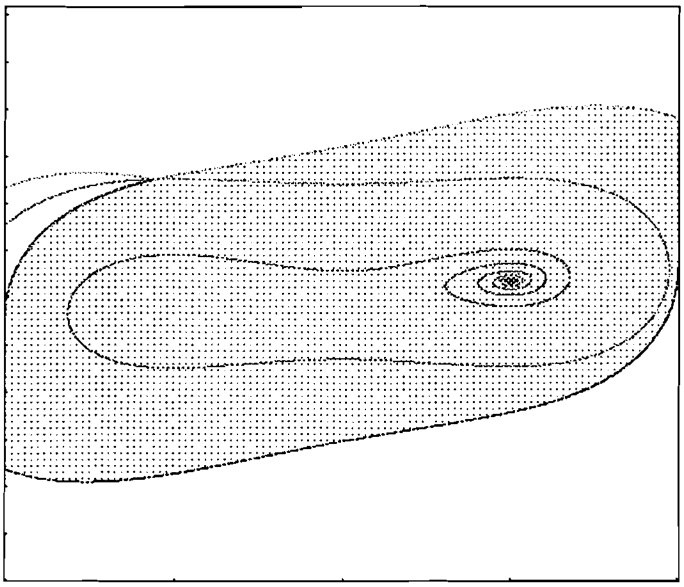
defined on $[0, \infty[$ with $x(0) = x_0$, $u(0) = u_0$, and viable in $Graph(R)$.

Example — Figure 3 shows a minimally forced solution for the Duffing oscillatory differential inclusion:

$$(39) \qquad \begin{cases} x''(t) - \delta x'(t) + x^3(t) - x(t) \in [-c, c], \\ \text{where } x(t) \in [-b, b] \text{ for all } t \geq 0. \end{cases}$$

The state–control solution oscillates freely in the phase space until arriving at the boundary of the graph of the maximal regulation map, where it has to be forced with maximal force $|c|$ not to leave $Graph(R)$, i. e., not to leave the constraint set.

Figure 3: Minimally forced solution for the Duffing oscillatory differential inclusion



5 A two–dimensional example

In the following we want to characterize the maximal regulation map for the special two–dimensional affine oscillatory control system

$$(40) \qquad \begin{cases} i. & x'(t) = g(x(t)) - h(x(t))u(t) \\ ii. & u'(t) + \omega(x(t)) \in [-c, c], \end{cases}$$

subjected to the constraints

$$(41) \qquad \forall t \geq 0, \quad x(t) \in [-b, b].$$

The constraint map U is defined by

$$(42) \qquad \forall\, x(t) \in [-b, b], \quad U(x) = I\!R,$$

and we assume in the following that

$$(43) \begin{cases} \text{i. } g, h, \omega : I\!R \longrightarrow I\!R \text{ are Lipschitz,} \\ \text{ii. } h(x) > 0 \text{ for all } x \in [-b, b]. \end{cases}$$

An efficient tool are monotonic cells, which are presented in [3] [4]. In order to denote the qualitative states and to characterize the monotonic behaviour of the solutions of the affine oscillatory differential inclusion (40), we introduce the three determinate signs $+, -$ and 0 and the indeterminate sign i, and set

Definition 5.1 *A* monotonic cell *for the affine oscillatory differential inclusion (40) is a set*

$$K_{s_1,s_2} := \{(x,u) \in \mathbb{R} \times \mathbb{R}; \quad sign(g(x) - h(x)u) = s_1 \text{ and } sign(-\omega(x) + v) = s_2 \text{ for all } v \in [-c, c]\},$$

where $s_1, s_2 \in \{+, -, 0, i\}$.

The monotonic cell $K_{0,0}$ is the set of all equilibria of (40), and $K_{i,i}$ is the whole constraint set.

Example — Figure 4 shows a partition of the plane in monotonic cells for the oscillatory differential inclusion

$$(44) \qquad \begin{cases} i. & x'(t) = 2ax(t) - u(t) \\ ii. & u'(t) - \omega^2 x(t) \in [-c, c], \end{cases}$$

where $0 < a < \omega$ and $c > 0$ are given. We use the symbols $\uparrow\rightarrow$ for $++$, $\updownarrow\rightarrow$ for $+i$ and so on.

