Decison Making Under Uncertainty

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Foreword

IIASA celebrated its twentieth anniversary on May 12–13 with its fourth general conference, *IIASA '92: An International Conference on the Challenges to Systems Analysis in the Nineties and Beyond.* The conference focused on the relations between environment and development and on studies that integrate the methods and findings of several disciplines. The role of systems analysis, a method especially suited to taking account of the linkages between phenomena and of the hierarchical organization of the natural and social world, was also assessed, taking account of the implications this has for IIASA's research approach and activities.

This paper is one of six IIASA Collaborative Papers published as part of the report on the conference, an earlier instalment of which was *Science and Sustainability*, published in 1992.

When policy advisors come to appreciate that real uncertainty will affect the application of their recommendations they usually respond in one of two ways:

1. They may say that there are many possibilities, and then prepare a scenario for each; knowing the options advances the policymaker a little but his real decision making is not advanced, and on that he is left without advice.
2. They say that the uncertainties are so great that action had better be delayed until more is known; this recommendation for inaction is often very attractive to a policymaker, especially if getting more knowledge will mean waiting to enact some unpopular measure until a successor takes over the office.

Since there are no situations in which data is complete and exact, what can be done? That question is specially relevant to environmental decisions. At least policy can avoid what is called the prisoners' dilemma, where two people making rational decisions independently, i.e., each not knowing what the other will decide, can put themselves into a worse condition than if they make certain decisions that from the individual viewpoint are irrational, and much worse than if they participate in a collective decision. French indicative planning, now less favored than it was, aims to spread knowledge to each enterprise in an industry of what the competitors are intending, in the hope that that that will mean better decisions all around. A special case is what economists call externality, where a piece of common property is exploited by independent individuals, solitary choosers. One of the questions is whether the prisoner's dilemma and externalities can be handled by dissemination of information alone, or whether some form of compulsion is required, for example compulsion in the form of required pollution permits. These would give those who choose to buy them a marketable permission to exploit, and it can be shown that the outcome is economically superior to any instruction from above.

Professor Krasovskii provides some models, simple in form but sophisticated in substance, that show the nature of the problem of uncertainty in decision making, and how at least in theory it can be dealt with.

Committee for IIASA '92
Nathan Keyfitz (Chair)*

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Decision Making Under Uncertainty

Nikolai N. Krasovskii

This paper deals with models of dynamic phenomena that are subjected to control actions (decisions) and unknown disturbances (uncertainties). The terminology that I have started to use in the first phrase is quite common in contemporary theoretical research and applications. Its substantial meaning is heavily overloaded, however. On the one hand, the terminology may be felt to include large-scale multiparametric models of systems analysis and synthesis related to pretentious "global" projects whilst, on the other, it includes modest problems that could be successfully placed within the framework of low-dimensional models of the classical types known, for example, in analytical mechanics.

In this paper we will concentrate on models of a simple form to illustrate some basic concepts in the field. Of course, some of these models may seem to be of a rather fabulous nature and on the verge of being amusing. The author would like to beg the reader's forgiveness for this small weakness of his. At the same time, I wish to remark that a mathematical model (of which I will only speak in this paper) is, after all, only a caricature of reality. However, this is precisely one of the basic features and possibilities of mathematical models. Doesn't a good caricature indeed emphasize at least some of the essential features of a phenomenon?

Before passing on to the main text of this paper I would also like to mention some considerations which I always use when teaching in

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college and high school. I believe that education is something that has an especially high priority at this moment. You are all aware of the "brain drain" problem in our country. I will allow myself to suggest that this phenomenon is not as destructive as it seems and it may even turn out to be useful in some sense provided, however, that there is a good educational system which will reproduce the "brains" and which will start to operate with children at a fairly early age. Human experience indicates that education in applied issues should often start by using model problems, assuming, of course, that simplification does not erase the basic properties of reality.

Let us now consider a typical scheme of a controlled dynamic system that evolves in time t. We thus have a controlled process (a "plant") L, an information system Y that forms the on-line informational image $y[t]$, the decision-making (control) block and the uncertain environment V – the source of a disturbance $v[t]$. We presume that the on-line description of the state of the process is given by a variable $x[t_*]$ in a framework sufficient to define the future course of the process provided the initial state is measured and the decision variable $u[t]$ and the disturbance variable $v[t]$ are known for $t > t_*$. The information on the on-line and future values of $x[t_*]$ and v may be unknown to the decision maker, who is unable to penetrate beyond the informational image of the process, which in turn may be regularly refined through an on-line process of collecting additional information. The formal nature of the variables $x[t]$, $y[t]$, $u[t]$ and $v[t]$ is naturally defined by the type of real system and the model selected for its description. These variables may be represented by scalars, vectors, functions, probabilistic distributions, sets, logical statements, etc. It should be understood, however, that the decision maker is actually dealing not with the real process but only with its informational image and forming a complementary vision of the whole process $x[t]$ through available knowledge. Let me illustrate this by means of an example of the evolution of a fabulous model system whose computer simulation was a blitz-task at a high-school competition in informatics. Although simplistic in appearance, the system does indicate some substantial issues in decision-making. The evolution of the process described by the model is as follows.

The fairyland Cyclonia has the form of a circumference. Its population consists of n property owners. They are located along the circumference and numerated counterclockwise from 1 to n. On the first day of the first month of the reforms the ith owner had a "prosperity" sum estimated by i monetary units or "muns". On that day an Alien suggested to the population of the country that it should vote on the following transformation (change): each owner should give away his "prosperity" to the next neighbor in a clockwise direction. Each owner whose prosperity increases after this transaction must divert a part $\varepsilon\Delta$ of the increment Δ of his prosperity to the Alien. The transformation is accepted if at least kn owners ($0 < k < 1$) vote with "yes".

It is known that each owner voted with "yes" if and only if the transaction increased both his own prosperity and that of his nearest neighbors in both directions. The numbers n and k are such that under the above conditions the Alien's suggestion was accepted and the transformation did take place.

It is known that after the transformation each owner whose prosperity (after the payment of the ε-increment) constituted the value b ($b > a$, a given) invests $b - a$ "muns" in external commerce. By the end of the month this gives him an additional income $\beta(b - a) = 1 - p_1 - p_2$ with probability p_1. With probability p_2 he wins or loses nothing and with probability p_3 he loses $\beta(b - a)$.

On the first day of the second month the Alien suggested voting again on a similar transformation. The conditions for the evolution of the system again remain the same. The process is then continued throughout the following months as long as the voting is positive.

