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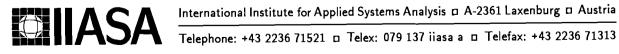
A Differential Game with Two
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Continuous Case

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Abstract

We study a two-players differential game in which one player wants the state of the system to reach an open target while the other player wants the state of the system to avoid this target. The aim of this paper is to show that, if the first players plays "Carathéodory strategies" and the second player plays controls, then the game is not well-defined, i.e., either the "alternative" or the "causality" is not satisfied for that game.

Résumé

Nous étudions un jeux différentiel dans lequel un des joueurs cherche à faire en sorte que l'état du système atteigne une cible donnée tandis que l'autre joueur agit sur l'état du système afin que celui-ci évite la cible. L'objet de ce travil est de montrer que, si Ursule joue des stratégies Carathéodory et Victor des contrôles, alors le jeu n'est en général pas bien défini : soit il ne vérifie pas le principe "d'alternative", soit il ne satisfait pas le principe de "causalité".

Keywords : Differential Games, Pursuit and Evasion Games, Viability Theory.

A.M.S. classification : 49J24, 49J52, 90J25, 90J26.

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1 Introduction

We study a $N-$dimensional dynamical system governed by two controls:

$$x' = f(x, u, v), \quad \text{where } u \in U, \ v \in V$$

We assume that Ursula plays u and that Victor plays v. Let Ω be an open target. We investigate the game where Ursula wants the state of the system $x(\cdot)$ to reach Ω while Victor wants the state of the system $x(\cdot)$ to avoid Ω forever.

Note that this game is equivalent with the following:
Let $K := I\!R^N \backslash \Omega$ be a closed subset of $I\!R^N$. In the new game, Ursula wants the state of the system to leave K, while Victor wants the state of the system to remain in K. For technical reasons, we shall in fact study this last game.

We study this problem in the framework of the Carathéodory strategies. This "continuous" case is interesting because it makes clear the difficulties of the problem. We shall prove bellow that, in some cases, there is no satisfying definition of the game for Carathéodory strategies: Either the game does not satisfy the alternative principle (a point x may belong neither to Ursula's victory domain, nor to Victor's victory domain), or the game does not satisfy the causality principle, i.e., Victor (for instance) needs the knowledge of Ursula's future strategy to win.

We describe the kind of strategies we use:

Definition 1.1 *A strategy $\tilde{u}(\cdot, \cdot) : I\!R^+ \times I\!R^N \to U$ is a Carathéodory strategy if $\tilde{u}(\cdot, \cdot)$ is measurable, and $\tilde{u}(t, \cdot)$ is continuous for almost every $t \geq 0$.*
We shall denote by $\mathcal{U}$ the set of Carathéodory strategies.

The class of counter-strategies Victor shall use is the set of time-measurable controls:

$$\mathcal{N} = \{v(\cdot) : [0, +\infty) \to V, \text{ measurable application }\}$$

In the sequel, we shall keep the following notation:

$$f(\cdot, \tilde{u}(\cdot, \cdot), V) := \bigcup_{v \in V} f(\cdot, \tilde{u}(\cdot, \cdot), v)$$

Recall that, if $\tilde{u}(\cdot,\cdot) \in \mathcal{U}$, a map $x(\cdot)$ is a solution of

$$(1) \qquad \begin{cases} x'(t) \in f(x(t), \tilde{u}(t,x(t)), V) \text{ for almost every } t \geq 0 \\ x(0) = x_0 \end{cases}$$

if and only if there is a control $v(\cdot) \in \mathcal{N}$ such that $x(\cdot)$ is a solution of

$$\begin{cases} x'(t) = f(x(t), \tilde{u}(t,x(t)), v(t)) \text{ for almost every } t \geq 0 \\ x(0) = x_0 \end{cases}$$

We summarize the assumptions on f: Throughout this paper, we assume that the map $f : I\!\!R^N \times U \times V \to I\!\!R^N$ satisfies

$$(2) \qquad \begin{cases} 1) & \text{U and V are metric compact spaces,} \\ & \text{and } V \text{ is a convex subset of } I\!\!R^d. \\ 2) & f : I\!\!R^n \times U \times V \to I\!\!R^n \text{ is continuous.} \\ 3) & f(\cdot, u, v) \text{ is a } \ell-\text{Lipschitz map for any } u \text{ and } v. \\ 4) & f \text{ is affine in } v. \end{cases}$$

Throughout this paper, we denote by B the closed unit ball of the state space $I\!\!R^N$.