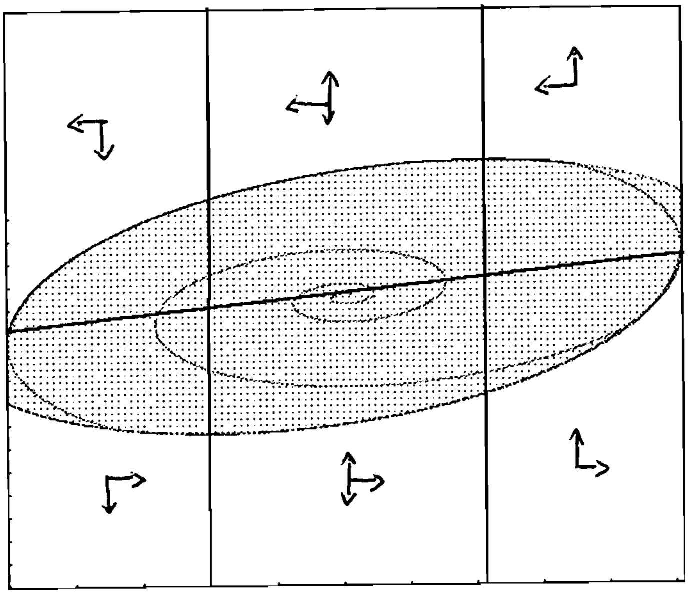
We define the closed sets

$$(45) \qquad \begin{array}{lclcl} Cell(1) & = & \overline{K}_{++} & = & [\omega(x) \leq -c] \cap [u \leq \frac{g(x)}{h(x)}], \\ Cell(2) & = & \overline{K}_{+i} & = & [-c \leq \omega(x) \leq c] \cap [u \leq \frac{g(x)}{h(x)}], \\ Cell(3) & = & \overline{K}_{+-} & = & [\omega(x) \geq c] \cap [u \leq \frac{g(x)}{h(x)}], \\ Cell(4) & = & \overline{K}_{--} & = & [\omega(x) \geq c] \cap [u \geq \frac{g(x)}{h(x)}], \\ Cell(5) & = & \overline{K}_{-i} & = & [-c \leq \omega(x) \leq c] \cap [u \geq \frac{g(x)}{h(x)}], \\ Cell(6) & = & \overline{K}_{-+} & = & [\omega(x) \leq -c] \cap [u \geq \frac{g(x)}{h(x)}]. \end{array}$$

The set $K_{0,0}$ of all equilibria is the curve segment

$$(46) \qquad K_{0,0} = \{(x,u) \in \mathbb{R} \times \mathbb{R}; \ -c \leq \omega(x) \leq c \text{ and } u = \frac{g(x)}{h(x)}\}.$$

Figure 4: Monotonic cells



To describe the viability kernel of $Graph(U)$ for (40), let us consider for all $(x_0, u_0) \in Graph(U)$ the solutions $(x^\sharp(\cdot), u^\sharp(\cdot))$ and $(x^\flat(\cdot), u^\flat(\cdot))$ of the differential equations

$$(47) \quad \begin{cases} i. & x'(t) = g(x(t)) - h(x(t))u(t), & x(0) = x_0 \\ ii. & u'(t) + \omega(x(t)) = c, & u(0) = u_0 \end{cases}$$

and

$$(48) \quad \begin{cases} i. & x'(t) = g(x(t)) - h(x(t))u(t), & x(0) = x_0 \\ ii. & u'(t) + \omega(x(t)) = -c, & u(0) = u_0 \end{cases}$$

being viable on some intervall $[0, t^\sharp]$ and $[0, t^\flat]$ respectively. The explicit curve equations

$$(49) \quad \begin{cases} \rho^\sharp : u \mapsto \rho^\sharp(u), & \rho^\sharp(u(t)) = x(t) \;\; \forall t \in [0, t^\sharp] \\ \rho^\flat : u \mapsto \rho^\flat(u), & \rho^\flat(u(t)) = x(t) \;\; \forall t \in [0, t^\flat] \end{cases}$$

and their respective inverses

$$(50) \quad \begin{cases} \pi^\sharp : x \mapsto \pi^\sharp(x), & \pi^\sharp(x(t)) = u(t) \;\; \forall t \in [0, t^\sharp] \\ \pi^\flat : x \mapsto \pi^\flat(x), & \pi^\flat(x(t)) = u(t) \;\; \forall t \in [0, t^\flat] \end{cases}$$