One has to write a computer program that allows one to simulate the evolution of economics. The program comes to a stop when one of two conditions is fulfilled: either the voting in a new month produces a negative result or the number of months reaches a given number m.

The input data: n, $0 < \varepsilon < 1$, $0 < k < 1$, $a > 0$, $\beta > 0$, p_1, p_2, m.

The output data: the number of the month for the end of the process, the prosperity of each owner, the total prosperity of Cyclonia, the total prosperity of the Alien (all of these at the end of the transformations).

The respective evolution was simulated with the following input data:

$$n = 40, \quad \varepsilon = 0.2, \quad k = 0.3, \quad a = 20, \quad \beta = 0.2,$$
$$p_1 = 0.4, \quad p_2 = 0.3, \quad p_3 = 0.5, \quad m = 50 \ .$$

The total prosperity of the population of Cyclonia at the beginning of the process was 820, that of individual owners not more than 40 (the individual maximum). The prosperity of the Alien was 0. After 50 transformations the total prosperity of the population had become 488.8, with the individual maximum amounting to 26.9 and that of the Alien to 539.1.

I believe that despite its humoristic form the model contains some important tutorial information for the student. Namely, it indicates, perhaps too explicitly, that a restriction on the informational horizon and on the memory concerning the process, as well as the subjectivity of decision-making with limitations on the objective knowledge of reality, may lead to highly undesirable results due to a bad choice of decision parameters and despite the overall good intentions.

Let us now return to the main direction of the presentation. We presume that the law that governs the evolution of the variable $x[t]$ and also perhaps the law that governs the evolution $y[t]$ is of a differential type. We will therefore deal with systems described by differential or recurrence equations with a minor discrete time increment in the latter case. Our aim will be to integrate these infinitesimal or difference generators of the evolution process, aiming at a best possible result in our interest by selecting appropriate decision or control rules.

Our ideal case would be if the lack of information on the system were only fictitious and a sufficient amount of information could be obtained through properly organized measurement and the processing of its result. The information processing would thus incorporate the integration of the infinitesimal or recurrence generators of the process. Such a situation is not realistic, however, and would be encountered perhaps only in a fairy tale. Let us nevertheless illustrate this option. I will now describe a model example that demonstrates to the student how small perturbations of the conditions of the problem change the effect of the informational image on the decision. The latter may then have to be changed from a vigorous deterministic rule to a modified rule that would, perhaps, incorporate some game-theoretic or statistical schemes and a presumption that the game would be repetitive.

The problem is to simulate on a computer a trick well known in literature. It was suggested at one of our olympiads for high-school students and formulated as follows.

One has to prepare a computer program that demonstrates the following trick through an interactive man-machine procedure. On three seats numerated as 1, 2, 3 one places three objects (for example, a glass, a plate and a spoon). The numbers and the names of the objects are known to the computer but the correspondence between the numbers (the seats) and the objects is unknown.

Stage 1. The man tells the machine of only one correspondence (seat – object) of his choice.

Stage 2. After that one performs several pairwise permutations of the objects by interchanging the numbers (for example, the object from N1 goes to N3 and the object from N3 to N1; the object from N3 goes to N2 and the one from N2 to N3, etc.).

Stage 3. After that the man choses one object and makes the two others change places. Which the other two objects are that change places (by changing seats or changing numbers) is not communicated to the machine.

Stage 4. One again performs several pairwise permutations of the objects by interchanging the numbers.

Stage 5. The machine has to guess the object chosen at stage 3. In order to do this the machine asks the number of the seat for one of the objects that it chooses.

Stage 6. The machine tells us the object chosen at stage 3.

The last stages were deliberately formulated in an ambiguous way. Here we have to distinguish two interpretations of the situation. The first one would mean the following: "The machine tells the number of the seat and asks which object is on that seat". This is the canonical formulation. Here an algorithm exists that interactively processes the information and allows a unique solution. This algorithm is usually well discovered by students.

There is, however, a second interpretation of the phrase, namely: "The machine names the object and asks for the number of the seat it occupies". Then the information available to the machine may turn to be insufficient for a unique answer on what the respective object chosen at stage 3 is. The process of guessing the answer may be based on a matrix game. Namely, the one who makes the choice at stage 3 may select one of the three strategies V1, V2, V3, each of which at stage 1 stands for naming an object and naming the corresponding number and at stage 3 stands for the choice of one of the objects. On the other hand, the machine may select one of the three strategies U1, U2, U3, each of which at stage 5 stands for naming an object and asking for the seat it occupies. At stage 6 each of these strategies names the chosen object depending on the answer at stage 5 which was either n1, n2 or n3. If the cost for a correct guess is taken to be 1 or 0 is taken for a wrong guess, then the matrix game discussed here does not have a saddle point and, therefore, a value in the given class of pure strategies.

The situation is different, however, in the class of mixed strategies where a saddle point, and therefore a value of the game, does exist. A computer simulation of the situation in the class of mixed strategies is a good methodological exercise that would also serve as a good introduction to more complicated problems of decision-making under uncertainty.

Let us return to the basic text. Here we have to admit that the statistical approach is not flawless and may not work well in the absence, for example, of good learning statistics, or in a condition when the available informational image is due to a single evolution trajectory. Then the cost criteria for control under uncertainty is selected in the form of an index

$$\rho = \gamma\,(x[t], \quad t_0 \leq t \leq \Theta)$$

for which one has to *secure a guaranteed value.* Otherwise we should estimate a control strategy S_u through the upper bound

$$\rho^* = \sup_{y[\cdot],v[\cdot]} g$$

of the values $y[\cdot], v[\cdot]$ that are consistent with S_u.

The optimal ("minimax" or "guaranteed") strategy is then determined as the one S_u^0 for which the value ρ^* turns out to be minimal and which, therefore, *guarantees* the value

$$\rho^0 = \min_{S_u} g^* \quad .$$

The guaranteed strategy is sometimes criticized as very cautious, being related to the worst case realization, which may not be the actual case. The counterargument is that here we are not dealing with an "open loop control" or otherwise, with a rigidly fixed solution programmed in advance. The decision is made due to a feedback strategy, so that the realization $u[t], t_0 \leq t \leq \Theta$, is calculated in an on-line procedure due to strategy S_u^0 that tracks the evolution of the informational image $y[t]$ that reflects the actual evolution $x[t]$ and that ensures a flexible reaction to any deviation of the external environment from the anticipated worst case situation. More precisely, the optimal strategy that defines the decision rule should be universal relative to the on-line informational images $y[t]$ that may be met throughout the process.