Recall that, under assumptions (2), differential inclusion (1) has always solutions (See for instance [4]).

2 The victory domains

Let us define the "natural" victory domains for such a game. We study only the case when Victor plays by retorting. The case when Ursula plays by retorting is more difficult and shall not be treated here.

Victor's (retorting) victory domain is the set of initial positions $x_0 \in K$, such that, for any Ursula's strategy $\tilde{u}(\cdot,\cdot) \in \mathcal{U}$, there is a solution $x(\cdot)$ of (1) which remains in K on $[0, +\infty)$.

The complement of Victor's retorting victory domain is called **Ursula's (blind) victory domain**. It is the set of initial positions x_0 of K, for which Ursula can find a Carathéodory strategy $\tilde{u}(\cdot,\cdot) \in \mathcal{U}$, such that any solution

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of (1) leaves K is finite time.

With the previous definition of the game, the "alternative" is obviously satisfied: A point x_0 of K either belongs to Victor's victory domain, or to Ursula's victory domain.

To give another formulation of Victor's victory domain, let us denote by $Viab_{f(\cdot,\tilde{u}(\cdot,\cdot),V)}(K)$ (where $\tilde{u}(\cdot,\cdot) \in \mathcal{U}$) the set of initial positions x_0 for which there exists a solution of (1) which remains in K on $[0, +\infty)$. Then Victor's victory domain is equal to:

$$\bigcap_{\tilde{u}(\cdot,\cdot) \in \mathcal{U}} Viab_{f(\cdot,\tilde{u}(\cdot,\cdot),V)}(K)$$

The main drawback of the definition of Victor's victory domain is that it does not satisfy a priori the principle of causality: Victor chooses his strategy by using, not only the knowledge of Ursula's past and present strategy, but also that of her future strategy. Indeed, from assumption, Ursula chooses her strategy at the beginning of the game, and cannot change it throughout the game.

We shall not provide a rigorous definition of Victor's victory domain in the case when Victor plays in a causal way. The main idea is the following: If he plays in a causal way, Victor has to ensure to remain in his victory domain. So his causal victory domain is equal to the largest set contained in K in which Victor can ensure to remain.
It is proved in ([2]) that such the closure D of such a set satisfies the following tangential condition:

$$\forall x \in D, \ \forall u \in U, \ \exists v \in V, \ f(x,u,v) \in T_D(x)$$

where $T_D(x)$ denotes the contingent cone to D at x (See [1]). Conversely, a closed set D satisfying this tangential condition is also a set in which, whatever Carathéodory strategy Ursula plays, Victor can ensure the state of the system to remain (It is a consequence of the Measurable Viability Theorem ([4]). A closed set satisfying this tangential condition is called a *discriminating domain*.

In ([1]), it is proved that any closed subset K of $I\!\!R^N$ contains a largest discriminating domain: This set - which is a closed discriminating domain - contains any closed discriminating domain contained in K. It is called the *discriminating kernel* of K for f and it is denoted by $Disc_f(K)$.

Thus it is not difficult to show that Victor's causal victory domain is equal to the discriminating kernel of K for f. This result can be proved in a rigorous way, but we shall not do so because it is rather long, and not very interesting because of the following result:

Proposition 2.1 *There are some dynamics f satisfying (2) and some closed set K such that:*

$$(3) \qquad Disc_f(K) \neq \bigcap_{\tilde{u}(\cdot,\cdot)\in\mathcal{U}} Viab_{f(\cdot,\tilde{u}(\cdot,\cdot),V)}(K)$$

Next section is devoted to an example of such a situation.