of the solution curves $t \mapsto (x^\sharp(t), u^\sharp(t))$ and $t \mapsto (x^\flat(t), u^\flat(t))$ satisfy on the set $Int(Cell(1) \cup Cell(2))$ the differential equation

$$\frac{d\rho^\sharp}{du}(u) = \frac{g(\rho^\sharp(u)) - h(\rho^\sharp(u))u}{-\omega(\rho^\sharp(u)) + c}, \qquad \rho^\sharp(u_0) = x_0 \tag{51}$$

and on the set $Int(Cell(4) \cup Cell(5))$

$$\frac{d\rho^\flat}{du}(u) = \frac{g(\rho^\flat(u)) - h(\rho^\flat(u))u}{-\omega(\rho^\flat(u)) - c}, \qquad \rho^\flat(u_0) = x_0, \tag{52}$$

and respectively on the set $Int(Cell(2) \cup Cell(3))$

$$\frac{d\pi^\sharp}{dx}(x) = \frac{-\omega(x) + c}{g(x) - h(x)\pi^\sharp(x)}, \qquad \pi^\sharp(x_0) = u_0 \tag{53}$$

and on $Int(Cell(5) \cup Cell(6))$

$$\frac{d\pi^\flat}{dx}(x) = \frac{-\omega(x) - c}{g(x) - h(x)\pi^\flat(x)}, \qquad \pi^\flat(x_0) = u_0. \tag{54}$$

Let $\overline{\rho}^\sharp$ be the solution of (51) passing through $(\frac{g(b)}{h(b)}, b)$ and $\overline{\rho}^\flat$ the solution of (52) passing through $(\frac{g(-b)}{h(-b)}, -b)$. The functions $\overline{\rho}^\sharp$ and $\overline{\rho}^\flat$ being increasing, we can introduce their respective inverse functions $\overline{\pi}_1^\sharp = \overline{\rho}^{\sharp -1}$ and $\overline{\pi}_1^\flat = \overline{\rho}^{\flat -1}$. Furthermore, we fix two points $(x^\sharp, \overline{\pi}_1^\sharp(x^\sharp))$ and $(x^\flat, \overline{\pi}_1^\flat(x^\flat))$ on the curves $\overline{\pi}_1^\sharp$ and $\overline{\pi}_1^\flat$ respectively and consider the solutions of (53) and (54) respectively starting there. We posit the following additional assumption:

$$-b \in Dom(\overline{\pi}_2^\sharp) \quad \text{and} \quad b \in Dom(\overline{\pi}_2^\flat), \tag{55}$$

so that both curves $\overline{\pi}_2^\sharp$ and $\overline{\pi}_2^\flat$ do not reach the curve $u = \frac{g(x)}{h(x)}$ but the boundary of the constraint set. We can hence define for all $x \in [-b, b]$

$$\pi_{last}^\sharp : \quad x \mapsto \begin{cases} \overline{\pi}_1^\sharp(x), & \text{if } x \in Dom(\overline{\pi}_1^\sharp) \\ \overline{\pi}_2^\sharp(x), & \text{if } x \in Dom(\overline{\pi}_2^\sharp), \end{cases} \tag{56}$$

$$\pi_{last}^\flat : \quad x \mapsto \begin{cases} \overline{\pi}_1^\flat(x), & \text{if } x \in Dom(\overline{\pi}_1^\flat) \\ \overline{\pi}_2^\flat(x), & \text{if } x \in Dom(\overline{\pi}_2^\flat). \end{cases} \tag{57}$$

Note that $\pi_{last}^\sharp$ and $\pi_{last}^\flat$ are well defined and that they satisfy the respective differential equations (53) and (54) on the open interval $]-b, b[$.

Proposition 5.1 *The viability kernel of $Graph(U)$ for the differential inclusion (40) is the graph of the regulation map R defined by*

$$\forall x \in [-b, b], \qquad R(x) = [\pi^{\sharp}_{last}(x), \pi^{\flat}_{last}(x)].$$

