Working Paper

An Endogenous Growth Model for Technological Leading-Following: an Asymptotical Analysis

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Preface

The paper presents the mathematical part of a joint research on an endogenous growth theory and alternative approaches performed by Gernot Hutschenreiter (the Austrian Institute of Economic Research, WIFO), Yuri Kaniovski (the project on Systems Analysis of Technological and Economic Dynamics) and Arkadii Kryazhimskii (the Dynamic Systems project). A substantial part of the research is presented in Hutschenreiter, et. al., 1995, together with relevant references, an entire explanation of all variables and parameters, and economic interpretations of the results, whose formal justification is the goal of the present paper.

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1 Model. Result Formulations

1.1 Equation of technological leading-following

We deal with a differential equation for technological leading-following (suggested in Hutschenreiter, et. al., 1995,)

$$\dot{n}^A(t) = \bar{g}^A n^A(t) \tag{1.1}$$

$$\dot{n}^B(t) = \left[c^B(n^B(t) + \gamma(n^B(t))\delta n^A(t)) - \frac{\alpha}{v^B(t)}\right]_+ \tag{1.2}$$

$$\dot{v}^B(t) = \rho v^B(t) - \frac{1-\alpha}{n^B(t)} . \tag{1.3}$$

In (1.2) we set $[p]_+ = \max\{p, 0\}$. The equation generalizes the brand proliferation model due to Grossman and Helpman, 1991. It is intended to model the economic development of a large country A, a technological leader, and a smaller country B, a technological follower. Variables $n^A(t)$ and $n^B(t)$ stand for the knowledge stocks which originate at time t in countries A and B respectively, and variable $v^B(t)$ stands for the value at t of the representative firm in country B. It is assumed that country B "taps" a part of the knowledge stock accumulated in country A. The varying coefficient $\gamma(n^B(t))$ in (1.2) is responsible for the rate of "tapping". Function $\gamma(n^B)$ is defined on $[0, \infty)$. It is assumed to be continuously differentiable, strictly increasing, and satisfying the conditions

$$\gamma(0) = 0, \quad 0 \leq \gamma(n^B) \leq 1 .$$

Coefficients $\bar{g}^A$, c^B, ρ, δ, α are positive constants, and $\alpha < 1$. One has

$$\bar{g}^A = (1-\alpha)\frac{L^A}{a} - \alpha\rho, \quad c^B = \frac{L^B}{a} \tag{1.4}$$

where L^A and L^B are exogenous labor supplies in countries A and B respectively. Coefficient α is related to the elasticity of substitution, ϵ, by $\epsilon = 1/(1-\alpha)$, ρ is the discount rate, and $1/a$ stands for the amount of labor needed to develop a unit of new product. In what follows, time t varies from zero to infinity.

1.2 Goals. Outline of results

As it is seen from (1.1),

$$n^A(t) = \exp(\bar{g}^A t)n_0^A \tag{1.5}$$

where $n_0^A = n^A(0)$. Our goal is to compare $n^B(t)$ with $n^A(t)$ at large t. Namely, we shall be interested in finding an asypmptotics for the ratio

$$r(t) = \frac{n^B(t)}{n^A(t)} \tag{1.6}$$

as t grows to infinity. The three asymptotics are radically different:

$$\lim_{t\to\infty} r(t) = r_\infty = 0 \tag{1.7}$$

$$\lim_{t\to\infty} r(t) = r_\infty = \infty \tag{1.8}$$

$$\lim_{t\to\infty} r(t) = r_\infty, \quad 0 < r_\infty < \infty\ . \tag{1.9}$$

Relation (1.7) implies that in the long run the knowledge stock of the follower asymptotically vanishes in comparison to that of the leader. Relation (1.8) reflects, conversely, the situation where the follower's knowledge stock grows so rapidly that in the long run the knowledge stock of the leader vanishes in comparison to it. In case of (1.9) we have approximately the same rates of growth for the knowledge stocks of the leader and the follower.

Relations (1.7) – (1.9) can also be interpreted in terms of $\tau(t)$, the follower's time lag defined through

$$n^B(t) = n^A(t - \tau(t)) = n_0^A \exp(\bar{g}^A(t - \tau(t)))\ . \tag{1.10}$$

If one of the relations (1.7) – (1.9) takes place, then $\exp(-\bar{g}^A \tau(t)) \to r_\infty$ as $t \to \infty$, and there exists the limit

$$\lim_{t\to\infty} \tau(t) = \tau_\infty\ .$$

In case of (1.7) we have $\tau_\infty = \infty$, that is, the follower's lag grows to infinity. In case of (1.8) it holds that $\tau_\infty = -\infty$, that is, the follower overtakes the leader, and the leader's lag grows infinitly. In case of (1.9) the limit τ_∞ is bounded, and we have

$$\tau_\infty = -\frac{1}{\bar{g}^A} \ln r_\infty\ . \tag{1.11}$$

Our main result states the following. For every solution to equation (1.1) – (1.3) starting at a state with positive coordinates the ratio $r(t)$ has a limit r_∞ (thus $r(t)$ cannot move periodically or chaotically). For the limit r_∞, which depends on a solution, only three values are admissible. We compute these values explicitly. The largest is either ∞, or positive; the smallest is zero; the intermediate one is positive and bounded. We call the corresponding asymptotics *upper*, *lower* and *intermediate*. We prove that the upper and lower asymptotics are feasible, that is, each of them is realized by some solutions. The feasibility of the intermediate asymptotics is proved under an extra constraint on the parameters (see below (1.13)).

As justified in Hutschenreiter et. al., 1995, the intermediate asymptotics is of special interest. Further economic arguments lead to the following question: can $r(t)$ corresponding to the intermediate asymptotics be strictly increasing? A sharper setting: can such $r(t)$ starting at $r(0) < 1$ end up with $r_\infty > 1$? In the first case the follower steadily approaches the leader. In the second case the follower overtakes the leader starting at a lagged position. Under appropriate constraints on the parameters, we provide positive answers to the above questions. (The second one is provided under stricter constraints.)

The accurate result formulations are given in the next subsection. Sections 2, 3, 4 and 5 contain the proofs.

In some episodes, we shall use the following inequalities:

$$c^B + \rho \;<\; \frac{\bar{g}^A + \rho}{1-\alpha} \tag{1.12}$$

$$\alpha c^B \;<\; \frac{1-\alpha}{\bar{g}^A + \rho} \tag{1.13}$$

$$c^B \;>\; \bar{g}^A \; . \tag{1.14}$$

Referring to (1.4) rewrite these inequalities in other notations. We have that (1.12) is equivalent to

$$L^B < L^A \; , \tag{1.15}$$

(1.13) is equivalent to

$$\alpha \left(\frac{L^A}{a} + \rho \right) \frac{L^B}{a} < 1 \tag{1.16}$$

and (1.14) is equivalent to

$$\frac{L^B}{a} > \bar{g}^A \; . \tag{1.17}$$

Inequality (1.14) being rewritten as

$$c^B + \rho > \bar{g}^A + \rho$$

is in a sense complementary to (1.12); we shall utilize its equivalent form

$$(c^B + \rho)\frac{1-\alpha}{\bar{g}^A + \rho} + \alpha > 1 \; . \tag{1.18}$$

1.3 Result formulations

In what follows, the term *solution* (to (1.1) — (1.3)) means a solution $(n^A(t), n^B(t), v^B(t))$ to equation (1.1) — (1.3) defined on the half-interval $[0, \infty)$ and starting at positive values

$$n^A(0) = n_0^A, \quad n^B(0) = n_0^B, \quad v^B(0) = v_0^B \; .$$

Let $(n^A(t), n^B(t), v^B(t))$ be an arbitrary solution. Let ratio $r(t)$ be defined by (1.6), and time lag $\tau(t)$ be defined by (1.10). Let, finally,

$$\gamma_\infty = \lim_{n^B \to \infty} \gamma(n^B) \; . \tag{1.19}$$

The next Proposition presents our main result. (Below, for better coordination with Hutschenreiter, et. al, 1995, values depending on c^B and, sometimes, $\bar{g}^A$ are doubled by their expressions computed through (1.4).)

Proposition 1.1 *For an arbitrary solution $(n^A(t), n^B(t), v^B(t))$, one and only one of the following three cases takes place:*

the lower asymptotics:

$$n^B(t) = \text{const} \quad \textit{for large } t$$

$$\lim_{t\to\infty} v^B(t) = -\infty$$

$$\lim_{t\to\infty} r(t) = r_\infty = 0$$

$$\lim_{t\to\infty} \tau(t) = \tau_\infty = \infty \ ;$$

the upper asymptotics:

$$\lim_{t\to\infty} n^B(t) = \infty$$

$$\lim_{t\to\infty} v^B(t) = \infty$$

$$\lim_{t\to\infty} r(t) = r_\infty = \begin{cases} \infty, & \bar{g}^A \le c^B = L^B/a \\ \gamma_\infty \delta \frac{\bar{g}^A}{\bar{g}^A - c^B} = \gamma_\infty \delta \frac{\bar{g}^A}{\bar{g}^A - L^B/a}, & \bar{g}^A > c^B = L^B/a \end{cases}$$

$$\lim_{t\to\infty} \tau(t) = \tau_\infty = \begin{cases} -\infty, & \bar{g}^A \le c^B = L^B/a \\ -\frac{1}{\bar{g}^A}\ln r_\infty & \bar{g}^A > c^B = L^B/a \end{cases} ;$$

the intermediate asymptotics:

$$\lim_{t\to\infty} n^B(t) = \infty$$

$$\lim_{t\to\infty} v^B(t) = 0$$

$$\lim_{t\to\infty} n^B(t) v^B(t) = \frac{1-\alpha}{\bar{g}^A + \rho} = \frac{1}{L^A/a + \rho}$$

$$\lim_{t\to\infty} r(t) = r_\infty = \gamma_\infty \delta \frac{c^B(1-\alpha)}{(\bar{g}^A + \rho) - (c^B + \rho)(1-\alpha)} = \gamma_\infty \delta \frac{1}{L^A/L^B - 1}$$

$$\lim_{t\to\infty} \tau(t) = \tau_\infty = -\frac{1}{\bar{g}^A}\ln r_\infty \ .$$

Remark 1.1 In the case of the intermediate asymptotics the denominator in the expression for r_∞ is necessarily positive; consequently inequality (1.12) is satisfied. Therefore, if (1.12) is violated, then there is no solution having the intermediate asymptotics.

Remark 1.2 The labels for the above three asymptotics, *lower, upper* and *intermediate*, are motivated by the fact that the limit r_∞ of $r(t)$ for the intermediate asymptotics lies, as one can easily verify, between zero (that is, the limit of $r(t)$ for the lower asymptotics) and the limit r_∞ for the upper asymptotics.

Proposition 1.2 *For each of the two asymptotics, lower and upper, there exists a solution having this asymptotics.*

Let us give sufficient conditions for a solution to have the lower or upper asymptotics. Introduce the two curves where the right hand sides of equations (1.2) and (1.3) vanish. Call them, respectively, the n^B- *barrier* and the v^B- *barrier*. As one can easily see, the equations for the n^B- and v^B- barriers are, respectively,

$$v^B = \mu_n(n^B, n^A), \quad v^B = \mu_v(n^B)$$

where

$$\mu_n(n^B, n^A) = \frac{\alpha}{c^B(n^B + \gamma(n^B)\delta n^A)} \tag{1.20}$$

$$\mu_v(n^B) = \frac{1-\alpha}{\rho n^B} \ . \tag{1.21}$$

We shall write

$$\mu_n[t] = \mu_n(n^B(t), n^A(t)) \tag{1.22}$$

$$\mu_v[t] = \mu_v(n^B(t)) \ . \tag{1.23}$$

Take a $t^A \geq 0$ such that

$$4\gamma(n_0^B)\delta n_0^A \exp(\bar{g}^A t^A) \geq 1 \ . \tag{1.24}$$

Clearly, one can put

$$t^A = \left[-\frac{1}{\bar{g}^A} \ln(4\gamma(n_0^B)\delta n_0^A) \right]_+ \ ;$$

in particular $t^A = 0$ if n_0^A and n_0^B are large enough, namely

$$4\gamma(n_0^B)\delta n_0^A \bar{g}^A \geq 1 \ . \tag{1.25}$$

Proposition 1.3 *If for a certain $t_0 \geq t^A$ it holds that*

$$v^B(t_0) \leq \mu_n[t_0], \quad \dot{v}^B(t_0) \leq \dot{\mu}_n[t_0] \ ,$$

then $(n^A(t), n^B(t), v^B(t))$ has the lower asymptotics.

Proposition 1.4 *If for a certain $t_0 \geq 0$ it holds that*

$$v^B(t_0) \geq \mu_v[t_0] \ ,$$

then $(n^A(t), n^B(t), v^B(t))$ has the upper asymptotics.

Let us pass to the intermediate asymptotics. Referring to Remark 1.1, we list (1.12) among conditions sufficient for the feasibility of the intermediate asymptotics.

Proposition 1.5 *If (1.12), (1.13) (or, equivalently, (1.15), (1.16)) are satisfied, then there exists a solution with the intermediate asymptotics.*

As justified in Hutschenreiter, et. al, 1995, the intermediate asymptotics is of primary interest from an economic point of view. In particular, the follower's trajectories $(n^B(t), v^B(t))$ corresponding to the intermediate asymptotics satisfy the so called perfect-foresight condition (see Grossman and Helpman, 1991), whereas those corresponding to the lower and upper asymptotics do not satisfy this condition. The existence of a solution with the intermediate asymptotics and $r(t)$ strictly increasing is the next issue in our analysis. Below, we formulate sufficient existence conditions.

Introduce

Condition 1.1 For every time interval $[t_1, t_2]$ of nonzero length and every solution $(n^A(t), n^B(t), v^B(t))$ such that $v^B(t)$ satisfies $\mu_n[t] < v^B(t) < \mu_v[t]$ $(t \in [t_1, t_2])$, the product $n^B(t)v^B(t)$ is nonconstant on $[t_1, t_2]$.

Condition 1.2 For every interval $[p_1, p_2]$ of nonzero length with $p_1 \geq 0$ one cannot find any positive σ, β such that

$$\gamma'(p) = \sigma p^{-\beta}$$

for all $p \in [p_1, p_2]$.

Remark 1.3 Here are two examples of $\gamma(p)$ satisfying Condition 1.2:

$$\gamma(p) = 1 - \exp(-p)$$

$$\gamma(p) = \frac{2}{\pi}\mathrm{arctg}(p) \ .$$

Proposition 1.6 *Condition 1.2 implies Condition 1.1.*

Let us call a solution $(n^A(t), n^B(t), v^B(t))$ *catching up* if $r(t)$ (1.6) is stricly increasing, and *overtaking* if it is catching up, and

$$r(0) < 1 < r_\infty = \lim_{t\to\infty} r(t) \ .$$

Proposition 1.7 *Let one of Conditions 1.1, 1.2 and inequalities (1.12), (1.13), (1.18) (or, equivalently, (1.15), (1.16), (1.17)) be satisfied. Then*

(i) there exists a catching up solution with the intermediate asymptotics;

(ii) if (see Proposition 1.1)

$$r_\infty = \gamma_\infty \delta \frac{c^B(1-\alpha)}{(\bar{g}^A + \rho) - (c^B + \rho)(1-\alpha)} = \gamma_\infty \delta \frac{1}{L^A/L^B - 1} > 1 \ ,$$

then there exists an overtaking solution with the intermediate asymptotics.

A criterion of the intermediate asymptotics is as follows.

Proposition 1.8 *A solution $(n^A(t), n^B(t), v^B(t))$ has the intermediate asymptotics if and only if there exists a $t_0 \geq 0$ such that*

$$\mu_n[t] < v^B(t) < \mu_v[t]$$

for all $t \geq t_0$.

The next characterizations specifying Propositions 1.5 and 1.7 are given in terms of variables

$$x(t) = n^B(t)v^B(t), \quad \xi(t) = n^A(t)v^B(t) \ .$$

Set

$$x_0 = x(0), \quad \xi_0 = \xi(0)$$

and introduce

$$\begin{aligned} x_\xi &= \frac{1-\alpha}{\bar{g}^A + \rho} \qquad (1.26) \\ x_v &= \frac{1-\alpha}{\rho} \ . \end{aligned}$$

Obviously,

$$x_\xi < x_v \ . \qquad (1.27)$$

Proposition 1.9 *Let inequalities (1.12), (1.13) be true. Let* $\xi_0 > 0$ *and* $v_0^{B-} > 0$ *satisfy*

$$(c^B + \rho)x_\xi + c^B\delta\gamma_\infty\xi_0 - 1 < 0 \tag{1.28}$$

$$\min\left\{4\delta\bar{g}^A\gamma\left(\frac{x_0}{v_0^{B-}}\right)\frac{\xi_0}{v_0^{B-}} \ : \ x_0 \in [x_\xi, x_v]\right\} > 1 \ . \tag{1.29}$$

Then there exists an $x_0 \in [x_\xi, x_v)$ *such that solution* $(n^A(t), n^B(t), v^B(t))$ *with the initial state*

$$n^A(0) = \frac{\xi_0}{v_0^{B-}}, \quad n^B(0) = \frac{x_0}{v_0^{B-}}, \quad v^B(0) = v_0^{B-}$$

has the intermediate asymptotics.