Note that the inclusion

$$Disc_f(K) \subset \bigcap_{\tilde{u}(\cdot,\cdot)\in\mathcal{U}} Viab_{f(\cdot,\tilde{u}(\cdot,\cdot),V)}(K)$$

always holds true thanks to the Measurable Viability Theorem ([4]).
So Proposition 2.1 states that, if Victor plays in a causal way, alternative property does not hold: There are some initial positions for which Ursula cannot surely win if she plays Carathéodory strategies, and where Victor cannot surely win if he plays in a causal way.

This result is rather surprising because we shall prove in ([3]) that, for some class $\mathcal{U}'$ of discontinuous strategies, there is an equality in equation (3). This means in particular that there is no way to approximate by a Carathéodory feedback such discontinuous strategies.

3 A counter-example

We study here an example of Proposition 2.1. Our problem is in the plane. Set: $K := [-1,1] \times I\!R$, and

$$f(x,y,u,v) := \{\phi(y)u + \phi(-y)v + 2x(\phi(y) - \phi(-y))\} \times \{-1\} \text{ if } y \geq 0$$
$$f(x,y,u,v) := \{\phi(y)u + \phi(-y)v\} \times \{-1\} \text{ if } y \leq 0$$

where u and v belong to $[-1,1]$ and $\phi(t) := e^t/(1 + e^t)$. Let us point out that f is Lipschitz, and linear in u and v. In particular, f satisfies (2).

Note also that ϕ is an increasing map and that $\phi(t) > \phi(-t)$ if $t > 0$. Moreover, $\lim_{t \to -\infty} \phi(t) = 0$, while $\lim_{t \to +\infty} \phi(t) = 1$. Thus $\phi(t) \in [0,1]$ for any t.

We shall denote by f_x the projection of f onto the horizontal axis, while f_y denoted the projection of f onto the vertical one. We shall also denote by $\mathcal{S}_{f(\cdot,\tilde{u}(\cdot,\cdot),V)}(x_0,y_0)$ the set of solutions to:

$$\begin{cases} (x'(t),y'(t)) \in f(x(t),y(t),\tilde{u}(t,x(t),y(t)),V) \text{ for almost every } t \geq 0 \\ x(0) = x_0 \text{ and } y(0) = y_0 \end{cases}$$

Proposition 3.1 *For the map f and the closed set K previously defined, one has:*

$$Disc_f(K) = \emptyset$$

but

$$\bigcap_{\tilde{u}(\cdot,\cdot) \in \mathcal{U}} Viab_{f(\cdot,\tilde{u}(\cdot,\cdot),V)}(K) \neq \emptyset$$

We prove in a first step that $Disc_f(K)$ is empty. In a second step, we prove that, for any Carathéodory strategy $\tilde{u}(\cdot,\cdot)$,

$$Viab_{f(\cdot,\tilde{u}(\cdot,\cdot),V)}(K) \cap ([-1,1] \times \{0\}) \neq \emptyset$$

We conclude in a third step by showing the existence of a point (x_0,y_0) (with $y_0 < 0$) from where, for any Carathéodory strategy $\tilde{u}(\cdot,\cdot)$, one can reach any point of $[-1,1] \times \{0\}$.

3.1 First step : $Disc_f(K) = \emptyset$

Let T be the smallest positive real such that

$$\forall t \geq T, \ \phi(t) - \phi(-t) \geq 1/2.$$

Recall that $\phi(t) - \phi(-t) \rightarrow_{t \rightarrow +\infty} 1$, so such a T exists.

Note that

$$Disc_f(K) \neq \emptyset \iff Disc_f(K) \cap [-1,1] \times [T, +\infty) \neq \emptyset.$$

Indeed, if a point (x_0, y_0) belongs to $Disc_f(K)$ then there is a solution of the differential inclusion for $f(\cdot, 1, V)$ (for $u = 1$) which remains in $Disc_f(K)$ on $[0, +\infty)$. Since $y(t) = y_0 + t$ (because $f_y \equiv 1$ for any $(x, y) \in I\!\!R^2$),

$$\forall t \geq T - y_0, \ (x(t), y(t)) \in Disc_f(K) \cap [-1,1] \times [T, +\infty)$$

and thus $Disc_f(K) \cap [-1,1] \times [T, +\infty)$ is not empty.

