Working Paper

Persistent Unstable Equilibria in Wonderland

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ABSTRACT

Models of the interactions between population, economy, and environment often contain nonlinear functional relationships and variables that vary at different speeds. These properties foster apparent unpredictabilities in system behavior. Using a simple deterministic model of demographic, economic and environmental interactions, we illustrate the usefulness of geometric singular perturbation theory and local bifurcation theory. In particular we show how it is possible to obtain analytic expressions for: (1) the level of emissions above which environmental deterioration begins, (2) the time it takes from reaching the critical level of emissions to the beginning of rapid environmental deterioration, and (3) the level of emissions at the time that rapid deterioration begins. Because our results are analytic, they make the outcomes of demographic, economic, and environmental interactions more predictable, and, therefore, potentially more manageable.

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1 Introduction

One of the difficulties in managing complex environments is the appearance of phases of surprising behavior. One example of this is the appearance of the stratospheric 'ozone hole' over Antartica. Surprises can result because we do not know the structure producing environmental changes, or because, although we know the structure, we cannot accurately estimate its parameters. Surprises can also occur when the structure and parameters are known.

In a recent paper, Milik et al. (1995) studied the seemingly unpredictable occurrence of a rapid environmental deterioration in the context of a deterministic nonlinear model of the interactions between population, economic development, and the environment. Using local bifurcation theory, they were able to state the specific demographic, economic, and environmental conditions that separate the region where the environment could cleanse itself from the region where emissions were causing environmental damage. These conditions determine what they called the *critical border.* Even when an increasing emission flow causes the critical border to be crossed, rapid environmental deterioration might begin only many years later.

Surprising periods of rapid environmental deterioration do not have to occur in reality or in the simple model that we use here (see Sanderson (1994a) and Milik et al. (1995)). In this paper we assume that the model has parameter values that imply, after an initial period of environmental stability, a rapid environmental deterioration, and determine analytically the time it takes from crossing the critical border to the onset of rapid environmental change. We call this interval the *environmental grace period.* The environmental grace period is a time of persistent, but unstable equilibrium. From an environmental manager's viewpoint, it is a dangerous period because the system has become unstable, but changes in environmental quality are not yet signalling the seriousness of the pollution problem. Moreover, in the grace period we assume that there are no policy interventions. This corresponds roughly to the R-policies analysed in Rinaldi et al. (1995).

Our calculations are based mainly on a paper of Rinaldi and Muratori (1992) on cyclic behaviour of predator-prey systems. In particular, we set up an integral equation relating the point where the system's trajectories cross the critical border and the point where rapid environmental deterioration begins. Once we know the date of the end of the grace period, it is easy to compute the polluton flow at that point.

The paper is organized as follows. In Section 2 we present the model and its assumptions. Section 3 gives a first insight into the system's dynamics. The analysis of the model in terms of slow-fast dynamics is presented in Section 4. Section 5 illustrates the calculation of the integral equation and the length of the grace period. How this length depends on the initial conditions of the environment and per capita output and the parameters of the model is presented in Section 6. Numerical simulations and discussions of sensitivity are given in Section 7. We close with some conclusions and a suggestion for further research (Section 8). Finally, the appendixes document some of the mathematical calculations.

2 The model

The world we shall step into now is a continuous version of Sanderson's *Wonderland* model (Sanderson 1994a) of economic, demographic and environmental interactions that has been studied in Milik et al. (1995) (Appendix A). To facilitate analytical derivation of the length of the grace period, we

1. replace concave functions involving exponential terms with Monod type functions and
2. omit the dynamics of the pollution flow per unit of output.

The first modification poses no difficulties since the qualitative behaviour of Wonderland remains the same (see Section 3). The second assumption guarantees that the grace period will exist and have a strictly positive, finite duration (see Milik et al. 1995, p. 13). As the focus of the study is to determine the length of the grace period, the second assumption does not really constitute a severe restriction.

The dynamics in our modified Wonderland model is characterized by three state variables:
$x(t)$ ... population
$y(t)$ ... per capita output
$z(t)$ ... quality of environment (stock of natural capital)
which evolve according to [1]

$$\dot{x} = xn(y,z) \tag{1}$$

[1] For notational convenience we omit the time argument t in the following.

$$\dot{y} = y\left[\gamma - (\gamma+\eta)(1-z)^{\lambda}\right] \tag{2}$$
$$\dot{z} = \nu z(1-z)\left[-\frac{\omega f(x,y,z) - \delta z^{\rho}}{\omega f(x,y,z) - \delta z^{\rho} + 1}\right] \tag{3}$$

where

$n(y,z) = b(y,z) - d(y,z)$	population growth rate
$b(y,z) = \beta_1\left[\beta_2 - \frac{1}{2}(1 + \frac{\beta\bar{y}(y,z)}{\beta\bar{y}(y,z)+1})\right]$	crude birth rate
$d(y,z) = \delta_1\left[\delta_2 - \frac{1}{2}(1 + \frac{\alpha\bar{y}(y,z)}{\alpha\bar{y}(y,z)+1})\right](1 + \delta_3(1-z)^{\vartheta})$	crude death rate
$\bar{y}(y,z) = y - c(y,z)$	net per capita output
$c(y,z) = \varphi(1-z)^{\mu}y$	pollution control
$f(x,y,z) = pxy - \frac{\kappa}{2}\frac{\epsilon c(y,z)x}{\epsilon c(y,z)x+1}$	pollution flow

with 20 positive parameters, which can be grouped as follows:

population: $\beta_1, \beta_2, \beta, \delta_1, \delta_2, \delta_3, \alpha, \vartheta$

economy: γ, η, λ, p

environment: $\kappa, \epsilon, \delta, \rho, \omega, \nu$

environmental policy: φ, μ

and where the inequality $1 \geq \delta$ has to hold (Appendix B).

Equation (1) states that population growth $n(y,z)$ depends endogenously on per capita output y and the level of natural capital z. Natural capital is assumed to be bounded in the interval $[0,1]$. If natural capital is not polluted at all, it takes on the value $z = 1$. On the other extreme, when the environment is so polluted that it produces the maximum possible damage to human health and to the economy, $z = 0$.

The endogenous population growth rate is defined by the difference between the crude birth rate b (ratio of births to the population per unit of time) and the crude death rate d (ratio of deaths to the population per unit of time). Both crude rates decrease with increases in net per capita output $\bar{y}(y,z)$. Additionally, the death rate rises as the stock of natural capital decreases.

Net per capita output $\bar{y}(y,z)$ is defined as per capita output, net of per capita expenditures on pollution control $c(y,z)$. How polluted the environment is can be determined by the value of z. The lower the value z, the more polluted is the environment. Pollution control expenditures are assumed to depend on how polluted the environment is (i.e. on z) and not on the current flow of emissions. For example, we spend money on reducing the amount of particulate matter in the air because the environment is polluted and we have difficulty in breathing. If we lived in a place where the wind always blew the particles away and we were always left with clean air, we would not spend any money on pollution control, even though there were pollution flows. In addition, per capita spending on pollution control increases with per capita output.