The given two basic features secure a certain guaranteed result from possible unfavorable circumstances and allow for improvement of the situation if the worst case does not occur. They seem to justify the applicability of this approach to the design of feedback-type rules for decision under uncertainty. Nevertheless we often encounter advertisements for rigidly fixed "scenarios" that are supposed to secure a successful performance of the system.

The requirement of *universality* for the decision rule may generally be formulated as follows. The optimal strategy S_u^0 should ensure for any possible realization $y[t_*]$ $t_* \epsilon [t_0, \Theta]$ a minimal value over all admissible strategies S_u, namely

$$\rho^0(y[t_*]) = \min_{S_u} \sup_{y[\cdot],v[\cdot]} \rho \quad ,$$

where the upper bound is taken over all possible future realizations $y[t]$, $v[t]$, consistent with the given informational image $y[t_*]$. In each specific case a precise formulation is available within a rigorous formal scheme. The key point here is a rational description of the informational image and its evolution. Here I have in mind the minimax game-theoretic approaches in the form of dual games of feedback control and observation, developed, particularly, in Sverdlovsk (now Yekatherinburg).

In connection with the present discussion I will mention a specific game-theoretic formalization of a control problem for a differential system in the class of *mixed strategies*. This allows us to propagate the classical notion of mixed strategies for matrix games to dynamic systems. The approach combines the problem of finding a minimax estimate related to the uncertainty of the system with some average probabilistic procedures that may correspond to the statistics of the disturbance (perhaps, even unknown), together with artificially introduced statistics of the generated decision rule (the control).

It is important, however, to emphasize that with the incorporation of mixed strategies the total result of the control process is evaluated not by an average cost taken over the realizations, but in the form of a guaranteed probability of performance that turns to be close to unity over every individual realization of the process. This is due to the fact that throughout the evolution of the dynamic system the law of large numbers holds for a large number of small sequential time intervals over each of which a separate control decision is taken. The actual scheme is such that, together with the actual controlled x–object, we introduce a computer simulation y–model (a "leader" in terms of control optimization) or a z–model (if we are to simulate the worst case disturbance). Both models are incorporated into the control loop. Abstract models that form some fictitious motions $w_y[\cdot]$ and $w_z[\cdot]$ that correspond to $y[\cdot]$ and $z[\cdot]$ are also formed. The evolution of all the mentioned motions is organized in a mode that ensures motion of the leader that is optimal in some sense and also a stable tracking of the leader by the actual motion. This therefore ensures a suboptimal performance of the actual process.

All the given features are thus synthesized into the overall scheme of feedback control through an appropriate mixed strategy. A successful incorporation of the scheme requires the solution of quite a number of auxiliary problems of stability, optimal control, etc. This also includes the utilization of the so-called open-loop synthesis method. The idea of the method consists of coupling the on-line element $w[t]$ that follows the informational image $y[t]$ with an auxiliary stochastic procedure that acts as a sort of locator that predicts the future course of the game on the basis of some fictitious time $\tau > t$ and some conceivable controlled stochastic process, based on an auxiliary probabilistic space, constructed independently of the space for the control and the disturbance.

Under rather general conditions for a differential evolutionary system we prove the existence of a saddle point $\{S_u^0, S_v^0\}$ that is formed through mixed minimax (S_u^0) and maxmin (S_v^0) strategies yielding a value $\rho^0(t_*, x_*)$ of the respective differential game that is equal to the guaranteed results $\rho_u^0(t_*, x_*)$ and $\rho_v^0(t_*, x_*)$. These results should be interpreted as the limit values for the guarantees, which could be approached as close as we wish by increasing (regulating) the frequency of the random tests that determine the stochastic mechanism of generating the decision rule (the control).

The given conceptual framework for decision-making (control) under uncertainty was suggested and developed in Sverdlovsk (now Yekatherinburg) for dynamic systems, described by differential equations. This concept presumes a game-theoretic nature of the problem and therefore lies within the framework of differential games. Let us illustrate these general considerations by the following specific material. Suppose the controlled system is described by the equation

$$\dot{x} = A(t)x + B(t)u + C(t)v$$

and we are to minimize the cost

$$\gamma = |x[\vartheta]| \ .$$

Here ϑ is a given time for the end of the process and the symbol $|x|$ stands for the Euclid norm of the vector x. The process starts at time t_0. Suppose the informational image $y[t]$ has the form

$$y[t] = \{x_0^*, q[\eta], t_0 \leq \eta \leq t; u[\eta], t_0 \leq \eta < t\} \ .$$

Here x_0^* is the observed initial state where $x_0^* = x_0 + \Delta x_0$, and where x_0 is the actual initial state. The second element of $y[t]$ is the history of the observed variable $q[t]$ which reflects the measurement of the state space vector $x[t]$ with an error, namely

$$q[t] = K[t]x[t] + \Delta q[t],$$

where K is a given matrix. The third element of $y[t]$ is the history of the control $u[t]$. It may also be convenient to include into $y[t]$ or $u(y[t])$ some other observable parameters. The given values of $u[t]$ and the possible values of $v[t]$, Δx_0 and $\Delta q[t]$ satisfy the given constraints. However, it is more convenient to restrict the variables $u[t], v[t], \Delta x_0, \Delta q[t]$ by adding penalty functions to the value γ. Then the problem may be formulated as follows. Let the restrictions on $u[t], v[t], \Delta x_0$ and $\Delta q[t]$ be of quadratic nature. One has to devise the strategy $u^0(y[t], \varepsilon)$ that minimizes the guaranteed result

$$\tilde{\rho}(y[t_0]) = \sup_{q[\ ], v[\ ], x[\ ]} \tilde{\gamma}(x_0^*, q[\cdot], v[\cdot], x[\cdot])$$

for the functional $\tilde{\gamma}$ of type

$$\begin{aligned} \tilde{\gamma} \ &= \ |x(\vartheta)| + \int_{t_0}^{\vartheta} (u'[\tau]\Phi[\tau]u[\tau] - v'[\tau]\Psi[\tau]v[\tau] \\ &\quad -\Delta q'[\tau]Q[\tau]\Delta q[\tau])d\tau - \Delta x_0' P \Delta x_0, \end{aligned}$$