So, to prove that $Disc_f(K)$ is empty, it is sufficient to show that $Disc_f(K) \cap [-1,1] \times [T, +\infty)$ is empty.

For that purpose, let $y \geq T$ and x belong to $[-1, \frac{1}{2})$. If we set $u = -1$, for any $v(\cdot) \in \mathcal{N}$:

$$
\begin{aligned}
f_x(x, y, u, v(t)) &= -\phi(y) + \phi(-y)v(t) + 2x(\phi(y) - \phi(-y)) \\
&\leq -\phi(y) + \phi(-y) + 2x(\phi(y) - \phi(-y)) \\
&= (\phi(y) - \phi(-y))(2x - 1) \\
&\leq x - \tfrac{1}{2}
\end{aligned}
$$

So, for any initial point (x_0, y_0) with $y_0 \geq T$ and $x_0 \in [-1, \frac{1}{2})$, for any control $v(\cdot) \in \mathcal{N}$, any solution $(x(\cdot), y(\cdot))$ of

$$\begin{cases} (x'(t), y'(t)) = f(x(t), y(t), -1, v(t)) \text{ for almost every } t \geq 0 \\ (x(0), y(0)) = (x_0, y_0) \end{cases}$$

satisfies: $y(t) = y_0 + t \geq T$ and

$$x'(t) \leq x(t) - 1/2 \leq 0$$

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for almost every $t \geq 0$. Gronwall's Lemma yields:

$$\forall t \geq 0, \; x(t) \leq (x_0 - \frac{1}{2})e^t + \frac{1}{2}$$

and thus $x(t) < -1$ for t sufficiently large because $x_0 < 1/2$.

In the same way, for $x_0 \in (\frac{1}{2}, 1]$ and $y_0 \geq T$, Ursula plays $\tilde{u}(\cdot, \cdot) \equiv 1$. Then for any solution $(x(\cdot), y(\cdot))$ of (1), there is a $t \geq 0$ with $x(t) > 1$.

So for any $y_0 \geq T$ and any $x_0 \in [-1, 1]$ there is a constant strategy u such that any solution of (1) leaves K in finite time. In particular, $Disc_f(K) \cap [-1, 1] \times [T, +\infty) = \emptyset$, and thus $Disc_f(K)$ is empty.

3.2 Second step

We first prove the following Lemma:

Lemma 3.1 *Let $\tilde{u}(\cdot, \cdot)$ be a Carathéodory strategy.*
For any $z \in [-1, 1]$, for any $t \geq 0$, there are $x_0 \in [-1, 1]$ and a solution $(x(\cdot), y(\cdot)) \in \mathcal{S}_{f(\cdot, \tilde{u}(\cdot, \cdot), V)}((x_0, 0))$ such that:
1) $(x(s), y(s)) \in K$ for every $s \in [0, t]$,
2) $x(t) = z$.

Proof : Let $\tilde{u}_1(\cdot, \cdot) \in \mathcal{U}$ be any Carathéodory strategy. We first prove that the locally compact set $[-1, 1] \times I\!\!R_*^+$ (i.e., K restricted to $y > 0$) is viable for the set-valued map $-f(\cdot, \tilde{u}_1(\cdot, \cdot), V)$. For that purpose, we prove that the tangential condition

$$-f(x, y, \tilde{u}_1(s, x, y), V) \cap T_{[-1,1] \times I\!\!R_*^+}(x, y) \neq \emptyset$$

is fulfilled for any $x \in [-1, 1]$ and $y > 0$.