The availability of natural capital also influences the growth rate of the economy as indicated by equation (2). The lower the stock of natural capital is, the lower the rate of per capita output growth will be. When the environment is totally polluted, i.e. $z = 0$, per capita output shrinks at the rate $-\eta$, while per capita output increases at the rate γ if environment is not polluted at all, i.e. $z = 1$.

The growth of natural capital (see equation 3) is assumed to be logistic. The speed at which natural capital regenerates (indicated by the term $\nu\left[-\frac{\omega f(x,y,z)-\delta z^\rho}{\omega f(x,y,z)-\delta z^\rho+1}\right]$) depends positively on the level of natural capital z and is negatively influenced by the amount of pollution flow $f(x, y, z)$, while ν represents a positive scaling factor. This specification is based on the idea that nature has the ability to cleanse itself, but that the strength of this ability diminishes as the stock of natural capital decreases. The function $g(z) = \frac{\delta}{\omega} z^\rho$ transforms the stock of natural capital into a flow of cleansing services measured in the same units as the pollution flow; this cleansing flow diminishes as the stock of natural capital decreases. The difference between the two flows, $\left[\frac{\delta}{\omega} z^\rho - f(x, y, z)\right]$, is the net effect of natural and human forces on the environment. When this net flow is zero, the level of natural capital remains constant. The pollution flow $f(x, y, z)$ that sets the net flow equal to zero, and which, therefore, maintains constant the stock of natural capital, is called the *critical pollution flow*. Clearly, the critical pollution flow is $\frac{\delta}{\omega} z^\rho$. If the actual amount of the pollution flow is above the critical flow the environment deteriorates, while, if it is below, the environment regenerates. The fraction $\frac{\delta}{\omega}$ is the level of the critical flow when the environment is unpolluted ($z = 1$). The parameter ρ determines how quickly the critical flow decreases as the stock of natural capital falls.

The pollution flow $f(x, y, z)$ - which refers to the emission of pollutants into the environment per unit of time - is determined by the impact on resources as given by the well known I-PAT identity [2] pxy, where the constant p denotes the pollution flow per unit of output, and by the amount spent on pollution control $c(y, z)$. If the environment is not polluted at all, the second term vanishes and the pollution flow equation becomes the I-PAT identity.

[2] The I-PAT identity states that the impact on natural resources and environment, I, is related to the size of the population, P, to per capita output (affluence), A, and to technology, T, which refers to pollution flow generated per unit of output (Ehrlich and Holdren 1971).

3 Numerical results

To illustrate that the change in the functional forms as stated in the previous section does not alter the qualitative behaviour of Wonderland and to highlight the possibility of a sudden deterioration of the environment we illustrate the system dynamics given the same parameter values as in Sanderson (1994b, p.22):

Population: $\beta_1 = 0.04, \beta_2 = 1.375, \beta = 0.16, \delta_1 = 0.01, \delta_2 = 2.5, \delta_3 = 4, \alpha = 0.18, \vartheta = 15$

Economy: $\gamma = 0.02, \eta = 0.1, \lambda = 2, p = 1$

Environment: $\kappa = 2, \epsilon = 0.02, \delta = 1, \rho = 2, \omega = 0.1, \nu = 1$

Environmental policy: $\varphi = 0.5, \mu = 2$

with the initial conditions set at the values: $x = y = 1, z = 0.7$ and the time specified in years.[3] We observe (Figure 1 a,b) that the system variables change with very different velocities, i.e. they exhibit *slow-fast* dynamics. Initially, the quality of environment z increases very fast to a high level while population x and output y remain nearly constant. After this phase of fast evolution, population and output increase with a slow speed, while the environment remains on its high level. Once output and consequently the actual pollution flow $f(x, y, z)$ are too high, the environmental quality drops very fast. This environmental collapse stops at a very low value near $z = 0$ and is followed by a slow decline of population and the economy. Surprisingly, even when we have crossed the critical pollution flow (at time T_1), the decrease in natural capital, which will be unavoidable, might occur many time steps later (at time T_2). This is very important for practical purposes, because noticeable changes in the stock of natural capital may come only after the pollution flow is significantly above the critical value. Since the actual pollution flow always stays above the critical pollution flow from there on, environment continually deteriorates and we end up with no natural capital being left, i.e. near $z = 0$.[4] The time span between T_1 and T_2 is exactly what we denote as the environmental grace period.

4 Analysis of the model in terms of slow-fast dynamics

The observed slow-fast dynamics suggest analyzing the model using concepts of *geometric singular perturbation theory.* This has been already done in detail in Milik et al. (1995, p. 9ff). We only summarize those results here.

[3]The numerical calculations have been performed using the LOCalBIFurcation program LOCBIF (Khibnik et al. 1993). For numerical integration we used a stiff fifth-order solver.

[4]In Milik et al. (1995) it is shown that the actual pollution flow might well fall below the critical pollution flow again, if pollution flow per unit of output decreases over time and pollution control expenditures are increased.