Proof —

i. We show first that $Viab(Graph(U)) \subset Graph(R)$ by contradiction. We fix $(x_0, u_0) \in Graph(U) \backslash Graph(R)$, we can assume without restriction that $u_0 < \pi^{\sharp}_{last}(x_0) \leq \frac{g(x_0)}{h(x_0)}$. Let $(x(\cdot), u(\cdot))$ be a solution of (40) starting at (x_0, u_0) and viable on $[0, t_0[$ in the interior of the constraint set. The curve $t \mapsto (u(t), x(t))$ having the explicit curve equation $\rho : u \mapsto \rho(u)$, $\rho(u(t)) = x(t)$ $\forall t \in [0, t_0[$ satisfies

$$\begin{aligned} \frac{d\rho}{du}(u(t)) &= \frac{x'(t)}{u'(t)} \\ &\geq \frac{g(\rho(u(t)) - h(\rho(u(t))u(t)}{-\omega(\rho(u(t)) + c} \\ &= \frac{d\rho^{\sharp}}{du}(u(t)) \end{aligned}$$

for all $t \in [0, t_0[$. Hence $x(t) = \rho(u(t)) \geq \rho^{\sharp}(u(t)) > \rho^{\sharp}_{last}(u(t))$ for all $t \in [0, t_0[$ and therefore $\frac{g(x(t))}{h(x(t))} \geq \pi^{\sharp}_{last}(x(t)) > u(t)$ for all $t \in [0, t_0[$. In particular when $x(t_0) = b$, we obtain $\frac{g(b)}{h(b)} = \pi^{\sharp}_{last}(b) > u(t_0)$. Since the velocity $x'(t_0) = g(b) - h(b)u(t_0)$ is positive, the solution is not viable.

ii. We show that $Graph(R) \subset Viab(Graph(U))$ by constructing particular viable solutions, namely the minimally forced solutions. We fix $(x_0, u_0) \in Graph(R)$. If $u_0 = \frac{g(x_0)}{h(x_0)}$ with $-c \leq \omega(x_0) \leq c$, we can take the equilibrium solution $x(t) \equiv x_0, u(t) \equiv u_0$. If not, we take the solution of

$$(58) \qquad \begin{cases} i. & x'(t) \;=\; g(x(t)) - h(x(t))u(t) \\ ii. & u'(t) + \omega(x(t)) \;=\; 0 \end{cases}$$

as long as it remains in $Graph(R)$. When for some time t_1 the solution curve reaches the boundary of $Graph(R)$, we have to damp the oscillation with maximal force $|c|$. We can assume without restriction that it reaches the boundary in the point $(x_1, \pi^\natural_{last}(x_1))$. Therefore we take the solution $(x^\natural(\cdot), u^\natural(\cdot))$ of

$$(59) \qquad \begin{cases} i. & x'(t) = g(x(t)) - h(x(t))u(t), \quad x(t_1) = x_1 \\ ii. & u'(t) + \omega(x(t)) = c, \qquad\qquad\qquad u(t_1) = u_1, \end{cases}$$

ranging over the curve $u^\natural(t) = \pi^\natural_{last}(x^\natural(t))$ which runs in the half space set $[u \leq \frac{g(x)}{h(x)}]$. According to the differential equation (59), we see that $x(t)$ increases to b where it arrives at time t_2 with velocity 0, and $u(t)$ increases until it arrives at the value $\frac{g(b)}{h(b)}$. It is sufficient to show that all points $(x_0, \frac{g(x_0)}{h(x_0})$, where $x \in [-b, b]$, are viable. We have to consider three cases:

(a) If $-c \leq \omega(x_0) \leq c$, this is obvious, because $(x_0, \frac{g(x_0)}{h(x_0})$ is an equilibrium.

(b) Suppose now that $\omega(x_0) > c$. Because of condition (55), the point x_0 lies in $Dom(\pi^\natural_{last})$, and therefore $\frac{g(x_0)}{h(x_0} > \pi^\natural_{last}(x_0)$. The solution $(x(\cdot), u(\cdot))$ of

$$(60) \quad \begin{cases} i. & x'(t) = g(x(t)) - h(x(t))u(t), \quad x(0) = x_0 \\ ii. & u'(t) + \omega(x(t)) = c, \qquad\qquad\qquad u(0) = \frac{g(x_0)}{h(x_0)} \end{cases}$$

satisfies hence $u(t) > \pi^\natural_{last}(x(t))$ as long as $u(t) \leq \frac{g(x(t))}{h(x(t))}$. Therefore there exists $t_1 \geq 0$ such that $x(t_1) = x_1$ and $u_1 = u(t_1) = \frac{g(x_1)}{h(x_1)}$. If $-c \leq \omega(x_0) \leq c$, we can take the equilibrium solution $x(t) \equiv x_1, u(t) \equiv u_1$ for all $t \geq t_1$. If not, we continue as in case (c).