If, on one hand, $y > 0$ and $x := 1$, then, for any time s,

$$\begin{aligned}
-f_x(1, y, \tilde{u}_1(s, (1, y)), 1) \\
= -\phi(y)\tilde{u}_1(s, (1, y)) - \phi(-y) - 2(\phi(y) - \phi(-y)) \\
\leq -(\phi(y) - \phi(-y)) \;\; < 0
\end{aligned}$$

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So $-f(1, y, \tilde{u}_1(s, (1, y)), 1) \in T_K(1, y)$. On the other hand, if $y > 0$ and $x = -1$, then

$$
\begin{aligned}
&-f_x(-1, y, \tilde{u}_1(s, (-1, y)), -1) \\
&\quad = -\phi(y)\tilde{u}_1(s, (-1, y)) + \phi(-y) + 2(\phi(y) - \phi(-y)) \\
&\quad \geq (\phi(y) - \phi(-y)) > 0
\end{aligned}
$$

So $-f(-1, y, \tilde{u}_1(s, (-1, y)), -1) \in T_K(-1, y)$, and $[-1, 1] \times I\!R_*^+$ is (locally) viable for $-f(\cdot, \tilde{u}_1(\cdot, \cdot), V)$.

Let $t > 0$ and $z \in [-1, 1]$. We set $\tilde{u}_1(s, (x, y)) := \tilde{u}(t - s, (x, y))$. Since $[-1, 1] \times I\!R_*^+$ is (locally) viable for $-f(\cdot, \tilde{u}_1(\cdot, \cdot), V)$, there is a solution $(x(\cdot), y(\cdot))$ of the differential inclusion for $-f(\cdot, \tilde{u}_1(\cdot, \cdot), V)$, starting from (z, t), which remains in K as long as $y(s) \geq 0$ (thanks to the Measurable Viability Theorem ([4])). Note that $y(s) = t - s$.

The map $(x_1(\cdot), y_1(\cdot))$ defined by: $x_1(s) = x(t - s)$ and $y_1(s) = s$, is a solution of the differential inclusion for $f(\cdot, \tilde{u}(\cdot, \cdot), V)$, starting from $(x_1(0), 0) \in [-1, 1] \times \{0\}$, which remains in K on $[0, t]$, and which satisfies moreover: $x_1(t) = z$. So the Lemma is proved. $\square$

Corollary 3.1 *For any* $\tilde{u}(\cdot, \cdot) \in \mathcal{U}$,

$$
([-1, 1] \times \{0\}) \cap Viab_{f(\cdot, \tilde{u}(\cdot, \cdot), V)}(K) \neq \emptyset
$$

Proof : For any $t \geq 0$, let us denote by A_t the closed set:

$$
A_t := \{x \in [-1, 1] \mid \exists (x(\cdot), y(\cdot)) \in \mathcal{S}_{f(\cdot, \tilde{u}(\cdot, \cdot), V)}((x, 0))
$$
$$
\text{which remains in } K \text{ on } [0, t] \quad \}
$$

Then A_t is nonempty from Lemma 3.1. Moreover, A_t is a compact subset of $[-1, 1]$, and $A_{t'} \subset A_t$ whenever $t' \leq t$. So $A_\infty := \bigcap_{t \geq 0} A_t$ is not empty.

It is easy to show that $A_\infty \times \{0\}$ is the set of initial conditions $(x, 0)$ for which there is a solution $(x(\cdot), y(\cdot))$ of the differential inclusion for $f(\cdot, \tilde{u}(\cdot, \cdot), V)$, which remains in K on $[0, +\infty)$. Thus

$$
([-1, 1] \times \{0\}) \cap Viab_{f(\cdot, \tilde{u}(\cdot, \cdot), V)}(K) = A_\infty
$$

is not empty. $\square$

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3.3 Third step

Since $[-1, 1] \times \{0\} \cap Viab_{f(\cdot, \tilde{u}(\cdot, \cdot), V)}(K)$ is not empty for any $\tilde{u}(\cdot, \cdot) \in \mathcal{U}$, it is sufficient to prove the following Lemma to conclude:

Lemma 3.2 *There is $(x_0, y_0) \in K$, with $y_0 < 0$, such that: For any $\tilde{u}(\cdot, \cdot) \in \mathcal{U}$, for any $z \in [-1, 1]$, a solution $(x(\cdot), y(\cdot)) \in \mathcal{S}_{f(\cdot, \tilde{u}(\cdot, \cdot), V)}(x_0, y_0)$ exists, such that:*
1) $\forall s \le |y_0|$, $(x(s), y(s)) \in K$
2) $x(|y_0|) = z$.