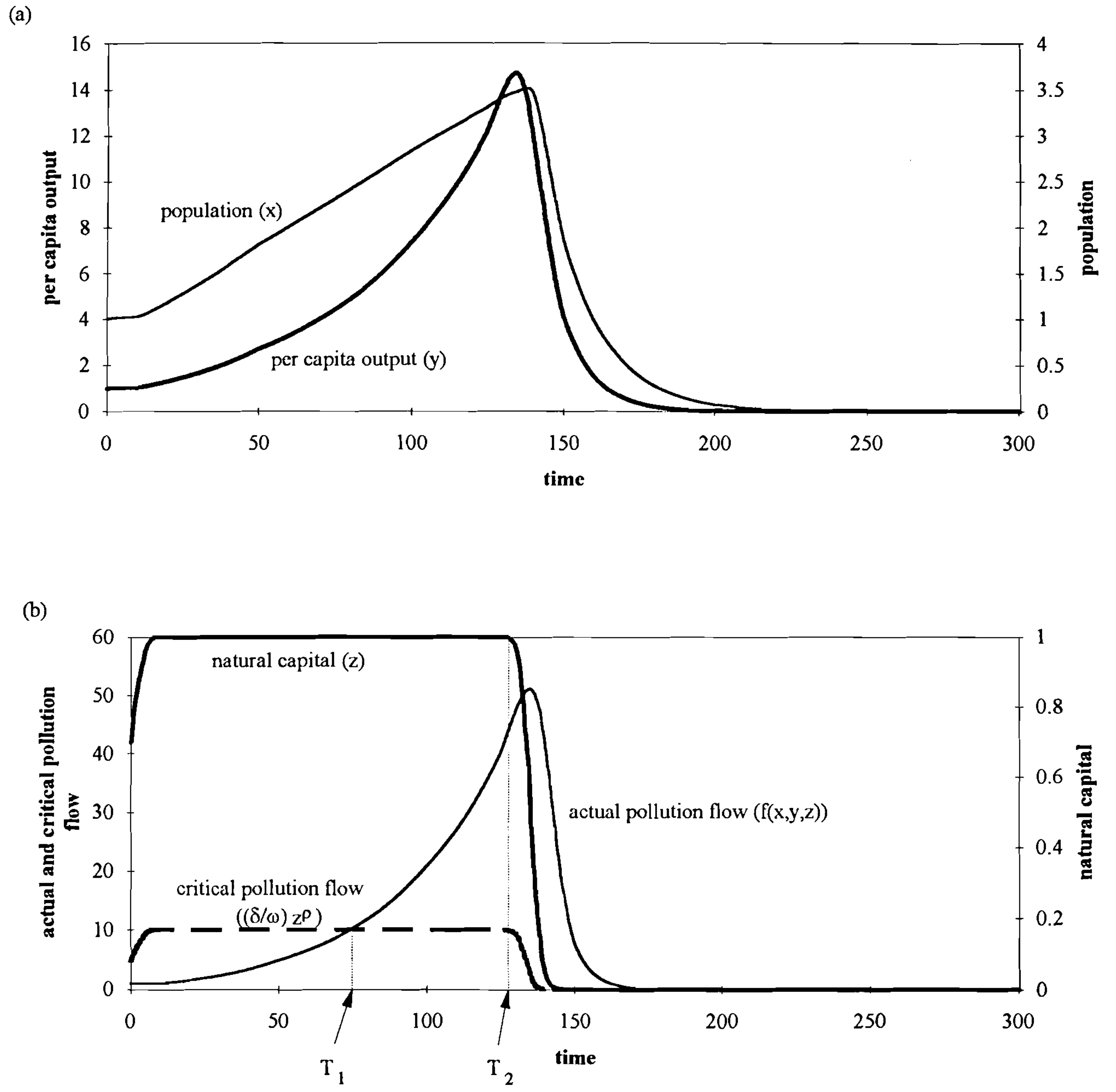

Figure 1. Time series of a: population and per capita output, b: natural capital, actual and critical pollution flow, with initial condition $x = 1, y = 1, z = 0.7$. Parameters are at their reference values.

Corresponding to the orders of magnitude of the rate of change of the variables, natural capital can be regarded as the fast variable while population and economy move slowly over time. With an appropriate rescaling of parameters ($\gamma \to \varepsilon\gamma$, $\eta \to \varepsilon\eta$, $\beta_1 \to \varepsilon\beta_1$, $\delta_1 \to \varepsilon\delta_1$) the system can be written as a singularly perturbed problem on the fast time scale,

$$\dot{x} = \varepsilon x n(y,z) \tag{4}$$

$$\dot{y} = \varepsilon y\left[\gamma - (\gamma+\eta)(1-z)^{\lambda}\right] \tag{5}$$

$$\dot{z} = \nu z(1-z)\left[-\frac{\omega f(x,y,z) - \delta z^{\rho}}{\omega f(x,y,z) - \delta z^{\rho} + 1}\right] \tag{6}$$

with ε being the small perturbation parameter.

Consideration of their *slow manifolds* is important for the analysis of such systems. These surfaces are defined by the equilibrium solutions of the *layer problem* given by system (4)-(6) with $\varepsilon = 0$. They are Z_r, Z_0 and Z_1 defined, respectively, by $z = z(x,y)$, $z = 0$ and $z = 1$ where $z = z(x,y)$ is implicitly given as the solution of the equation

$$\frac{\delta}{\omega} z^{\rho} - f(x,y,z) = 0. \tag{7}$$

As can be seen in Figure 2, Z_r intersects Z_0 along the x and y axes and intersects Z_1 along a critical curve C given by the solution of equation (7) with $z = 1$: [5]

$$xyp = \frac{\delta}{\omega}. \tag{8}$$

This last intersection corresponds to a *transcritical bifurcation* of equilibria for the layer problem. Crossing this curve, the equilibrium $z = 1$ changes its stability: for $xyp < \frac{\delta}{\omega}$ it is stable, while it is repelling for $xyp > \frac{\delta}{\omega}$. Similarly, it can be shown that the equilibrium $z = 0$ is stable for all $x > 0$ and $y > 0$. The basins of attraction of the two manifolds (Z_0 and the stable part of Z_1) are separated by the repelling manifold Z_r given by equation (7).

For those parts of phase space where the manifolds Z_0, Z_1 and Z_r are hyperbolic, i.e. away from bifurcation curves like C and for ε sufficiently small the layer problem captures the fast evolution of the variable z towards or away from the slow manifolds while the slow variables x and y can be regarded as slowly varying 'parameters' moving on the slow manifold.

Around the transcritical bifurcation curve we observe an interesting and somewhat counter-intuitive phenomenon (see Figure 2). The trajectory is not repelled from Z_1 immediately but follows closely the repelling part of the manifold Z_1 for a while until it is ultimately repelled away. It seems as if the singularly perturbed system needs some time to 'feel' the

[5] We have visualized the set of equilibria of the layer problem with numerical computations using the program package Mathematica (Wolfram, 1991).