(c) Suppose that $\omega(x_0) < -c$. Analogously to case (b), we take first the solution of

$$(61) \quad \begin{cases} i. & x'(t) = g(x(t)) - h(x(t))u(t), \quad x(0) = x_0 \\ ii. & u'(t) + \omega(x(t)) = -c, \qquad\qquad\quad u(0) = \frac{g(x_0)}{h(x_0)}, \end{cases}$$

which satisfies $u(t) < \pi^\flat_{last}(x(t))$ as long as $u(t) \geq \frac{g(x(t))}{h(x(t))}$. There exists $t_1 \geq 0$ such that $x(t_1) = x_1$ and $u_1 = u(t_1) = \frac{g(x_1)}{h(x_1)}$. If $-c \leq \omega(x_0) \leq c$, we take the equilibrium solution $x(t) \equiv x_1, u(t) \equiv u_1$ for all $t \geq t_1$, if not, we go back to case (b).

Hence we have shown that we can construct sucessively a solution regulated by an alternating control $+c$ or $-c$, which remains in $Graph(R)$ and which may converge finally to an equilibrium or oscillate forever. □

Since the functions $\pi^{\sharp\omega}_{last}$ and $\pi^{\flat\omega}_{last}$ converge pointwise on $[-b, b]$ and hence uniformely to the functions $\pi^{\sharp 0}_{last}$ and $\pi^{\flat 0}_{last}$ respectively when $\|\omega\|_\infty$ tends to 0, we obtain the following

Proposition 5.2 (Convergence of viability kernels) *The viability kernels $Viab_\omega(K)$ of the differential inclusion (40) submitted to the constraints (41) under the conditions (43) converges to the viability kernel $Viab_0(K)$ of the differential inclusion*

$$\left\{ \begin{array}{ll} i. & x'(t) = g(x(t)) - h(x(t))u(t) \\ ii. & u'(t) \in [-c, c], \end{array} \right. \tag{62}$$

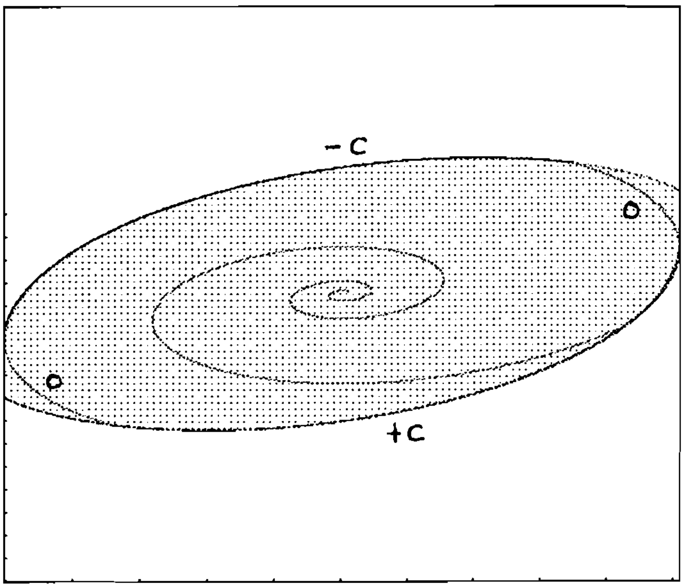
where $x(t) \in [-b, b]$ for all $t \geq 0$, when $\|\omega\|_\infty$ tends to 0:

$$Lim_{\|\omega\|_\infty \longrightarrow 0} Viab_\omega(K) \quad = \quad Viab_0(K).$$

Example (Continuation) — We use the result above to compute the maximal regulation map of the oscillatory differential inclusion (44), where $x(t) \in [-b, b]$ for all $t \geq 0$. (See figure 5.) We computed also a minimally forced solution starting in the interior of the viability kernel near the unstable equilibrium $(0, 0)$. It is oscillating with increasing amplitude until arriving at the boundary of the graph of the maximal regulation map at $\pi^{\sharp}_{last}$. There it is forced with maximal force c, and remains on the boundary until it arrives at the point $(b, 2ab)$, which is not an equilibrium. Starting from there it oscillates on the following cycle forever:

- In the first phase, it oscillates freely without forcing until arriving at the boundary of $Graph(R)$ on the upper limiting curve $\pi^{\flat}_{last}$.
- In the second phase, it decreases on the curve $\pi^{\flat}_{last}$ to the point $(-b, -2ab)$ forced by minimal force $-c$.
- In the third phase, it starts from $(-b, -2ab)$ an unforced oscillation, until arriving at the boundary of $Graph(R)$ on the lower limiting curve $\pi^{\sharp}_{last}$.