Thanks to Lemma 3.2, we can achieve the proof of Proposition 3.1:
Proof of Proposition 3.1 : Let (x_0, y_0) be as in Lemma 3.2. Let us show that:

$$(x_0, y_0) \in \bigcap_{\tilde{u}(\cdot, \cdot) \in \mathcal{U}} Viab_{f(\cdot, \tilde{u}(\cdot, \cdot), V)}(K)$$

Indeed, from Corollary 3.1, for any fixed $\tilde{u}(\cdot, \cdot) \in \mathcal{U}$,

$$([-1, 1] \times \{0\}) \cap Viab_{f(\cdot, \tilde{u}(\cdot, \cdot), V)}(K) \ne \emptyset.$$

Let $(z, 0)$ belong to this set.

From Lemma 3.2, there is a solution $(x(\cdot), y(\cdot))$ of the differential inclusion for $f(\cdot, \tilde{u}(\cdot, \cdot), V)$, starting from (x_0, y_0), which remains in K on $[0, |y_0|]$, and such that $x(|y_0|) = z$. Note that $y(|y_0|) = 0$. Moreover, from the very definition of z, there is another solution $(x_1(\cdot), y_1(\cdot))$ of the differential inclusion for $f(\cdot, \tilde{u}(\cdot, \cdot), V)$, starting from $(z, 0)$, which remains in K on $[0, +\infty)$. Concatenating the solutions $(x(\cdot), y(\cdot))$ and $(x_1(\cdot), y_1(\cdot))$, we obtain a viable solution of the differential inclusion for $f(\cdot, \tilde{u}(\cdot, \cdot), V)$ starting from (x_0, y_0). $\square$

Proof of Lemma 3.2 :

The proof is set in four steps:
• In the first set, we show that the lines $\{1\} \times \mathbb{R}_*^-$ and $\{-1\} \times \mathbb{R}_*^-$ are (locally compact) viability domains for $f(\cdot, \tilde{u}(\cdot, \cdot), V)$.
• In the second step, we show that, for any $(\bar{x}, \bar{y})$ of K, with $\bar{y} < 0$, the reachable set for $f(\cdot, \tilde{u}(\cdot, \cdot), V)$ at time $|\bar{y}|$, starting from $(\bar{x}, \bar{y})$ is an interval contained in $\mathbb{R} \times \{0\}$.

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- In the third step we prove the existence of $(\bar{x}, \bar{y})$ of K, for which the previous reachable set contains $[-1, 1] \times \{0\}$.
- In the fourth step, we show that the same reachable set, but with the constraint that the state of the system remains in K, still contains $[-1, 1] \times \{0\}$.

The first step : We make the proof for $\{1\} \times I\!R_*^-$, the other case been symmetrical. Let $(1, y) \in \{1\} \times I\!R_*^-$. Recall that

$$f_x(1, y, \tilde{u}(t, (1, y)), V) = \phi(y)\tilde{u}(t, (1, y)) + \phi(-y)[-1, 1]$$

where $\phi(-y) > \phi(y)$, and $|\tilde{u}(t, (1, y))| \leq 1$. Thus $0 \in f_x(1, y, \tilde{u}(t, (1, y)), V)$.

Since $T_{\{1\} \times I\!R_*^-}(1, y) = \{0\} \times I\!R$, we have proved that:

$$\forall y < 0, \ f(1, y, \tilde{u}(t, (1, y)), V) \cap T_{\{1\} \times I\!R_*^-}(1, y) \neq \emptyset$$

Thus $\{1\} \times I\!R_*^-$ is viable for $f(\cdot, \tilde{u}(\cdot, \cdot), V)$.