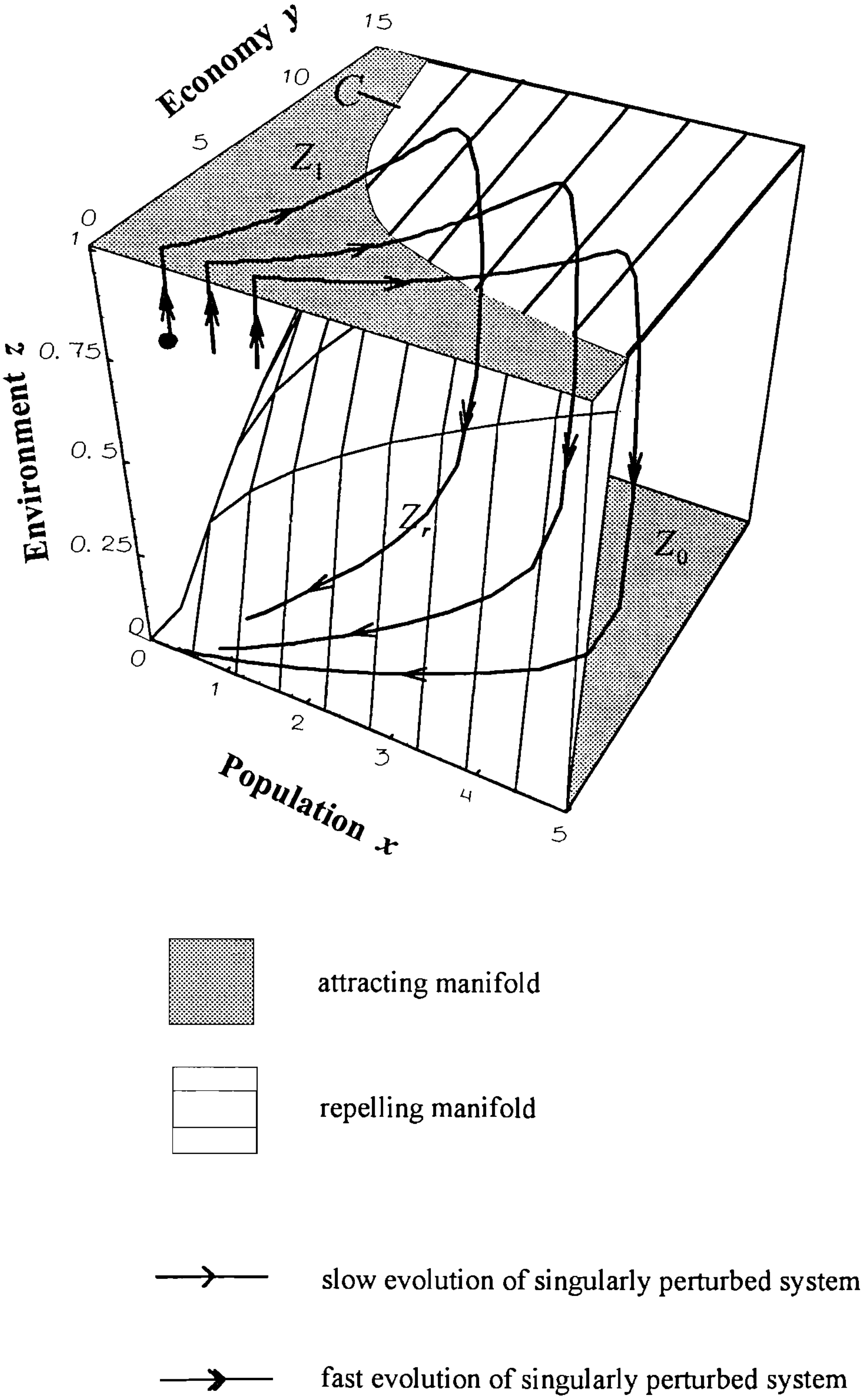

Figure 2. The invariant slow manifolds Z_0, Z_1 and Z_r together with orbits of the singularly perturbed system obtained for three different initial conditions, i.e. $(0.5, 1, 0.8), (1, 1, 0.8), (1.5, 1, 0.8)$. Parameters are at their reference values.

change in stability. This is related to the exponential attractiveness of the stable part of the slow manifold Z_1. The trajectory is exponentially close to the slow manifold. Therefore it takes a considerable time interval $O(1)$ until the trajectory leaves a small $O(\varepsilon)$ neighborhood of the now unstable slow manifold. This phenomenon is studied rigorously in Schecter (1985) for two dimensional singularly perturbed problems with a transcritical bifurcation for the layer problem.

In the following section we study the specific values of the slow variables x and y and the time needed until the fast variable z drops down once the manifold Z_r has been crossed.

5 Length of the environmental grace period in Wonderland

The time span between the crossing of the orbit of the system with the manifold Z_r and the beginning of the rapid environmental deterioration, when the trajectories are ultimately repelled away from the unstable manifold Z_1, the grace period, can be calculated as follows. Equations (4)-(6) can be written as

$$\dot{x} = \varepsilon G(x,y,z) \quad (9)$$
$$\dot{y} = \varepsilon H(y,z) \quad (10)$$
$$\dot{z} = z(1-z)F(x,y,z) \quad (11)$$

where $G(x,y,z) = xn(y,z)$, $H(y,z) = y\left[\gamma - (\gamma+\eta)(1-z)^{\lambda}\right]$ and $F(x,y,z) = \nu\left[-\frac{\omega f(x,y,z)-\delta z^{\rho}}{\omega f(x,y,z)-\delta z^{\rho}+1}\right]$.

Consider the orbit of system (9) - (11) that starts at $A^{\varepsilon} = (x_{IN}, y_{IN}, 1-\varepsilon)$ and where $f(x_{IN}, y_{IN}, 1-\varepsilon) < \frac{\delta}{\omega}(1-\varepsilon)^{\rho}$ holds, i.e. we start above the manifold Z_r. In addition, in the following, the population growth rate is assumed to be positive. If $\varepsilon > 0$ is small, this curve tends towards the manifold Z_1, moves slowly to the right of the manifold Z_r and leaves the ε-tube of the manifold Z_1 crossing the surface $Z_{1-\varepsilon}$ at the point $B^{\varepsilon} = (x^{\varepsilon}_{OUT}, y^{\varepsilon}_{OUT}, 1-\varepsilon)$ (see Figure 3). The point of intersection of the orbit with the manifold Z_r has been denoted by $\tilde{D}^{\varepsilon} = (\tilde{x}^{\varepsilon}, \tilde{y}^{\varepsilon}, \tilde{z})$. Even though $\tilde{z}$ depends upon ε, for notational convenience, we have omitted the superscript ε.

To calculate the time span between $\tilde{D}^{\varepsilon}$ and B^{ε} we use the approach illustrated in Rinaldi and Muratori (1992) with one distinction. Since we are interested in the dependence of the grace period on the critical value $\tilde{z}$ of the environment, i.e. the value of z where the orbit of the system crosses the manifold Z_r, we consider the path $\tilde{D}^{\varepsilon}B^{\varepsilon}$ in the following calculations, and not the path $A^{\varepsilon}B^{\varepsilon}$ as done in Rinaldi and Muratori (1992).

We view $\tilde{z}$ as the long-run value of the stock of natural capital in the absence of anthropogenic pollution. In the following we study only the time from $\tilde{D}^{\varepsilon}$ to B^{ε}. Had we begun

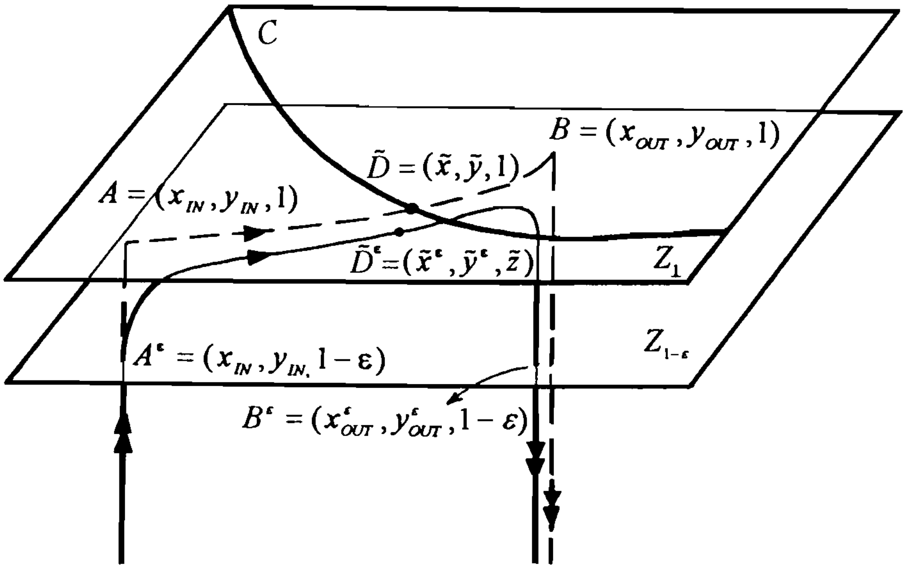

Figure 3. The trajectory from $\tilde{D}^{\varepsilon}$ to B^{ε} along which the system is integrated.

the analysis prior to $\tilde{D}^{\varepsilon}$, it would have been impossible to derive the relationship between the grace period and the parameters of the model, holding $\tilde{z}$ fixed.