Remark 1.4 Inequality (1.28) is satisfied for ξ_0 close to zero, since, as follows from (1.26) and (1.12),

$$(c^B + \rho)x_\xi = (c^B + \rho)\frac{1-\alpha}{\bar{g}^A + \rho} < 1 \ .$$

If $\xi_0 > 0$ is fixed, (1.29) is satisfied for sufficiently small v_0^{B-}. It is seen from the relations

$$\gamma\left(\frac{x_0}{v_0^{B-}}\right) \geq \gamma\left(\frac{x_\xi}{v_0^{B-}}\right) \to \gamma_\infty > 0 \quad \text{as} \quad v_0^{B-} \to 0$$

$$\frac{\xi_0}{v_0^{B-}} \to \infty \quad \text{as} \quad v_0^{B-} \to 0 \ .$$

Figure 1 schematically illustrates Proposition 1.9. The upper curve symbolizes the set of initial pairs (x_0, ξ_0) corresponding to solutions with the intermediate asymptotics (v_0^{B-} is fixed); this curve lies between x_ξ and x_v. The bold curves ended by arrows show corresponding solutions in coordinates $(x(t), \xi(t))$. They converge to $(x_\xi, x_\xi/r_\infty)$. Condition (1.28) is reflected in the fact that the curve for (x_0, ξ_0) lies on the left of ξ^+; condition (1.29) is illustrated by the fact that x_ξ lies above the curve $\gamma(x/v_0^{B-}) = q/\xi$.

Proposition 1.10 *Let one of Conditions 1.1, 1.2, and inequalities (1.12), (1.13), (1.18) (or, equivalently, (1.15), (1.16), (1.17)) be satisfied. For every* $n_0^{B-} > 0$ *there exist a* $\xi_0 > 0$ *and an* $x_0 \in (0, x_\xi)$ *such that solution* $(n^A(t), n^B(t), v^B(t))$ *with the initial state*

$$n^A(0) = \frac{\xi_0}{v_0^{B+}}, \quad n^B(0) = \frac{x_0}{v_0^{B+}}, \quad v^B(0) = v_0^{B+} \tag{1.30}$$

$$v_0^{B+} = \frac{\alpha - \gamma(n_0^{B-})\delta\xi_0}{c^B n_0^{B-}} \tag{1.31}$$

is catching up and has the intermediate asymptotics (an additional characterization of (x_0, ξ_0) *is that* $\alpha - c^B\gamma(n_0^{B-})\delta\xi_0$ *is positive and sufficiently small).*

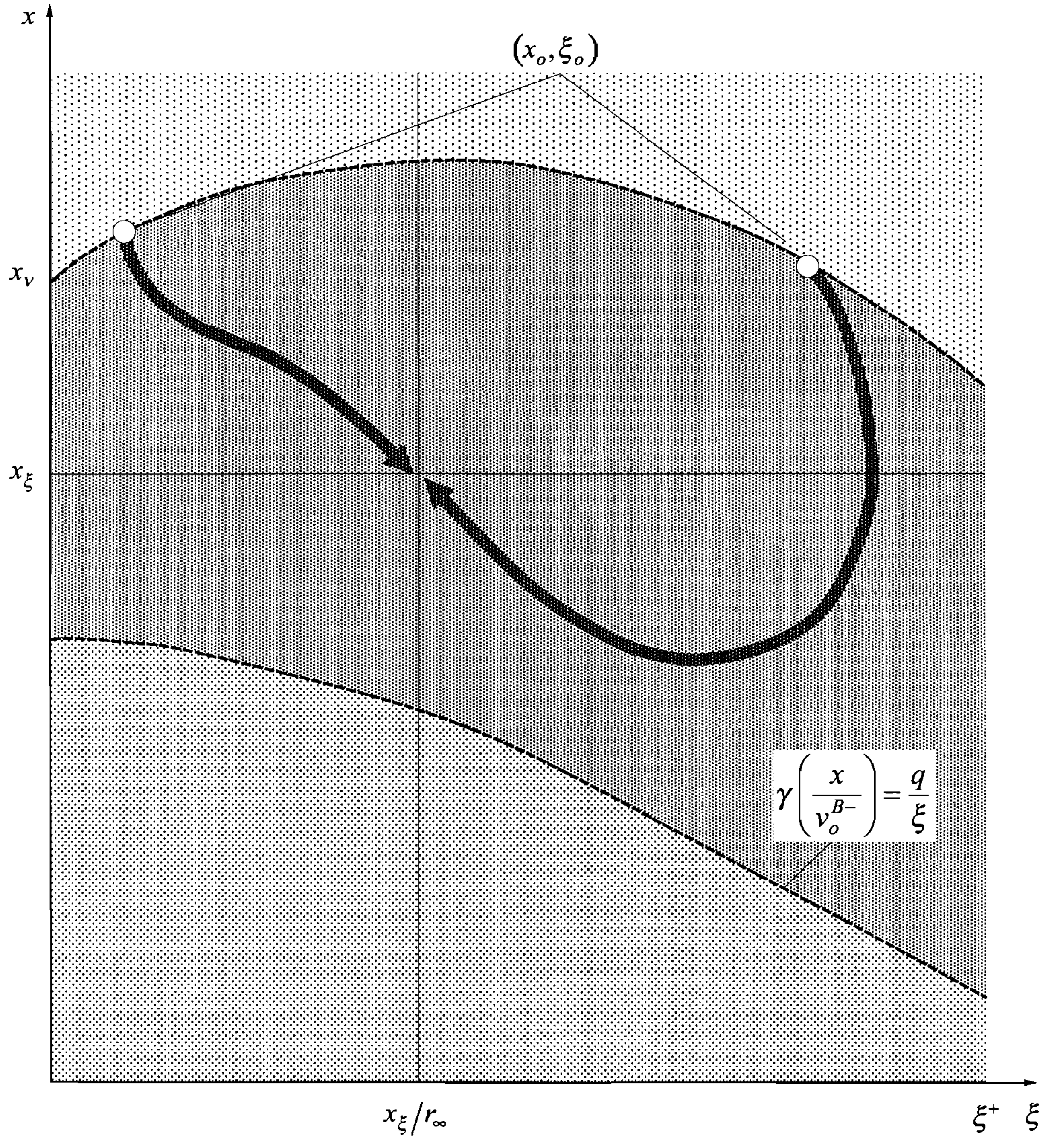
Moreover, if

$$r_\infty = \gamma_\infty\delta\frac{c^B(1-\alpha)}{(\bar{g}^A + \rho) - (c^B + \rho)(1-\alpha)} > 1$$

and

$$\frac{x_\xi\gamma(n_0^{B-})\delta}{\alpha} < 1 \ , \tag{1.32}$$

then ξ_0 *and* x_0 *can be selected so that the above solution is overtaking.*



$$\xi^+ = \frac{1-(c^B+\rho)x_\xi}{c^B\delta\gamma_\infty}$$

$$q = \frac{v_o^{B-}}{4\delta c^A}$$

Figure 1: A schematic illustration of Proposition 1.9. The upper curve represents the set of initial points (x_0, ξ_0) on the (x, ξ)-plane corresponding to solutions with the intermediate asymptotics.

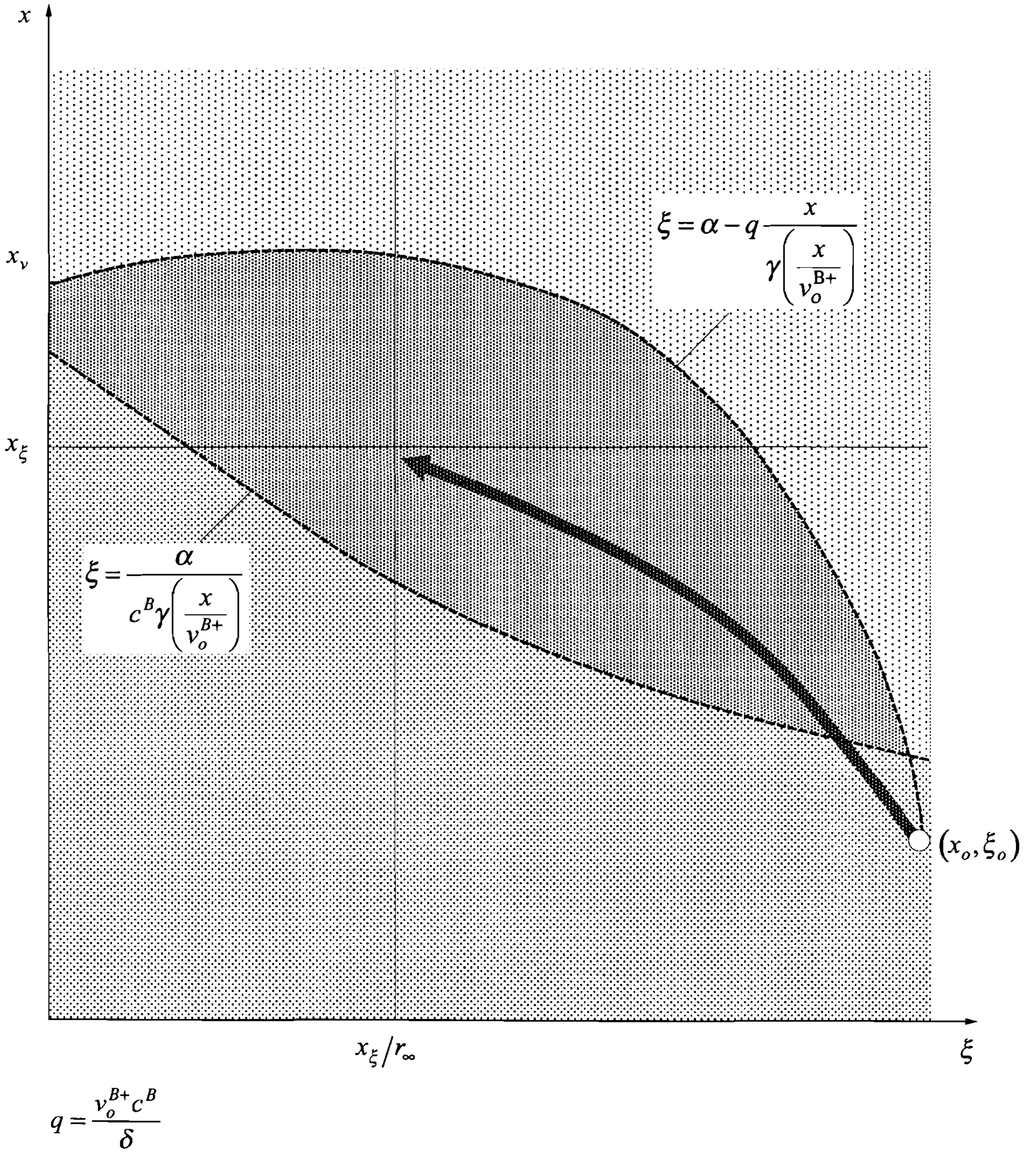


Figure 2: A schematic illustration of Proposition 1.10. Point (x_0, ξ_0) on the (x, ξ)-plane corresponds to a catching up solution with the intermediate asymptotics.

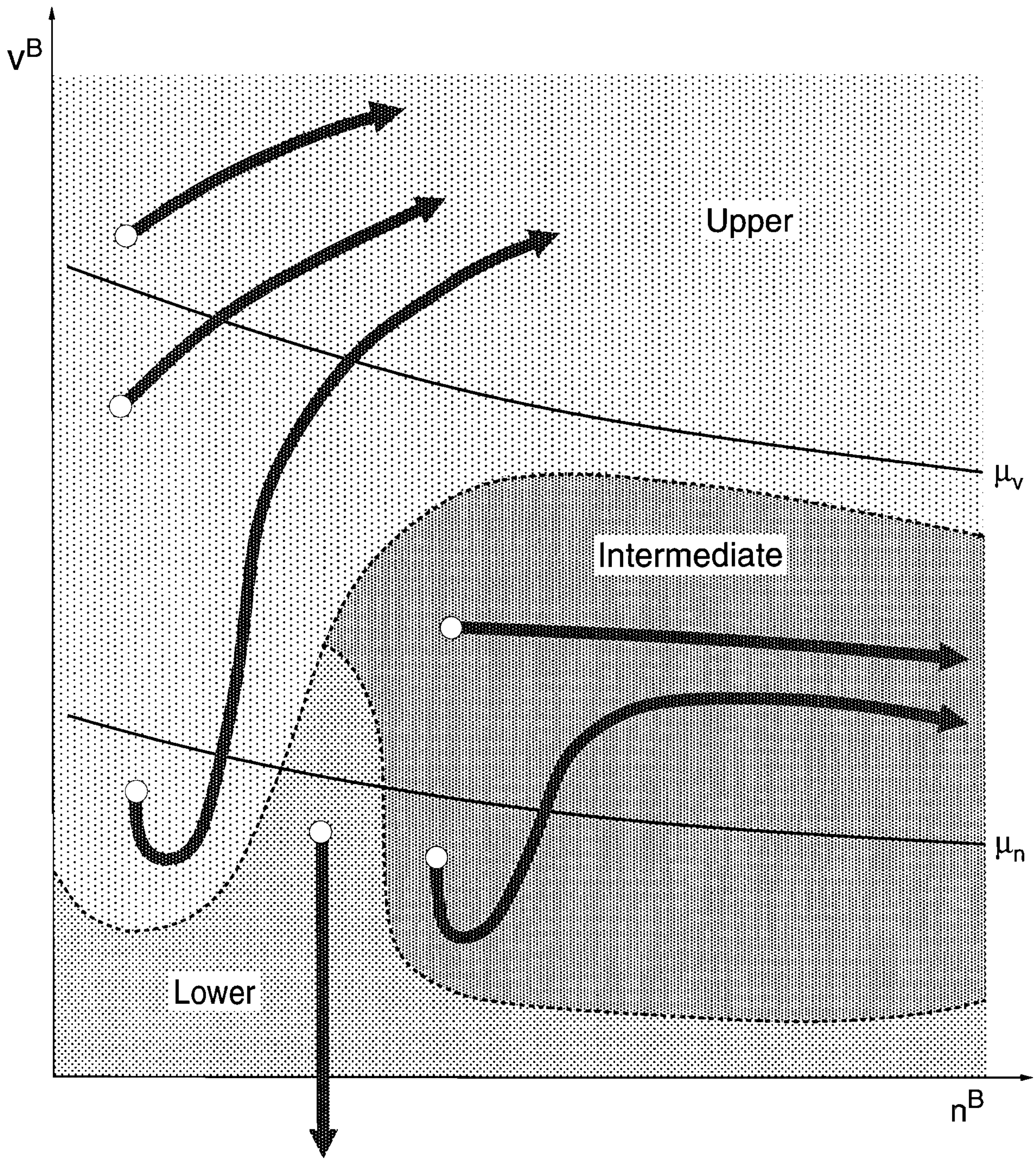


Figure 3: A schematic representation of the n^B- and v^B-barriers and the separation of the n^B-, v^B-plane into domains which lead to the lower, intermediate and upper asymptotics.

Remark 1.5. The existence of $n_0^{B-} > 0$ satisfying (1.32) follows from the fact that $\gamma(0) = 0$.

Figure 2 illustrates Proposition 1.10. The upper curve symbolizes relation (1.31) (with n_0^{B-} expressed through x_0 as in (1.30)) between the components of pairs (x_0, ξ_0) corresponding to catching up solutions with the intermediate asymptotics; v_0^{B+} is fixed. The lower curve illustrates the relation $\alpha - c^B\gamma(n_0^{B-})\delta\xi_0 = 0$ (see the additional characterization of (x_0, ξ_0)). The two curves cross under x_ξ and on the right of x_ξ/r_∞. An initial point (x_0, ξ_0) lies on the first curve, slightly below the point of crossing. The bold curve ended by the arrow shows a corresponding solution in coordinates $(x(t), \xi(t))$. It converges to $(x_\xi, x_\xi/r_\infty)$ moving North-West, which implies that $r(t) = n^B(t)/n^A(t) = x(t)/\xi(t)$ is strictly increasing.

Propositions 1.9 and 1.10 imply obviously Propositions 1.5 and 1.7, respectively. Therefore, we shall prove Propositions 1.9 and 1.10 only.

The paper is organized as follows. In section 2 we prove Propositions 1.3, 1.4 providing sufficient conditions for the lower and upper asymptotics, and establish the important fact that for an arbitrary solution $(n^A(t), n^B(t), v^B(t))$ either one of these sufficient conditions is satisfied, or $v^B(t)$ stays between the barriers, that is, the condition of Proposition 1.8 holds true. In section 3 we consider an arbitrary solution satisfying the condition of Proposition 1.8 and prove that it has the intermediate asymptotics. This proves Proposition 1.8 and completes the proof of the main Proposition 1.1. In section 4 we prove Proposition 1.9 on the feasibility of the intermediate asymptotics; as it was said above, this proves Proposition 1.5, too. In section 5, Proposition 1.10 on the existence of catching up and overtaking solutions having the intermediate asymptotics is justified (thus proving Proposition 1.7); the proof of Proposition 1.6 finalizes the analysis.

Thus, sections 4 and 5 are devoted to the existence statements, whereas the main qualitative results are concentrated in sections 2 and 3. These results are illustrated schematically on Figure 3 for a fixed n_0^A. The Figure shows the borders between domains of (n_0^B, v_0^B) starting different asymptotics. The bold lines with arrows illustrate behaviors of $(n^B(t), v^B(t))$. In accordance with Proposition 1.3 the lower asymptotics starts below the n^B-barrier μ_n; $(n^B(t), v^B(t))$ goes vertically down. By Proposition 1.4 the upper asymptotics starts above the v^B-barrier μ_v; both $n^B(t)$ and $v^B(t)$ go to infinity. We do not exclude that some points (n_0^B, v_0^B) above μ_n (not above μ_v) start the lower asymptotics, and some points (n_0^B, v_0^B) below μ_v or μ_n start the upper asymptotics; the corrsponding trajectories are sketched out on the Figure. Points (n_0^B, v_0^B) starting the intermediate asymptotics lie either between the barriers μ_v and μ_n, or (we do not exclude this) below the barriers. Point $(n^B(t), v^B(t))$ with the intermediate asymptotics enters the corridor between the barriers from below, or starts in this corridor, and stays there forever.