Figure 5: The largest regulation map for the oscillatory differential inclusion in the example



- In the fourth phase, it increases on the curve $\pi^{\sharp}_{last}$ to the point $(b, 2ab)$ forced by maximal force c, where it goes back to the first phase of the cycle.

When ω tends to 0, the viability kernel of $Graph(U)$ tends to the viability kernel of $Graph(U)$ under the differential inclusion

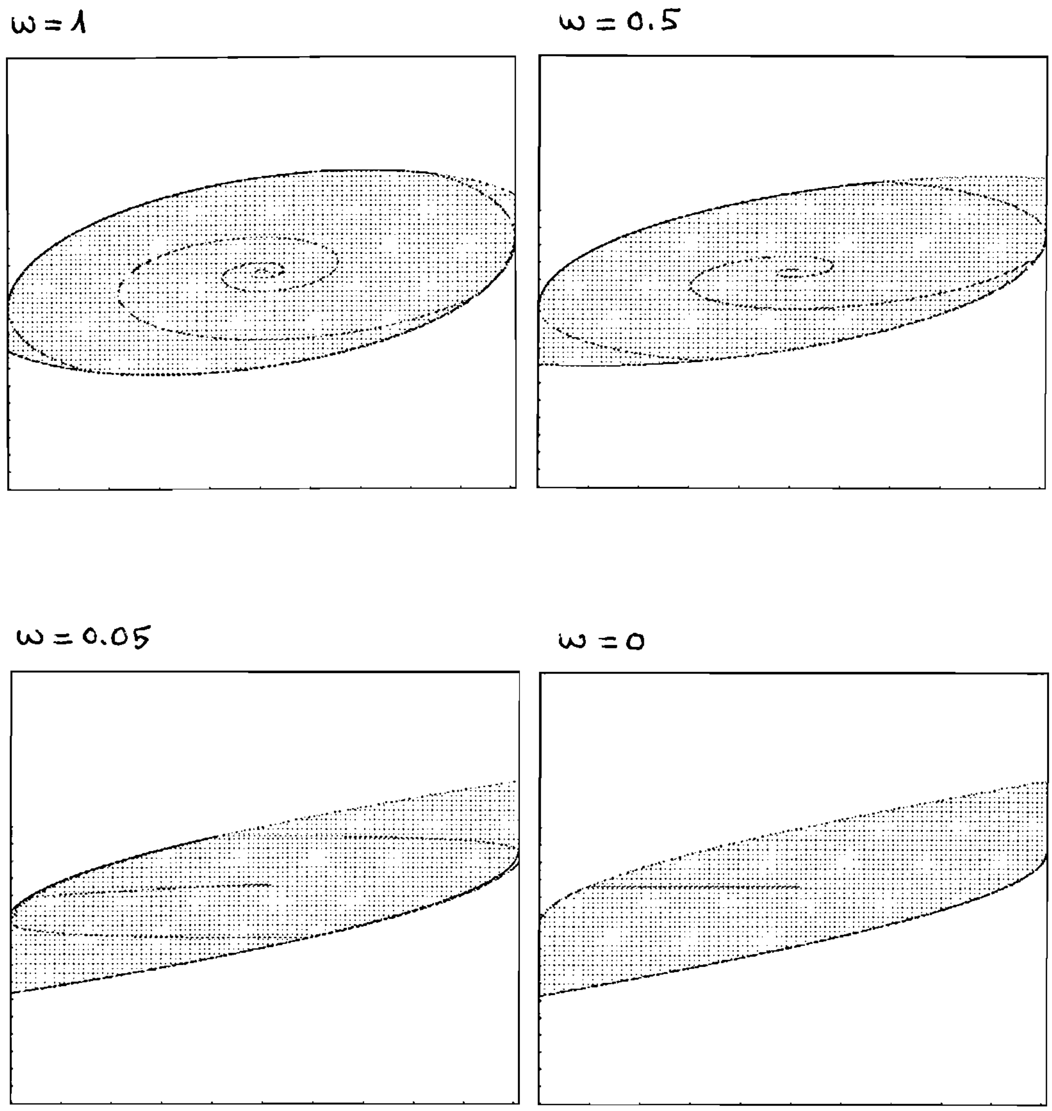
$$(63) \qquad \left\{ \begin{array}{ll} i. & x'(t) = 2ax(t) - u(t) \\ ii. & u'(t) \in [-c, c] \end{array} \right.$$

according to Proposition 5.2.

