Working Paper

Minimax Control of Constrained Parabolic Systems

Boris S. Mordukhovich and Kaixia Zhang

WP-95-70
July 1995

IIASA International Institute for Applied Systems Analysis □ A-2361 Laxenburg □ Austria
Telephone: +43 2236 807 □ Fax: +43 2236 71313 □ E-Mail: info@iiasa.ac.at

Minimax Control of Constrained Parabolic Systems

Boris S. Mordukhovich and Kaixia Zhang

WP-95-70
July 1995

The first author is a guest scholar from the Department of Mathematics, Wayne State University, Detroit, MI 48202, USA. E-mail: boris@math.wayne.edu

IIASA International Institute for Applied Systems Analysis □ A-2361 Laxenburg □ Austria
Telephone: +43 2236 807 □ Fax: +43 2236 71313 □ E-Mail: info@iiasa.ac.at

Abstract

In this paper we formulate and study a minimax control problem for a class of parabolic systems with controlled Dirichlet boundary conditions and uncertain distributed perturbations under pointwise control and state constraints. We prove an existence theorem for minimax solutions and develop effective penalized procedures to approximate state constraints. Based on a careful variational analysis, we establish convergence results and optimality conditions for approximating problems that allow us to characterize suboptimal solutions to the original minimax problem with hard constraints. Then passing to the limit in approximations, we prove necessary optimality conditions for the minimax problem considered under proper constraint qualification conditions.

Keywords: Parabolic equations, uncertain disturbances, Dirichlet boundary controls, minimax criterion, state constraints, approximations, constraint qualification, and variational inequalities.

Minimax Control of Constrained Parabolic Systems

Boris S. Mordukhovich and Kaixia Zhang

1 Introduction

This paper is devoted to the study of a *minimax* optimal control problem for a class of distributed-parameter parabolic systems with uncertain disturbances (perturbations) and boundary controls. Our motivations partially come from applications to *robust control* of constrained parabolic systems with uncertainty conditions; see [18–20, 22]. It is natural that uncertain disturbances frequently occur in many control systems operating in real-life settings, although we are not familiar with any results in the literature directly related to the minimax control problem under consideration.

From a mathematical viewpoint, we consider a control problem for linear parabolic systems with uncertain disturbances, state constraints, and bounded controllers in the *Dirichlet boundary conditions.* The objective is to compensate undesirable effects of system disturbances through boundary control actions such that a nonlinear cost functional achieves its minimum for the worst (maximal) disturbances. The optimization problem under consideration appears to be essentially *nonsmooth* and requires special methods for its variational analysis. To provide such an analysis in this paper we systematically use *smooth approximation procedures.*

The main results of the paper include an existence theorem and necessary conditions for evaluating both the worst distributed perturbations and optimal boundary controllers under *pointwise ("hard") control and state constraints.* Actually we split the original minimax problem into two interrelated optimal control problems for distributed perturbations and boundary controllers with *moving* state constraints. Then we approximate state constraints in each of these problems by effective penalizations involving C^∞-approximations of maximal monotone operators. We establish strong convergence results for such processes and obtain characterizations of optimal solutions to the approximating problems. Finally imposing proper constraint qualifications, we come up to necessary optimality conditions for the worst perturbations and optimal controllers in the original state-constrained minimax problem. Some results and special cases have been presented in [21, 23].

This paper is organized as follows. In Section 2 we formulate the minimax control problem for our study taking into account a *semigroup model* for parabolic equations with the Dirichlet boundary conditions. We define an optimal solution to the minimax problem as a *saddle point* for a certain integral functional. To obtain necessary optimality conditions for the minimax problem we split it into two separate (but interconnected) optimization problems for disturbance and control functions. The first one involves a system with uncertain disturbances and homogeneous boundary conditions. The second problem deals with optimization of boundary controllers in the absence of disturbances. Both systems are subject to moving state constraints that depend on space and time variables and reflect the nature of the minimax problem.

In Section 3 we present some important properties of *mild solutions* to parabolic systems with the Dirichlet boundary conditions and related continuity/regularity results that are crucial in our approach. Using these properties, we prove an *existence theorem* for optimal solutions (saddle points) to the minimax control problem under consideration.

In Section 4 we treat uncertain disturbances as *distributed controllers* in an auxiliary optimal control problem with bounded control functions and pointwise state constraints. To remove the latter constraints we use a penalization procedure involving C^∞-approximations of multivalued maximal monotone operators with *nonregular* functions in approximating cost functionals. Empoying such a procedure and a detailed variational analysis of the approximating problems, we obtain strong convergence results and necessary *suboptimality* conditions to characterize the worst perturbations in the original minimax problem. Some constructions and results of this section are related to those in Barbu [2], Bonnans and Tiba [7], Friedman [11], He [12], and Neittaanmäki and Tiba [24, 25] in the framework of parabolic variational inequalities with bounded operators.

In Section 5 we study the *Dirichlet boundary control problem* with control and state constraints corresponding to the second subsystem under the worst disturbances. There are many publications devoted to various boundary control problems for parabolic systems; see, e.g., Balakrishnan [1], Barbu [2], Fattorini and Murphy [9, 10], Lasiecka and Triggiani [13, 14], Lions [15], Mackenroth [17], Tröltzsch [28], Washburn [29], and references therein. The main complications in our case arise from the presence of *pointwise* state constraints simultaneously with hard constraints on *measurable* (L^∞) control functions acting within the Dirichlet boundary conditions. It is well known that the latter conditions provide the *lowest regularity properties* of solutions and are related to *unbounded* operators in the framework of variational inequalities. The Dirichlet boundary control case turns out to be the most challenging and less developed; cf. [2, 9, 13, 14] and references therein. Variational analysis of such problems is more difficult in comparison with the case of control functions acting through the Neumann boundary conditions which ensure higher regularity properties of the corresponding solutions.

To develop such an analysis in the case of nonregular Dirichlet boundary controllers we use properties of mild solutions studied in Section 3 and effective smooth approximation procedures. In Section 5 we prove the strong convergence of approximations ensuring suboptimality of optimal controllers to the approximating problems in the original problem with state constraints. Then we provide a variational analysis of the approximating problems with hard constraints only on Dirichlet boundary controllers. In this line we obtain necessary optimality conditions for the approximating problems that can be written in the form of *bang-bang principle.*

In the final Section 6 we impose proper *constraint qualifications* that allow us to pass to the limit in the necessary optimality conditions for the approximating problems and characterize both worst disturbances and optimal controllers to the original minimax control problem with state constraints. These constraint qualifications fit the nature of the minimax problem under consideration being different from the classical Slater interiority condition. Developing the limiting procedure, we obtain necessary optimality conditions for the original state-constrained problem that involve measure-type Lagrange multipliers.

Our notation is basically standard; cf. [2, 15, 26]. Recall that L^* always denotes the dual (adjoint) operator to a linear operator L between Banach spaces.

2 Problem Formulation and Splitting

We consider the following system

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay = Bw + f \text{ a.e. in } Q := (0,T) \times \Omega, \\ y(0,x) = y_0(x), \; x \in \Omega, \\ y(t,\xi) = u(t,\xi), \; (t,\xi) \in \Sigma := (0,T] \times \Gamma \end{cases} \tag{2.1}$$

under the pointwise constraints:

$$a \le y(t,x) \le b \text{ a.e. } (t,x) \in Q, \tag{2.2}$$

$$c \le w(t,x) \le d \text{ a.e. } (t,x) \in Q, \tag{2.3}$$

$$\mu \le u(t,\xi) \le \nu \text{ a.e. } (t,\xi) \in \Sigma \tag{2.4}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded open set with sufficiently smooth boundary Γ and each of the intervals $[a,b]$, $[c,d]$, and $[\mu,\nu]$ contains 0.