2 Upper and Lower Asymptotics

2.1 Elementary properties of solutions and barriers

Note first that, as follows from equation (1.2),

$$n^B(t) \quad \text{is} \quad \text{nondecreasing} . \tag{2.1}$$

The next properties of $\mu_n[t]$ and $\mu_v[t]$ (see (1.22), (1.23)) are obvious:

$$\mu_n[t] \quad \text{is strictly} \quad \text{decreasing} \tag{2.2}$$

$$\mu_n[t] \geq 0 \tag{2.3}$$

$$\mu_n[t] \to 0 \quad \text{as} \quad t \to \infty \tag{2.4}$$

$$\mu_v[t] \quad \text{is nonincreasing} \ . \tag{2.5}$$

The properties follow from (1.5) and (2.1).

The signs of the derivatives $\dot{n}^B(t)$, and $\dot{v}^B(t)$ are determined by a position of $v^B(t)$ with respect to $\mu_n[t]$ and $\mu_v[t]$, respectively. Namely,

$$\dot{n}^B(t) = 0 \quad \text{if} \quad v^B(t) \leq \mu_n[t] \tag{2.6}$$

$$\dot{n}^B(t) = c^B(n^B(t) + \gamma(n^B(t))\delta n^A(t)) - \frac{\alpha}{v^B(t)} > 0 \quad \text{if} \quad v^B(t) > \mu_n[t] \tag{2.7}$$

$$\dot{v}^B(t) < 0 \quad \text{if} \quad v^B(t) < \mu_v[t] \tag{2.8}$$

$$\dot{v}^B(t) > 0 \quad \text{if} \quad v^B(t) > \mu_v[t] \tag{2.9}$$

$$\dot{v}^B(t) = 0 \quad \text{if} \quad v^B(t) = \mu_v[t] \ . \tag{2.10}$$

From (2.6) and (1.21) follows:

$$\dot{\mu}_v[t] = 0 \quad \text{if} \quad v^B(t) \leq \mu_n[t] \ . \tag{2.11}$$

Properties (2.2) and (2.4) of function $\mu_n[t]$ are supplemented by the following:

$$\frac{d^2\mu_n[t]}{dt^2} > 0 \quad \text{if} \quad t > t^A, \quad v^B(t) \leq \mu_n[t] \ . \tag{2.12}$$

Indeed, let us compute $d^2\mu_n[t]/dt^2$. By (1.22), (1.20)

$$\dot{\mu}_n[t] = -\sigma\frac{\dot{g}(t)}{g^2(t)}$$

where

$$\sigma = \frac{\alpha}{c^B}$$

$$g(t) = n^B(t) + \gamma(n^B(t))\delta n^A(t) \ .$$

Hence

$$\begin{aligned} \frac{d^2\mu_n[t]}{dt^2} &= -\frac{\sigma}{g^2(t)}\frac{d^2g(t)}{dt^2} + 4\sigma\left(\frac{\dot{g}(t)}{g(t)}\right)^2 \\ &= \frac{\sigma}{g^2(t)}\left[4(\dot{g}(t))^2 - \frac{d^2g(t)}{dt^2}\right] \ . \end{aligned} \tag{2.13}$$

We have

$$\dot{g}(t) = \dot{n}^B(t)[1 + \gamma'(n^B(t))\delta n^A(t)] + \gamma(n^B(t))\delta\dot{n}^A(t) \ .$$

By (2.6) $\dot{n}^B(t) = 0$; therefore

$$\dot{g}(t) = \gamma(n^B(t))\delta\dot{n}^A(t) \ .$$

The next differentiation with the usage of $\dot{n}^B(t) = 0$ provides

$$\frac{d^2g(t)}{dt^2} = \gamma(n^B(t))\delta\frac{d^2n^A(t)}{dt^2} \ .$$

Substituting in (2.13) and using (1.5), we get

$$\begin{aligned} \frac{d^2\mu_n[t]}{dt^2} &= \frac{\sigma}{g^2(t)}\left[4\gamma^2(n^B(t))\delta^2(\dot{n}^{A2}(t)) - \gamma(n^B(t))\delta\frac{d^2n^A(t)}{dt^2}\right] \\ &= \frac{\sigma}{g^2(t)}\gamma(n^B(t))\delta[4\gamma(n^B(t))\delta c^{A2}n_0^{A2}\exp(2\bar{g}^At) - c^{A2}n_0^A\exp(\bar{g}^At)] \\ &= \frac{\sigma}{g^2(t)}\gamma(n^B(t))\delta c^{A2}n_{0A}^2\exp(\bar{g}^At)[4\gamma(n^B(t))\delta n_0^A\exp(\bar{g}^At) - 1] \ . \end{aligned}$$

Since $n^B(t)$ is nondecreasing (see (2.1)), $\gamma(n^B(t))$ is nondecreasing too; hence for $t > t^A$ the last square bracket, in view of (1.24), is positive, and we obtain (2.12).

2.2 The three initial cases

Fix a

$$t_0 \geq 0 \ .$$

We shall consider separately the three positions of $v^B(t_0)$ with respect to the n^B- and v^B-barriers. These are:

$$v^B(t_0) \geq \mu_v[t_0] \tag{2.14}$$

$$v^B(t_0) \leq \mu_n[t_0], \quad v^B(t_0) < \mu_v[t_0] \tag{2.15}$$

$$\mu_n[t_0] < v^B(t_0) < \mu_v[t_0] \ . \tag{2.16}$$

Note that the above cases do not overlap, and one of them holds true. We denote

$$c = \frac{c^B}{\bar{g}^A} \ . \tag{2.17}$$

2.3 Case 1 (above the v^B-barrier). Absolute asymptotics

Consider case (2.14). Here and in the next subsection we assume that inequality (2.14) is satisfied strictly, i.e.

$$v^B(t_0) > \mu_v[t_0] \ . \tag{2.18}$$

Let us show the "absolute" asymptotics

$$v^B(t) \to \infty \quad \text{as} \quad t \to \infty \tag{2.19}$$

$$n^B(t) \to \infty \quad \text{as} \quad t \to \infty \ . \tag{2.20}$$

We start with (2.19). ¿From equation (1.3) for $v^B(t)$ and the fact that $n^B(t)$ is nondecreasing (see (2.1), we get

$$\dot{v}^B(t) \geq \rho v^B(t) - \rho\mu_v[t_0] \ ;$$

also (2.18) implies that

$$\rho v^B(t_0) - \rho\mu_v[t_0] = \rho v^B(t_0) - \rho\mu_v[t_0] > t_0 \ .$$

Hence

$$v^B(t) \geq v_*^B(t)$$

where

$$\dot{v}_*^B(t) = \rho v_*^B(t) - \rho\mu_v[t_0], \quad v_*^B(t_0) = v^B(t_0) \ .$$

We have

$$v_*^B(t) = (v^B(t_0) - \mu_v[t_0])\exp(\rho(t-t_0)) + \mu_v[t_0] \ .$$

Since

$$v_*^B(t) \to \infty \quad \text{as} \quad t \to \infty \ ,$$

we get (2.19).

Let us prove (2.20). From (2.19) and (2.5) we deduce that for all sufficiently large t the first inequality in (2.7) is satisfied. Hence the second inequality in (2.7) is true for all large t. For all large t the right hand side of this inequality is greater than a positive ϵ; this follows easily from (2.19) and the fact that $n^B(t)$ and $n^A(t)$ are nondecreasing. Therefore

$$\dot{n}^B(t) \geq \epsilon > 0$$

for all large t. This obviously yields (2.20).

2.4 Case 1 (above the v^B-barrier). Relative asymptotics

Basing on (2.19), (2.20), we shall find an asymptotics for the ratio $r(t)$ (1.6). Recall that inequality (2.18) is assumed. Note first that by (2.20) and by the definition of γ_∞ (see (1.19)) it holds that

$$\gamma(n^B(t)) \to \gamma_\infty \quad \text{as} \quad t \to \infty \ . \tag{2.21}$$

Let $\nu(n^A)$ be the inverse to $n^A(t)$, that is,

$$\nu(n^A) = \frac{1}{\bar{g}^A}\ln n^A \quad (n_0^A \leq n^A < \infty) \ .$$

Denote

$$n^B[n^A] = n^B(\nu(n^A))$$

$$r[n^A] = \frac{n^B[n^A]}{n^A} \ .$$

Obviously

$$\lim_{t\to\infty} r(t) = \lim_{n^A\to\infty} r[n^A] \ . \tag{2.22}$$

Let us find the limit. Differentiate $n^B[n^A]$. Introducing

$$\nu = \nu(n^A) \ ,$$

using equation (1.1) and inequality (2.7) (we showed above that it is satisfied for all large t), we obtain that for all sufficiently large n^A

$$\begin{aligned} n^{B'}[n^A] &= \frac{\dot{n}^B(\nu)}{\dot{n}^A(\nu)} \\ &= \frac{1}{\bar{g}^A n^A(\nu)}\left(c^B(n^B(\nu) + \gamma(n^B(\nu))\delta n^A(\nu)) - \frac{\alpha}{v^B(\nu)}\right) \\ &= \frac{c^B}{\bar{g}^A}\left[\frac{n^B(\nu)}{n^A(\nu)} + \gamma(n^B(\nu))\delta\right] - \frac{\alpha}{\bar{g}^A n^A(\nu) v^B(\nu)} \\ &= \frac{\bar{g}^A}{c^B}\left[\frac{n^B[n^A]}{n^A} + \gamma_\infty\delta\right] + \sigma[n^A] \end{aligned} \tag{2.23}$$

where

$$\sigma[n^A] = \frac{c^B}{\bar{g}^A}(\gamma(n^B[n^A]) - \gamma_\infty)\delta - \frac{\alpha}{\bar{g}^A n^A v^B(\nu)} \ .$$

By (1.5) and (2.19)

$$0 \geq \sigma[n^A] \to 0 \quad \text{as} \quad n^A \to \infty \ . \tag{2.24}$$

Treating (2.23) as a linear differential equation for $n^B[n^A]$ and integrating it from an arbitrary $n^A_* \geq n^A_0$ to an arbitrary $n^A \geq n^A_*$ we get

$$\begin{aligned}
n^B[n^A] &= n^B_* \exp\left(c\int_{n^A_*}^{n^A}\frac{dn}{n}\right) + \\
&\quad \gamma_\infty\delta\int_{n^A_*}^{n^A}\exp\left(c\int_{n_*}^{n^A}\frac{dn}{n}\right)dn_* + \int_{n^A_*}^{n^A}\exp\left(c\int_{n_*}^{n^A}\frac{dn}{n}\right)\sigma[n_*]dn_* \\
&= n^B_*\exp\left(c\ \ln\frac{n^A}{n^A_*}\right) + \\
&\quad \gamma_\infty\delta\int_{n^A_*}^{n^A}\exp\left(c\ \ln\frac{n^A}{n_*}\right)dn_* + \int_{n^A_*}^{n^A}\exp\left(c\ \ln\frac{n^A}{n_*}\right)\sigma[n_*]dn_* \\
&= n^B_*\left(\frac{n^A}{n^A_*}\right)^c + \gamma_\infty\delta(n^A)^c\int_{n^A_*}^{n^A}\frac{1}{n^c_*}dn_* + (n^A)^c\int_{n^A_*}^{n^A}\frac{1}{n^c_*}\sigma[n_*]dn_* \ .
\end{aligned}$$

Here we use notation (2.17) and set $n^B_* = n^B[n^A_*]$. Specify the second and the third terms on the right. For the second term we have

$$\gamma_\infty\delta(n^A)^c\int_{n^A_*}^{n^A}\frac{1}{n^c_*}dn_* = \begin{cases} \gamma_\infty\delta(n^A)^c\frac{1}{1-c}[(n^A)^{1-c} - (n^A_*)^{1-c}], & c \neq 1 \\ \gamma_\infty\delta n^A \ln\frac{n^A}{n^A_*}, & c = 1 \end{cases} .$$

For the third term, using the inequality from (2.24), we get

$$0 \geq n^{Ac}\int_{n^A_*}^{n^A}\frac{1}{n^c_*}\sigma[n_*]dn_* \geq \begin{cases} -n^{Ac}\epsilon[n^A_*]\frac{1}{1-c}[(n^A)^{1-c} - (n^A_*)^{1-c}], & c \neq 1 \\ -n^{Ac}\epsilon[n^A_*]\ln\frac{n^A}{n^A_*}, & c = 1 \end{cases}$$

where

$$\epsilon[n^A_*] = \sup\{|\ \sigma[n^A]\ | :\ n^A \geq n^A_*\} \ .$$

Note that convergence (2.24) implies

$$\epsilon[n^A_*] \to 0 \quad \text{as} \quad n^A_* \to \infty \ . \tag{2.25}$$

Dividing $n^B[n^A]$ by n^A, we arrive at the following estimates:

$$\begin{aligned}
r[n^A] &\leq \frac{n^B_*}{(n^A_*)^c(n^A)^{1-c}} + \begin{cases} \gamma_\infty\delta\frac{1}{1-c}\left[1 - \frac{(n^A_*)^{1-c}}{(n^A)^{1-c}}\right], & c \neq 1 \\ \gamma_\infty\delta\ln\frac{n^A}{n^A_*}, & c = 1 \end{cases} \\
r[n^A] &\geq \frac{n^B_*}{(n^A_*)^c(n^A)^{1-c}} + \begin{cases} (c\gamma_\infty\delta - \epsilon[n^A_*])\frac{1}{1-c}\left[1 - \frac{(n^A_*)^{1-c}}{(n^A)^{1-c}}\right], & c \neq 1 \\ (\gamma_\infty\delta - \epsilon[n^A_*])\ln\frac{n^A}{n^A_*}, & c = 1 \end{cases} .
\end{aligned}$$

Hence

$$\limsup_{n^A\to\infty} r[n^A] \leq \lim_{n^A\to\infty}\frac{n^B_*}{(n^A_*)^c(n^A)^{1-c}} + \begin{cases} \gamma_\infty\delta\frac{1}{1-c}\left[1 - \frac{(n^A_*)^{1-c}}{(n^A)^{1-c}}\right], & c \neq 1 \\ \gamma_\infty\delta\ln\frac{n^A}{n^A_*}, & c = 1 \end{cases}$$

$$
\begin{aligned}
&= \gamma_\infty \delta \frac{1}{1-c} \\
\liminf_{n^A\to\infty} r[n^A] &\geq \lim_{n^A\to\infty} \frac{n_*^B}{(n_*^A)^c (n^A)^{1-c}} + \begin{cases} (c\gamma_\infty\delta - \epsilon[n_*^A])\frac{1}{1-c}\left[1 - \frac{(n_*^A)^{1-c}}{(n^A)^{1-c}}\right] & c \neq 1 \\ (\gamma_\infty\delta - \epsilon[n_*^A])\ln\frac{n^A}{n_*^A}, & c = 1 \end{cases} \\
&= (\gamma_\infty\delta - \epsilon[n_*^A])\frac{1}{1-c} \,.
\end{aligned}
$$

Letting $n_*^A \to \infty$ and referring to convergence (2.24), we obtain:

$$
\lim_{n^A\to\infty} r[n^A] = \begin{cases} \infty, & c \geq 1 \\ \gamma_\infty \delta \frac{1}{1-c}, & c < 1 \end{cases}
$$

or, in the initial notations (see (2.22), (2.17)),

$$
\lim_{t\to\infty} r(t) = \begin{cases} \infty, & c \geq 1 \\ \gamma_\infty\delta\frac{1}{1-c} = \gamma_\infty\delta\frac{\bar{g}^A}{\bar{g}^A - c^B}, & c < 1 \end{cases} \,. \tag{2.26}
$$

Relations (2.26), (2.19), (2.20) show that for the case where (2.18) is satisfied the statement of Proposition 1.4 is true.

2.5 Case 1 (at the v^B-barrier)

In this subsection we consider the case where inequality (2.14) turns into the equality, i.e.

$$
v^B(t_0) = \mu_v[t_0] \; ; \tag{2.27}
$$

thus we complete the analysis of case (2.14). We reduce the situation to that considered in the previous two subsections, and thus show that asymptotics (2.26), (2.19), (2.20) takes place. This – together with the observation given in the last paragraph of subsection 2.4 – complete the proof of Proposition 1.4.

Like in subsection 2.3, we get

$$
v^B(t) \geq v_*^B(t) = (v_0^B - \mu_v[t_0])\exp(\rho t) + \mu_v[t_0] \,.
$$

Hence, due to (2.27),

$$
v^B(t) \geq \mu_v[t_0] \,. \tag{2.28}
$$

Since $\mu_v[t]$ is nonincreasing (see (2.5)), we have

$$
v^B(t) \geq \mu_v[t] \quad (t \geq t_0) \,. \tag{2.29}
$$

It is sufficient to show that for a certain $t_1 \geq t_0$, the inequality (2.29) is satisfied strictly. In this case we repeat the argument of the previous two subsections starting with

$$
v^B(t_1) > \mu_v[t_1]
$$

instead of (2.18), and arrive at (2.26).

Let us suppose, to the contrary, that (2.29) is untrue, that is,

$$
v^B(t) = \mu_v[t] \quad (t \geq t_0) \,. \tag{2.30}
$$

¿From convergence (2.4), inequalities (2.28) and the inequality $\mu_v[t_0] > 0$, we deduce that

$$
v^B(t) \geq \mu_v[t_0] > \mu_n[t]
$$

for all t greater than a sufficiently large t_*. Hence by (2.7) $n^B(t)$ is strictly increasing after t_*. Therefore, as it is seen from (1.23), (1.21), $\mu_v[t]$ is strictly decreasing after t_*. Now (2.28) and (2.5) imply that for $t > t_*$

$$v^B(t) \geq \mu_v[t_0] > \mu_v[t]$$

contradicting the assumption (2.30). The contradiction completes the proof.