The second step : Since the set-valued map $f(\cdot, \tilde{u}(\cdot, \cdot), V)$ is Carathéodory, the reachable set at some time t is always connected. Since $f_y \equiv 1$, the reachable set for $f(\cdot, \tilde{u}(\cdot, \cdot), V)$ at time $|\bar{y}|$ from $(\bar{x}, \bar{y})$ (with $\bar{y} < 0$) is contained in $I\!R \times \{0\}$ and is connected. Thus it is an interval of $I\!R \times \{0\}$.

Third step : Let $(0, \bar{y})$ be an initial point, with $\bar{y} < 0$. For $v(\cdot) \equiv 1$, we denote by $(x(\cdot), y(\cdot))$ a solution of the differential equation for $f(\cdot, \tilde{u}(\cdot, \cdot), 1)$, starting from $(0, \bar{y})$. Recall that $y'(t) = 1$, and so $y(t) = t + \bar{y}$. Thus

$$x'(t) = \phi(t + \bar{y})\tilde{u}(t, (x(t), t + \bar{y})) + \phi(-\bar{y} - t) \geq -\phi(t + \bar{y}) + \phi(-\bar{y} - t)$$

and so:

$$x(|\bar{y}|) \geq \int_0^{|\bar{y}|} [-\phi(t + \bar{y}) + \phi(-\bar{y} - t)] dt$$

Since $-\phi(y) + \phi(-y) \to_{y \to -\infty} 1$, there is some $\bar{y} < 0$ such that $x(|\bar{y}|) \geq 1$. Since the game is symmetrical, setting $v(\cdot) \equiv -1$, we obtain $x(|\bar{y}|) \leq -1$. Thus the reachable set of $f(\cdot, \tilde{u}(\cdot, \cdot), V)$ starting from $(0, \bar{y})$ at time $|\bar{y}|$ contains $[-1, 1] \times \{0\}$, because it is a connected set contained in $I\!R \times \{0\}$.

Fourth step : Let $(\bar{x}, \bar{y})$ as in the previous step. For any $z \in [-1, 1]$, there is a solution $(x(\cdot), y(\cdot))$ starting from $(\bar{x}, \bar{y})$, such that $x(|\bar{y}|) = z$. The problem is that $(x(\cdot), y(\cdot))$ does not necessarily remain in K.

To obtain a solution which enjoys the same properties than $(x(\cdot), y(\cdot))$, and which remains in K, let us define: $(x_1(\cdot), y_1(\cdot))$ by: $x_1(t) = x(t)$ if $x(t) \in [-1, 1]$, $x_1(t) = 1$ if $x(t) \geq 1$, $x_1(t) = -1$ if $x(t) \leq -1$, and $y_1(t) = y(t)$.

It remains to prove that $(x_1(\cdot), y_1(\cdot))$ is a solution of the differential inclusion for $f(\cdot, \tilde{u}(\cdot, \cdot), V)$. Let us point out first that $(x_1(\cdot), y_1(\cdot))$ is Lipschitz, and so, almost everywhere derivable.

The map $(x_1(\cdot), y_1(\cdot))$ is obviously a solution for any $t \geq 0$ such that $x_1(t) \in (-1, 1)$, because on some $(t - h, t + h)$ (with $h > 0$), $x(\cdot) \equiv x_1(\cdot)$.

Let t be a point where $x_1(t) = 1$ and where $x_1'(t)$ exists. Since $x_1(\cdot)$ remains in $[-1, 1]$, one has: $x_1'(t) \leq 0$. If $x_1'(t) = 0$, then the first step of the proof yields that

$$(x_1'(t), y_1'(t)) = (0, 1) \in f(x_1(t), y_1(t), \tilde{u}(t, (x_1(t), y_1(t))), V).$$

Otherwise, $x_1'(t) < 0$. There is some $h < 0$ such that $x_1(s) \in (-1, 1)$ for $s \in (t, t + h)$. In particular, $x_1(s) = x(s)$ for $s \in (t, t + h)$. Since $x(\cdot)$ is a solution of the differential inclusion for $f(\cdot, \tilde{u}(\cdot, \cdot), V)$, $x_1'(t)$ belongs to $f(x_1(t), y_1(t), \tilde{u}(t, (x_1(t), y_1(t))), V)$. So the proof is complete. $\square$

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