Let $X := L^2(\Omega;\mathbf{R})$, $U := L^2(\Gamma;\mathbf{R})$, and $W := L^2(\Omega;\mathbf{R})$ be, respectively, spaces of states, controls, and disturbances. (In what follows we remove $\mathbf{R}$ from the latter and similar space notation for real-valued functions). Denote by

$$U_{ad} := \{u \in L^p(0,T;U) \mid \mu \le u(t,\xi) \le \nu \text{ a.e. } (t,\xi) \in \Sigma, u(0,\xi) = 0\}$$

the set of admissible controls where $L^p(0,T;U)$ is the space of U-valued functions $u(\cdot) = u(\cdot,\xi)$ on $[0,T]$ with the norm

$$\|u\|_{L^p(0,T;U)} := (\int_0^T \|u(t)\|_U^p dt)^{1/p} = (\int_0^T (\int_\Gamma |u(t,\xi)|^2 d\xi)^{p/2} dt)^{1/p}.$$

Similarly we define the set of admissible disturbances

$$W_{ad} := \{w \in L^2(0,T;W) \mid c \le w(t,x) \le d \text{ a.e. } (t,x) \in Q\}.$$

A pair $(u,w) \in U_{ad} \times W_{ad}$ is called a *feasible solution* to system (2.1) if the corresponding trajectory y satisfies the state constraints (2.2). We always assume that problem (2.1)–(2.4) admits at least one feasible pair (u,w).

Although $W_{ad} \subset L^\infty(Q)$ and $U_{ad} \subset L^\infty(\Sigma)$, we prefer to use the standard norms of the spaces $L^2(0,T;W)$ and $L^p(0,T;U)$ for finite p *sufficiently big*; see Section 3 for more detials. The reason is that we would like to take advantages of the *differentiability* of the latter norms away from the origin to perform our variational analysis.

Throughout the paper we impose the following hypotheses:

(H1) $A = -\sum_{i,j=1}^N \dfrac{\partial}{\partial x_i}(a_{ij}(x)\dfrac{\partial}{\partial x_j}) + a_0(x)$ is a strongly uniformly symmetric elliptic operator on Ω with real-valued smooth coefficients $a_{ij}(x) = a_{ji}(x)$ and $a_0(x) \ge 0$ satisfying

$$\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \ge \beta_0 \sum_{i=1}^N \xi_i^2, \; \beta_0 > 0, \; \forall x \in \Omega \text{ and } (\xi_1,\cdots,\xi_N) \in \mathbf{R}^N.$$

(H2) $f \in L^\infty(Q)$ and $y_0(x) \in H_0^1(\Omega) \cap H^2(\Omega)$ with $a \le y_0(x) \le b$ a.e. $x \in \Omega$.

(H3) $B : L^2(0,T;W) \to L^2(0,T;X)$ is a bounded linear operator

One can always assume that the operator $-A$ generates a strongly continuous analytic semigroup $S(\cdot)$ on X satisfying the exponential estimate

$$\|S(t)\| \leq M_1 e^{-\omega t} \tag{2.5}$$

for some constants $\omega > 0$ and $M_1 > 0$ where $\|\cdot\|$ denotes the standard operator norm from X to X. Otherwise one can introduce the stable translation $\tilde{A} = A + \tilde{\omega} I$ that possesses these properties; see, e.g., [26].

Note that since $w \in L^2(0,T;W)$ and $u \in L^p(0,T;U)$, system (2.1) may *not* have strong or classical solutions for some $(u,w) \in U_{ad} \times W_{ad}$; cf., [2, 15, 26]. In this case, principal difficulties come from discontinuous boundary controls in the Dirichlet conditions. Taking advantages of the semigroup approach to parabolic equations, we are going to use for our analysis a concept of *mild solutions* to Dirichlet boundary problems.

Let us consider the *Dirichlet map D* defined by $y = Du$ where y satisfies

$$\begin{cases} Ay = 0 \ \text{ in } Q, \\ y(t,\xi) = u(t,\xi), \ (t,\xi) \in \Sigma. \end{cases}$$

It is well known (see, e.g., [13–15]) that the operator

$$D : L^2(\Gamma) \to \mathcal{D}(A^{1/4-\delta}) = H^{1/2-2\delta}(\Omega), \ 0 < \delta \leq 1/4, \tag{2.6}$$

is linear and continuous.

2.1. Definition. A continuous function $y : [0,T] \to X$ is said to be a *mild solution* of system (2.1) corresponding to $(u,w) \in L^p(0,T;U) \times L^2(0,T;W)$ if for all $t \in [0,T]$ one has

$$\begin{aligned} y(t) = S(t)y_0 + \int_0^t S(t-\tau)(Bw(\tau) + f(\tau))d\tau - \int_0^t AS(t-\tau)Du(\tau)d\tau = \\ S(t)y_0 + \int_0^t S(t-\tau)(Bw(\tau) + f(\tau))d\tau - \int_0^t A^{3/4+\delta}S(t-\tau)A^{1/4-\delta}Du(\tau)d\tau \end{aligned} \tag{2.7}$$

where D is the Dirichlet operator defined in (2.6) with $\delta \in (0, 1/4]$.

We refer the reader to [1, 13, 14, 29] for various properties and applications of mild solutions. It is essential for this paper that the assumptions made ensure the existence and uniqueness of a mild solution to (2.1) for any $w \in L^2(0,T;W)$ and $u \in L^p(0,T;U)$ with big p.

Let us observe that while the X-valued function $y(t)$ in (2.7) is continuous by definition, the real-valued function $y(t,x)$ of two variables is merely *measurable* (due to $X = L^2(\Omega)$) that distinguishes mild solutions from other concepts of solutions to parabolic equations. The mild solution approach allows us to deal with *nonregular* (measurable) data of parabolic equations and the Dirichlet boundary conditions considered in the paper. On the other hand, the absence of continuity creates substantial difficulties that we are going to overcome in what follows.

We also note that δ in Definition 2.1 may be any fixed number from the interval $(0, 1/4]$ (usually $\delta < 1/4$). Although the first equality in representation (2.7) does not depend on δ at all, this number explicitely appears in some estimations below that are the better the closer δ is to zero.

Now let us introduce the cost functional

$$J(u,w) := \iint_Q g(t,x,y(t,x))dtdx + \iint_Q \varphi(t,x,w(t,x))dtdx + \iint_\Sigma h(t,\xi,u(t,\xi))dtd\xi \tag{2.8}$$

where y is a trajectory (mild solution) for system (2.1) generated by u and w. We always suppose that functional (2.8) is well defined and finite for all admissible processes in (2.1)–(2.4). Some additional assumptions on integrands g, φ, and h will be imposed in Sections 3–5.

In this paper we study a minimax control problem as follows:

(P) find an admissible control $\bar{u} \in U_{ad}$ and a disturbance $\bar{w} \in W_{ad}$ such that $(\bar{u}, \bar{w})$ is a *saddle point* for the functional $J(u, w)$ subject to system (2.1) and state constraints (2.2).

This means that

$$J(\bar{u}, w) \le J(\bar{u}, \bar{w}) \le J(u, \bar{w}) \ \ \forall u \in U_{ad}, \ \forall w \in W_{ad} \tag{2.9}$$

under relations (2.1) and (2.2). Such a pair $(\bar{u}, \bar{w})$ is called an *optimal solution* to (P).

For studying optimal solutions to problem (P) we are going to use the following *splitting procedure* based on the linearity of system (2.1).

Let us split the original system (2.1) into two subsystems with separated disturbances and boundary controls. The first system

$$\begin{cases} \dfrac{\partial y_1}{\partial t} + Ay_1 = Bw + f \ \text{ a.e. in } \ Q, \\ y_1(0, x) = y_0(x), \ x \in \Omega, \\ y_1(t, \xi) = 0, \ (t, \xi) \in \Sigma \end{cases} \tag{2.10}$$

has zero (homogeneous) boundary conditions and depends only on disturbances. The second one

$$\begin{cases} \dfrac{\partial y_2}{\partial t} + Ay_2 = 0 \ \text{ a.e. in } \ Q, \\ y_2(0, x) = 0, \ x \in \Omega, \\ y_2(t, \xi) = u(t, \xi), \ (t, \xi) \in \Sigma \end{cases} \tag{2.11}$$

is generated by boundary controls and does not involve disturbances. It is easy to see that for any $(u, w) \in U_{ad} \times W_{ad}$ one has

$$y(t, x) = y_1(t, x) + y_2(t, x) \ \ \forall (t, x) \in Q \tag{2.12}$$

for the corresponding trajectories of systems (2.1), (2.10), and (2.11).