2.6 Case 2 (below the n^B-barrier). Divergence

Let us pass to case (2.15). In this subsection we assume that

$$t_0 \geq t^A \tag{2.31}$$

and the velocity of $v^B(t)$ at $t = t_0$ is no greater than that of $\mu_n[t]$ ($v^B(t)$ "diverges" from $\mu_n[t]$ at $t = t_0$), that is,

$$\dot{v}^B(t_0) = \rho(v^B(t_0) - \mu_v[t_0]) \leq \dot{\mu}_n[t_0] . \tag{2.32}$$

We shall prove that

$$\begin{aligned} v^B(t) &\leq \mu_n[t] \quad (t \geq t_0) & (2.33)\\ v^B(t) &< \mu_n[t] \quad (t > t_0) & (2.34)\\ v^B(t) &\leq \mu_v[t] \quad (t \geq t_0) & (2.35) \end{aligned}$$

$$\lim_{t\to\infty} v^B(t) = -\infty \tag{2.36}$$

Note that (2.33) yields by (2.6) that

$$n^B(t) = \text{const} \quad (t \geq t_0) ;$$

combining this with (1.5), we arrive at the lower asymptotics

$$\lim_{t\to\infty} r(t) = 0 . \tag{2.37}$$

This proves Proposition 1.3.

Let us prove (2.33) and (2.35). Introduce

$$t^* = \sup\{ t \geq t_0 \ : \ v^B(\tau) - \mu_n[\tau] \leq v(t_0) - \mu_n[t_0] \quad \text{for all } \tau \in [t_0, t] \} . \tag{2.38}$$

By (2.15) and (2.11)

$$\mu_v[t] = \text{const} = d \quad (t_0 \leq t < t^*) .$$

Hence (see also the second inequality in (2.15))

$$\begin{aligned} v^B(t) &= (v^B(t_0) - \mu_v[t_0])\exp(\rho(t-t_0)) + \mu_v[t_0] \\ &= (v^B(t_0) - d)\exp(\rho(t-t_0)) + d \\ &\leq v^B(t_0) \leq d = \mu_v[t] \quad (t_0 \leq t < t^*) . & (2.39) \end{aligned}$$

Therefore, in order to prove (2.33), (2.35), it is sufficient to show that

$$t^* = \infty .$$

Suppose, to the contrary, that

$$t^* < \infty .$$

Note that (2.39) implies

$$\begin{aligned} \dot{v}^B(t) &= \rho(v^B(t_0) - d)\exp(\rho(t - t_0)) \\ &\leq \rho(v^B(t_0) - d) \\ &\leq \dot{v}^B[t_0] \quad (t_0 \leq t \leq t^*) \end{aligned}$$

and by (2.12)

$$\dot{\mu}_n[t] > \dot{\mu}_n[t_0] \quad (t_0 \leq t \leq t^*)$$

yielding (see (2.32) and (2.15))

$$v^B(t) < \mu_n[t] \quad (t_0 < t \leq t^*) . \tag{2.40}$$

Hence

$$v^B(t) < \mu_n(t) \quad (t_0 < t \leq t^* + \epsilon)$$

for a sufficiently small positive ϵ, which contradicts to the definition of t^*. The contradiction completes the proof of (2.33), (2.35). Repeating the above proof of (2.40) for $t^* = \infty$, we obtain (2.34) and (2.36).

2.7 Case 2 (below the n^B-barrier). Convergence

Continuing the analysis of the previous subsection, we study case (2.15) under the assumption (2.31). Here, instead of (2.32), we assume the opposite inequality, that is,

$$\dot{v}^B(t_0) = \rho(v^B(t_0) - \mu_v[t_0]) > \dot{\mu}_n[t_0] \tag{2.41}$$

(implying "convergence" of $v^B(t)$ to $\mu_n[t]$ at $t = t_0$). Introduce

$$t^* = \sup\{\ t \geq t_0\ : v^B(\tau) \leq \mu_n[\tau],\ v^B(\tau) < \mu_v[\tau] \text{ for all } \tau \in [t_0, t]\ \} .$$

Like in (2.39) we state that

$$v^B(t) = (v^B(t_0) - d)\exp(\rho(t - t_0)) + d \quad (t_0 \leq t < t^*)$$

$$\mu_v[t] = \text{const} = d \quad (t_0 \leq t < t^*) .$$

In view of the second inequality in (2.15), $v^B(t)$ is decreasing on $[t_0, t^*)$, and

$$\dot{v}^B(t) = \rho(v^B(t) - d) \leq \dot{v}^B(t_0) < 0 \quad (t_0 \leq t < t^*) .$$

On the other hand

$$\lim_{t \to \infty} \dot{\mu}_n(t) = 0$$

(indeed, (2.12) implies the existence of the above limit, (2.2) shows that it is nonpositive, and (2.4) proves that in cannot be strictly negative). Hence, if $t^* = \infty$, then there exists a $t_1^* \geq t_0$ such that we have (2.15), (2.32) (see subsection 2.6) with t_0 replaced by t_1^*. As it is shown in subsection 2.6, in this case the lower asymptotics (2.37) takes place. Let

$$t^* < \infty .$$

Then either

$$v^B(t^*) = \mu_v[t^*] \tag{2.42}$$

or

$$v^B(t^*) = \mu_n[t^*] \; . \tag{2.43}$$

If (2.42) is satisfied, then we have case (2.14) with t_0 replaced by t^*; consequently, the conclusion of subsection 2.5 is true, that is, the asymptotics (2.26) takes place.

If (2.42) is untrue, (2.43) is satisfied, and we have

$$v^B(t^*) = \mu_n[t^*] < \mu_v[t^*] \; .$$

If

$$\dot{v}^B(t^*) \leq \dot{\mu}_n[t^*] \; ,$$

then we again have the case considered in subsection 2.6 (with t_0 replaced by t_*), and the lower asymptotics (2.37) takes place. If

$$\dot{v}^B(t^*) > \dot{\mu}_n[t^*] \; ,$$

then evidently

$$v^B(t) > \mu_n[t] \quad (t^* < t < t^* + \epsilon)$$

for a small positive ϵ, and we have case (2.16) with t_0 replaced by a $t_0^* \in (t^*, t^* + \epsilon)$. This case is considered in the next subsection.

2.8 Case 3 (between the barriers)

Let us consider case (2.16). With no loss of generality assume (2.31).

There are the only three possibilities, that is,

$$\mu_n[t] < v^B(t) < \mu_v[t] \quad (t_0 \leq t < t_1), \quad v^B(t_1) = \mu_v[t_1] \tag{2.44}$$

$$\mu_n[t] < v^B(t) < \mu_v[t] \quad (t_0 \leq t < t_1), \quad v^B(t_1) = \mu_n[t_1] \tag{2.45}$$

$$\mu_n[t] < v^B(t) < \mu_v[t] \quad (t \geq t_0) \; . \tag{2.46}$$

If (2.44) holds, then we have case (2.14) with t_0 replaced by t_1, and the conclusion of subsection 2.5 is true, that is, the asymptotics (2.26) takes place. If (2.45) holds, then obviously

$$\dot{v}^B(t_1) \leq \dot{\mu}_n[t_1]$$

and we have (2.15), (2.32) (see subsection 2.6) with t_0 replaced by t_1; by the statement of subsection 2.6 the asymptotics (2.37) takes place. In case of (2.46), where $v^B(t)$ stays in the corridor between the n^B- and v^B-barriers starting from $t = t_0$, we shall call $(n^A(t), n^B(t), v^B(t))$ an *intermediate trajectory.*

2.9 Summary: upper and lower asymptotics, and intermediate trajectories

Let us sum up the results obtained in this section. In subsections 2.3 – 2.5 it was stated that in case of (2.14) $(n^A(t), n^B(t), v^B(t))$ has the upper asymptotics (see Proposition 1.1). In subsection 2.8 it was stated that in case of (2.16) there are the only three outcomes, that is,

(i) $(n^A(t), n^B(t), v^B(t))$ has the lower asymptotics;

(ii) $(n^A(t), n^B(t), v^B(t))$ has the upper asymptotics;

(iii) $(n^A(t), n^B(t), v^B(t))$ is an intermediate trajectory.

The same three outcomes are the only admissible ones in case of (2.15); this was shown in subsection 2.6, 2.7 (in subsection 2.7 we refer to case (2.16) considered in subsection 2.8).

So far as for a solution $(n^A(t), n^B(t), v^B(t))$ one and only one of cases (2.14), (2.15), (2.16) may occur, one and only one of situations (i), (ii), (iii) takes place. Therefore, in order to complete the proof of Proposition 1.1, it is now sufficient to show that in situation (iii) $(n^A(t), n^B(t), v^B(t))$ has the intermediate asymptotics. In other words, one should prove Proposition 1.8. This is our goal in the next section.

Recall that Propositions 1.3 and 1.4 characterizing some of the solutions having the lower and upper asymptotics were proved, respectively, in subsection 2.6 and subsections 2.4, 2.5.

3 Intermediate Asymptotics

3.1 Introductory comments

As it was said in subsection 2.9, the goal of the present section is to complete the proof of Proposition 1.1 by proving Proposition 1.8. In fact only the sufficiency conjecture of this Proposition (stating that an intermediate trajectory has the intermediate asymptotics) should be justified. The necessity conjecture follows from what was said in subsection 2.9. Indeed, if a solution $(n^A(t), n^B(t), v^B(t))$ has the intermediate asymptotics, it has neither the lower, nor the upper asymptotics (it is easily seen from the characterization of these asymptotics given in Proposition 1.1); in other words, cases (i) and (ii) of subsection 2.9 cannot take place. Consequently (iii) takes place yielding that $(n^A(t), n^B(t), v^B(t))$ is an intermediate trajectory.

Thus, in this section $(n^A(t), n^B(t), v^B(t))$ is an intermediate trajectory, or, equivalently, (2.46) is satisfied. We shall prove that it has the intermediate asymptotics.

3.2 New variables

Along with variables $n^A(t)$, $n^B(t)$, $v^B(t)$ introduce

$$x(t) = n^B(t)v^B(t), \quad \xi(t) = n^A(t)v^B(t) \ . \tag{3.1}$$

Note that

$$r(t) = \frac{n^B(t)}{n^A(t)} = \frac{x(t)}{\xi(t)} \ . \tag{3.2}$$

Multiply (1.2) by $v^B(t)$ and (1.3) by $n^B(t)$ and add; we get an equation for $x(t)$:

$$\begin{aligned} \dot{x}(t) &= \dot{n}^B(t)v^B(t) + n^B(t)\dot{v}^B(t) \\ &= c^B n^B(t)v^B(t) + c^B\gamma(n^B(t))\delta n^A(t)v^B(t) - \alpha + \rho n^B(t)v^B(t) - 1 + \alpha \end{aligned} \tag{3.3}$$

or

$$\dot{x}(t) = (c^B + \rho)x(t) + c^B\gamma(n^B(t))\delta\xi(t) - 1 \ . \tag{3.4}$$

A transformation of (1.3) leads to an equation for $\xi(t)$:

$$\begin{aligned}\dot{\xi}(t) &= \dot{n}^A(t)v^B(t) + n^A(t)\dot{v}^B(t) \\ &= \bar{g}^A n^A(t)v^B(t) + \rho n^A(t)v^B(t) - \frac{1-\alpha}{n^B(t)v^B(t)} n^A(t)v^B(t)\end{aligned}$$

or

$$\dot{\xi}(t) = \left(\bar{g}^A + \rho - \frac{1-\alpha}{x(t)}\right)\xi(t) \ . \tag{3.5}$$

The criterion (2.46) for an intermediate trajectory is obviously equivalent to

$$x_n(t) < x(t) < x_v \quad (t \geq t_0) \tag{3.6}$$

where

$$x_v = \mu_v[t]n^B(t) = \frac{1-\alpha}{\rho n^B(t)} n^B(t) = \frac{1-\alpha}{\rho} \tag{3.7}$$

$$x_n(t) = \mu_n[t]n^B(t) = \frac{\alpha c^B n^B(t)}{n^B(t) + \gamma(n^B(t))\delta n^A(t)} \ . \tag{3.8}$$

So far as for $v^B(t) = \mu_n[t]$ we have $\dot{n}^B(t) = 0$, the differentiation of $x_n(t)$ provides

$$\dot{x}_n(t) = \dot{\mu}_n[t]n^B(t) + \mu_n[t]\dot{n}^B(t) = \dot{\mu}_n[t]n^B(t) < 0 \ . \tag{3.9}$$

(we have also used (2.2)). Note that since the left hand side of (3.6) is positive (see (3.8)), (3.1) implies that

$$v^B(t) > 0, \quad \xi(t) > 0 \quad (t \geq t_0) \ . \tag{3.10}$$

Come back to the characterization of the intermediate asymptotics given in Proposition 1.1. To complete the proof of Proposition 1.9, it is sufficient to establish the convergences

$$\lim_{t\to\infty} x(t) = x_\xi = \frac{1-\alpha}{\bar{g}^A + \rho} \tag{3.11}$$

$$\lim_{t\to\infty} r(t) = r_\infty = \frac{c^B\delta\gamma_\infty(1-\alpha)}{(\bar{g}^A + \rho) - (c^B + \rho)(1-\alpha)} \ ; \tag{3.12}$$

note that (3.11) and (3.16) imply the equality

$$\lim_{t\to\infty} v^B(t) = 0$$

mentioned in Proposition 1.1. Also observe that

$$x_v > x_\xi \ . \tag{3.13}$$

Thus, in what follows $(x(t), \xi(t))$ satisfies (3.1), (3.4), (3.5), (3.6). We shall also refer to the monotonicity conditions for $\xi(t)$ obviously following from (3.5) and the definition of x_ξ (1.26). Namely, we have

$$\dot{\xi}(t) > 0 \quad \text{if} \quad x(t) > x_\xi \tag{3.14}$$

$$\dot{\xi}(t) < 0 \quad \text{if} \quad x(t) < x_\xi \ . \tag{3.15}$$

3.3 Absolute asymptotics

In this subsection we state that

$$\lim_{t\to\infty} n^B(t) = \infty \tag{3.16}$$

thus proving the first convergence in the characterization of the intermediate asymptotics given in Proposition 1.1.

Suppose, to the contrary, that

$$\lim_{t\to\infty} n^B(t) = n_*^B < \infty \ . \tag{3.17}$$

Note that since $n^B(t)$ is nondecreasing (see (2.1)),

$$n^B(t) \le n_*^B \quad (t \ge t_0) \ .$$

Hence (see (1.21), (1.23))

$$\mu_v[t] = \frac{1-\alpha}{\rho n^B(t)} \ge \frac{1-\alpha}{\rho n_*^B} = \mu_{v*}. \tag{3.18}$$

Suppose that

$$v^B(t_*) < \mu_{v*} \tag{3.19}$$

for some $t_* \ge t_0$.

By (1.3) and (3.18)

$$\dot{v}^B[t] \le \rho(v^B(t) - \mu_{v*}) \ .$$

Consequently

$$v^B(t) \le v_*^B(t) \tag{3.20}$$

where

$$\dot{v}_*^B(t) = \rho(v_*^B(t) - \mu_{v*}), \quad v_*^B(t_0) = v^B(t_0) \ .$$

We have

$$v_*^B(t) = (v(t_0) - \mu_*)\exp(\rho(t-t_0) + \mu_* \ .$$

In view of (3.19)

$$\lim_{t\to\infty} v_*^B(t) = -\infty \ .$$

Hence and by (3.20)

$$\lim_{t\to\infty} v^B(t) = -\infty \ .$$

So far as $\mu_n[t]$ is bounded (see (2.4)), there is a $t_1 \ge t_0$ such that

$$v^B(t_1) < \mu_n[t_1]$$

which contradicts the assumption (2.46). The contradiction proves (3.16) under the condition that (3.19) holds for some $t_* \ge t_0$.

To complete the proof, it remains to show that the latter holds indeed. Assume the contrary, that is,

$$v^B(t) \ge \mu_{v*} \quad (t \ge t_0).$$

In view of (2.46) we have

$$\mu_v[t] \ge v^B(t) \ge \mu_{v*} \quad (t \ge t_0).$$

By (3.17)

$$\lim_{t\to\infty} \mu_v[t] = \mu_{v*} \ .$$

Consequently

$$\lim_{t\to\infty} v^B(t) = \mu_{v*} \ .$$

This together with (3.1) and the inequality $\mu_{v*} > 0$, imply that

$$\lim_{t\to\infty} \xi(t) = \infty \ .$$

Hence in view of (3.4) we get

$$\lim_{t\to\infty} x(t) = \infty \ .$$

On the other hand, (3.17) and (3.1) lead to

$$\lim_{t\to\infty} x(t) = n_*^B \mu_{v*} < \infty \ .$$

The obtained contradiction finalizes the proof of equality (3.16).

3.4 Specification of monotonicity conditions. 1

In this and the next two subsections we shall specify the monotonicity criteria (3.14), (3.15) for the variable $\xi(t)$. Namely we shall prove that $x(t)$ can cross the critical level x_ξ only once, and only from above.