where the prime stands for the transpose, and Φ, Ψ, P and Q are the matrices of positive-definite quadratic forms. The exact setting of the problem requires us to explain the notations. We will be brief in this. Let $\varepsilon > 0$ characterize the neighborhood in which the respective elements are selected. The given relations are then treated as the limits for a certain time-discretized approximation scheme when ε and Δt tend to zero. While forming the motions $y[t]$ of the information system it is convenient to include the observed variable $\tilde{y}[t] = \{\tilde{x}[t], \tilde{y}_{n+1}, [t]\}$ described by the equations

$$\begin{aligned} \dot{\tilde{x}}[t] &= X[\vartheta, t]u(y(t), \varepsilon), \\ \dot{\tilde{y}}_{n+1}[t] &= u(y[t], \varepsilon)\Phi[t]u(y[t], \varepsilon) \\ \tilde{x}[t_0] &= 0, \quad y_{n+1}[t_0] = 0 \end{aligned}$$

into the informational image of the system. Here $X[t, \tau]$ is the fundamental matrix solution of the equation

$$\dot{x} = A(t)x \ .$$

We will also use the variable

$$g[t] = q[t] - K\int_{t_0}^{t} X[t, \nu]B[\nu]u[\nu]d\nu.$$

Finally we have

$$y[t] = \{x_0^* : g[\eta], t_0 \leq \eta \leq t; \tilde{y}[t]\}.$$

Suppose the realization of the process resulted at time t in the state $y[t]$, the accompanying element is chosen in the form

$$z[t] = \{x_0^* : g[\eta], t_0 \leq \eta \leq t; \tilde{z}[t]\}$$

$$\tilde{z} = \{w, \tilde{z}_{n+1}\}.$$

This element generates for $\tau > t$ a certain fictitious stochastic motion

$$z[\tau, \omega] = \{x_0 : g(\nu, \omega), t_0 \leq \nu \leq \tau, \tilde{z}(\tau, \omega)\}.$$

This is constructed on an appropriate probability space and is generated by an appropriate stochastic process $\xi(\tau, \omega)$ and an appropriate stochastic differential equation

$$\dot{w}(\tau, \omega) = X[\theta, \tau]B(\tau)u(\tau, \omega),$$

$$\dot{\tilde{z}}_{n+1}(\tau, \omega) = u'(\tau, \omega)\Phi[\tau]u(\tau, \omega) \ .$$

Here $u(\tau, \omega)$ is a stochastic program nonanticipative relative to $\xi(\tau, \omega)$. A nonanticipative random function $g(\tau, \omega)$ is a prolongation for $\delta > t$ of the informational error $g[\eta]$, $t_0 \leq \eta \leq t$, that had realized before time t. Moreover the construction includes random variables $w(\omega), v(\tau,\omega), t_0 \leq \tau \leq \theta$. In an additional stochastic construction they imitate the real disturbance and the real initial state x_0 that are actually unknown throughout the process.

For the stochastic motion we have an additional minimax problem where the additional cost functional γ^* arrives from $\tilde{\gamma}$ after a substitution of all the realized motions by those of the fictitious motion. The minimax is calculated by minimization over a stochastic program $u(\tau,\omega)$ and by maximizing over $g(\tau,\omega)$, $v(\tau,\omega)$ and $w(\omega)$. The value of this minimax ρ^* gives a desired estimate for the optimal guaranteed result $\rho^0(z[t])$ for the primal control problem over the neighborhood of the actual informational image $y[t]$. After that the control action $u^0[t] = u^0(y[t],\varepsilon)$ is selected by applying the condition of steepest descent to a certain Liapunov function λ, namely

$$\left\{\frac{d\lambda}{dt}\right\}_{u^0} = min_u \left\{\frac{d\lambda}{dt}\right\}_u \quad .$$

Symbol $\frac{d\lambda}{dt}$ stands for the full derivative of λ in time t along the motions of the system which is calculated due to the rules of Liapunov's stability theory. The Liapunov function could be selected as a quadratic form of the difference

$$\tilde{y}[t] - \tilde{z}[t] \quad .$$

In compliance with Liapunov's stability theory the perturbed motion is the one that describes the evolution of the informational image $y[t]$. The unperturbed motion is the one that is formed as the evolution of the accompanying element $z[t]$.

Let us emphasize an interesting point. If we know in advance that our control will be optimal, then the worst case disturbances could be specified in advance as a deterministic program – a function of time t. However, if we deviate from the optimal control situation, then the worst-case "antioptimal" disturbances should be, in general, organized through a feedback correction procedure. Let us now proceed with another model illustration taken from the curriculum of the University of Ural at Yekatherinburg. This illustration is of a mechanical nature.

Take an object of variable mass that satisfies the Meschersky equation, that is driven by a reactive control force and also subjected to a central force, a friction and a disturbance. The disturbance consists of an irregular additional external force and a certain clearance that deviates the control force. The cost criteria is the estimate for the distance of the object from a preappointed "target" point at given time Θ.

The given informational image is the current position $\{t, x[t]\}$ of the system and the velocity $\overset{\circ}{x}[t]$. The optimal mixed strategies – the minimizing S_u^0 and the maximizing S_v^0 – were synthesized according to the scheme loosely described in the above. The process was simulated on an IBM PC-type computer. Under the selected starting data the optimal guaranteed results were as follows:

The value of the game:

$$\rho^0(t_*, x_*) = 1.065 \ .$$

The value with simulated optimal control and disturbance

$$\gamma = 1,\ 161 \approx \rho^0 \ .$$

With optimal control and nonoptimal disturbance

$$\gamma = 0.466 < \rho^0 \ .$$

With nonoptimal decision, optimal disturbance

$$\gamma = 1,\ 614 > \rho^0 \ .$$

The nonoptimal control forced the acceleration toward the origin at each instant t, while the nonoptimal disturbance forced it from the origin.

Discussion

Gustav Feichtinger

Professor Nikolai Krasovskii's paper begins with models of a simple form, of rather fabulous nature and on the verge of being amusing. It seems to us that such types of models provide an efficient way to transport basic concepts and insights. To put it in another way: Bourbakism is didactically *out*. Some mathematical models are only caricatures of reality: but good caricatures indeed emphasize at least some of the essential features of a phenomenon.

Furthermore Krasovskii stresses the importance of education, which should have high priority, especially at this stage of the historical process, i.e., the rapid changes in Eastern Europe and the dramatic developments in the North–South conflict.