Let $\bar{y}_1$ and $\bar{y}_2$ be, respectively, the (unique) trajectories of systems (2.10) and (2.11) corresponding to $\bar{w}$ and $\bar{u}$. Consider the cost functionals

$$J_1(w, y_1) := \iint_Q [g(t, x, y_1(t, x) + \bar{y}_2(t, x)) + \varphi(t, x, w(t, x))] dt dx \tag{2.13}$$

for disturbances w and

$$J_2(u, y_2) := \iint_Q g(t, x, \bar{y}_1(t, x) + y_2(t, x)) dt dx + \iint_\Sigma h(t, \xi, u(t, \xi)) dt d\xi \tag{2.14}$$

for boundary controls u.

Let us define two optimization problems corresponding to the cost functionals introduced. The first one is:

(P_1) maximize $J_1(w, y_1)$ in (2.13) over $w \in W_{ad}$ subject to system (2.10) and the state constraints

$$a - \bar{y}_2(t, x) \le y_1(t, x) \le b - \bar{y}_2(t, x) \ \text{ a.e. } (t, x) \in Q. \tag{2.15}$$

The second problem is:

(P$_2$) minimize $J_2(u, y_2)$ in (2.14) over $u \in U_{ad}$ subject to system (2.11) and the state constraints

$$a - \bar{y}_1(t,x) \le y_2(t,x) \le b - \bar{y}_1(t,x) \text{ a.e. } (t,x) \in Q. \tag{2.16}$$

The following assertion shows that the original minimax problem (P) can be splitted into two optimization problems (P$_1$) and (P$_2$) separated on disturbances and controls.

2.2. Proposition. *Let $(\bar{u}, \bar{w})$ be an optimal solution to problem* (P), *and let $\bar{y}_1$ and $\bar{y}_2$ be the corresponding trajectories of systems* (2.10) *and* (2.11). *Then $\bar{w}$ solves problem* (P$_1$) *and $\bar{u}$ solves problem* (P$_2$).

Proof. Using (2.12), one can immediately conclude that $\bar{w}$ is a feasible solution to (P$_1$), i.e., the corresponding trajectory $\bar{y}_1$ of (2.10) satisfies the state constraints (2.15). Now the left-hand side of (2.9) implies, due to (2.8) and (2.13), that $\bar{w}$ is an optimal solution to (P$_1$). Arguments for $\bar{u}$ are similar. □

Therefore, to obtain necessary conditions for a given optimal solution $(\bar{u}, \bar{w})$ to the minimax problem (P), we can consider the separate problems (P$_1$) for $\bar{w}$ and (P$_2$) for $\bar{u}$ with the connecting state constraints (2.15) and (2.16). Note that these constraints depend on (t,x), i.e., turn out to be *moving*. The latter property is essential for studying the minimax problem under consideration.

3 Properties of Mild Solutions and Existence Theorem for Minimax Problem

In this section we present some important properties of mild solutions to system (2.1) and prove an existence theorem for the minimax control problem under consideration.

Let $S(t)$ be an analytic semigroup on X generated by the operator $-A$ and satisfying the exponential extimate (2.5), and let D be the Dirichlet operator with the continuity property (2.6). In what follows we use the important estimates

$$\|A^{\delta} D\| \le M_2, \;\; \|A^{3/4+\delta} S(t)\| \le \frac{M_3}{t^{3/4+\delta}} \text{ for any } \delta \in (0, 1/4] \tag{3.1}$$

where $\|\cdot\|$ represents the corresponding operator norms. These estimates were established by Balakrishnan and Washburn [1, 29]; see also Lasiecka and Triggiani [13, 14] for related considerations.

Looking at representation (2.7) of mild solutions, one can observe that the main complications are created by the last term reflecting the Dirichlet boundary conditions. To study this term we consider an operator $\mathcal{L}$ from $L^p(0,T;U)$ into $L^r(0,T;H^{1/2-\varepsilon}(\Omega))$ defined by the formula

$$\mathcal{L}u = (\mathcal{L}u)(t) := -A \int_0^t S(t-\tau) D u(\tau) d\tau = -\int_0^t A^{3/4+\delta} S(t-\tau) A^{1/4-\delta} D u(\tau) d\tau \tag{3.2}$$

where p, $r \in [1, \infty]$, $\delta \in (0, 1/4]$, and $\varepsilon \in (0, 1/2]$. Here $H^{1/2-\varepsilon}(\Omega) \subset L^2(\Omega) = X$ is the Sobolev space whose norm $\|y\|_{1/2-\varepsilon}$, being stronger than $\|y\|_X$, can be defined by $\|y\|_{1/2-\varepsilon} := \|A^{1/4-\varepsilon/2} y\|_X$; cf. [15, p. 21]. Note that $H^0(\Omega) = L^2(\Omega)$.

It is well known that the operator $\mathcal{L}$ in (3.2) may be *unbounded* for some p and r. However, this operator enjoys nice regularity/continuity properties for p sufficiently big, as one can see from the following assertion. Similar but somewhat different results are proved in [14, Theorem 2.5]; see also references therein.

3.1. Proposition. *Let $p > 4/\varepsilon$ for some $\varepsilon \in (0, 1/2]$. Then $\mathcal{L}u \in C([0,T]; H^{1/2-\varepsilon}(\Omega))$ for any $u \in L^p(0,T;U)$. Moreover, the operator $\mathcal{L} : L^p(0,T;U) \to C([0,T]; H^{1/2-\varepsilon}(\Omega))$ is linear and continuous.*

Proof. Obviously $\mathcal{L}$ is linear. To show that $\mathcal{L}$ is continuous we should prove its boundedness; that is,

$$\|\mathcal{L}u\|_{C([0,T];H^{1/2-\varepsilon}(\Omega))} \le K\|u\|_{L^p(0,T;U)} \text{ for some } K > 0. \tag{3.3}$$

It follows from (3.1) and (3.2) that for any $t \in [0,T]$ one has

$$\|(\mathcal{L}u)(t)\|_{1/2-\varepsilon} = \|\int_0^t A^{1/4-\varepsilon/2} AS(t-\tau)Du(\tau)d\tau\|_X =$$
$$\|\int_0^t A^{1-\varepsilon/4}S(t-\tau)A^{1/4-\varepsilon/4}Du(\tau)d\tau\|_X \le M_2M_3\int_0^t (t-\tau)^{-(1-\varepsilon/4)}\|u\|_U d\tau \le$$
$$M_2M_3(\int_0^t (t-\tau)^{-(1-\varepsilon/4)q}d\tau)^{1/q}\|u\|_{L^p(0,T;U)}$$

where $1/p + 1/q = 1$. Since $p > 4/\varepsilon$ infers $q < \frac{4}{4-\varepsilon}$, we get

$$\|(\mathcal{L}u)(t)\|_{1/2-\varepsilon} \le M_2M_3(\frac{1}{1-(1-\varepsilon/4)q})^{1/q} t^{\frac{1-(1-\varepsilon/4)q}{q}}\|u\|_{L^p(0,T;U)}. \tag{3.4}$$

Let us prove that $\mathcal{L}u \in C([0,T]; H^{1/2-\varepsilon}(\Omega))$, i.e., $(\mathcal{L}u)(t)$ is continuous at any point $t_0 \in [0,T]$ in the norm of $H^{1/2-\varepsilon}(\Omega)$. Indeed, taking for definiteness $t \ge t_0$, one has

$$(\mathcal{L}u)(t) - (\mathcal{L}u)(t_0) = -\int_{t_0}^t AS(t-\tau)Du(\tau)d\tau -$$
$$(S(t-t_0) - I)\int_0^{t_0} AS(t-\tau)Du(t)d\tau.$$