Thus, in this and the next two subsection we assume that $x(t)$ crosses x_ξ. Denote by t_1 the smallest time t (no smaller than t_0) such that $x(t) = x_\xi$. So, we have

$$x(t_1) = x_\xi \ . \tag{3.21}$$

In this subsection we state an auxilliary property of $x(t)$. Namely, we assume

$$\dot{x}(t_1) \geq 0 \tag{3.22}$$

and prove that

$$x(t) \geq x(t_1) = x_\xi \quad (t \geq t_1) \tag{3.23}$$

and for some positive ϵ and α^* it holds that

$$x(t) \geq x(t_1) + \alpha^* = x_\xi + \alpha^* \quad (t \geq t_1 + \epsilon) \ . \tag{3.24}$$

Differentiating (3.4) at $t = t_1$, we get

$$\begin{aligned}\frac{d^2x(t_1)}{dt^2} &= (c^B + \rho)\dot{x}(t_1) \\ &+ c^B\delta\gamma'(n^B(t_1))\dot{n}^B(t_1)\xi(t_1) \\ &+ c^B\delta\gamma(n^B(t_1))\dot{\xi}(t_1) \ .\end{aligned}$$

The first term on the right is nonnegative in view of (3.22). The second term is positive, since $\gamma(n^B)$ is strictly increasing (subsection 1.1) and $\dot{n}^B(t_1) > 0$; the latter follows from (2.46) and (2.7) (with $t = t_1$). The third term is zero, since, as follows from (3.14), (3.15) and (3.21), we have $\dot{\xi}(t_1) = 0$. Thus

$$\frac{d^2x(t_1)}{dt^2} > 0 \ .$$

Consequently, by (3.22)

$$\dot{x}(t) > 0 \quad (t_1 < t \leq t_1 + \epsilon)$$

for a certain positive ϵ. This together with (3.21) imply

$$x(t) > x_\xi \quad (t_1 < t < t_1 + \epsilon) \ . \tag{3.25}$$

Let

$$0 < \alpha^* < x(t_1 + \epsilon) - x_\xi \ .$$

Taking into account (3.25), note that in order to prove (3.23) and (3.24), it is sufficient to show that

$$x(t) > x_\xi + \alpha^* \quad (t \geq t_1 + \epsilon) \ .$$

Suppose, to the contrary, that there is a $t_2 \geq t_1 + \epsilon$ such that

$$x(t_2) = x_\xi + \alpha^*$$

and obtain a contradiction. The supposition implies that there exists a $t_* \in [t_1, t_2]$ such that

$$x(t) > x_\xi = x(t_1) \quad (t_1 < t \leq t_*) \tag{3.26}$$

and

$$\dot{x}(t_*) < 0 \ .$$

Using (3.22) and (3.4) we deduce

$$\begin{aligned} 0 > \dot{x}(t_*) - \dot{x}(t_1) &= (c^B + \rho)(x(t_*) - x(t_1)) \\ &+ c^B \delta \gamma(n^B(t_*)) \xi(t_*) \\ &- c^B \delta \gamma(n^B(t_1)) \xi(t_1) \ . \end{aligned} \tag{3.27}$$

The first term on the right is positive in view of (3.26). Since $n^B(t)$ is increasing,

$$\gamma(n^B(t_*)) \geq \gamma(n^B(t_1)) \ .$$

By (3.26) and (3.14),

$$\xi(t_*) > \xi(t_1) \ .$$

Hence the difference of the last two terms on the right hand side of (3.27) is positive. Therefore the right hand side of (3.27) is positive, which is in fact not the case (see the left hand side of (3.27)). The contradiction completes the proof.

Copying the above argument, we establish the following more general statement. Let for an arbitrary (not necessarily intermediate) solution $(n^A(t), n^B(t), v^B(t))$, variables $x(t)$, $\xi(t)$ satisfy

$$x(t_1^*) \geq x_\xi$$

and (see (3.4))

$$\dot{x}(t_1^*) = (c^B + \rho)x(t_1^*) + c^B \gamma(n^B(t_1^*)) \delta \xi(t_1^*) - 1 \geq 0 \ .$$

Then for some positive ϵ and α^* it holds that

$$x(t) \geq x(t_1^*) + \alpha^* \quad (t \geq t_1^* + \epsilon) \ .$$

This statement will be used in subsection 4.6, 4.7.

3.5 Crossing the critical level from the right

In this subsection, we establish that in fact the inequality (3.22) does not hold. That means that $x(t)$ crosses the critical level x_ξ the first time only from above.

Assume, to the contrary, that (3.22) takes place. Prove that

$$\lim_{t\to\infty} x(t) = \infty \; ; \tag{3.28}$$

this contradicts the assumption (3.6) (the contradiction shows that (3.22) is not possible).

According to subsection 3.4 (3.22) implies (3.24). Hence by (3.5)

$$\dot{\xi}(t) \geq b\xi(t) \quad (t \geq t_1 + \epsilon)$$

where

$$b = \bar{g}^A + \rho - \frac{1-\alpha}{x_\xi + \alpha^*} \; ;$$

note that by the definition of x_ξ (see (1.26))

$$b > 0 \; .$$

Then we have

$$\xi(t) \geq \xi_*(t) \quad (t \geq t_1 + \epsilon)$$

where

$$\dot{\xi}_*(t) = b\xi_*(t), \quad \xi_*(t_1 + \epsilon) = \xi(t_1 + \epsilon) \; .$$

In accordance with (3.10), $\xi_*(t_1 + \epsilon) > 0$; consequently

$$\lim_{t\to\infty} \xi_*(t) = \infty \; .$$

Therefore

$$\lim_{t\to\infty} \xi(t) = \infty \; .$$

Hence, for all sufficiently large t, the right hand side of (3.4) is no smaller than $(c^B + \rho)x(t) + 1$. Then by the theorem of comparison

$$x(t) \geq x_*(t) \quad (t \geq t_*)$$

where t_* is sufficiently large, and

$$\dot{x}_*(t) = (c^B + \rho)x_*(t) + 1, \quad x_*(t_*) = x(t_*) \; .$$

So far as

$$\lim_{t\to\infty} x_*(t) = \infty$$

we have the desired equality (3.28). The proof is completed.

In a similar way, we prove the following more general statement: relations

$$t_2 \geq t_0, \quad x(t_2) \geq x_\xi, \quad \dot{x}(t_2) \geq 0$$

imply (3.28) and, consequently, are not possible for an intermediate trajectory $(n^A(t), n^B(t), v^B(t))$.

3.6 Specification of monotonicity conditions. 2

In the previous subsection we stated that (3.22) is not possible. Therefore we have

$$\dot{x}(t_1) < 0 \ . \tag{3.29}$$

Let us show that

$$x(t) < x_\xi \quad (t > t_1) \ . \tag{3.30}$$

Indeed, we evidently have either (3.30), or

$$x(t) < x_\xi \quad (t_1 < t < t_2), \quad x(t_2) = x_\xi \ .$$

The latter implies

$$\dot{x}(t_2) = \lim_{t \to t_2 - 0} \frac{x(t_2) - x(t)}{t_2 - t} \geq 0 \ .$$

This is not possible by the last statement of subsection 3.5. Thus (3.30) holds.

3.7 Classification of monotonicity intervals

The results of subsections 3.4 – 3.6 lead to a classification of monotonicity intervals for $\xi(t)$. Namely, one and only one of the following three outcomes is admissible:

$$x(t) > x_\xi \quad (t \geq t_0) \tag{3.31}$$

$$x(t) \geq x_\xi \quad (t_0 \leq t \leq t_1), \quad x(t) < x_\xi \quad (t > t_1) \tag{3.32}$$

$$x(t) < x_\xi \quad (t \geq t_0) \ . \tag{3.33}$$

Indeed let

$$x(t_0) \geq x_\xi \ .$$

Then we have either (3.31), or (3.29), where t_1 is the time of the first crossing of x_ξ by $x(t)$. In case of (3.29), due to the statement of subsection 3.6, (3.30) is fulfilled. The latter is evidently equivalent to (3.32). Let

$$x(t_0) < x_\xi \ .$$

Then we have either (3.33), or

$$x(t) < x_\xi \quad (t_0 \leq t < t_2), \quad x(t_2) = x_\xi \ .$$

If the latter holds, then

$$\dot{x}(t_2) = \lim_{t \to t_2 - 0} \frac{x(t_2) - x(t)}{t_2 - t} \geq 0 \ ;$$

this is not possible by the last statement of subsection 3.5. The proof is completed.

Combining the above classification with the monotonicity criteria (3.14), (3.15), we obtain the following:

in case of (3.31) : $\quad \dot{\xi}(t) > 0 \quad (t \geq t_0)$ (3.34)

in case of (3.32) : $\quad \dot{\xi}(t) > 0 \quad (t_0 \leq t < t_1), \quad \dot{\xi}(t) < 0 \quad (t > t_1)$ (3.35)

in case of (3.33) : $\quad \dot{\xi}(t) < 0 \quad (t \geq t_0) \ .$ (3.36)

3.8 ξ-Boundedness

Note that $\xi(t)$ is bounded, namely

$$0 < \xi(t) \leq a^* < \infty \quad (t \geq t_0) \ . \tag{3.37}$$

Indeed, from equation (3.5) for $\xi(t)$ follows

$$\xi(t) > 0 \ .$$

Hence, if (3.37) is violated, then

$$\lim_{t\to\infty} \xi(t) = \infty \ ;$$

the latter yields, like in subsection 3.5, the limit equality (3.28) which is not possible by (3.6). The contradiction proves (3.37).

3.9 x-Limit. 1

In this and the next two subsections we employ the previously obtained results for proving the equality (3.11) (see also (1.26)). As it was noted in the last paragraph of subsection 3.2, this equality provides one of the characterizations of the intermediate asymptotics.

According to the classification of subsection 3.7, one of cases (3.31), (3.32), (3.33) takes place. In this subsection we prove (3.11) for case (3.31). Suppose that (3.11) is violated. Then (3.31) implies that

$$x(t_i) > x_\xi + \beta, \quad \beta > 0, \quad t_i \to \infty \ . \tag{3.38}$$

Equation (3.4), assumption (3.31) and the lower bound in (3.37) yield that $\dot{x}(t)$ is bounded from below, that is,

$$\dot{x}(t) \geq b = (c^B + \rho)x_\xi - 1 \quad (t \geq t_0) \ .$$

This and (3.38) imply that

$$x(t) \geq x_\xi + \beta/2 \quad (t_i \leq t \leq t_i + \epsilon)$$

for a sufficiently small positive ϵ. Then by equation (3.5)

$$\dot{\xi}(t) = \left(c^A + \rho - \frac{1-\alpha}{x(t)} \right) \xi(t) > l\xi(t) \quad (t_0 \leq t \leq t_i + \epsilon)$$

where

$$l = c^A + \rho - \frac{1-\alpha}{x_\xi + \beta/2} > 0$$

(the inequality follows from the expression (1.26) for x_ξ). So far as by (3.34) $\xi(t)$ is increasing, we have

$$\begin{aligned} \dot{\xi}(t) &\geq 0 \quad (t \notin [t_i, t_i + \epsilon]) \\ \dot{\xi}(t) &\geq l\xi(t_0) = \text{const} > 0 \quad (t \in [t_i, t_i + \epsilon]) \ . \end{aligned}$$

Hence

$$\lim_{t\to\infty} \xi(t) = \infty$$

which is not possible due to (3.37). The contradiction proves (3.11).

3.10 x-Limit. 2

Let one of cases (3.32), (3.33) take place. To justify (3.11) (see the next subsection) we need the following statement:

$$\xi(t) \geq a_* > 0 \quad (t \geq t_0) . \tag{3.39}$$

Prove this statement. Suppose that (3.39) is untrue. Then by (3.35), (3.36)

$$\lim_{t\to\infty} \xi(t) = 0 . \tag{3.40}$$

We shall show that the latter yields that

$$\lim_{t\to\infty} x(t) = \frac{1}{c^B + \rho} . \tag{3.41}$$

Note that the right hand side of (3.41) is greater than x_ξ (1.26); this follows from (1.12). Therefore (3.41) contradicts (3.32) and (3.33). The contradiction proves (3.39).

Thus, in the rest of this subsection we concentrate on stating (3.41). Suppose that (3.41) is violated. Let, first,

$$x(t_i) > \frac{1}{c^B + \rho} + \sigma, \quad \sigma > 0, \quad t_i \to \infty . \tag{3.42}$$

Then by (3.4) and (3.37)

$$\begin{aligned} x(t) &\geq \exp((c^B + \rho)(t - t_i))x(t_i) - \int_{t_i}^{t} \exp((c^B + \rho)(t - \tau))d\tau \\ &= \exp((c^B + \rho)(t - t_i)) \left(x(t_i) - \frac{1}{c^B + \rho} \right) + \frac{1}{c^B + \rho} . \end{aligned}$$

Referring to (3.42), we see that

$$\lim_{t\to\infty} x(t) = \infty$$

which contradicts (3.6). Thus (3.42) is not possible. Suppose that

$$x(t_i) < \frac{1}{c^B + \rho} - \sigma, \quad \sigma > 0, \quad t_i \to \infty . \tag{3.43}$$

Due to (3.40), (3.37), for a large i, we have

$$\kappa(t) = c^B \delta\gamma(n^B(t))\xi(t) < \sigma/2 \quad (t \geq t_i) .$$

Fix such i. Then by (3.4)

$$\begin{aligned} x(t) &= \exp((c^B + \rho)(t - t_i))x(t_i) - \\ &\quad \int_{t_i}^{t} \exp((c^B + \rho)(t - \tau)d\tau + \int_{t_i}^{t} \exp((c^B + \rho)(t - \tau)\kappa(\tau)d\tau \\ &\leq \exp((c^B + \rho)(t - t_i))x(t_i) + \int_{t_i}^{t} \exp(c^B + \rho)(t - \tau)(-1 + \sigma/2)d\tau \\ &= \exp(c^B + \rho)(t - t_i) \left(x(t_i) - \frac{1 - \sigma/2}{c^B + \rho} \right) - \frac{1}{c^B + \rho} \end{aligned}$$

The expression in brackets is, as follows from (3.43), no greater than $-(\sigma/2)(c^B + \rho)$. Consequently,

$$\lim_{t\to\infty} x(t) = -\infty$$

contradicting to (3.6). The contradiction shows that (3.43) is not possible. Convergence (3.41) is stated, and the proof of (3.39) is completed.

3.11 x-Limit. 3

Prove (3.11) for cases (3.32) and (3.33). Suppose that (3.11) is untrue. Then (3.32) and (3.33) imply that

$$x(t_i) < x_\xi - \beta, \quad \beta > 0, \quad t_i \to \infty \ . \tag{3.44}$$

Using (3.4), (3.6) and (3.37), we get

$$\dot{x}(t) \leq b = (c^B + \rho)x_v + c^B \delta \gamma_\infty a^* - 1 \ .$$

Hence, in view of (3.44),

$$x(t) \leq x_\xi - \beta/2 \quad (t_i \leq t \leq t_i + \epsilon)$$

for a certain positive ϵ. Then by (3.5)

$$\dot{\xi}(t) = \left(\bar{g}^A + \rho - \frac{1-\alpha}{x(t}\right)\xi(t) < l\xi(t) \quad (t_i \leq t \leq t_i + \epsilon)$$

where

$$l = \bar{g}^A + \rho - \frac{1-\alpha}{x_\xi - \beta/2} < 0 \ ;$$

the inequality is seen from the expression (1.26) for x_ξ. Consequently,

$$\xi(t_i + \epsilon) < q\xi(t_i)$$

where

$$q = \exp(-l\epsilon) < 1 \ .$$

Hence fixing k, we get

$$\xi(t_i + \epsilon) < q^{i-k}\xi(t_k) \to 0 \ .$$

We have obtained a contradiction to property (3.39) stated in the previous subsection. The proof of (3.11) is completed.

3.12 ξ-Limit

In this subsection we make use of (3.11) for stating that

$$\lim_{t\to\infty} \xi(t) = \xi_\infty = \frac{1 - (c^B + \rho)x_\xi}{c^B \delta \gamma_\infty} \ . \tag{3.45}$$

Note that the above expression for ξ_∞ is derived by formal letting the right hand side of equation (3.4) for $x(t)$ be equal to zero "at infinity", that is,

$$(c^B + \rho)x_\xi + c^B \delta \gamma_\infty \xi_\infty - 1 = 0 \ .$$

Substituting the expression (1.26) for x_ξ, we specify ξ_∞ as

$$\xi_\infty = \frac{(\bar{g}^A + \rho) - (c^B + \rho)(1 - \alpha)}{(\bar{g}^A + \rho)c^B \delta \gamma_\infty} \ . \tag{3.46}$$

Prove (3.45). According to subsection 3.7 (see (3.34) – (3.36)), $\xi(t)$ is monotonical at large times. Hence there is the limit

$$\xi_\infty = \lim_{t\to\infty} \xi(t) \ .$$

By (3.37)

$$0 \le \xi_\infty < \infty .$$

Combining this with (3.11), observing equation (3.4) for $x(t)$, and taking into account convergence (3.16), we conclude that

$$\lim_{t\to\infty} \dot{x}(t) = (c^B + \rho)x_\xi + c^B\delta\gamma_\infty\xi_\infty - 1 = \dot{x}_\infty .$$

If $\dot{x}_\infty > \epsilon > 0$, then

$$\dot{x}(t) > \epsilon/2$$

for all sufficiently large t; consequently

$$\lim_{t\to\infty} x(t) = \infty$$

which contradicts (3.6). Similiarly, we arrive at a contradiction assuming $\dot{x}_\infty < 0$. Therefore

$$\dot{x}_\infty = 0$$

yielding (3.45).

3.13 Intermediate asymptotics for intermediate trajectory

Limits (3.11) and (3.45) provide the desired asymptotics of ratio $r(t)$ (1.6) for an intermediate trajectory $(n^A(t), n^B(t), v^B(t))$ (characterized by (2.46)). Namely, we have (3.12). Indeed, using, sequentially, (1.6), (3.2), (3.11) and (3.45), we obtain:

$$\begin{aligned}\lim_{t\to\infty} r(t) &= \lim_{t\to\infty} \frac{n^B(t)}{n^A(t)} = \lim_{t\to\infty} \frac{x(t)}{\xi(t)} = \frac{x_\xi}{\xi_\infty} \\ &= \frac{\frac{1-\alpha}{\bar{g}^A+\rho}}{\frac{1-(c^B+\rho)\frac{1-\alpha}{\bar{g}^A+\rho}}{c^B\delta\gamma_\infty}} \\ &= \frac{c^B\delta\gamma_\infty(1-\alpha)}{(\bar{g}^A+\rho)-(c^B+\rho)(1-\alpha)} .\end{aligned}$$

Proposition 1.8 is proved. In accordance with the summary given in subsection 2.9, the proof of Proposition 1.1 is completed.