6 Free oscillation cells

In this section, we want to compare free oscillations to oscillations regulated by a minimal forcing term: the forcing term is 0 as long as the viability of the oscillation is not at stake. We consider the general oscillatory control

Figure 6: The largest regulation map for the oscillatory differential inclusion in the example, when ω tends to 0



system

$$(64)\qquad \begin{cases} i. & x'(t) \; = \; f(x(t), u(t)) \\ ii. & u'(t) + \omega(x(t), u(t)) \; \in \; \varphi(x(t), u(t))B \end{cases}$$

under the constraints

$$(65)\qquad u(t) \in U(x(t)) \qquad \forall t \geq 0.$$

According to Theorem 3.2, we fix the largest subregulation map R associated with given φ for the oscillatory control system (64). Besides the control system (64), we consider the differential equation

$$(66)\qquad \begin{cases} i. & x'(t) \; = \; f(x(t), u(t)) \\ ii. & u'(t) + \omega(x(t)) \; = \; 0, \end{cases}$$

where $(u(t)) \in U(x(t))$ for all $t \geq 0$. This differential equation describes the free oscillations of (64). According to Theorem 3.2, we fix the largest subregulation map R^0 associated with the zero function. The graph of R^0 is the viability kernel of the differential equation (66) and is called the *free oscillation cell* of the oscillatory control system (64). Obviously $Graph(R^0) \subset G_R^{-1}(0)$.

We observe the following behaviour of a minimally forced solution of the oscillatory control system (64):
If for some $t_1 \geq 0$ the solution $(x(\cdot), u(\cdot))$ enters the subset $G_R^{-1}(0)$, then $(x(\cdot), u(\cdot))$ oscillates freely. Furthermore, we obtain the following alternative:

i. If $(x(t_1, u(t_1)))$ is contained in the free oscillation cell $Graph(R^0)$, then $(x(t), u(t))$ continue the unforced oscillation for all $t \geq t_1$, and $(x(t), u(t))$ remains in the free oscillation cell for all $t \geq t_1$.

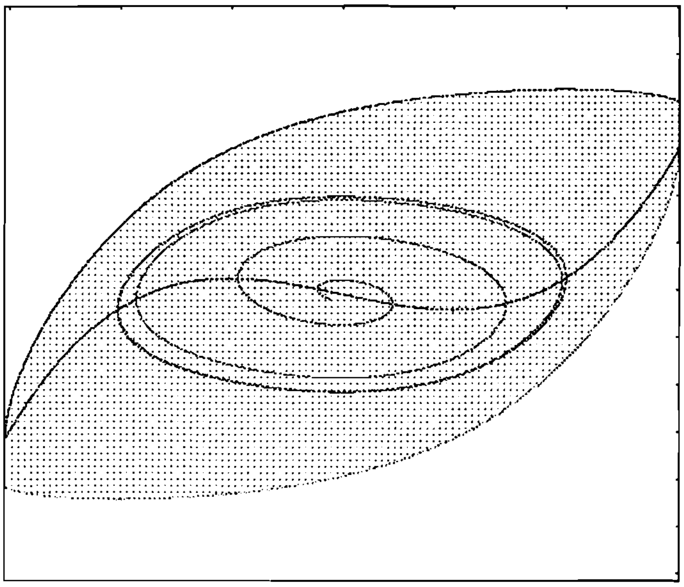
ii. If $(x(t_1), u(t_1)) \notin Graph(R^0)$, then $(x(t), u(t))$ must eventually leave $G_R^{-1}(0)$ in finite time [2, prop. 4.1.4]. After that, the solution $(x(\cdot), u(\cdot))$ has to be forced to maintain viability.

Example (Van der Pol oscillatory differential inclusion) — The figure below shows the largest regulation map of the *Van der Pol oscillatory differential inclusion* represented in the decomposition

$$(67)\qquad \begin{cases} i. & x'(t) \; = \; a(\frac{x^3}{3} - x) - u \\ ii. & u'(t) - \epsilon x \; \in \; [-c, c], \end{cases}$$

where $x(t) \in [-b, b]$ for all $t \geq 0$. The free oscillation set is bounded by the α–limit of any free trajectory starting near the stable equilibrium $(0, 0)$.