The latter implies that

$$\|(\mathcal{L}u)(t) - (\mathcal{L}u)(t_0)\|_{1/2-\varepsilon} \to 0 \text{ as } t \to t_0$$

by virtue of estimate (3.4) and the strong continuity of $S(\cdot)$. Moreover, from (3.4) and the definition of the norm in $C([0,T]; H^{1/2-\varepsilon}(\Omega))$ we immediately get inequality (3.3) with

$$K := M_2M_3(\frac{1}{1-(1-\varepsilon/4)q})^{1/q} T^{\frac{1-(1-\varepsilon/4)q}{q}}$$

that ensures the required continuity of $\mathcal{L}$. This ends the proof of the proposition. □

3.2. Corollary. *Let ε and p satisfy the assumptions in Proposition 3.1. Then the operator $\mathcal{L}$ in (3.2) acting from $L^p(0,T;U)$ into $C([0,T]; H^{1/2-\varepsilon}(\Omega))$ is weakly continuous. This implies that for any sequence $u_n \to u$ weakly in $L^p(0,T;U)$ one has $\mathcal{L}u_n \to \mathcal{L}u$ weakly in $C([0,T]; H^{1/2-\varepsilon}(\Omega))$ as $n \to \infty$.*

Proof. It is well known from standard functional analysis that any linear continuous operator between normed spaces is automatically weakly continuous. Therefore, the results in this corollary follow directly from Proposition 3.1. □

3.3. Remark. Taking into account the results obtained above and Definition 2.1 of mild solutions to system (2.1), we can conclude that the strong (respectively, weak) convergence of boundary controls in $L^p(0,T;U)$ implies the strong (weak) convergence of the corresponding trajectories (2.7) in $C([0,T];H^{1/2-\varepsilon}(\Omega))$ if p is sufficiently big. Observe that if there is no boundary term in (2.7), then any mild solution to (2.1) turns out to be a solution to (2.1) in the usual (strong) sense; see, e.g., [2, p. 26]. In the latter case, the *weak* convergence of disturbances $w_n \to w$ in $L^p(0,T;W)$ implies the *strong* convergence of the corresponding trajectories $y_n \to y$ in $C([0,T];X)$ as $n \to \infty$ for any $p \geq 1$; see [2, 11].

Now we are going to show that in the general case of mild solutions corresponding to the Dirichlet boundary conditions, the *weak* convergence of controls $u_n \to u$ in $L^p(0,T;U)$ for big p implies the *pointwise* convergence of a subsequence of solutions $y_n(t,x) \to y(t,x)$ a.e. in Q. This fact follows from the next proposition and turns out to be crucial for proving the main results of the paper.

3.4. Proposition. *Let ε and p satisfy the assumptions in Proposition* 3.1 *and let $\mathcal{L}$ be defined in* (3.2). *Then the weak convergence of $u_n \to u$ in $L^p(0,T;U)$ implies*

$$\mathcal{L}u_n \to \mathcal{L}u \;\; \textit{strongly in} \;\; L^2(Q) \;\; \textit{as} \;\; n \to \infty. \tag{3.5}$$

Moreover, there is a subsequence of $\{(\mathcal{L}u_n)(t,x)\}$ which converges to $(\mathcal{L}u)(t,x)$ a.e. in Q.

Proof. It follows from Corollary 3.2 that

$$\mathcal{L}u_n \to \mathcal{L}u \;\text{ weakly in }\; C([0,T];H^{1/2-\varepsilon}(\Omega)).$$

This infers that $(\mathcal{L}u_n)(t,\cdot \to (\mathcal{L}u)(t,\cdot)$ weakly in $H^{1/2-\varepsilon}(\Omega)$ for each t and also that the sequence $\{\mathcal{L}u_n\}$ is bounded in $C([0,T];H^{1/2-\varepsilon}(\Omega))$. Moreover, from [16, Theorem 16.1, p. 99] we know that the embedding of $H^{1/2-\varepsilon}(\Omega)$ into X is compact. So the weak convergence of $(\mathcal{L}u_n)(t,\cdot) \to (\mathcal{L}u)(t,\cdot)$ in $H^{1/2-\varepsilon}(\Omega)$ for each t and the property of compact operators (see, e.g., [30, Theorem 10.7.1, p. 226]) yield that $(\mathcal{L}u_n)(t,\cdot) \to (\mathcal{L}u)(t,\cdot)$ *strongly* in X for each t. Hence we obtain the following results:

(i) $(\mathcal{L}u_n)(t,\cdot)$ are uniformly bounded in X, i.e., there exists $M \geq 0$ such that

$$\|(\mathcal{L}u_n)(t)\|_X \leq M \;\; \forall t \in [0,T] \text{ and } n = 1,\ 2,\ldots;$$

(ii) $\|(\mathcal{L}u_n)(t) - (\mathcal{L}u)(t)\|_X \to 0$ for every $t \in [0,T]$ as $n \to \infty$.

Let us consider a sequence of real-valued nonnegative functions ϕ_n on $[0,T]$ defined by

$$\phi_n(t) := \int_\Omega |(\mathcal{L}u_n)(t,x) - (\mathcal{L}u)(t,x)|^2 dx \;\; \forall t \in [0,T].$$

Then (i) and (ii) imply, respectively, that ϕ_n are uniformly bounded on $[0,T]$ and $\phi_n(t) \to 0$ pointwisely in $[0,T]$ as $n \to \infty$.

Now using the Lebesgue dominated convergence theorem, we arrive at

$$\int_0^T \phi_n(t)dt \to 0 \;\text{ as } n \to \infty.$$

This means that (3.5) holds. Therefore, $\{(\mathcal{L}u_n)(t,x)\}$ contains a subsequence that converges to $(\mathcal{L}u)(t,x)$ for a.e. $(t,x) \in Q$. □

The convergence/continuity results presented above are crucial to justify approximations and limiting procedures developed in this paper. Hereafter we always assume that p is sufficiently big to ensure the convergence property in Proposition 3.4.

To go ahead we need to impose proper assumptions on the integrands in the cost functional (2.8) that ensure semicontinuity properties of (2.8) in the corresponding topologies. The main assumptions are as follows:

(H4a) $g(t,x,y)$ satisfies the Carathéodory condition, i.e., $g(t,x,y)$ is measurable in $(t,x) \in Q$ for all $y \in \mathbf{R}$ and continuous in $y \in \mathbf{R}$ for almost all $(t,x) \in Q$. Moreover, there exist a nonnegative function $\eta(\cdot) \in L^1(Q)$ and a constant $\zeta \geq 0$ such that

$$|g(t,x,y)| \leq \eta(t,x) + \zeta|y|^2 \text{ a.e. } (t,x) \in Q, \ \forall y \in \mathbf{R}. \tag{3.6}$$

(H5a) $\varphi(t,x,w)$ is measurable in $(t,x) \in Q$, continuous and *concave* in $w \in [c,d]$, and for some function $\kappa(\cdot) \in L^1(Q)$ one has
$\varphi(t,x,w) \leq \kappa(t,x)$ a.e. $(t,x) \in Q$, $\forall w \in [c,d]$.

(H6a) $h(t,\xi,u)$ is measurable in $(t,\xi) \in \Sigma$, continuous and *convex* in $u \in [\mu,\nu]$, and for some function $\gamma(\cdot) \in L^1(\Sigma)$ one has
$h(t,\xi,u) \geq \gamma(t,\xi)$ a.e. $(t,\xi) \in \Sigma$, $\forall u \in [\mu,\nu]$.

Let us discuss hypotheses (H4a)-(H6a). The meaning of (H4a) becomes apparent through the following result by Polyak [27, Theorem 2] that is frequently used in the sequel for furnishing limiting processes.

3.5. Proposition. *Let $g(t,x,y)$ satisfy the Carathéodory condition in* (H4a). *Then the growth condition* (3.6) *is necessary and sufficient for the continuity of the functional*

$$G(y) := \iint_Q g(t,x,y)dtdx$$

in the strong topology of $L^2(Q)$.