4 Existence of Intermediate Trajectories

4.1 Introductory comments

In this section we shall prove Proposition 1.9 (on the feasibility of the intermediate asymptotics subject to condition (1.13)); this will prove Proposition 1.7, too.

Like in section 3, we use variables $x(t)$, $\xi(t)$ (3.1). Recall that in subsection 3.2 it was noted that the fact that a solution $(n^A(t), n^B(t), v^B(t))$ to (1.1) – (1.3) is an intermediate trajectory is equivalent to inequalities (3.6). Therefore, our goal is to prove that under conditions of Proposition 1.9 there exists a solution $(n^A(t), n^B(t), v^B(t))$ (whose initial state is specified in this Proposition) satisfying (3.6).

Note that equation (3.4) for $x(t)$ and equation (3.5) for $\xi(t)$, as well as the monotonicity criteria (3.14), (3.15) remain true provided $x(t) > x_n(t)$ (recall the notation $[\cdot]_+$ in (1.2)). We shall use these facts in our analysis.

In this section the inequality (1.13) is assumed to be satisfied. Note that this inequality yields that

$$x_\xi > x_n(t) \tag{4.1}$$

(see (1.26), (3.8)).

4.2 A nonexistence domain

We start with specifying positions

$$x_0 = x(0), \quad \xi_0 = \xi(0)$$

preventing $x(t)$ to satisfy (3.6). In this subsection we point out pairs (x_0, ξ_0) such that corresponding $x(t)$ breaks (3.6) by crossing $x_n(t)$. Namely, we shall state now that if

$$t^A = 0 \tag{4.2}$$

$$x_0 \leq x_\xi \tag{4.3}$$

and

$$(c^B + \rho)x_0 + c^B \delta \gamma_\infty \xi_0 - 1 < 0 \ , \tag{4.4}$$

then

$$\dot{x}(t) \leq -\beta, \quad \beta > 0 \quad (0 \leq t < t_{1,n}) \tag{4.5}$$

$$x(t) < x_n(t) \quad (t \geq t_{1,n}) \tag{4.6}$$

where

$$t_{1,n} = \inf\{\tau \geq 0 \ : \ x(t) \leq x_n(t)\} \ .$$

Let us assume (4.3), (4.4), and prove (4.5). By (4.4) and (3.4) we have

$$\dot{x}(0) < 0 \ .$$

Hence for certain positive β_0 and ϵ

$$\dot{x}(t) < \beta_0 \quad (0 \leq t \leq \epsilon) \ . \tag{4.7}$$

Due to (4.3)

$$x(\epsilon) < x_\xi - \sigma, \quad \sigma = \beta_0 \epsilon \ . \tag{4.8}$$

Referring to (3.15), we conclude that

$$\xi(\epsilon) < \xi_0 - \kappa \tag{4.9}$$

for a certain positive κ. Let

$$t^* = \sup\{ \ \tau \geq \epsilon \ : \ \xi(t) \leq \xi_0 - \kappa/2 \ (\epsilon \leq t \leq \tau) \ \} \ .$$

Then by (3.4)

$$\dot{x}(t) \leq bx(t) + \sigma \quad (\epsilon \leq t < t^*) \tag{4.10}$$

where

$$b = c^B + \rho$$

$$\sigma = \sup\{ \ c^B \delta \gamma(n^B(t)) \xi(t) - 1 \ : \ \epsilon \leq t \leq t^* \ \} \ .$$

By (4.4), (4.8), and the definition of t^* we have

$$\begin{aligned} bx(\epsilon)+\sigma &\leq (c^B+\rho)x_0 + c^B\delta\gamma_\infty(\xi_0-\kappa/2) \\ &\leq -c^B\delta\gamma_\infty\kappa/2 \\ &= -\beta_1 < 0 \; . \end{aligned} \tag{4.11}$$

From (4.10) follows

$$\begin{aligned} x(t) &\leq \exp(b(t-\epsilon))x(\epsilon) + \int_\epsilon^t \exp(b(t-\tau))\sigma d\tau \\ &= \exp(b(t-\epsilon))\left(x(\epsilon)+\frac{\sigma}{b}\right) - \frac{\sigma}{b} \quad (\epsilon \leq t < t^*) \; . \end{aligned}$$

Using again (4.10), and taking into account (4.11), we get

$$\dot{x}(t) \leq \exp(b(t-\epsilon))(bx(\epsilon)+\sigma) < -\beta_1 \quad (\epsilon \leq t < t^*) \; .$$

This in combination with (4.7) gives

$$\dot{x}(t) \leq -\beta, \quad (0 \leq t < t^*) \tag{4.12}$$

where

$$\beta = \min\{\beta_0, \beta_1\} \; .$$

Thus, in order to reach (4.5), it is sufficient to show that

$$t^* = \infty \; .$$

Suppose, to the contrary, that

$$t^* < \infty \; .$$

Then, as follows from the definition of t^*, for an arbitrary $\nu > 0$ there is a $t_* \in [t^*, t^*+\nu]$ such that

$$\dot{\xi}(t_*) > 0 \; . \tag{4.13}$$

For ν sufficiently small (4.3) and (4.12) imply

$$x(t_*) < x_\xi \; .$$

By (3.15) this contradicts (4.13). The contradiction completes the proof of (4.5).

Let us prove (4.6). Obviously, we have

$$\dot{x}(t_{1,n}) \leq \dot{x}_n(t_{1,n}).$$

This, due to the fact that $\dot{n}^B(t_{1,n}) = 0$, implies that

$$\dot{v}^B(t_{1,n}) \leq \dot{\mu}_v(t_{1,n}).$$

Now recall (4.2) and make use of the statement of subsection 2.6 putting $t_0 = t_{1,n}$. Thus we straightforwardly obtain (4.6).

4.3 Lower bound trajectory

Fix a $\xi_0 > 0$ satisfying

$$(c^B + \rho)x_\xi + c^B \delta \gamma_\infty \xi_0 - 1 < 0 \tag{4.14}$$

(i.e. (4.4) with $x_0 = x_\xi$). Consider a solution

$$(n^A(t), n^B(t), v^B(t)) = (n^{A-}(t), n^{B-}(t), v^{B-}(t))$$

to (1.1) – (1.3) such that $x(t)$, $\xi(t)$ defined by (3.1) satisfy

$$x(0) = x_\xi, \quad \xi(0) = \xi_0 \ .$$

For these variables we use the notations

$$x(t) = x^-(t), \quad \xi(t) = \xi^-(t) \ .$$

By $x_n^-(t)$ we denote $x_n(t)$ (see (3.8)) corresponding to the above solution $(n^A(t), n^B(t), v^B(t))$. Assumption (4.14) means that conditions (4.3) and (4.4) of the previous subsection are satisfied. Therefore (4.5), (4.6) hold; using the above notations rewrite these properties as

$$x^-(t) \quad \text{crosses } x_\xi \text{ and } x_n^-(t) \tag{4.15}$$

$$x^-(t) \le x_\xi \quad \text{after crossing } x_\xi \text{ and before crossing } x_n(t)$$

$$x_n^-(t) \le x_n(t) \quad \text{after crossing } x_n(t) \ .$$

The last two properties and (1.27) imply that

$$x^-(t) \le x_\xi \quad \text{after crossing } x_\xi \ . \tag{4.16}$$

In this section we shall call $x^-(t)$ the *lower bound trajectory* (for intermediate trajectories).

4.4 Specification of parameters

Specify

$$v_0^{B-} = v^{B-}(0)$$

so as to ensure

$$t^A = 0$$

(see (1.24)) for all solutions $(n^A(t), n^B(t), v^B(t))$ with initial states

$$n^A(0) = n_0^A = \frac{\xi_0}{v_0^{B-}}, \quad n^B(0) = n_0^B = n_0^B[x_0] = \frac{x_0}{v_0^{B-}}, \quad v^B(0) = v_0^{B-} \tag{4.17}$$

where x_0 varies in the interval $[x_\xi, x_v]$. More precisely, assume v_0^{B-} to satisfy

$$4\delta \bar{g}^A \gamma(n_0^B[x_0]) n_0^A > 1 \quad (x_0 \in [x_\xi, x_v]) \tag{4.18}$$

(recall that $x_v > x_\xi$; see (3.13)). This property will allow us to apply the result of subsection 2.6 to a solution with a particular initial state (4.17).

As it was noticed in subsection 1.3, property (4.18) takes place for a sufficiently small v_0^{B-}, since, as follows from (4.17), we have

$$\gamma(n_0^B[x_0]) = \gamma\left(\frac{x_0}{v_0^{B-}}\right) \ge \gamma\left(\frac{x_\xi}{v_0^{B-}}\right) \to \gamma_\infty > 0 \quad \text{as} \quad v_0^B \to 0$$

$$n_0^A = \frac{\xi_0}{v_0^{B-}} \to \infty \quad \text{as} \quad v_0^B \to 0 \ .$$

4.5 Definition of a trajectory

In this subsection we define a particular solution $(n^A(t), n^B(t), v^B(t))$; in the next subsections it will be proved that this solution is an intermediate trajectory.

Come back to solution $(n^{A-}(t), n^{B-}(t), v^{B-}(t))$, defined in subsection 4.3. In accordance with notations (3.1), we have

$$x^-(0) = x_\xi = n^{B-}(0)v^{B-}(0), \quad \xi_-(0) = \xi_0 = n^{A-}(0)v^{B-}(0) .$$

For every

$$x_0 \in [x_\xi, x_v]$$

there exists the single

$$n_0^B = n_0^B[x_0] \in [x_\xi/v_0^{B-}, x_v/v_0^{B-}]$$

such that

$$x_0 = n_0^B[x_0]v^{B-}(0) .$$

Solution $(n^A(t), n^B(t), v^B(t))$ to (1.1) - (1.3) determined by the initial conditions (4.17) will be said to *correspond to* x_0; making it explicit that variables $x(t)$, $\xi(t)$ (3.1) determined by the above solution depend on x_0, we shall write

$$x(t) = x(t \mid x_0), \quad \xi(t) = \xi(t \mid x_0) .$$

Obviously

$$x(t \mid x_\xi) = x^-(t) . \tag{4.19}$$

Note also that

$$\dot{x}(0 \mid x_v) > 0 . \tag{4.20}$$

Indeed, for the solution $(n^A(t), n^B(t), v^B(t))$ corresponding to x_v by (3.3) we have

$$\dot{x}(0) = \dot{n}^B(0)v^B(0) + n^B(0)\dot{v}^B(0)$$

where $x(t) = x(t \mid x_v)$. By (2.7), (1.27)

$$\dot{n}^B(0) > 0 ;$$

(3.7), (3.1) imply

$$v^B(0) = \mu_v[t_0] ;$$

hence by (2.10)

$$\dot{v}^B(0) = 0 .$$

These relations yield (4.20).

According to (4.15), (4.16)

$$x^-(t) = x(t \mid x_\xi) \quad \text{crosses } x_\xi \text{ and } x_n^-(t) \tag{4.21}$$

$$x^-(t) = x(t \mid x_\xi) \le x_\xi \text{ after crossing } x_\xi . \tag{4.22}$$

We supplement these conditions by

$$\max\{\dot{x}^-(t) \ : \ 0 \le t \le t_\xi^-\} < 0; \quad t_\xi^- \text{ is the time of first crossing } x_\xi . \tag{4.23}$$

The latter holds automatically by the statement of subsection 4.2, as long as $t_\xi^- = 0$.

Introduce the set X of all $x_0 \in [x_\xi, x_v]$ such that

$$x(t) = x(t \mid x_0)$$

satisfyies the similar conditions, that is,

$$x(t) \text{ crosses } x_\xi \text{ and } x_n(t) \tag{4.24}$$

$$x(t) \leq x_\xi \text{ after crossing } x_\xi \tag{4.25}$$

$$\max\{\dot{x}(t) \ : \ 0 \leq t \leq t_\xi\} < 0; \quad t_\xi \text{ is the time of first crossing } x_\xi \ . \tag{4.26}$$

¿From the definition (4.19) of $x^-(t)$ and (4.21), (4.22), (4.23) follows

$$x_\xi \in X \ ;$$

From (4.20) we get

$$x_v \notin X \ .$$

Therefore for

$$x_* = \sup X \tag{4.27}$$

we have

$$x_* \in [x_\xi, x_v) \ . \tag{4.28}$$

From now on, let solution $(n^A(t), n^B(t), v^B(t))$ correspond to to x_*. In what follows

$$x(t) = x(t \mid x_*) \ . \tag{4.29}$$

In the next subsections we shall show that $(n^A(t), n^B(t), v^B(t))$ is an intermediate trajectory, or, equivalently (see subsection 3.2), $x(t)$ satisfies inequalities (3.6) with $t_0 = 0$. This will prove Proposition 1.9.

4.6 Monotonicity before crossing x_ξ

The proof comprises several statements. In this subsection the following is established: either

$$x(t) \text{ is strictly decreasing on } [0, \infty) \text{ and does not cross } x_\xi \ , \tag{4.30}$$

or $x(t)$ satisies (4.26).

Assume that the statement is untrue. Then there is a $t_1 \geq 0$ such that

$$x(t_1) \geq x_\xi \tag{4.31}$$

$$\dot{x}(t_1) \geq 0 \ .$$

Then by the last statement of subsection 3.4

$$x(t_1 + \epsilon) \geq x(t_1) + \alpha^* \tag{4.32}$$

for some positive ϵ, α^*. By the definition of x_* (see (4.27)) there exists a sequence

$$x_{0i} \to x_* - 0, \quad x_{0i} \in X \ . \tag{4.33}$$

Hence

$$x_i(t) = x(t \mid x_{0i}) \to x(t) \tag{4.34}$$

uniformly on $[0, t_1 + \epsilon]$. Take i so large that

$$x_i(t_1) \leq x(t_1) + \alpha^*/2 \ .$$

As follows from the definition of X (see (4.25) and (4.26) where $x(t)$ is replaced by $x_i(t)$), it holds that

$$\begin{aligned} x_i(t_1+\epsilon) &\leq \max\{x_i(t_1), x_\xi\} \\ &\leq \max\{x(t_1)+\alpha^*/2, x_\xi\} \\ &= x(t_1)+\alpha^*/2 \\ &\leq x(t_1+\epsilon)-\alpha^*/4 \end{aligned}$$

(we have also utilized (4.31), (4.32)); this contradicts to (4.34). The contradiction completes the proof.

If (4.30) takes place, then in accordance with (4.29), (4.28),

$$x_\xi < x(t) < x_v$$

and in view of (1.27), we have the desired inequality (3.6) (with $t_0 = 0$). Thus, in what follows we shall study the case where $x(t)$ crosses x_ξ, and (4.26) is satisfied.

4.7 Staying under x_ξ after crossing

Let us show that if $x(t_\xi) = x_\xi$ ($t_\xi \geq 0$), then

$$x(t) < x_\xi \quad (t > t_\xi) \ . \tag{4.35}$$

By the last assumption of the previous subsection, (4.26) is fulfilled. Hence

$$x(\nu) < x_\xi \tag{4.36}$$

for all $\nu > t_\xi$ sufficiently close to t_ξ. Take a ν satisfying

$$t_\xi < \nu < t_\xi + \epsilon$$

with an arbitrarily small ϵ. Define $x_i(t)$ like in the previous subsection (see (4.33), (4.34)). Note that convergence (4.34) is uniform at every bounded interval. In view of (4.36), for large i

$$x_i(\nu) < x_\xi \ ;$$

Then by the definition of X (replace in (4.25) $x(t)$ by $x_i(t)$)

$$x_i(t) \leq x_\xi \quad (t \geq \nu) \ .$$

This together with (4.34) give

$$x(t) \leq x_\xi \quad (t \geq \nu) \ .$$

Since ν is arbitrarily close to t_ξ, we conclude that

$$x(t) \leq x_\xi \quad (t \geq t_\xi) \ . \tag{4.37}$$

Now to prove (4.35) it is sufficient to verify that there does not exist a $t > t_\xi$ such that

$$x(t) = x_\xi \ .$$

Suppose that such a t exists; let t_1 be the minimum of all such t. We have

$$x(t) < x_\xi \quad (t_\xi < t < t_1)$$

$$x(t_1) = x_\xi \ .$$

Hence, as one can easily see,

$$\dot{x}(t_1) \geq 0 \ .$$

Observing the last two relations and exploiting the last statement of subsection 3.4, we derive that

$$x(t) > x_\xi + \alpha^* \quad (t \geq t_1 + \epsilon)$$

for some positive ϵ, α^*. This contradicts to (4.37). The proof is completed by contradiction.

4.8 Staying under $x_n(t)$ after crossing

State the following. If $x(t)$ crosses $x_n(t)$ at time $t_n \geq 0$, then

$$x(t) < x_n(t) \quad (t > t_n) \ . \tag{4.38}$$

So far as

$$x(t) > x_n(t) \quad (0 \leq t < t_n), \quad x(t_n) = x_n(t_n) \ ,$$

we have by (3.8), (3.1)

$$v^B(t) > \mu_n[t] \quad (0 \leq t < t_n), \quad v^B(t_n) = \mu_n[t_n] \ .$$

Hence conditions (2.15), (2.32) of subsection 2.6 are satisfied for $t_0 = t_n$ (recall that in subsection 4.4 we guaranteed that $t^A = 0$). By the statement of subsection 2.6 inequality (2.34) holds with $t_0 = t_n$. Multiplying it by $n^B(t)$, we get (4.38) (see (3.1), (3.8)).