The *IV Viennese Workshop on Optimal Control and Dynamical Systems* (Feichtinger, 1992) contains also a few papers which are at first glance humoristic, but deal with problems which are also of interest in real applications. This is a good tradition, already taken up in the first Viennese Workshop. To mention a few topics, we refer to the "dynamics of extramarital affairs", "optimal drinking behavior at a party", "optimal slidesmanship", "corruption of politicians", etc.

A good example for models of this type is provided by the author in his fairyland Cyclonia model. It illustrates in humoristic form how a model may contain important tutorial information for students. The message in this case is that subjectivity of decision-making together with restrictions on the information structure of the process *may lead to highly undesirable results despite the overall*

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good intentions. More precisely, the "Alien" acting as coordinator becomes richer at the cost of the agents who are exploited by the Alien.

This remark leads us to a second point we would like to deal with. We consider, as Krasovskii did, an action selected by a decision-maker which runs counter to its good intention.

Let us just mention an example to clarify what we mean. Long and Siebert (1985) and Steindl *et al.* (1986) analyze the impact of an *incentive* scheme in which a firm receives a reward (or pays a penalty) when it deviates above (below) its normal employment level. The latter concept is introduced as a weighted average of past employment levels. It turns out that such an institutional setting (more precisely, an incentive scheme of a certain kind) may imply cyclical demand in labor. Thus, institutional arrangements may be responsible for business cycles. Clearly, alternative regimes of hiring and firing are beyond good intention.

We conclude our remarks on the discussion by briefly sketching a related situation. As above, we consider a case in which the actions of a decision-maker may be counter-productive to the society as a whole.

In Dockner and Feichtinger (1992) the role of aggressive environmentalists in the process of exploiting a renewable resource (like a fish population) is studied. In particular, the harvesting activities of a profit-maximizing firm are interrupted by the actions of environmentalists who aim at reducing total catch by the firm in order to protect the fish population. The resulting differential game model has been formulated to answer the question, "Does society gain from the actions of environmentalists?"

It is interesting to mention that a similar paradigm is applicable to "explain" cyclical patterns in ancient Chinese history, i.e., periodic alternatives between despotism and anarchy. The open-loop Nash solution of the differential game provides, up to now, the simplest case of a three-dimensional canonical system exhibiting a stable limit cycle as an "optimal" solution (see Feichtinger and Novak, 1992).

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Discussion

Eberhard P. Hofer

In modern terminology the term uncertainty is often related to chaos, fuzziness, and neural nets. In Nikolai Krasovskii's paper uncertainties are unknown disturbances or parameters. The methodology aspect is covered from an upper level and is rather global.

The range of applications for decision-making under uncertainties in technical and non-technical linear and non-linear dynamical systems spreads from low-dimensional to large-scale systems. What is the salient feature of the problem and what could an engineering approach to the solution look like?

Rather than assuming a stochastic process, a deterministic treatment of uncertainty seems to be very attractive, e.g., as outlined in George Leitmann's work: one requires *certain* performance in the presence of *uncertain* information. Under the assumption that nature itself is not uncertain, but only our information, our knowledge, and our methods are uncertain, the deterministic approach seems to be appropriate.

As already mentioned, decision-making very much depends on the underlying performance index. Therefore, optimization problems and differential games have to play a dominating role in this framework. The importance of the so-called auxiliary problems such as stability, control and observer design is given by the fact that their solutions form a basis for a – let me call it – "methodology box". A box full of tools available for practical problems marked with recommendations and indications about their limits in use.

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The reader would have liked to hear more about the methods, analytical and numerical ones. I would have liked a list of references in order to participate in the rich experience of the author.

As far as numerical methods are concerned, comments on the implementation of efficient and reliable software for more complex systems, as well as good price/performance relations for implemented software, would be important. Interval methods (algebraic methods), model reduction methods as an important tool to reduce system order (singular perturbations), and knowledge-based decision making with appropriate algorithms have to be discussed within this framework. In modeling, which plays a key role in dynamical systems, first include the governing physical phenomena, then intuition and heuristics.

All our really important problems are of an interdisciplinary nature. We have to bring together the knowledge and experience of our participating researchers from the various disciplines at the very beginning. This is our best chance for reducing complexity in the system and handling uncertainties, in order to be prepared for good decisions. We need problem-oriented research to be specific, flexible and efficient.

When we talk about the enormous challenges of the nineties and beyond we have to face the population problem, economic problems, and health problems, with special emphasis on the immunology problem. For solving these problems I ask:

Do we have the appropriate models?
Do we have the right performance index?
Do we have the powerful methods?

I think there is a strong demand for our contributions in these fields for a good future on our planet earth.

Rapporteur's Report

Yuri M. Ermoliev

Decision-making under uncertainty is as old as mankind. Caesar crossing the Rubicon said *Alea iacta est* (the dice are cast), implying that our decisions are affected by chance.

Uncertainty is an essential feature for any study directed toward the future. The tendency in decision-making involving uncertainty is to postpone decisions until uncertainties are resolved. Unfortunately, uncertainties are inherent in virtually all systems related to economics, meteorology, demography, ecology, etc., and will never be resolved. In order to encourage a decision-maker to act we need appropriate tools to explicitly treat the uncertainties involved.

We cannot predict or optimize everything unless we make some assumptions about patterns of the phenomena: create a model. Sometimes the assumptions are made in an explicit form that lead to mathematical models and a rigorous analysis. Sometimes they are made implicitly and leads to a verbal analysis, often providing more questions than answers. We cannot rely on such statements as "market forces bring demand and prices into equilibrium" or "adaptation and incremental improvements will find an optimal decision", since they are true only when underlying processes have certain properties, for instance, smoothness, nontoxicity, concavity, product homogeneity and divisibility, unchanging "environment", full information (certainty), lack of collusion or strategic behavior, and absence of externalities. Otherwise the "convergence" cannot be proved even on paper (mathematically) and such mechanisms

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without the help of a regulatory body may create unpredictable behavior and dangerous "locked-in" structures, caused even by small disturbances at the initial stage.

The discussion of the section "Decision-Making Under Uncertainty" concentrated on various possible approaches for dealing with the uncertainty, one of which is described in the paper written for the conference by Professor Nikolai N. Krasovskii. The report below reflects the discussions, with a subjective view by the rapporteur.

Uncertainty enters various steps of a model building process, starting from the model structure: purely verbal or with some precise details; which variables are driving?; levels of aggregation and decomposition (region, country, the universe) coinciding with uncertainty of available information; relations between variables, for instance, diffusion equations or only simple "predictors" such as transfer coefficients are needed in pollution management if we take into account uncertainties in the weather conditions, "dose–effect" relations, current and projected emission patterns, etc.