From equation (11) it follows that

$$\frac{1}{1-z}dz = zF(x,y,z)dt \tag{12}$$

while from equation (9) and (10) we have

$$\frac{dx}{\varepsilon G(x,y,z)} = dt \tag{13}$$

and

$$\frac{dy}{\varepsilon H(y,z)} = dt \tag{14}$$

so that by substitution of (13) or (14) into (12), we obtain, respectively

$$\frac{\varepsilon}{1-z}dz = z\frac{F(x,y,z)}{G(x,y,z)}dx \tag{15}$$

or

$$\frac{\varepsilon}{1-z}dz = z\frac{F(x,y,z)}{H(y,z)}dy \tag{16}$$

Thus, we have rearranged terms such that the left and the right hand sides of equations (15) and (16) are of order 1.

Integrating (15) or (16) from $\tilde{D}^\varepsilon$ to B^ε yields respectively

$$\int_{\tilde{z}}^{1-\varepsilon} \frac{\varepsilon}{1-z} dz = \int_{\tilde{x}^\varepsilon}^{x^\varepsilon_{OUT}} z \frac{F(x,y,z)}{G(x,y,z)} dx \tag{17}$$

or

$$\int_{\tilde{z}}^{1-\varepsilon} \frac{\varepsilon}{1-z} dz = \int_{\tilde{y}^\varepsilon}^{y^\varepsilon_{OUT}} z \frac{F(x,y,z)}{H(y,z)} dy \tag{18}$$

The right hand sides of equations (17) and (18) can be approximated with their limits for ε going to zero, since each term of the integrant is of order 1, yielding

$$\varepsilon \ln(\frac{1-\tilde{z}}{\varepsilon}) \approx \int_{\tilde{x}}^{x_{OUT}} \frac{F(x,y,1)}{G(x,y,1)} dx \tag{19}$$

or

$$\varepsilon \ln(\frac{1-\tilde{z}}{\varepsilon}) \approx \int_{\tilde{y}}^{y_{OUT}} \frac{F(x,y,1)}{H(y,1)} dy \tag{20}$$

where, for equation (19), $y = y(x)$ is the solution of $\frac{dy}{dx} = \frac{H(y,1)}{G(x,y,1)}$ given the initial condition $\tilde{x}$ or $\tilde{y}$ [6], while, for equation (20), $x = x(y)$ is the solution of $\frac{dx}{dy} = \frac{G(x,y,1)}{H(y,1)}$ given the same initial condition. We have therefore approximated, only in the right hand sides of equations (17) and (18), the trajectory from A^ε to B^ε with the dashed line in Figure 3 passing through points $A = (x_{IN}, y_{IN}, 1)$ and $B = (x_{OUT}, y_{OUT}, 1)$, where $\tilde{D} = (\tilde{x}, \tilde{y}, 1)$ is its point of intersection with the critical curve C.

Equations (19) and (20) can be used to express x_{OUT} and y_{OUT} - which are an approximation of x^ε_{OUT} and y^ε_{OUT} for $\varepsilon \to 0$ - as a function of $\tilde{z}, \tilde{x}$ or $\tilde{z}, \tilde{y}$ respectively. In principle, it is sufficient to determine only either x_{OUT} or y_{OUT} and calculate the other value using the explicit form $y(x)$ or $x(y)$.

Once we have calculated the values of x_{OUT} and y_{OUT}, we can determine the time span, T, between the point of intersection, $\tilde{D}^\varepsilon$, of the trajectory with the manifold Z_r and the point B^ε of rapid environmental deterioration (see Figure 3). In particular we can calculate the grace period T as follows.

$$T = \int_0^T dt = \int_{\tilde{x}^\varepsilon}^{x^\varepsilon_{OUT}} \frac{1}{\varepsilon G(x,y,z)} dx \approx \frac{1}{\varepsilon} \int_{\tilde{x}}^{x_{OUT}} \frac{1}{G(x,y,1)} dx \tag{21}$$

or

$$T = \int_0^T dt = \int_{\tilde{y}^\varepsilon}^{y^\varepsilon_{OUT}} \frac{1}{\varepsilon H(y,z)} dy \approx \frac{1}{\varepsilon} \int_{\tilde{y}}^{y_{OUT}} \frac{1}{H(y,1)} dy \tag{22}$$

Summing up, we can either use equations (19) and (21) or (20) and (22) to obtain x_{OUT} and T or y_{OUT} and T, respectively.

[6] We need either $\tilde{x}$ or $\tilde{y}$ since we can calculate the other value by solving $F(\tilde{x}, \tilde{y}, 1) = 0$.

6 Sensitivity analysis

Using the results of the previous section we can investigate the dependence of the grace period on the system's parameters and the critical values of the economy, $\tilde{y}$ and the environment, $\tilde{z}$. In particular, there are 11 parameters affecting the grace period which can be grouped as follows:

population: $\beta_1, \beta_2, \beta, \delta_1, \delta_2, \alpha$

economy: γ, p

environment: δ, ω, ν

Regarding the parameters defining the population growth rate we shall restrict our sensitivity analysis to the most important ones, which are the levels of the crude birth and death rate, β_1 and δ_1 respectively.

For analytical convenience we fix $\tilde{y}$ in the following. To proceed we first calculate $x(y)$ as the solution of $\frac{dx}{dy} = \frac{xn(y,1)}{y\gamma}$ given the initial condition $\tilde{y}$. This yields

$$x = \tilde{x}h(y, \tilde{y}) \tag{23}$$

where

$$h(y, \tilde{y}) = \exp \int_{\tilde{y}}^{y} \frac{n(\eta, 1)}{\eta\gamma} d\eta \tag{24}$$

and $\tilde{x} = \frac{\delta}{\omega p \tilde{y}}$.

We start by analyzing the effect of the parameters defining the critical border, i.e. p, δ and ω.

The derivative of the grace period, T, w.r.t. the parameter p can be calculated as follows. Differentiating equation (20) w.r.t. the parameter p yields

$$0 \approx \frac{F(x_{OUT}, y_{OUT}, 1)}{H(y_{OUT}, 1)} \frac{dy_{OUT}}{dp} + \int_{\tilde{y}}^{y_{OUT}} \frac{1}{H(y,1)} \frac{\partial F(x, y, 1)}{\partial p} dy \tag{25}$$

where $F(x, y, 1) = \nu[-\frac{\omega p x y - \delta}{\omega p x y - \delta + 1}]$ and $H(y, 1) = y\gamma$. Substituting equation (23) one can show that the integrant in (25) equals zero. Since $F(x_{OUT}, y_{OUT}, 1)$ is negative and $H(y_{OUT}, 1)$ is positive, it follows that $\frac{dy_{OUT}}{dp}$ equals zero.