4.9 Staying between barriers

This subsection finalizes the proof. Namely, we shall show that inequalities (3.6) (with $t_0 = 0$) hold.

Suppose, to the contrary, that (3.6) ($t_0 = 0$) is violated. Property (4.26) (subsection 4.6), the statement of subsection 4.7, and the inequality $x(0) = x_* < x_v$ (4.28) imply

$$x(t) < x_v \quad (t \geq 0) \ .$$

Then as follows from the relations

$$x(0) = x_* > x_\xi > x_n(0)$$

(see (4.28) and (4.1)) we have

$$x(t_*) = x_n(t_*)$$

for a certain $t_* \geq 0$. Obtain a contradiction. Namely, state that

$$x_* + \epsilon \in X \tag{4.39}$$

for all sufficiently small positive ϵ. This indeed contradicts to the definition of x_* (see (4.27)). Let

$$x_\epsilon(t) = x(t \mid x_* + \epsilon), \quad \epsilon > 0 \ .$$

From (4.26) follows that for small ϵ, $x_\epsilon(t)$ crosses x_ξ, and

$$\max\{\dot{x}_\epsilon(t) \ : \ 0 \leq t \leq t_{\xi,\epsilon}\} < 0; \quad t_{\xi,\epsilon} \text{ is the time of first crossing } x_\xi \ . \tag{4.40}$$

The statement of subsection 4.8 yields that

$$x(t_n + 1) < x_n(t_n + 1)$$

(recall that $x(t)$ crosses $x_n(t)$ at t_n). Hence for small ϵ, we have

$$x_\epsilon(t_n + 1) < x_{n,\epsilon}(t_n + 1) \ ;$$

here and in what follows $x_{n,\epsilon}(t)$ is the analogue of $x_n(t)$ for the solution $(n_\epsilon^A(t), n_\epsilon^B(t), v_\epsilon^B(t))$ corresponding to $x_* + \epsilon$, that is, $x_{n,\epsilon}(t)$ is given by (3.8) where $n^A(t)$, $n^B(t)$ are replaced by $n_\epsilon^A(t)$, $n_\epsilon^B(t)$. Thus $x_\epsilon(t)$ crosses $x_{n,\epsilon}(t)$, and, consequently, x_ξ:

$$x_\epsilon(t) \text{ crosses } x_\xi \text{ and } x_{n,\epsilon}(t) \ . \tag{4.41}$$

Moreover, for the time $t_{n,\epsilon}$ of the first crossing with $x_{n,\epsilon}(t)$, we obviously have

$$x_\epsilon(t_{n,\epsilon}) = x_{n,\epsilon}(t_{n,\epsilon}), \quad t_{n,\epsilon} \leq t_n + 1 \ . \tag{4.42}$$

From (4.40), (4.26) follows that for small ϵ, $t_{\xi,\epsilon}$ is arbitrarily close to t_ξ, and

$$\dot{x}_\epsilon(t) \leq -\beta < 0, \quad \dot{x}(t) \leq -\beta \quad (\mid t - t_\xi \mid \leq \sigma)$$

where $\beta > 0$ and $\sigma > 0$ do not depend on ϵ. Combining this with (4.35), (4.42), we obtain

$$x_\epsilon(t) < x_\xi \quad (t_{\xi,\epsilon} < t \leq t_{n,\epsilon}) \tag{4.43}$$

for small ϵ. From here we deduce, like in subsection 4.8 (replacing $x(t)$ by $x_\epsilon(t)$), that

$$x_\epsilon(t) < x_{n,\epsilon}(t) \quad (t > t_{n,\epsilon}) \ .$$

This together with (4.43) yield that

$$x_\epsilon(t) \leq x_\xi \text{ after crossing } x_\xi \ . \tag{4.44}$$

We see that (4.41), (4.44), (4.40) prove (4.39). The desired contradiction is obtained. The inequalities (3.6) characterizing an intermediate trajectory are proved.

5 Existence of Catching Up and Overtaking Intermediate Trajectories

5.1 Introductory comments

In this section we justify Propositions 1.7 and 1.10, which provide conditions sufficient for a solution $(n^A(t), n^B(t), v^B(t))$ to have the intermediate asymptotics (that is, to satisfy (3.6)) and, simultaneously, be catching up or overtaking. Recall that in subsection 1.3 we called a solution catching up if the corresponding ratio $r(t)$ is strictly increasing, and overtaking if it is catching up and $r(t)$ grows from $r(0) < 1$ to $r_\infty > 1$. As it was noted in subsection 1.3, Proposition 1.10 implies Proposition 1.7. Therefore we shall prove Proposition 1.10 only.

Principally, we follow the logic of section 4. However the requirement that $r(t)$ increases implies some modifications. The main modification consists in setting (in subsection 5.5) the initial value x_* for the variable $x(t)$ corresponding to an intermediate trajectory to be not greater (like in subsection 4.5) but smaller than the critical value x_ξ. This analytic pattern requires Condition 1.1 (not needed in section 4), which claims that $x(t)$ cannot be constant at any interval.

Thus, in this section the condtions of Proposition 1.10 are assumed to hold. Namely we assume inequalities (1.13), (1.18) and (everywhere with the exception of subsection 5.9) Condition 1.1. Using notations (3.1) we rewrite this Condition in the following equivalent form: for every time interval $[t_1, t_2]$ of nonzero length such that solution $(n^A(t), n^B(t), v^B(t))$ is such that $x(t)$ (3.1) satisfies $x_n(t) < x(t) < x_v$ for $t \in [t_1, t_2]$, the function $x(t)$ cannot be constant on $[t_1, t_2]$. We shall utilize this in subsection 5.6. In subsection 5.9 we shall show that this condition is fulfilled if Condition 1.2 holds (thus proving Proposition 1.6).

5.2 A nonexistence domain

We start with specifying positions

$$x_0 = x(0), \quad \xi_0 = \xi(0)$$

preventing $x(t)$ to satisfy (3.6) with $t_0 = 0$. In this subsection we point out pairs (x_0, ξ_0) such that the corresponding $x(t)$ breaks (3.6) by crossing x_v. Namely, we shall state now that if

$$x_0 \geq x_\xi \tag{5.1}$$

and

$$(c^B + \rho)x_0 + c^B \delta \gamma(n_0^B)\xi_0 - 1 > 0 \ , \tag{5.2}$$

then

$$\dot{x}(t) \geq \beta, \quad \beta > 0 \quad (0 \leq t < t_{1,v}) \tag{5.3}$$

$$x(t) > x_v \quad (t \geq t_{1,v}) \tag{5.4}$$

where

$$t_{1,v} = \inf\{\tau \geq 0 \ : \ x(t) \geq x_v\} \ .$$

Note that (5.3) implies obviously

$$t_{1,v} < \infty \ .$$

As long as

$$x(t_{1,v}) = x_v$$

or, equivalently, (see (3.7), (3.1))

$$v^B(t_{1,v}) = \mu_v \ ,$$

we have the case considered in subsections 2.3, 2.4 (with t_0 replaced by $t_{1,v}$). As it was proved in subsection 2.3, (2.19), (2.20) hold. Hence $x(t)$ crosses x_v. Consequently (3.6) is violated.

Let us assume (5.1), (5.2) and prove (5.3). By (5.2) and (3.4) we have

$$\dot{x}(0) > 0 \ .$$

Hence for some positive β_0 and ϵ

$$\dot{x}(t) > \beta_0 \quad (0 \le t \le \epsilon) \ . \tag{5.5}$$

In accordance with (5.1)

$$x(\epsilon) > x_\xi + \epsilon_1, \quad \epsilon_1 = \beta_0 \epsilon \ . \tag{5.6}$$

Referring to (3.14), we conclude that

$$\xi(\epsilon) > \xi_0 + \kappa \tag{5.7}$$

for a certain positive κ. Let

$$t^* = \sup\{ \ \tau \ge \epsilon \ : \ \xi(t) \ge \xi_0 + \kappa/2 \ (\epsilon \le t \le \tau) \ \} \ .$$

Then by (3.4)

$$\dot{x}(t) \ge bx(t) + \sigma \quad (\epsilon \le t < t^*) \tag{5.8}$$

where

$$b = c^B + \rho$$

$$\sigma = \inf\{ \ c^B \delta \gamma(n^B(t))\xi(t) - 1 \ : \ \epsilon \le t \le t^* \ \} \ .$$

By (5.2), (5.6) and the definition of t^* we have

$$\begin{aligned} ax(\epsilon) + \sigma &\ge (c^B + \rho)x_0 + c^B \delta \gamma(n_0^B)(\xi_0 - \kappa/2) \\ &\ge c^B \delta \gamma_\infty \kappa/2 \\ &= \beta_1 > 0 \ . \end{aligned} \tag{5.9}$$

From (5.8) follows

$$\begin{aligned} x(t) &\ge \exp(b(t-\epsilon))x(\epsilon) + \int_\epsilon^t \exp(b(t-\tau))\sigma d\tau \\ &= \exp(b(t-\epsilon))\left(x(\epsilon) + \frac{\sigma}{b}\right) - \frac{\sigma}{b} \quad (\epsilon \le t < t^*) \ . \end{aligned}$$

Using again (5.8), and taking into account (5.9), we get

$$\dot{x}(t) \ge \exp(b(t-\epsilon))(bx(\epsilon) + \sigma) > \beta_1 \quad (\epsilon \le t < t^*) \ .$$

This in combination with (5.5) gives

$$\dot{x}(t) \ge \beta \quad (0 \le t < t^*) \tag{5.10}$$

where

$$\beta = \min\{\beta_0, \beta_1\} \ .$$

Thus, in order to reach (5.3), it is sufficient to show that

$$t^* = \infty \ .$$

Suppose, to the contrary, that

$$t^* < \infty \ .$$

Then, as follows from the definition of t^*, for an arbitrary $\nu > 0$ there is a $t_* \in [t^*, t^* + \nu]$ such that

$$\dot{\xi}(t_*) < 0 \ . \tag{5.11}$$

If ν is sufficiently small, then, as follows from (5.1) and (5.10),

$$x(t_*) > x_\xi \ .$$

This together with (3.14) contradict to (5.11). The contradiction completes the proof of (5.3).

5.3 Upper bound trajectory

Fix a $\xi_{00} > 0$ and a $n_{00}^B > 0$ satisfying

$$c^B \gamma(n_{00}^B)\delta\xi_{00} < \alpha \tag{5.12}$$

$$(c^B + \rho)x_\xi + c^B\delta\gamma(n_0^B)\xi_0 - 1 > 0 \quad \text{for all} \quad n_0^B \geq n_{00}^B,\ \xi_0 \geq \xi_{00} \tag{5.13}$$

(see (5.2) where $x_0 = x_\xi$). Such ξ_{00} and n_{00}^B exist due to assumption (1.18); the latter shows that it is sufficient to take ξ_{00} and n_{00}^B so that

$$\inf\{c^B\gamma(n_0^B)\delta\xi_0 \ :\ \xi_0 \geq \xi_{00},\ n_0^B \geq n_{00}^B\} = c^B\gamma(n_{00}^B)\delta\xi_{00}$$

is smaller than α being sufficiently close to it. In what follows $\xi_{**} > \xi_{00}$ (see (5.12)) is determined by

$$c^B\gamma(n_{00}^B)\delta\xi_{**} = \alpha\ . \tag{5.14}$$

Consider the solution

$$(n^A(t), n^B(t), v^B(t)) = (n^{A+}(t), n^{B+}(t), v^{B+}(t))$$

to (1.1) – (1.3) with the initial state

$$\begin{aligned} n^{A+}(0) &= n_0^{A+} = \frac{\xi_0}{v_0^{B+}} \\ n^{B+}(0) &= n_0^{B+} = \frac{x_\xi}{v_0^{B+}} > n_{00}^B \\ v^{B+}(0) &= v_0^{B+} > 0 \end{aligned}$$

where

$$\xi_0 \in [\xi_{00}, \xi_{**})\ . \tag{5.15}$$

Particular values for ξ_0 and v_0^{B+} will be specified in the next subsection. For the variables $x(t)$, $\xi(t)$ (3.1) corresponding to this solution we shall write $x^+(t)$, $\xi^+(t)$. According to the previous subsection, (5.3), (5.4) hold; using the above notations, rewrite these properties as

$$x^+(0) = x_\xi, \quad \dot{x}^+(0) > 0, \quad x^+(t) \text{ crosses } x_v\ . \tag{5.16}$$

The latter implies that the above solution is not an intermediate trajectory. Let us call it the *upper bound trajectory*.

5.4 Specification of parameters

Fix an $n_0^{B-} > 0$. Recall that by the definition of n_0^{B+}

$$n_0^{B+}v_0^{B+} = x_\xi \tag{5.17}$$

We set

$$v_0^{B+} = \frac{\alpha - c^B\gamma(n_0^{B-})\delta\xi_0}{c^B n_0^{B-}}\ . \tag{5.18}$$

Note that the fact that $\xi_0 < \xi_{**}$ (see (5.14), (5.15)) implies that $v_0^{B+} > 0$. We let ξ_0 (5.15) be sufficiently close to ξ_{**}. Observe that as ξ_0 approaches ξ_{**} from below, v_0^{B+} (5.18) approaches zero. Assume ξ_0 to be so close to ξ_{**} that

$$n_0^{B+} = \frac{x_\xi}{v_0^{B+}} > \max\{n_0^{B-}, n_{00}^B\}\ . \tag{5.19}$$

Note that (5.18) implies

$$v_0^{B+} = v^{B-}(0) = \mu_n[0] = \frac{\alpha}{c^B(n_0^{B-} + \gamma(n_0^{B-})\delta n_0^{A+}} \tag{5.20}$$

(see (1.20)). Indeed transform (5.18) as follows:

$$c^B(n_0^{B-}v_0^{B+} + \gamma(n_0^{B-})\delta\xi_0) = \alpha$$

$$v_0^{B+} = \frac{\alpha v_0^{B+}}{c^B(n_0^{B-}v_0^{B+} + \gamma(n_0^{B-})\delta\xi_0)} .$$

Dividing the right hand side through v_0^{B+}, we get (5.20).

Introduce the solution

$$(n^A(t), n^B(t), v^B(t)) = (n^{A-}(t), n^{B-}(t), v^{B-}(t))$$

with the initial state

$$\begin{aligned} n^{A-}(0) &= n_0^{A+} = \frac{\xi_0}{v_0^{B+}} \\ n^{B-}(0) &= n_0^{B-} \\ v^{B-}(0) &= v_0^{B-} = v_0^{B+} . \end{aligned} \tag{5.21}$$

Call it the *lower bound trajectory.* For the corresponding variables $x(t)$, $\xi(t)$ defined by (3.1) we shall write $x^-(t)$, $\xi^-(t)$.

Let us show that choosing ξ_0 sufficiently close to ξ_{**} (5.14), we can ensure that for the lower bound trajectory $(n^A(t), n^B(t), v^B(t)) = (n^{A-}(t), n^{B-}(t), v^{B-}(t))$ it holds that

$$\dot{v}^{B-}(0) < \dot{\mu}_n[0] . \tag{5.22}$$

It will be important for us that (5.20) and (5.22) (together with the equality $\dot{n}^{B-}(0) = 0$ guaranteed by (5.20)) imply

$$x^-(0) = x_n(0), \quad \dot{x}^-(0) < \dot{\mu}_n[0]n^{B-}(0) < 0 \tag{5.23}$$

for the lower bound trajectory $(n^A(t), n^B(t), v^B(t)) = (n^{A-}(t), n^{B-}(t), v^{B-}(t))$ (see (2.2)).