Further, let us consider hypothesis (H5a) in connection with the second integral term in (2.8) depending on $w(\cdot) \in L^2(Q)$ with $w(t,x) \in [c,d]$ a.e. in Q. It is well known that (H5a) ensures the upper semicontinuity of this functional in the weak topology of $L^2(Q)$ and, moreover, the concavity of $\varphi(t,x,\cdot)$ is a necessary condition for the weak upper semicontinuity; see, e.g., [27]. Symmetrically, the assumptions in (H6a) ensure the lower semicontinuity of the third integral functional in (2.8) in the weak topology of $L^2(\Sigma)$ with $u(t,\xi) \in [\mu,\nu]$ a.e. in Σ.

Now we are ready to prove an existence theorem of optimal solutions in the minimax problem (P).

3.6. Theorem. *Let hypotheses* (H1)-(H3) *and* (H4a)-(H6a) *hold and let, in addition, the integrand g be linear in y. Then the cost functional $J(u,w)$ in* (2.8) *has a saddle point $(\bar{u},\bar{w})$ on $U_{ad} \times W_{ad}$ subject to system* (2.1). *Moreover, if the corresponding trajectory of* (2.1) *satisfies the state constraints* (2.2), *then $(\bar{u},\bar{w})$ is an optimal solution to the original minimax problem* (P).

Proof. Let us consider the functional $J(u, w)$ defined on the set $U_{ad} \times W_{ad} \subset L^p(0, T; U) \times L^2(0, T; W)$ for big p. It is easy to conclude that both U_{ad} and W_{ad} are convex and weakly compact in $L^p(0, T; U)$ and $L^2(0, T; W)$, respectively. Moreover, one can always use the sequential weak topologies on these spaces by virtue of their reflexivity.

Furthermore, let us check that J is convex-concave on $U_{ad} \times W_{ad}$ by the convexity of h in u, concavity of φ in w, and linearity of g in y that implies the linear dependence of g in (u, w). First we show that J is weakly lower semicontinuous with respect to u in the space $L^p(0, T; U)$ for any fixed $w \in L^2(0, T; W)$.

Indeed, let $u_n \to \tilde{u}$ weakly in $L^p(0, T; U)$ as $n \to \infty$. According to the classical Mazur theorem, there is a sequence of convex combinations of u_n that converges to $\tilde{u}$ strongly in $L^p(0, T; U)$. It follows from the norm definitions in $L^p(0, T; U)$ and $U = L^2(\Sigma)$ that the latter sequence also converges to $\tilde{u}$ strongly in $L^2(\Sigma)$. Now employing the convexity of h in u and the arguments similar to [27, Theorem 1], we obtain

$$\iint_\Sigma h(t, \xi, \tilde{u}(t, \xi))dtd\xi \leq \liminf_{n\to\infty} \iint_\Sigma h(t, \xi, u_n(t, \xi))dtd\xi. \tag{3.7}$$

Let us consider the trajectories (mild solutions) y_n and $\tilde{y}$ of system (2.1) generated, respectively, by u_n and $\tilde{u}$ for any fixed w. Using Propositions 3.4 and 3.5, we conclude that

$$\iint_Q g(t, x, \tilde{y}(t, x))dtdx = \lim_{n\to\infty} \iint_Q g(t, x, y_n(t, x))dtdx \tag{3.8}$$

along a subsequence of $\{n\}$. Now relationships (3.7) and (3.8) ensure that the functional $J(\cdot, w)$ in (2.8) is weakly lower semicontinuous on U_{ad} for any fixed w.

To prove the weak upper semicontinuity of $J(u, \cdot)$ on W_{ad} for any fixed u, we use the same (symmetric) arguments taking into account that the weak convergence $w_n \to \tilde{w}$ in $L^2(0, T; W)$ implies even the strong convergence in $C([0, T]; X)$ of the corresponding trajectories $y_n \to \tilde{y}$; see Remark 3.3.

Therefore, the functional $J(u, w)$ in (2.8) is convex and weakly lower semicontinuous in u on the convex and weakly compact set $U_{ad} \subset L^p(0, T; U)$ as well as concave and weakly upper semicontinuous in w on the convex and weakly compact set $W_{ad} \subset L^2(0, T; W)$. Now the existence of a saddle point $(\bar{u}, \bar{w})$ for J on $U_{ad} \times W_{ad}$ subject to system (2.1) follows from the classical (von Neumann) minimax theorem in infinite dimensions (see, e.g., [3, Theorem 3.6 on p. 162]). Obviously, $(\bar{u}, \bar{w})$ is an optimal solution to the original minimax problem (P) if the corresponding trajectory $\bar{y}$ satisfies the state constraints (2.2). This ends the proof of the theorem. □

3.7. Remark. Hypotheses (H4a)–(H6a) on the integrands in (2.8) are required throughout the paper and play a substantial role in the subsequent sections to obtain the main results on the convergence of approximations and their variational analysis. On the contrary, a restrictive assumption about the linearity of g in y is made only in Theorem 3.6 to ensure the existence of a saddle point. This assumption can be removed if one considers saddle points in the framework of "mixed (relaxed) strategies"; cf., e.g., Berkovitz [5] in the context of ODE differential games.

4 Suboptimality Conditions for Worst Perturbations

This section is concerned with the first subproblem (P_1) formulated in Section 2. We can treat (P_1) as an optimal control problem with controls acting in the right-hand side of the parabolic equation. So one might employ optimal control theory for linear parabolic systems with distributed controls to find necessary optimality conditions for the maximal perturbations in (P). Note that the moving state constraints (2.15) involve the *nonregular* (measurable) function $\bar{y}_2(t,x)$, a mild solutions to the Dirichlet problem (2.11), that creates additional complications in the problem under consideration. Now we are going to use an *approximation method* to remove the latter constraints that allows us to obtain strong convergence results; cf. [2, 25]. After that we provide a detailed variational analysis of the approximating problems to derive necessary *suboptimality* conditions for the worst perturbations.

Let $\alpha : \mathbf{R} \rightrightarrows \mathbf{R}$ be a multivalued maximal monotone operator of the form

$$\alpha(r) = \begin{cases} [0,\infty) & \text{if } r = b, \\ (-\infty, 0] & \text{if } r = a, \\ 0 & \text{if } a < r < b, \\ \emptyset & \text{if either } r < a \text{ or } r > b. \end{cases} \tag{4.1}$$

Using the Yosida approximation

$$\epsilon^{-1}(r - (1+\epsilon\alpha)^{-1}r), \; r \in \mathbf{R} \text{ and } \epsilon > 0,$$

of $\alpha(\cdot)$ and then a C_0^∞–mollifier in $\mathbf{R}$, one can get a *smooth* approximation $\alpha_\epsilon(\cdot)$ of the mulitivalued operator (4.1). As noted in [2, p. 322], we may choose $\alpha_\epsilon(r)$ of the following form:

$$\alpha_\epsilon(r) = \begin{cases} \epsilon^{-1}(r-b) - 1/2 & \text{if } r \geq b+\epsilon, \\ (2\epsilon^2)^{-1}(r-b)^2 & \text{if } b \leq r < b+\epsilon, \\ \epsilon^{-1}(r-a) + 1/2 & \text{if } r \leq a-\epsilon, \\ -(2\epsilon^2)^{-1}(r-a)^2 & \text{if } a-\epsilon < r \leq a, \\ 0 & \text{if } a < r < b. \end{cases} \tag{4.2}$$

Then it is easy to check that

$$\epsilon\alpha'_\epsilon(r) = \begin{cases} 1 & \text{if } r \geq b+\epsilon, \\ \epsilon^{-1}(r-b) & \text{if } b \leq r < b+\epsilon, \\ 1 & \text{if } r \leq a-\epsilon, \\ -\epsilon^{-1}(r-a) & \text{if } a-\epsilon < r \leq a, \\ 0 & \text{if } a < r < b \end{cases} \tag{4.3}$$

with $|\epsilon\alpha'_\epsilon(r)| \leq 1$ for all $r \in \mathbf{R}$.