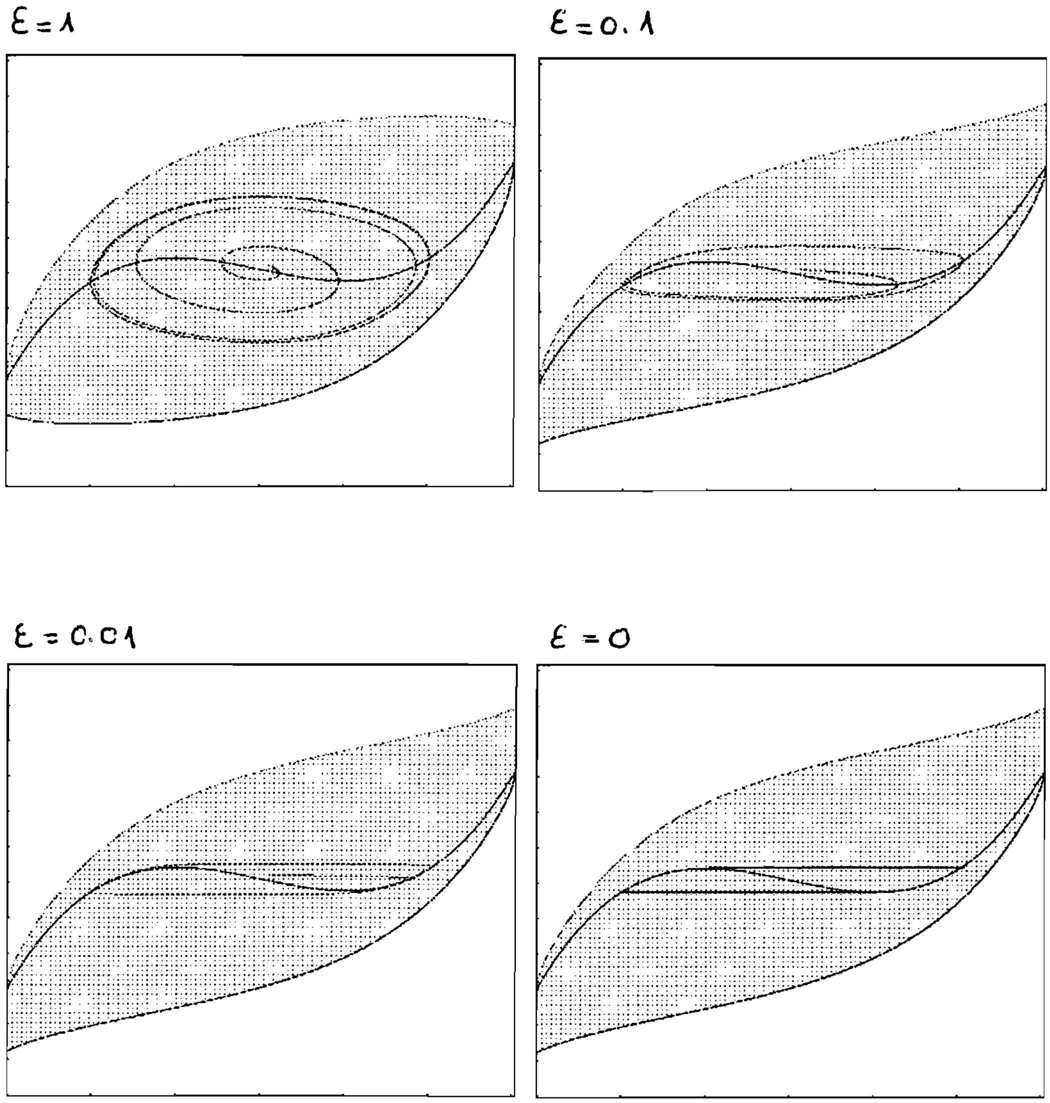
Figure 7: Free oscillation set of a Van der Pol differential inclusion



When ϵ tends to 0, the free oscillation set tends – according to Proposition 5.2 – to the inertial set of the differential inclusion

$$(68) \qquad \begin{cases} i. & x'(t) = a(\frac{x^3}{3} - x) - u \\ ii. & u'(t) \in [-c, c]. \end{cases}$$

Figure 8: Free oscillation cell, when the parameter ϵ tends to 0



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