We justify (5.22) as follows. As it was stated in subsection 2.1,

$$\begin{aligned} \dot{\mu}_n[t] &= -\frac{\alpha\gamma(n^B(t))\delta\dot{n}^A(t)}{c^B(n^B(t) + \gamma(n^B(t))\delta n^A(t))^2} \\ &= -\frac{\alpha\gamma(n^B(t))\delta\bar{g}^A n^A(t)}{c^B(n^B(t) + \gamma(n^B(t))\delta n^A(t))^2} . \end{aligned}$$

Hence by (1.3)

$$\dot{v}^B(0) - \dot{\mu}_n[0] = \rho v_0^{B+} - p_0 = \rho\frac{\alpha - c^B\gamma(n_0^{B-})\delta\xi_0}{c^B n_0^{B-}} - p_0 \tag{5.24}$$

$$p_0 = \frac{1-\alpha}{n_0^{B-}} - \frac{\alpha\gamma(n_0^{B-})\delta\bar{g}^A n_0^{A-}}{c^B(n_0^{B-} + \gamma(n_0^{B-})\delta n_0^{A-})^2} .$$

We have

$$
\begin{aligned}
p_0 &= \frac{1-\alpha}{n_0^{B-}} - \frac{\alpha\gamma(n_0^{B-})\delta\bar{g}^A n_0^{A-}}{c^B(n_0^{B-}+\gamma(n_0^{B-})\delta n_0^{A-})^2} \\
&\geq \frac{1-\alpha}{n_0^{B-}} - \frac{\alpha\bar{g}^A}{c^B(n_0^{B-}+\gamma(n_0^{B-})\delta n_0^{A-})} \\
&= \frac{1-\alpha}{n_0^{B-}} - \frac{\alpha\bar{g}^A v_0^{B+}}{c^B(n_0^{B-}v_0^{B+}+\gamma(n_0^{B-})\delta\xi_0)} \\
&\geq \frac{1-\alpha}{n_0^{B-}} - \frac{\alpha\bar{g}^A v_0^{B+}}{\gamma(n_0^{B-})\delta\xi_0} \\
&= \frac{1-\alpha}{n_0^{B-}} - \frac{\alpha\bar{g}^A}{\gamma(n_0^{B-})\delta\xi_0}\frac{\alpha - c^B\gamma(n_0^{B-})\delta\xi_0}{c^B n_0^{B-}} .
\end{aligned}
$$

The left hand side of (5.24) takes the form

$$
\dot{v}^B(0) - \dot{\mu}_n[0] = \rho v_0^{B+}\left(1 + \frac{\alpha\bar{g}^A}{\gamma(n_0^{B-})\delta\xi_0}\right)\frac{\alpha - c^B\gamma(n_0^{B-})\delta\xi_0}{c^B n_0^{B-}} - \frac{1-\alpha}{n_0^{B-}} .
$$

If ξ_0 is sufficiently close to ξ_{**}, or $\alpha - c^B\gamma(n_0^{B-})\delta\xi_0$ is sufficiently small, then the first term on the right is arbitrarily small, and we have (5.22). Namely, it is sufficient to let ξ_0 satisfy

$$
0 < \alpha - c^B\gamma(n_0^{B-})\delta\xi_0 < \frac{(1-\alpha)c^B}{\rho} . \tag{5.25}
$$

Note that equality (5.17) has the equivalent form

$$
n_0^{B+}\frac{\alpha - c^B\gamma(n_0^{B-})\delta\xi_0}{c^B n_0^{B-}} = x_\xi
$$

(see (5.18)), or

$$
n_0^{B+} = \frac{x_\xi c^B n_0^{B-}}{\alpha - c^B\gamma(n_0^{B-})\delta\xi_0} .
$$

hence in order to satisfy (5.19) it is sufficient to require

$$
0 < \alpha - c^B\gamma(n_0^{B-})\delta\xi_0 < \frac{\max\{n_0^{B-}, n_{00}^B\}}{x_\xi c^B n_0^{B-}} . \tag{5.26}
$$

In what follows, (5.18) (equivalent to (5.20)) is assumed, and ξ_0 is specified as above (see (5.25), (5.26)).

Thus we specified ξ_0 and v_0^{B+} so that for the lower bound trajectory $(n^A(t), n^B(t), v^B(t)) = (n^{A-}(t), n^{B-}(t), v^{B-}(t))$ we have the relations (5.23).

5.5 Definition of a trajectory

In this subsection we define a particular solution $(n^A(t), n^B(t), v^B(t))$; in the next subsections it will be proved that this solution is an intermediate trajectory with $r(t)$ strictly increasing (that is, $(n^A(t), n^B(t), v^B(t))$ is catching up).

Come back to the upper bound trajectory $(n^A(t), n^B(t), v^B(t)) = (n^{A+}(t), n^{B+}(t), v^{B+}(t))$. In accordance with the notations (3.1), we have

$$
x^+(0) = x_\xi = n^{B+}(0)v^{B+}(0), \quad \xi^+(0) = \xi_0 = n^{A+}(0)v^{B+}(0) .
$$

Let

$$x_0^- = n_0^{B-} v_0^{B+} = n_0^{B-} v^{B+}(0) \ .$$

For every

$$x_0 \in [x_0^-, x_\xi]$$

there exists the single

$$n_0^B = n_0^B[x_0] \in [n_0^{B-}, n_0^{B+}] = [x_0^-/v_0^{B+}, x_\xi/v_0^{B+}] \tag{5.27}$$

such that

$$x_0 = n_0^B[x_0] v_0^{B+} \ .$$

Solution $(n^A(t), n^B(t), v^B(t))$ determined by the initial conditions

$$n^A(0) = n_0^{A+}, \quad n_0^B = \frac{x_0}{v_0^{B+}}, \quad v_0^B = v_0^{B+}$$

will be said to *correspond to* x_0; making it explicit that variables $x(t)$, $\xi(t)$ (3.1) determined by the above solution depend on x_0, we shall write

$$x(t) = x(t \mid x_0), \quad \xi(t) = \xi(t \mid x_0) \ .$$

Obviously

$$x(t \mid x_\xi) = x^+(t), \quad x(t \mid x_0^-) = x^-(t). \tag{5.28}$$

In view of (5.16), for $x_0 < x_\xi$ sufficiently close to x_ξ we have

$$x(t \mid x_0) \quad \text{crosses } x_\xi \tag{5.29}$$

$$x(t \mid x_0) \text{ is strictly increasing on } [0, t_\xi); \quad t_\xi \text{ is the time of first crossing } x_\xi \ . \tag{5.30}$$

Consider the set X of all $x_0 \in [x_0^-, x_\xi]$ such that $x(t) = x(t \mid x_0)$ satisfyies the similar conditions:

$$x(t) \text{ crosses } x_\xi \tag{5.31}$$

$$x(t) \text{ is strictly increasing on } [0, t_\xi); \quad t_\xi \text{ is the time of first crossing } x_\xi \ . \tag{5.32}$$

¿From (5.28), (5.29), (5.30) follows that for $x_0 < x_\xi$ sufficiently close to x_ξ, we have

$$x_0 \in X \ ;$$

From (5.28), (5.23) follows

$$x_0^- = x_n(0) \notin X$$

where $x_n(t)$ corresponds to $x(t) = x^-(t) = x(t \mid x_0^-)$. Therefore for

$$x_* = \inf X \tag{5.33}$$

it holds that

$$x_* \in (x_0^-, x_\xi) \subset (x_n(0), x_\xi) \ . \tag{5.34}$$

Here and in the sequel we set

$$x(t) = x(t \mid x_*) \tag{5.35}$$

and assume $(n^A(t), n^B(t), v^B(t))$ to correspond to x_*. The ratio $r(t)$ is as usual defined by (1.6).

5.6 Monotonicity before crossing x_ξ

In this subsection the following is established: either

$$x(t) \text{ is strictly increasing on } [0, \infty) \text{ and does not cross } x_\xi \tag{5.36}$$

or $x(t)$ satisies (5.32).

Assume that the statement is untrue. Then there are $t_2 > t_1 \geq 0$ such that either

$$x(t_2) < x(t_1) < x_\xi \ , \tag{5.37}$$

or

$$x(t) = b = \text{const} < x_\xi \quad \text{on} \quad [t_1, t_2] \ . \tag{5.38}$$

By assumption (see subsection 5.1) (5.38) is not the case. Thus (5.37) holds. By definition (5.33) of x_*, there exists a sequence

$$x_{0i} \longrightarrow x_* + 0, \quad x_{0i} \in X \ . \tag{5.39}$$

Hence

$$x_i(t) = x(t \mid x_{0i}) \longrightarrow x(t) \tag{5.40}$$

uniformly on $[0, t_2 + \epsilon]$ with a certain $\epsilon > 0$. For large i, t_2 is smaller than the time of the first crossing of x_ξ by $x_i(t)$. So far as $x_i(t)$ is increasing on $[0, t_2]$, we get a contradiction with (5.37).

5.7 Strict monotonicity in a neighborhood of x_ξ

Let us prove that if $x(t_\xi) = x_\xi$ for a certain $t_\xi \geq 0$, then

$$\dot{x}(t_\xi) > 0 \ . \tag{5.41}$$

Suppose that (5.41) is untrue, and obtain a contradiction. Observe that from (5.32) follows

$$\dot{x}(t_\xi) \geq 0 \ .$$

Hence

$$\dot{x}(t_\xi) = 0 \ . \tag{5.42}$$

Note also that in view of (3.14), (3.15)

$$\dot{\xi}(t_\xi) = 0 \ .$$

The differentiation of $\dot{x}(t)$ with the usage of (3.4) gives

$$\begin{aligned} \frac{d}{dt}\dot{x}(t_\xi) &= (c^B + \rho)\dot{x}(t_\xi) + c^B\gamma'(n^B(t_\xi))\dot{n}^B(t_\xi)\delta\xi(t) + c^B\gamma(n^B(t))\delta\dot{\xi}(t) \\ &= c^B\gamma'(n^B(t_\xi))\dot{n}^B(t_\xi)\delta\xi(t) \ . \end{aligned}$$

Since

$$x(t_\xi) = x_\xi \in (x_n(t_\xi), x_v)$$

(which follows from (1.13)), we have $\dot{n}^B(t_\xi) > 0$. Hence

$$\frac{d}{dt}\dot{x}(t_\xi) > 0 \ .$$

Consequently

$$\frac{d}{dt}\dot{x}(t) > 0$$

for all $t > t_\xi - \epsilon$ with certain positive ϵ. Note that in view of $x_* < x_\xi$ (see (5.34)), $t_\xi > 0$. With no loss of generality assume $t_\xi - \epsilon > 0$. By (5.32)

$$\dot{x}(t_\xi - \epsilon) > 0 \ ;$$

therefore

$$\dot{x}(t_\xi) > 0 \ .$$

We obtained a contradiction with (5.42). Thus (5.41) is proved.

5.8 Staying between barriers

This subsection finalizes the proof of Proposition 1.10. Let us obtain the first statement of the Proposition. Namely, verify that inequalities (3.6) with $t_0 = 0$ hold, and $r(t)$ is strictly increasing.

Show that (3.6) is true with $t_0 = 0$. Prove first that the upper bound in (3.6) holds, that is,

$$x(t) < x_v \quad (t \geq 0) \ . \tag{5.43}$$

So far as $x_v > x_\xi$ (see (3.13)) and $x(0) = x_* < x_\xi$ (see (5.34)) it is sufficient to show that $x(t)$ does not cross x_ξ, i.e.

$$x(t) < x_\xi \quad (t \geq 0) \ . \tag{5.44}$$

Suppose, to the contrary, that $x(t)$ crosses x_ξ at some time t_ξ. Then, as it was stated in subsections 5.6 and 5.7, (5.32) and (5.41) hold. Let

$$x_\epsilon(t) = x(t \mid x_* - \epsilon), \quad \epsilon > 0 \ .$$

From (5.32) and (5.41) follows easily that for small ϵ,

$$x_\epsilon(t) \text{ crosses } x_\xi$$

and

$$x_\epsilon(t) \text{ is strictly increasing on } [0, t_{\xi,\epsilon}]; \quad t_{\xi,\epsilon} \text{ is the time of first crossing } x_\xi \ . \tag{5.45}$$

Therefore

$$x_* - \epsilon \in X$$

for all sufficiently small positive ϵ. This contradicts to the definition (5.33) of x_*. Thus $x(t)$ does not cross x_ξ (and consequently (5.43) holds). This fact and the statement of subsection 5.6 imply that (5.36) takes place. From (5.34) and

$$x(0) = x_* > x_n(0)$$

(see (5.34)) follows that

$$x(t) > x_n(0) \quad (t \geq 0).$$

Since $x_n(t)$ is decreasing (see (3.9)), we have

$$x(t) > x_n(t) \quad (t \geq 0) \ . \tag{5.46}$$

From (5.44), (5.46) and (3.13)) we get

$$x_n(t) < x(t) < x_\xi < x_v \quad (t \geq 0) \tag{5.47}$$

implying (3.6). Thus, $(n^A(t), n^B(t), v^B(t))$ is an intermediate trajectory.

Prove that $(n^A(t), n^B(t), v^B(t))$ is catching up. From (5.47) and (3.15) we deduce that $\xi(t)$ is strictly decreasing. This together with (5.36) yield that

$$r(t) = \frac{n^B(t)}{n^A(t)} = \frac{x(t)}{\xi(t)}$$

is strictly increasing.

Finally, pass to the second statement of Proposition 1.10 providing a condition sufficient for $(n^A(t), n^B(t), v^B(t))$ to be overtaking. If ξ_0 is such that

$$\frac{x_\xi}{\xi_0} < 1 , \tag{5.48}$$

then

$$r(0) = \frac{x_*}{\xi_0} < \frac{x_\xi}{\xi_0} < 1 .$$

If, moreover, for the limit of $r(t)$ corresponding to the intermediate asymptotics we have

$$r_\infty = \gamma_\infty \delta \frac{c^B(1-\alpha)}{(\bar{g}^A + \rho) - (c^B + \rho)(1-\alpha)} > 1,$$

then the above trajectory is overtaking, that is, $r(t)$ grows from a value smaller than 1 to a value bigger than 1. Let us assume that

$$\frac{x_\xi c^B \gamma(n_0^{B-})\delta}{\alpha} < 1 \tag{5.49}$$

(see (1.32)). Then (5.48) can be reached by choosing ξ_0 sufficiently close to ξ_{**} (5.14) (recall that the latter was assumed in subsection 5.4) Indeed, if ξ_0 is sufficiently close to ξ_{**}, then $c^B\gamma(n_0^{B-})\delta\xi_{**}$ is arbitrarily close to α; hence the difference

$$\frac{1}{\xi_0} - \frac{c^B\gamma(n_0^{B-})\delta}{\alpha}$$

is arbitrarily small, and the left hand sides of (5.48) and (5.49) are arbitrarily close to each other; thus inequality (5.48) is satisfied. To satisfy (5.48) as outlined above with keeping conditions (5.25), (5.26) imposed on ξ_0 in subsection 5.4 we move ξ_0, if necessary, closer to ξ_{**}. The proof of Proposition 1.10 is completed.

5.9 Proof of Proposition 1.6

Prove Proposition 1.6. Suppose that Condition 1.2 is fulfilled, that is, for any interval $[p_1, p_2]$ of nonzero length with $p_1 \geq 0$ there do not exist positive constants σ, β such that

$$\gamma'(p) = \sigma p^{-\beta}$$

for all $p \in [p_1, p_2]$. Let us show that Condition 1.1 (in the equivalent formulation of subsection 5.1) is true. Consider an arbitrary solution $(n^A(t), n^B(t), v^B(t))$ to (1.1) –

(1.3) such that for $x(t)$, $\xi(t)$ (3.1) it holds that $x_n < x(t) < x_v$ for $t \in [t_1, t_2]$ with $t_2 > t_1 \geq 0$. Prove that $x(t)$ cannot be constant on $[t_1, t_2]$.

Assume that

$$x(t)n^B(t)v^B(t) = b = \text{const} \quad (t \in [t_1, t_2]) \tag{5.50}$$

and obtain a contradiction. From (1.2), (1.3) and (5.50) we get

$$\dot{n}^B(t) = c^B(n^B(t) + \gamma(n^B(t))\delta n^A(t)) - \frac{\alpha}{b}n^B(t) \tag{5.51}$$

$$\dot{v}^B(t) = -d_0 v^B(t)$$

where

$$d_0 = \rho - \frac{1-\alpha}{b} .$$

Thus

$$v^B(t) = v_1^B \exp(-d_0(t - t_1)) ;$$

here and in what follows $v_1^B = v^B(t_1)$, $t \in [t_1, t_2]$. By (5.50)

$$n^B(t) = \frac{b}{v_1^B} \exp(d_0(t - t_1)) .$$

Substituting this and

$$n^A(t) = n_1^A \exp(\bar{g}^A(t - t_1))$$

where $n_1^A = n^A(t_1)$ (see (1.1)) in (5.51), we get

$$\begin{aligned} \frac{d_0 b}{v^B} \exp(d_0(t - t_1)) &= \left(c^B - \frac{\alpha}{b}\right) \frac{b}{v_1^B} \exp(d_0(t - t_1)) + \\ &\quad \gamma\left(\frac{b}{v_1^B} \exp(d_0(t - t_1))\right) \delta n_1^A \exp(\bar{g}^A(t - t_1)) . \end{aligned}$$

Hence

$$\begin{aligned} \gamma\left(\frac{b}{v_1^B} \exp(-d_0 t_1) \exp(d_0(t)\right) &= \frac{\exp(\bar{g}^A t_1)}{\delta n_1^A} \left(\frac{d_0 b}{v^B} - c^B + \frac{\alpha}{b}\right) - \\ &\quad \frac{b}{v_1^B} \exp(-d_0 t_1) \exp((d - \bar{g}^A)t) \end{aligned}$$

or in simpler natations

$$\gamma\left(d_1 \exp(d_0 t)\right) = d_2 \exp((d_0 - \bar{g}^A)t)$$

where

$$\begin{aligned} d_1 &= \frac{b}{v_1^B} \exp(-d_0 t_1) \\ d_2 &= \frac{\exp(\bar{g}^A t_1)}{\delta n_1^A} \left(\frac{d_0 b}{v^B} - c^B + \frac{\alpha}{b}\right) - \frac{b}{v_1^B} \exp(-d_0 t_1) . \end{aligned}$$

The differentiation results in

$$\gamma'(d_1 \exp(d_0 t)) d_1 d_0 \exp(d_0 t) = d_2(d_0 - \bar{g}^A) \exp((d - \bar{g}^A)t)$$

or

$$\begin{aligned}\gamma'(d_1 \exp(d_0 t)) &= \frac{d_2(d_0 - \bar{g}^A)}{d_1 d_0} \exp(-\bar{g}^A t) \\ &= \frac{d_2(d_0 - \bar{g}^A)}{d_1 d_0} d_1^{\bar{g}^A/d^0} (d_1 \exp(d_0 t))^{-\bar{g}^A/d^0}\end{aligned} \quad (5.52)$$

Introduce the new variable

$$p = d_1 \exp(d_0 t) \ ;$$

note that

$$p \in [p_1, p_2] = [d_1 \exp(d_0 t_1), d_1 \exp(d_0 t_2)] \ .$$

Rewrite (5.52) as

$$\gamma'(p) = a p^{-\beta}$$

where

$$a = \frac{d_2(d_0 - \bar{g}^A)}{d_1 d_0} d_1^{\bar{g}^A/d^0}, \quad \beta = -\frac{\bar{g}^A}{d^0} \ .$$

This holds for all $p \in [p_1, p_2]$, which contradics the initial assumption.

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