Let $(\bar{u}, \bar{w})$ be the given optimal solution to the minimax problem (P), and let $\bar{y}_1$ and $\bar{y}_2$ be the corresponding trajectories of systems (2.10) and (2.11), respectively. We consider the following parametric family of control problems with no state constraints that approximate the first subproblem (P_1) in Section 2 and depends on the given trajectory $\bar{y}_2$ of the Dirichlet system (2.11):

($\mathrm{P}_{1\epsilon}$) maximize $J_{1\epsilon}(w, y_1) := \iint_Q [g(t,x,y_1(t,x) + \bar{y}_2(t,x)) + \varphi(t,x,w(t,x))]dtdx -$

$\|w - \bar{w}\|^2_{L^2(0,T;W)} - \epsilon\|\alpha_\epsilon(y_1 + \bar{y}_2)\|^2_{L^2(0,T;X)}$

over $w \in W_{ad}$ subject to

$$\begin{cases} \dfrac{\partial y_1}{\partial t} + Ay_1 = Bw + f \;\text{ a.e. in } Q, \\ y_1(0,x) = y_0(x), \; x \in \Omega, \\ y_1(t,\xi) = 0, \; (t,\xi) \in \Sigma. \end{cases} \tag{4.4}$$

Note that $w \in W_{ad}$ and $f \in L^\infty(Q)$. The classical results ensure that system (4.4) has a unique strong solution $y_1 \in W^{1,2}([0,T];X)$ satisfying the estimate

$$\|\frac{\partial y_1}{\partial t}\|_{L^2(0,T;X)} + \|Ay_1\|_{L^2(0,T;X)} \le C(\|y_0\|_{H_0^1(\Omega)\cap H^2(\Omega)} + \|Bw + f\|_{L^2(0,T;X)})$$

(cf. Theorem 4.6 in [2, p. 27]). Let $\{w_n\} \subset W_{ad}$ and $\{y_{1n}\}$ be the corresponding sequence of strong solutions to system (4.4). Standard arguments show that if $w_n \to w \in W_{ad}$ weakly in $L^2(0,T;W)$, then $y_{1n} \to y_1$ strongly in $C([0,T];X)$ as $n \to \infty$ and y_1 is also a strong solution of (4.4) corresponding to w; cf. Remark 3.3 above.

To justify the approximation procedure in (4.2) and (4.4), first we have to show that the maximization problem $(\mathrm{P}_{1\epsilon})$ admits at least one optimal solution. To prove the existence theorem stated below we will follow the line of the classical Weierstrass theorem in infinite dimensions involving properness and upper semicontinuity of a cost functional on a compact feasible set. The main compications in our case are connected with the perturbation term of the cost functional that depends on the (nonregular) mild solution $\bar{y}_2$ of the Dirichlet system (2.11).

4.1. Proposition. *For each y_0 satisfying* (H2) *and each $\epsilon > 0$ problem* $(\mathrm{P}_{1\epsilon})$ *has at least one optimal solution* $(w_\epsilon, y_{1\epsilon}) \in W_{ad} \times W^{1,2}([0,T];X)$.

Proof. First we observe that the set of feasible solutions to problem (P_1) is nonempty because the pair $(\bar{w}, \bar{y}_1)$ is a feasible solution to $(\mathrm{P}_{1\epsilon})$ for any $\epsilon > 0$. Let us show that the cost functional $J_{1\epsilon}$ in $(\mathrm{P}_{1\epsilon})$ is *proper*, i.e., $J_{1\epsilon}(w, y_1) < \infty$ for any $w \in W_{ad}$ and the corresponding trajectory $y_1 \in W^{1,2}([0,T];X)$ of system (4.4). Clearly

$$\iint_Q g(t,x,y_1(t,x) + \bar{y}_2(t,x))dtdx + \iint_Q \varphi(t,x,w(t,x))dtdx < \infty \tag{4.5}$$

for all such (w, y_1) due to assumptions (H4a) and (H5a). Furthermore,

$$\|w - \bar{w}\|_{L^2(0,T;W)} < \infty \quad \forall w \in W_{ad}.$$

Now let us analyse the last term in $J_{1\epsilon}$ depending on $\bar{y}_2$. Due to (2.7) and (3.1) one has

$$\|\bar{y}_2(t)\|_X \le \frac{4M_2M_3\max\{|\mu|,\nu\}\sqrt{\mathrm{mes}(\Gamma)}}{1-4\delta}t^{\frac{1-4\delta}{4}} \quad \text{for any fixed } \delta \in (0,1/4). \tag{4.6}$$

To estimate $\|\alpha_\epsilon(y_1 + \bar{y}_2)\|_{L^2(0,T;X)}$ let us consider the sets

$$\begin{aligned}
\Omega_{1a}^t &:= \{x \in \Omega \mid a - \epsilon < y_1(t,x) + \bar{y}_2(t,x) \le a\};\\
\Omega_{2a}^t &:= \{x \in \Omega \mid y_1(t,x) + \bar{y}_2(t,x) \le a - \epsilon\};\\
\Omega_{1b}^t &:= \{x \in \Omega \mid b \le y_1(t,x) + \bar{y}_2(t,x) < b + \epsilon\};\\
\Omega_{2b}^t &:= \{x \in \Omega \mid y_1(t,x) + \bar{y}_2(t,x) \ge b + \epsilon\}
\end{aligned} \tag{4.7}$$

that are Lebesgue measurable with $\Omega = \Omega_{1a}^t \cup \Omega_{2a}^t \cup \Omega_{1b}^t \cup \Omega_{2b}^t$ for a.e. $t \in [0,T]$. Now taking into account the structure of $\alpha_\epsilon(\cdot)$ in (4.2), we obtain the following estimates:

$$\|\alpha_\epsilon(y_1 + \bar{y}_2)\|_{L^2(0,T;X)} = [\int_0^T \int_\Omega \alpha_\epsilon^2(y_1(t,x) + \bar{y}_2(t,x))dtdx]^{1/2} =$$

$$[\int_0^T (\int_{\Omega_{1a}^t} (2\epsilon^2)^{-2}(y_1(t,x) + \bar{y}_2(t,x) - a)^4 dx + \int_{\Omega_{2a}^t} (\epsilon^{-1}(y_1(t,x) + \bar{y}_2(t,x) - a) + 1/2)^2 dx +$$

$$\int_{\Omega^t_{1b}}(2\epsilon^2)^{-2}(y_1(t,x)+\bar{y}_2(t,x)-b)^4dx+\int_{\Omega^t_{2b}}(\epsilon^{-1}(y_1(t,x)+\bar{y}_2(t,x)-b)-1/2)^2dx)dt]^{1/2}\le$$
$$[\int_0^T(\frac{1}{4}\mathrm{mes}(\Omega^t_{1a})+\frac{1}{4}\mathrm{mes}(\Omega^t_{1b}))dt]^{1/2}+[\int_0^T\int_{\Omega^t_{2a}}(\epsilon^{-1}(y_1(t,x)+\bar{y}_2(t,x)-a)+1/2)^2dxdt]^{1/2}+$$
$$[\int_0^T\int_{\Omega^t_{2b}}(\epsilon^{-1}(y_1(t,x)+\bar{y}_2(t,x)-b)-1/2)^2dxdt]^{1/2}\le$$
$$\frac{3}{2}\sqrt{\mathrm{mes}(Q)}+\epsilon^{-1}[\int_0^T\int_{\Omega^t_{2a}}(\epsilon^{-1}(y_1(t,x)+\bar{y}_2(t,x)-a)^2dxdt]^{1/2}+$$
$$\epsilon^{-1}[\int_0^T\int_{\Omega^t_{2b}}(\epsilon^{-1}(y_1(t,x)+\bar{y}_2(t,x)-b)^2dxdt]^{1/2}\le$$
$$\frac{3}{2}\sqrt{\mathrm{mes}(Q)}+\epsilon^{-1}(|a|+b)\sqrt{\mathrm{mes}(Q)}+2\epsilon^{-1}\|y_1+\bar{y}_2\|_{L^2(0,T;X)}.$$

Combining this with (4.6) and the fact that $y_1\in W^{1,2}([0,T];X)$, we arrive at

$$\|\alpha_\epsilon(y_1+\bar{y}_2)\|_{L^2(0,T;X)}<\infty\ \ \forall\epsilon>0. \tag{4.8}$$