The existence of decision variables creates additional and essential sources of uncertainties in contrast with classical natural science models. The aim of a decision-making model is not only to make sense of a limited set of observable data, but also to change the current practice and structure for a better state which may be unlike anything that has been experienced in the past. The experiments or "trial and error" mechanisms may be dangerous, time-consuming or simply impossible – we have to rely only on models equipped with tools to treat arising uncertainties explicitly.

The goals of a decision-making process and constraints create more uncertainties in cases when good intentions may lead to bad results and public distrust. Constraints induce feedback. For instance, if carbon dioxide in water C_w and carbon dioxide in the atmosphere C_a satisfy a symbolic constraint (in a time interval)

$$\alpha C_w + \beta C_a = constant$$

with some positive coefficients α and β, then positive feedback occurs: warming diminishes C_w and thus increases C_a, which leads to more warming, diminishing C_w, and so on. [See also the discussion in Keyfitz (1992).]

Uncertainties in constraints such as in the values α and β influence a balance of positive and negative multiple feedback and may lead to opposite conclusions, for instance, on warming or on the ozone layer.

Modeling usually starts with an attempt to create a model which makes sense of observable data. From the formal point of view it can be interpreted as an approximation of available data by some functions, such as a polynomial, or other functions which may also be given implicitly as solutions of differential equations, or Petry and neural nets with unknown parameters to be identified. Since there may be errors of measurement, the "fitness" must also be understood in a certain way that often leads to non-unique solutions of the identification procedures and additional uncertainties in predictions. It is not true that any other, e.g., verbal, analyses avoid all the above mentioned difficulties: this is only possible by ignoring them.

Of course, the building of a decision-making model (conceptual or more formal, including details such as concrete values α and β) is not only a scientific task but also an art. The main purpose is not to take a picture of the situation, but to make a sketch – a laboratory world to examine possible concepts and alternatives. In the presence of essential uncertainties the most important task in the modeling seems to be a search for better solutions, comparative studies or optimization, rather then prediction and assessment. It is impossible to explore all details of the environment, biochemical, hydro-meteorological, genetic, etc., differences involving large numbers of variables. Therefore assessments and predictions will always yield poor absolute values. Despite this, the preference structure among decisions might be stable, which is similar to the difference between measuring exact weights of parcels and only guessing which is heavier.

The study of interactions between uncertainties, decisions and outcome is a methodologically challenging task. Relations among variables may be changed considerably with the change of decisions aggravating various risks. It is possible to speak about reliability of a decision: the best decision combined with "bad luck" may lead to negative effects, and a wrong decision with "good luck" to positive effects – at least for a while, as it was in the case of Chernobyl.

How can we choose decisions which are optimal and still robust against all eventualities? What are the enforcements of such decisions? For instance, decisions in the case of greenhouse effects might fundamentally endanger modern civilization, but on the other hand might be beneficial or perhaps even not big enough to matter (Clark, 1986).

As uncertainty is a broad concept, it is useful to approach it in many different ways [see, for example, von Winterfeldt and Edwards (1986)]. The most commonly used technique for dealing with long-term planning under uncertainty, scenario analysis, is seriously flawed. Although it can identify "optimal" solutions for each scenario, it does not provide any clue as to how these "optimal" solutions should be combined to produce a merely reasonable decision. The suggestion of mixing up all the best solutions with weights assigned to correponding scenarios may lead to wrong decisions, as it can be seen from the simplest situations. There is also a suggestion to use mixed strategies derived from an appropriate "pay-off" (decision/scenarios) matrix. In this case an optimal decision appears as a result of random choice among decisions which are optimal only for one scenario (e.g., best crop for "dry", "normal" or "wet" season). Of course, such a solution lacks the diversity (multiple crops, different energy sources, various products, etc.) needed for its robustness against all possible scenarios.

One clear and easy way to characterize uncertainty is by ranges or even sets of possible values (without identifyiing their likelihood): set-valued estimate. Since such a depiction of uncertainty does not provide any idea of more reasonable values, the choice of optimal decisions is usually based on the calculation of upper and lower bounds of outcomes with respect to all possible uncertainties; in particular, it is suggested for making decisions from the worst case situation – so called guaranteed decisions (strategies). In fact, such an approach for the case of dynamic systems modeled by differential equations is described in the paper written by Krasovskii.

In the case of dynamic systems modeled by differential equations, guaranteed strategies often lead to instabilities and irregular behavior of the corresponding trajectories. The absence of optimal trajectories in the commonly used sense requires generalization, and it is often suggested that the state of the system also be described as a set-valued estimate. Since nonlinear transformations of a dynamic system may map a simple set into a rather complicated domain, the set-valued estimates of the state are sometimes searched for among a rather simple approximation, for instance, ellipsoids.

A more specific assumption described by Krasovskii's approach is that the information image has to be generated by a differential equation (second player) and these equations are known to the decision maker (first player) and thus he can integrate them. ("The information processing would thus incorporate the integration of the infinitesimal or recurrent generators ... ", see p. 4.) Therefore, the search of guaranteed strategies requires the solution of optimal control subproblems that makes essential difficulties in a case of general-type constraints or nonlinearities. A typical application of such a scheme may be low-dimensional models of mechanics with two "players" modeled by differential equations.

The characterization of uncertainties by ranges and worst case analysis may provide useful insights. On the other hand, "because a range may be derived through a process of ruling out impossible values rather then through critical analysis of the relative likelihood of more reasonable values this depiction sometimes arouses scepticism and can appear non-scientific" (Finkel, 1990).

The following is a typical illustration (Finkel, 1990, p. xiii):

> Giving only the mean annual income ... or only the median and bounds would not reveal that a substantial proportion of the total national income accrues to a relatively small number of very wealthy persons.

The same applies for concentrations of pollutants, toxicants, energy demand for the next year, population, etc. For instance, the daily concentration of a toxicant may be within the normal level, but for five minutes it may vitally exceed the survival level. A simplified depiction of the uncertainty by decision makers may easily create a syndrome of "public concern" as the following example illustrates.

Consider a situation where two types of accidents might occur to a group of 10 people. The first type of accident will result in the death of all 10 persons in 1 out of 10 cases. The second type of accident will result in the death of each of the 10 persons (independently), also in 1 out of 10 cases. The range of possible deaths (set-valued estimate) and the averaged value are the same in both cases, that is 10 and $(1/10)\ 10 = 1$: but the chance of 10 deaths in the second type of accident is only $10^{-10} = 0.0000000001$ in contrast to 0.1 of the first type.