To determine the change in the grace period when the parameter p is varied, we differentiate T (22) w.r.t. the parameter p

$$\frac{dT}{dp} \approx \frac{1}{\varepsilon\gamma y_{OUT}} \frac{dy_{OUT}}{dp} \tag{26}$$

Since $\frac{dy_{OUT}}{dp} = 0$, the grace period does not depend on the parameter p.

Similarly, one can show that $\frac{dy_{OUT}}{d\omega}$ is equal to zero and consequently also $\frac{dT}{d\omega}$.

Contrary, differentiating (20) w.r.t. the parameter δ yields

$$0 \approx \frac{F(x_{OUT}, y_{OUT}, 1)}{H(y_{OUT}, 1)} \frac{dy_{OUT}}{d\delta} + \int_{\tilde{y}}^{y_{OUT}} \frac{1}{H(y,1)} \frac{\partial F(x,y,1)}{\partial \delta} dy \tag{27}$$

For $y > \tilde{y}$ the integrant is negative, so that $\frac{dy_{OUT}}{d\delta}$ has to be less than zero since $\frac{F(x_{OUT}, y_{OUT}, 1)}{H(y_{OUT}, 1)}$ is negative. Together with (22) it follows that the grace period decreases with δ.

In a similar way one can calculate $\frac{dy_{OUT}}{d\nu}$, $\frac{dy_{OUT}}{d\beta_1}$ and $\frac{dy_{OUT}}{d\delta_1}$. The same reasoning as before yields that the grace period decreases with increasing values of the speed at which natural capital regenerates, ν, and increasing values of the birth rate level, β_1, while it increases with increasing values of δ_1.

To determine the change in the grace period when the parameter γ varies we first differentiate equations (20) and (22) w.r.t. the parameter γ, obtaining

$$0 \approx \frac{F(x_{OUT}, y_{OUT}, 1)}{H(y_{OUT}, 1)} \frac{dy_{OUT}}{d\gamma} + \int_{\tilde{y}}^{y_{OUT}} F(x,y,1) \frac{\partial (1/H(y,1))}{\partial \gamma} dy \tag{28}$$

$$\frac{dT}{d\gamma} \approx -\frac{1}{\varepsilon \gamma^2} \int_{\tilde{y}}^{y_{OUT}} \frac{1}{y} dy + \frac{1}{\varepsilon \gamma} \frac{1}{y_{OUT}} \frac{dy_{OUT}}{d\gamma} \tag{29}$$

From the first equation we can explicitly solve for $\frac{1}{\gamma y_{OUT}} \frac{dy_{OUT}}{d\gamma}$. Plugging this expression into the second equation yields

$$\frac{dT}{d\gamma} \approx \frac{1}{\varepsilon \gamma^2} \int_{\tilde{y}}^{y_{OUT}} \frac{1}{y} \left[-1 + \frac{F(x,y,1)}{F(x_{OUT}, y_{OUT}, 1)} \right] dy \tag{30}$$

For $y > \tilde{y}$ the integrant is negative since $|F(x,y,1)| < |F(x_{OUT}, y_{OUT}, 1)|$. Therefore, an increase in γ decreases the grace period.

Similarly, one can determine the effect of changes in the level of the economy at the intersection point of the actual and critical pollution flow, $\tilde{y}$,on the grace period, which yields

$$\frac{dT}{d\tilde{y}} \approx \frac{1}{\varepsilon \gamma \tilde{y}} \left[-1 - \frac{\nu(\gamma + n(\tilde{y}, 1))}{F(x_{OUT}, y_{OUT}, 1)} \int_{\tilde{y}}^{y_{OUT}} \frac{1}{\gamma y} \frac{\omega p x y}{(\omega p x y - \delta + 1)^2} dy \right] \tag{31}$$

Since $F(x_{OUT}, y_{OUT}, 1)$ is negative and the integrant is positive, the sign of the derivative $\frac{dT}{d\tilde{y}}$ is indeterminate. Finally, the grace period changes with the critical level of the environment as follows

$$\frac{dT}{d\tilde{z}} \approx \frac{1}{F(x_{OUT}, y_{OUT}, 1)(\tilde{z} - 1)} \tag{32}$$

Each term in the denominator is negative implying that the grace period increases with increasing values of $\tilde{z}$.

7 Discussion

The sensitivity analysis in the previous section has revealed two surprising results. First, the grace period (T) does not depend on pollution flow per unit of output (p) and the parameter ω and second, the grace period depends negatively on the flow of cleansing services (δ). This means that once we have crossed the critical pollution flow, the time it takes until natural capital begins to collapse is independent of the pollution flow per unit of output and is even negatively influenced by the ability of nature to cleanse itself.

Obviously, these results must depend on the speed at which natural capital regenerates (in case of $\omega f(x,y,z) < \delta z^{\rho}$) or degenerates (in case of $\omega f(x,y,z) > \delta z^{\rho}$) as indicated by the term

$$\nu \left[-\frac{\omega f(x,y,z) - \delta z^{\rho}}{\omega f(x,y,z) - \delta z^{\rho} + 1} \right] \tag{33}$$

As the population, the economy and natural capital evolve over time, the speed of regeneration or degeneration of the environment will change. In particular, the derivative of (33) w.r.t. time is given by the following expression.

$$-\nu \left[\frac{\omega\left(\frac{\partial f(x,y,z)}{\partial x}\frac{dx}{dt} + \frac{\partial f(x,y,z)}{\partial y}\frac{dy}{dt} + \frac{\partial f(x,y,z)}{\partial z}\frac{dz}{dt}\right) - \delta\rho z^{\rho-1}\frac{dz}{dt}}{(\omega f(x,y,z) - \delta z^{\rho} + 1)^2} \right] \tag{34}$$

Evaluating (34) at $z = 1$ and $(x,y) \geq (\tilde{x},\tilde{y})$ yields

$$-\nu \left[\frac{\frac{\delta}{y} h(y,\tilde{y}) y (n(y,1) + \gamma)}{(\frac{\delta}{y} h(y,\tilde{y}) y - \delta + 1)^2} \right] \tag{35}$$

where we have used equation (23) to express x as a function of y. Intuitively, equation (35) indicates the acceleration of the degeneration of the environment at $z = 1$ and therefore implicitly influences the time it takes until an environmental collapse takes place. More specifically, at the onset of rapid environmental deterioration the system's dynamics change from slow to fast. But in order for this to happen, the speed at which the environment degenerates has to change. Since ω and p are no longer present in equation (35), they have no effect on the grace period. Contrary, changes in ν, δ, γ and the parameters characterizing the endogenous population growth rate n will change the grace period. The grace period will also depend on $\tilde{x}$ and $\tilde{y}$.

In order to determine the size of the effect of changes in the parameters and the critical values of the environment and the economy on the grace period, we have numerically solved equations (20) and (22) for selected parameters using the program package Mathematica (Wolfram, 1991). The parameter values are the same as given in Section 3 except the value of ν which is reduced to 0.5. Moreover $\tilde{y} = 1$ and $\tilde{z} = 0.999$.