So (4.5) and (4.8) yield

$$J_{1\epsilon}(w,y_1)<\infty\ \ \forall w\in W_{ad}$$

that ensures the properness of the cost functional in $(\mathrm{P}_{1\epsilon})$ for any $\epsilon>0$. Therefore, there exists a real number $j_{1\epsilon}$ such that

$$j_{1\epsilon}=\sup_{w\in W_{ad}}J_{1\epsilon}(w,y_1).$$

For each problem $(\mathrm{P}_{1\epsilon})$ let us consider a maximizing sequence $\{w_n,y_{1n}\}$ where ϵ is omitted for simplicity. ¿From the definition of supremum one has

$$j_{1\epsilon}-\frac{1}{n}\le J_{1\epsilon}(w_n,y_{1n})\le j_{1\epsilon}\ \ \forall n\in\mathbf{N}. \tag{4.9}$$

Recall that W_{ad} is bounded, closed, and convex in $L^2(0,T;W)$. Thus one can extract a subsequence of $\{w_n\}$ (without relabelling) that converges weakly in $L^2(0,T;W)$ to some function $\tilde{w}\in W_{ad}$. Let $\tilde{y}_1$ be a (strong) solution to (4.4) corresponding to $\tilde{w}$. According to the previous discussions we have

$$y_{1n}\to\tilde{y}_1\ \text{ strongly in } C([0,T];X)\text{ as } n\to\infty. \tag{4.10}$$

Furthermore, taking into account assumptions (H4a) and (H5a) as well as concavity and continuity of the function $-\|\cdot\|^2_{L^2(0,T;W)}$, we conclude (cf. the proof of Theorem 3.4) that

$$\limsup_{n\to\infty}(\int\int_Q[g(t,x,y_{1n}(t,x)+\bar{y}_2(t,x))+\varphi(t,x,w_n)]dtdx-\|w_n-\bar{w}\|^2_{L^2(0,T;W)})\le$$
$$\int\int_Q[g(t,x,\tilde{y}_1(t,x)+\bar{y}_2(t,x))+\varphi(t,x,\tilde{w})dtdx-\|\tilde{w}-\bar{w}\|^2_{L^2(0,T;W)}. \tag{4.11}$$

Then it follows from (4.10) and the continuity of (4.2) that

$$\lim_{n\to\infty}\alpha^2_\epsilon(y_{1n}+\bar{y}_2)=\alpha^2_\epsilon(\tilde{y}_1+\bar{y}_2)\ \text{ a.e. in } Q. \tag{4.12}$$

By virtue of (4.9) relationships (4.11) and (4.12) ensure the equality $j_{1\epsilon}=J_{1\epsilon}(\tilde{w},\tilde{y}_1)$ that ends the proof of the proposition. □

Next we need the following technical lemma that is important to justify the required convergence of approximation procedures in this and the subsequent sections.

4.2. Lemma. *Let $y_n(t,x)$, $n = 1,2,\ldots$, and $y(t,x)$ be nonnegative functions belonging to the space $L^1(Q)$. Given $c \geq 0$, consider the sets*

$$Q_n := \{(t,x) \in Q \mid y_n(t,x) > c + 1/n\}$$

defined for each $n = 1,2,\ldots$. Assume that $y_n(t,x) \to y(t,x)$ a.e. in Q and

$$\int\int_{Q_n} y_n(t,x)dtdx \to 0 \quad as \quad n \to \infty. \tag{4.13}$$

Then one has $0 \leq y(t,x) \leq c$ a.e. in Q.

Proof. Proving by contradiction, let us suppose that the conclusion of the lemma does not hold. Then for each small $\rho > 0$ there exists a measurable set $Q_\rho \subset Q$ such that $\text{mes}(Q_\rho) > 0$ and

$$y(t,x) > c + \rho \quad \text{whenever } (t,x) \in Q_\rho. \tag{4.14}$$

Now taking into account the convergence $y_n(t,x) \to y(t,x)$ a.e. in Q and using the classical Egorov theorem, we conclude that for each $\varepsilon > 0$ and $\rho > 0$ there exist a measurable set $Q_\varepsilon \subset Q$ and an integer $K > 0$ independent of (t,x) such that $\rho - 1/n > \rho/2 > 0$, $\text{mes}(Q \setminus Q_\varepsilon) < \varepsilon$, and

$$|y_n(t,x) - y(t,x)| < \rho/2 < \rho - 1/n \quad \text{whenever } n \geq K \text{ and } (t,x) \in Q_\varepsilon.$$

Let us choose $\varepsilon > 0$ provided that $\text{mes}(Q_\rho \cap Q_\varepsilon) \neq 0$. It follows from (4.14) that

$$y_n(t,x) > y(t,x) - \rho + 1/n > c + \rho - \rho + 1/n = c + 1/n \quad \text{whenever } n > K$$

for any $(t,x) \in Q_\rho \cap Q_\varepsilon$, i.e., $(Q_\rho \cap Q_\varepsilon) \subset Q_n$ for all $n > K$. Then from (4.13) and $y_n(t,x) \geq 0$ one has

$$\int\int_{Q_\rho \cap Q_\varepsilon} y_n(t,x)dtdx \to 0 \quad \text{as } n \to \infty.$$

The latter implies that

$$\int\int_{Q_\rho \cap Q_\varepsilon} y(t,x)dtdx = 0$$

by virtue of the uniform convergence $y_n(t,x) \to y(t,x)$ in $Q_\rho \cap Q_\varepsilon$ as $n \to \infty$. Due to the nonnegativity of y we arrive at the conclusion $y(t,x) = 0$ a.e. in $Q_\rho \cap Q_\varepsilon$ that constradicts (2.14). Therefore, we get $0 \leq y(t,x) \leq c$ a.e. in Q and complete the proof of the lemma. □

The next theorem ensures the *strong convergence* of the approximation procedure in this section and justifies *suboptimality* of optimal solutions to the approximating problems $(P_{1\epsilon})$ in the state-constrained problem (P_1) for the worst perturbations.

4.3. Theorem. *Let $(\bar{w}, \bar{y}_1)$ be the given optimal solution to problem (P_1) and let $\{(w_\epsilon, y_{1\epsilon})\}$ be a sequence of optimal solutions to problems $(P_{1\epsilon})$. Then there exists a subsequence of $\{\epsilon\}$ along which*

$$w_\epsilon \to \bar{w} \;\; strongly \;\; in \;\; L^2(0,T;W), \quad y_{1\epsilon} \to \bar{y}_1 \;\; strongly \;\; in \;\; C([0,T];X), \;\; and$$

$$J_{1\epsilon}(w_\epsilon, y_{1\epsilon}) \to J_1(\bar{w}, \bar{y}_1) \;\; as \;\; \epsilon \to 0.$$

Proof. Using the same weak-compactness arguments as in the proof of Proposition 4.1, we find a function $\tilde{w} \in W_{ad}$ and a subsequence of $\{w_\epsilon\}$ (without relabelling) along which

$$w_\epsilon \to \tilde{w} \text{ weakly in } L^2(0,T;W) \text{ as } \epsilon \to 0. \tag{4.15}$$

Moreover, there exists $\tilde{y}_1 \in W^{1,2}([0,T];X)$ satisfying (4.4) with $w = \tilde{w}$ such that

$$y_{1\epsilon} \to \tilde{y}_1 \text{ strongly in } C([0,T];X) \text{ as } \epsilon \to 0. \tag{4.16}$$

Let us show that the pair $(\tilde{w}, \tilde{y}_1)$ is a feasible solution to problem (P_1) in Section 2. To furnish this, it remains to show that $\tilde{y}_1$ satisfies the state constraints (2.15), i.e.,

$$a \le \tilde{y}_1(t,x) + \bar{y}_2(t,x) \le b \text{ a.e. in } Q. \tag{4.17}$$

First we note that $(\bar{w}, \bar{y}_1)$ is feasible to $(P_{1\epsilon})$ with $\alpha_\epsilon(\bar{y}_1 + \bar{y}_2) = 0$ a.e. in Q for all $\epsilon > 0$. Due to the optimality of $(w_\epsilon, y_{1\epsilon})$ in this problem one has

$$J_1(\bar{w}, \bar{y}_1) = J_{1\epsilon}(\bar{w}, \bar{y}_1) \le J_{1\epsilon}(w_\epsilon, y_{1\epsilon}) \;\; \forall \epsilon > 0. \tag{4.18}$$