A more general idea for depicting uncertainties is to assign weights to possible values of uncertainties (parameters, events), such as frequencies in the case of repetitive events, or confidence measures in the case of non-repetitive events (an accident at a particular plant). Such weights are often interpreted as a probabilistic measure (possibly of subjective nature). There may also be other versions, for instance when the support of the weight-function is interpreted as the "fuzzy set".

The difficulty of such an approach is that although the weights of initial data are known, their propagation through the system creates enormous computational difficulties for the analytical evaluation of outcome weights. The decision variables, as we can see, may dramatically affect these weights further and create a higher order source of uncertainties.

The interpretation of weights as a probabilistic measure has essential advantages compared with any other concepts since the study of the propagation in this case can be based on the Monte Carlo simulation techniques. In other words, it is possible to incorporate a Monte Carlo simulation model into an optimization process. Unfortunately it is impossible in other approaches such as the "fuzzy-set" theory. Besides, in such approaches there is no well-established, empirical method to quantify fuzziness (or vagueness, plausibility) similar to frequency analysis of real observations, experiments, results of questionnaires or expert judgements of the probability theory.

How can we design an optimal strategy by utilizing only weights for the initial data? The task of the Adaptation and Optimization

Project (1982-1985) at IIASA was to study the answers to this question. Extended discussions of motivations and developed tools with their possible applications and implementations has been published in the volume by Yu. Ermoliev and R. Wets (1988).

Let us note that the search procedure cannot be based on straightforward Monte Carlo simulations since even the evaluation of the best decision among two alternatives in such a case is equivalent to well-known problems of hypotheses testing. Since results of initial data propagation can be studied (in general) only by a sampling procedure, the above question is equivalent to the following:

> How can we find an optimal decision among an infinite number of feasible decisions without calculation of exact values of "objectives" and "constraints" and with a large number of decision variables and uncertainties?

The answer to this question leads us to stochastic optimization tools conceivable with only partially known distribution functions (and incomplete observations of unknown parameters), which have been successfully applied to a wide variety of problems.

There are differences between the typical formulation of the optimization problems that come from statistics and those from decision-making under uncertainty. Stochastic optimization models are mostly motivated by problems arising in so-called "here-and-now", or *ex ante* situations, when decisions must be made on the basis of existing or assumed *a priori* information about uncertainties. The situation is typical for problems of long-term planning (strategic behavior) that arise in systems analysis. In mathematical statistics we are mostly dealing with "wait-and-see", or *ex post* situations, when decisions are made on the basis of observations "during" the decision-making process. Such a situation is encountered in short-term planning or, say, in driving a car.

Generally speaking, in the case of uncertainties, nonstationarities or disequilibrium there are two major mechanisms for facilitating our response to uncertainty and changing conditions: the short-term adaptive adjustments (defensive driving, marketing, inventory control, emergency service, etc.) and long-term anticipative actions (engineering design, policy setting, investment strategies, insurance, pollution reduction strategies, land developments, even keeping an umbrella at the office, etc.). The anticipative, long-term perspectives are important for environmental problems.

The major challenge to the systems analyst is to develop an approach that combines both mechanisms (adaptive and anticipative) in the presence of a large number of uncertainties, and this in such a way that it is computationally tractable.

Dynamics and nonlinearities bring uncertainties of chaotic behavior similar to probabilistic and well-known behavior from the pseudo-random numbers generator theory. Again, often the famous question arises of whether the good play dice or nature is certain and can be predicted (contradicting even with W. Hasenberg's "uncertainty principle"). If this is the case, then one may become a very wealthy person by finding a deterministic equation describing fluctuations of prices on a stock exchange. How shall we characterize uncertainties of a chaotic nature, for instance, sequences of pseudo-random numbers – by ranges, trajectories, generating equations or remarkably stable frequences? Of course, the answer depends on the problem at hand.

Decisions create more uncertainties through externalities existing in economics and environment, when the decisions of a participant (firm, region, country) are uncertain until other participants reveal their decisions. Such uncertainties require not only exchange of information, but a concept of mutual interests and joint constraints which are not always understood and may often emerge only through a process of successive negotiations. Game theory exists: this is a theoretical framework with various concepts of possible equilibriums in the presence of the different interests of participants (of a cooperative or non-cooperative nature).

The treatment of the environment as this generation's "public good" brings uncertainties in the evaluation of real costs, benefits or efficiencies. Of course, the round table discussion will never produce values of damages and the negotiations must be supplemented by unified approaches to treat involved uncertainties. It is possible to imagine a model accepted by participants to serve the purpose of a witness or victim during a negotiation process.

Accumulative effects and the irreversibility of environmental damage when some aspect may be lost forever accentuates the long-term and anticipative nature of decisions and requires the establishing of "contacts" between current and future generations. What are the "taxes" of the current generation? What are "social" costs and benefits? The answer often calls for a regulatory body (or a number of competing authorities) to establish environmental quality in certain "norms" with appropriate monitoring regulations and verification procedures. It requires a dynamic approach to decision-making under uncertainty to enable observable episodes to be traced back to their sources.

A decision-making process can easily be directed toward desired results by manipulation of goals, constraints, parameters, relations between variables, etc. (with a model), that may lead to contradictory conclusions and public distrust. It is important to understand that the structure of a decision-making model is not as well defined as the structure of natural science models. The formulation of such a model always runs into difficulties in identifying objectives and constraints. Their understanding and quantification is usually achieved through a "dialogue" with the model when each run of the model provides new ideas for possible variables, constraints and goals. Such a process proceeds until a compromise decision emerges. Therefore, the model doesn't make decisions – they are made by the decision maker (participant) and the most important is not *what* he decides but *how* he decides.

We can say that in the decision-making process nothing can be claimed to be the true decision; it varies with changes in goals, alternatives, constraints, and information. The whole process must be viewed as a process of successive adjustments rather than a comprehensive choice. The study of such nonstationary decision-making processes is rather challenging. Important "adjustments" may not necessarily lead to incremental improvements even in the case of concave (or convex) but rapidly changing (nondifferentiable) objective (goal) functions.

The nature of uncertainty avoids certainty, so that even the best solution, as mentioned above, may have (with bad luck) a negative result. It requires transparent representation of scientific understanding of inherent uncertainty, which is often quite different from the public and the decision maker's perception.