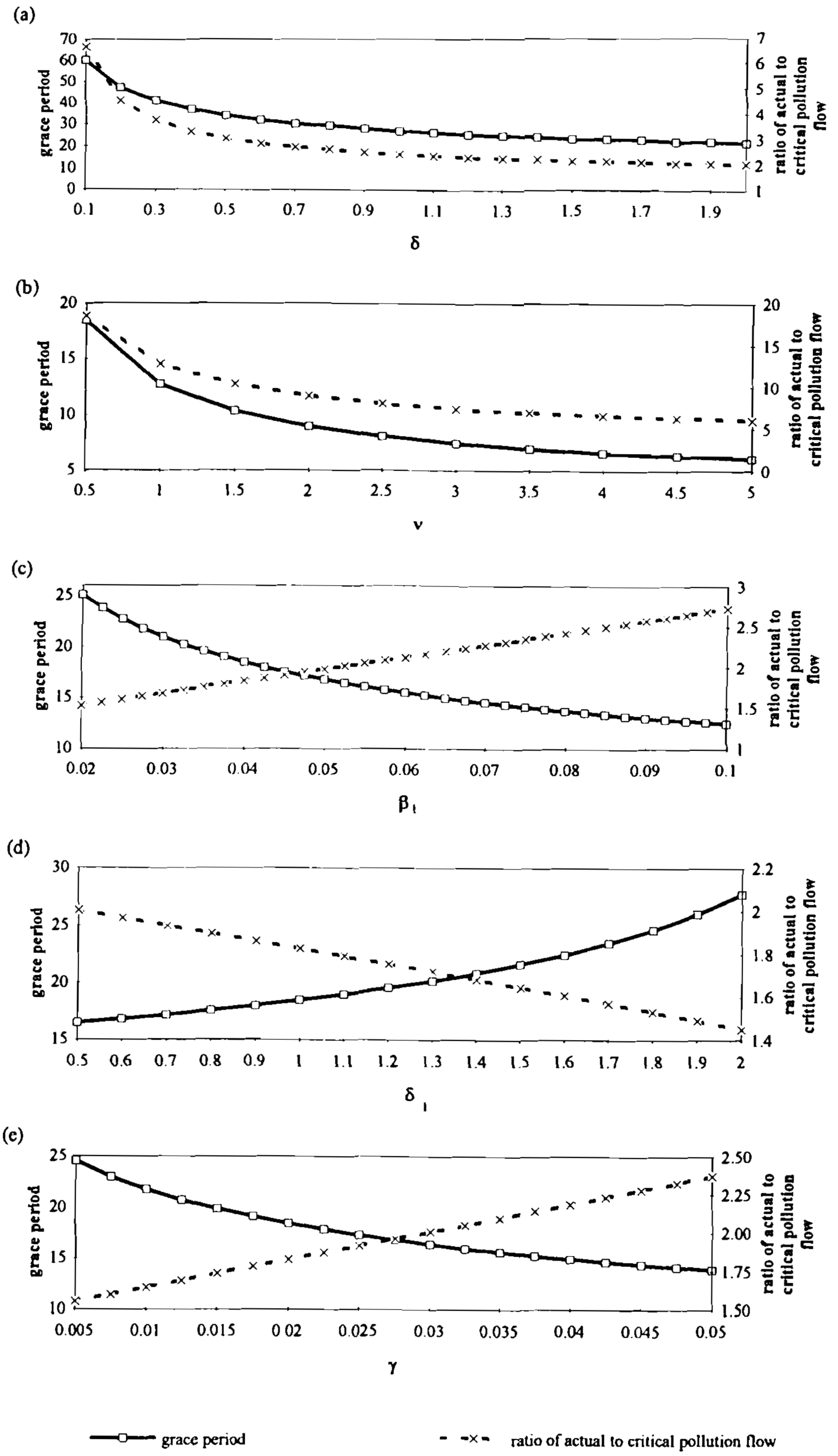

Figure 4. Grace period and ratio of actual to critical pollution flow versus the parameters a: δ, b: ν, c: β_1, d: δ_1 and e: γ. Parameters are at their reference values (except $\nu = 0.5$), $\tilde{y} = 1$ and $\tilde{z} = 0.999$.

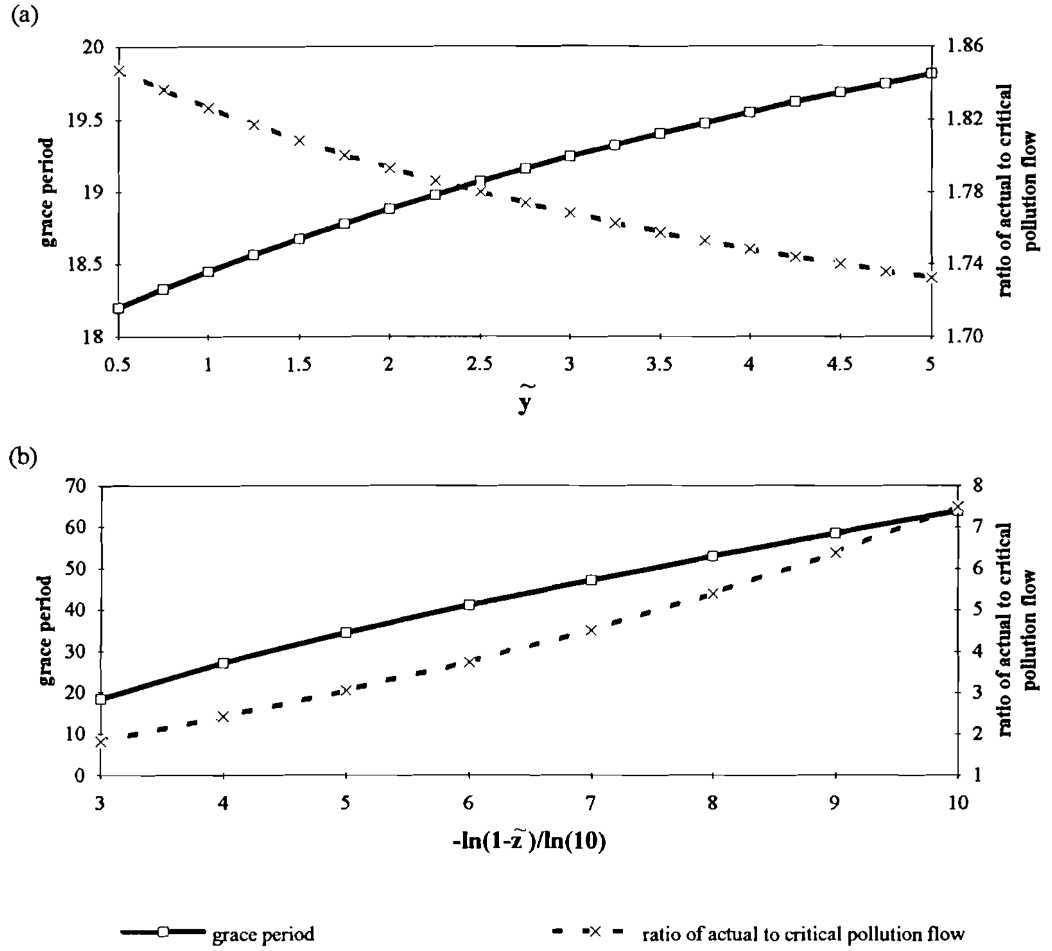

Figure 5. Grace period and ratio of actual to critical pollution flow versus the critical values of a: the economy $\tilde{y}$, and b: the environment $\tilde{z}$. Parameters are at their reference values (except $\nu = 0.5$), $\tilde{y} = 1$ and $\tilde{z} = 0.999$.

In Figure 4 and Figure 5 the grace period and the ratio of the actual to the critical pollution flow are plotted for the parameters discussed above and for the critical levels of the economy and environment. In varying the parameter values we have to take care not to change their order of magnitude since this would alter the time scale of the system's variables and therefore the analysis in the previous sections.

The numerical calculations confirm the analytical results. Notice that, for our choice of the paramter values, $\frac{dT}{d\tilde{y}}$, whose sign is not determined, is positive. Nevertheless, for other choices, it can either be negative or change its sign. Amazingly it might take up even to 60 years after we have crossed the critical border until a rapid environmental deterioration actually takes place (see, for example Figures 4.a and 5.b). As already mentioned several times, even when we have crossed the critical pollution flow, the collapse in natural capital might occur many years later. To get some feeling to what extent we might actually exceed the critical pollution flow before a rapid environmental deterioration takes place, we plot the ratio of the actual to the critical pollution flow at the point of the rapid environmental deterioration, namely $\frac{p x_{OUT} y_{OUT}}{\delta/\omega}$. One might expect that, the higher is this ratio, the longer will be the grace period, i.e. that the ratio and the grace period are posivitely correlated. In fact this happens in Figures 4.a, 4.b and 5.b where δ, ν and $\tilde{z}$ are varied. In the remaining figures, where $\beta_1, \delta_1, \gamma$ and $\tilde{y}$ are varied the ratio and the grace period are negatively correlated. For example, in Figure 4.e, an increase in γ implies an increase in the ratio and a decrease in the grace period. This positive or negative correlation can be intuitively explained considering that δ, ν, and $\tilde{z}$ do not change the dynamic of the system on the manifold Z_1, while β_1, δ_1, γ and $\tilde{y}$ change it (see equations (1)-(3)). In particular, the dynamics on Z_1 become faster by an increase in β_1 and γ and a decrease in δ_1, while the influence of a variation in $\tilde{y}$ is not determined. For example, an increase in γ decreases the grace period, but at the same time it increases the growth rate of the economy. As a result the growth rate of the actual pollution flow and therefore the ratio increase.

Moreover, as the numerical results show, the actual pollution flow might be even 20 times as large as the critical pollution flow (see for example Figure 4.b) when the rapid environmental deterioration begins.

8 Conclusions

The starting point for this paper is the nonlinear demographic, economic, and environmental interactions in an artificial world called Wonderland. A special feature of Wonderland, and some other environmental models as well (see Rinaldi et al. (1995)), is that the variables can exhibit slow-fast dynamics, where during certain periods some variables vary much more rapidly than others.

It turns out that even when an increasing pollution flow causes the system's trajectory to cross the critical border, so that the regenerative capacity of the environment is less than the pollution flow, nothing unusual seems to happen for a while. Only after a period of time, that in some cases can be quite long, does the environment suddenly begin to deteriorate. This grace period depends on parameters that influence the rate of economic growth, the crude birth rate, and the power of nature to detoxify itself.

Because of the time span between crossing the critical border and the beginning of a rapid environmental deterioration, the pollution flow has the time to increase far above the critical value. Therefore, if we adopt a policy of waiting until the first signs of environmental deterioration are observed before actions are taken to reduce the flow of pollution (*wait and see policies*), then, when action is initiated, actual pollution flows could be high compared with critical flows. This could make the reduction of the actual flows to their critical values more difficult. Drawing out the policy implications of these findings is a natural next step.

Appendix A - Continuous Wonderland model

(Milik et al. 1995)

The dynamics are given by four state variables
$x(t)$ ... population
$y(t)$ ... per capita output
$z(t)$ ... quality of environment (stock of natural capital)
$p(t)$ ... pollution flow per unit of output

which evolve according to [7]

$$\dot{x} = xn(y,z) \quad (36)$$
$$\dot{y} = y\left[\gamma - (\gamma+\eta)(1-z)^{\lambda}\right] \quad (37)$$
$$\dot{z} = \nu z(1-z)\left[e^{\delta z^{\rho} - w\, f(x,y,z)} - 1\right] \quad (38)$$
$$\dot{p} = -\chi p \quad (39)$$

where

$$n(y,z) = b(y,z) - d(y,z) \qquad \text{population growth rate}$$

$$b(y,z) = \beta_1\left[\beta_2 - \frac{e^{\beta\bar{y}}}{1+e^{\beta\bar{y}}}\right] \qquad \text{crude birth rate}$$

$$d(y,z) = \delta_1\left[\delta_2 - \frac{e^{\alpha\bar{y}}}{1+e^{\alpha\bar{y}}}\right](1+\delta_3(1-z)^{\vartheta}) \qquad \text{crude death rate}$$

$$\bar{y}(y,z) = y - c(y,z) \qquad \text{net per capita output}$$

$$c(y,z) = \varphi(1-z)^{\mu} y \qquad \text{pollution control}$$

$$f(x,y,z) = pxy - \kappa\left[\frac{e^{\epsilon c(y,z)x}}{1+e^{\epsilon c(y,z)x}} - 0.5\right] \qquad \text{pollution flow}$$

The meaning of the parameters is the same as in the text with the exception of χ, indicating the constant rate at which pollution flow per unit of output decreases in a time unit.

[7]For notational convenience we omit the time argument t in the following.

Appendix B - Restriction on δ

In order to replace

$$e^{\delta z^{\rho} - \omega f(x,y,z)} - 1 \tag{40}$$

in equation (38) with the function

$$-\frac{\omega f(x,y,z) - \delta z^{\rho}}{\omega f(x,y,z) - \delta z^{\rho} + 1} \tag{41}$$

we have to guarantee that both functions coincide on the range of variation of the term $t = \omega f(x,y,z) - \delta z^{\rho}$.

Since
$\lim_{t \to +\infty}(e^{-t} - 1) = -1$, $\lim_{t \to -\infty}(e^{-t} - 1) = +\infty$ for equation (40) and
$\lim_{t \to +\infty} -\frac{t}{t+1} = -1$, $\lim_{t \to -1^{+}} -\frac{t}{t+1} = +\infty$ for equation (41)
we have to guarantee that -1 is less than the minimum possible value of t, which yields, after some algebra, the inequality $1 \geq \delta$.

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