Uncertainty easily creates nonlinearity and even discontinuity, which in turn creates new uncertainty. In the case of guaranteed strategies, essential uncertainties are created by "inner" subproblems which are often not completely solved, and thus the guaranteed results are unknown. Sometimes this difficulty can be avoided by using nondifferentiable optimization techniques that have been a focal point of IIASA's research since the mid 1970s.

Stochastic approaches to decision-making under uncertainty aim to model situations for a risk, when each given decision may have both positive or negative results or externalities (win or lose, hit or miss, cost or profit, over- or under-estimating, etc.). The possibility of positive and negative externalities for the same solution results in nonlinearities and even discontinuities of corresponding risk indicators.

Risk-based environmental management provides many such examples. We can think of a typical "hit or miss" situation, that of reducing accumulative pollutants such as greenhouse gases and stratospheric ozone-destroying gases. Such problems are characterized by uncertain thresholds, which if exceeded may result in drastic losses. In this example, the discontinuity occurs due to irreversible environmental impacts. The presence of risks also creates a discontinuity in a rather "smooth" situation at first glance.

To illustrate this fact by verbal discussion is difficult. Therefore, let us consider a simple pollution control model, given by a linear inequality ("safety constraint"):

$$hx \leq 2 \ ,$$

where x is the level of the emission; h is a "predictor" which computes the average deposition level at a receptor point from the source of the emission. The norm of the "daily" depositions equals 2. Such safety constraints are important as a "surrogate indicator" in the case when the evaluation of real damages (costs) is impossible.

If h is known, then a permitted level of the emission x is defined easily from the inequality. Suppose now that h is a random variable which takes only two values, 0.5 and 1, with probability 0.5. Then for any $x \geq 0$ there may be two possibilities

$$hx \leq 2 \text{ or } hx > 2 \ ,$$

and the simplest indicator to characterize the risk of violating the safety constraint is (risk-function)

$$F(x) = Pr[hx > 2] \ .$$

The graph of this step function is shown in *Figure 1*. The accumulative effect (increasing level of pollution x) results in discontinuous changes, which is similar to the so-called "chemical time bomb" phenomena (see Stigliani *et al.*, 1991). They are unexpected and uncontrolled unless the rate of change is characterized. For instance, in a pollution control problem we might be interested in minimizing risk by a process of incremental improvements. Since the marginal value (derivatives) of $F(x)$ at any x is 0, the search process cannot be based on evaluations of these values. How one can characterize the increasing rate of two functions is shown in *Figure 2* in order to utilize it in the search for an optimal decision.

We can now imagine the possible nonlinearities and discontinuities of a similar risk function defined for the case where we have n emitters:

$$F(x) = Pr\left[h_1x_1 + h_2x_2 + \cdots + h_nx_n > q\right] \ ,$$

where $h_1, \ldots, h_n$ are random predictors (for each emitter $i = 1, \ldots, n$), $x_1, \ldots, x_n$ are levels of emission and q is the norm (critical load).

The discontinuity of $F(x)$ creates the uncertainty in the change or in the indication of improvements. In addition, there are critical uncertainties in the evaluation of the risk indicator $F(x)$ as a function of x. The choice of a decision x may dramatically affect the value $F(x)$: let us compare decisions $x = (1, 0, \ldots, 0)$ and $x = (1, 1, \ldots, 1)$. Although uncertainties in the initial data for (predictors) $h_1, \ldots, h_n$ are characterized by probabilistic measures, the probability of an outcome

$$h_1x_1 + \cdots + h_nx_n > q$$

Figure 1. Discontinuity of the risk function. The marginal value of the function is 0 but the risk increases with increasing x. An accumulative effect results in rapid and sudden changes.

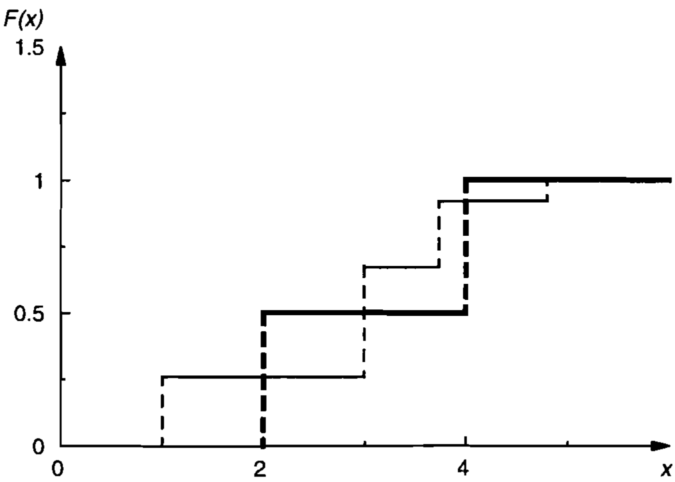

Figure 2. The rate of increase of two functions.

as a function of x is evaluated exactly only in exceptional cases (especially for more general risk indicators involving "damages").

The fundamental problem in the design of an optimal strategy is to bypass the above-mentioned discontinuities, uncertainties in the change and the evaluation of exact values of functions similar to $F(x)$, in general also involving costs, benefits, damages, etc.

Within the stochastic optimization framework such a search technique is being developed for problems with large numbers of decision variables and uncertainties, practically arbitrary "distributions" and rather general "objectives" and "constraints". The technique is essentially based on using only random observations of risk functions through Monte Carlo simulations or actual measurements.

A more rigorous and explicit formulation of assumptions may be a great advantage for decision making, even if uncertainty exists in the model. However, policy formation is a complicated process that involves political pressure, self-interest and the interest of various groups. The reluctance to reach firm (see Miller, 1980) conclusions enables a policy maker to use research to support or justify a pre-determined decision. Background and educational training is also important. Decision makers usually have difficulties assessing variable results and often only simple estimates and indicators are encouraged, which are often sources of public concern and distrust. For instance, such indicators as "life expectancy" and "collective dose" or "concentration level" alone are not able to depict the variations of effects within a population or country.

The complexity of the decision-making process itself creates additional uncertainties and constraints. We can also study the propagation of decisions through the "decision events tree", where an event may be "public rejection". The "dialogue" with the model at this stage involves more constraints and variables. For a "fair" solution to emerge at the end of such a process it may essentially depend on the dimensions of the problem (new perspectives) and the values of parameters (taxes, subsidies, discounting sales, safety levels, norms, etc.).

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