Using (4.18) and taking into account the structure of the cost functional in $(P_{1\epsilon})$ as well as assumptions (H4a) and (H5a), we conclude that the sequence $\{\epsilon^{1/2}\|\alpha_\epsilon(y_{1\epsilon} + \bar{y}_2)\|_{L^2(0,T;X)}\}$ is bounded. The latter yields

$$\epsilon\|\alpha_\epsilon(y_{1\epsilon} + \bar{y}_2)\|_{L^2(0,T;X)} \to 0 \text{ as } \epsilon \to 0. \tag{4.19}$$

Due to constructions (4.2) and (4.7) we obtain from (4.19) the limiting relationship

$$\begin{aligned} &\int_0^T \int_{\Omega_{1a}^t} (2\epsilon)^{-2}(y_{1\epsilon}(t,x) + \bar{y}_2(t,x) - a)^4 dtdx + \\ &\int_0^T \int_{\Omega_{2a}^t} ((y_{1\epsilon}(t,x) + \bar{y}_2(t,x) - a) + \epsilon/2)^2 dtdx + \\ &\int_0^T \int_{\Omega_{1b}^t} (2\epsilon)^{-2}(y_{1\epsilon}(t,x) + \bar{y}_2(t,x) - b)^4 dtdx + \\ &\int_0^T \int_{\Omega_{2b}^t} ((y_{1\epsilon}(t,x) + \bar{y}_2(t,x) - b) - \epsilon/2)^2 dtdx \to 0 \text{ as } \epsilon \to 0. \end{aligned} \tag{4.20}$$

Note that for almost all $t \in [0,T]$ one has $(y_{1\epsilon}(t,x) + \bar{y}_2(t,x) - a)^4 \le \epsilon^4$ a.e. in Ω_{1a}^t and $(y_{1\epsilon}(t,x) + \bar{y}_2(t,x) - b)^4 \le \epsilon^4$ a.e. in Ω_{1b}^t. This implies that the first and third integrals in (4.20) vanish when $\epsilon \to 0$. Now applying Lemma 4.2 to the second and fourth integrals in (4.20), we arrive at (4.17) and conclude that the pair $(\tilde{w}, \tilde{y}_1)$ is feasible to (P_1). This yields

$$J_1(\tilde{w}, \tilde{y}_1) \le J_1(\bar{w}, \bar{y}_1). \tag{4.21}$$

Using this fact, let us prove the desired strong convergence results of the theorem. First we rewrite (4.18) in the form

$$J_1(\bar{w}, \bar{y}_1) + \epsilon\|\alpha_\epsilon(y_{1\epsilon} + \bar{y}_2)\|^2_{L^2(0,T;X)} + \|w_\epsilon - \bar{w}\|^2_{L^2(0,T;W)} \le J_1(w_\epsilon, y_{1\epsilon}) \tag{4.22}$$

and take the upper limit in the both side of (4.22). Remember that under the assumptions made the functional $J_1(w,y)$ in (2.13) is upper semicontinuous in the weak topology of

$L^2(0,T;W)$ and the norm topology of $C([0,T];X)$; cf. the proof of Proposition 4.1. Empoying this fact together with (4.15), (4.16) and (4.21), we obtain

$$J_1(\bar{w},\bar{y}_1)+\limsup_{\epsilon\to 0}(\epsilon\|\alpha_\epsilon(y_{1\epsilon}+\bar{y}_2)\|^2_{L^2(0,T;X)}+\|w_\epsilon-\bar{w}\|^2_{L^2(0,T;W)})\le$$
$$\limsup_{\epsilon\to 0} J_1(w_\epsilon,y_{1\epsilon})\le J_1(\tilde{w},\tilde{y}_1)\le J_1(\bar{w},\bar{y}_1).$$

The latter yields

$$\lim_{\epsilon\to 0}\epsilon\|\alpha_\epsilon(y_{1\epsilon}+\bar{y}_2)\|^2_{L^2(0,T;X)}=0 \text{ and } \lim_{\epsilon\to 0}\|w_\epsilon-\bar{w}\|^2_{L^2(0,T;W)}=0, \tag{4.23}$$

i.e., $w_\epsilon\to\bar{w}$ strongly in $L^2(0,T;W)$ and, therefore, $y_{1\epsilon}\to\bar{y}_1$ strongly in $C[0,T];X)$ as $\epsilon\to 0$. Finally, the value convergence in the theorem follows from (4.23) and Proposition 3.5 due to assumptions (H4a) and (H5a). This ends the proof of the theorem. □

Now let us conduct a variational analysis of the approximating problems $(P_{1\epsilon})$ to obtain necessary conditions for their optimal solutions $(w_\epsilon,y_{1\epsilon})$. Due to Theorem 4.3 and the splitting procedure the results obtained in this way can be treated as suboptimality conditions for the worst perturbations in the original problem.

To furnish such an analysis let us impose the following additional assumptions:

(H4b) $g(t,x,y)$ is continuously differentiable in y for almost all $(t,x)\in Q$ and $\frac{\partial g}{\partial y}(t,x,y)$ is measurable in (t,x) for any $y\in\mathbf{R}$. Moreover, there is a nonnegative function $\eta_1\in L^2(Q)$ and a constant $\zeta_1\ge 0$ such that

$$|\frac{\partial g}{\partial y}(t,x,y)|\le\eta_1(t,x)+\zeta_1|y| \text{ a.e. } (t,x) \text{ in } Q,\ \forall y\in\mathbf{R}.$$

(H5b) $\varphi(t,x,w)$ is continuously differentiable in w for almost all $(t,x)\in Q$ with $\frac{\partial\varphi}{\partial w}(t,x.w)$ measurable in (t,x) for all $w\in[c,d]$. Moreover, there is a nonnegative function $\kappa_1\in L^1(Q)$ such that

$$|\frac{\partial\varphi}{\partial w}(t,x,w)|\le\kappa_1(t,x) \text{ a.e. } (t,x) \text{ in } Q,\ \forall w\in[c,d].$$

Let us consider the following *adjoint* parabolic equation with homogeneous terminal-boundary conditions:

$$\begin{cases}\frac{\partial\psi_1}{\partial t}-A\psi_1=-\frac{\partial g}{\partial y}(t,x,y_{1\epsilon}+\bar{y}_2)+2\epsilon\alpha'_\epsilon(y_{1\epsilon}+\bar{y}_2)\alpha_\epsilon(y_{1\epsilon}+\bar{y}_2) \text{ a.e. in } Q,\\ \psi_1(T,x)=0,\ x\in \text{cl}\,\Omega,\\ \psi_1(t,\xi)=0,\ (t,\xi)\in\Sigma\end{cases} \tag{4.24}$$

where $\text{cl}\,\Omega:=\Omega\cup\Gamma$ and the elliptic operator A in (H1) is self-adjoint. Clearly, (H4b) implies that $\frac{\partial g}{\partial y}(t,x,y_1(t,x)+\bar{y}_2(t,x))\in L^2(Q)$ for all $y_1\in C([0,T];X)$. Then it follows from Theorem 4.6 in [2, p. 27] that (4.24) has a unique strong solution $\psi_{1\epsilon}\in W^{1,2}([0,T];X)$ satisfying $\psi_{1\epsilon}\in C([0,T];X)\cap L^2(0,T;H^1_0(\Omega)\cap H^2(\Omega))$.

4.4. Theorem. *Let $(w_\epsilon,y_{1\epsilon})$ be an optimal solution to problem $(P_{1\epsilon})$ and let $\psi_{1\epsilon}$ be the corresponding strong solution to system (4.24). Then for any $w\in L^2(0,T;W)$ such that $w_\epsilon+\theta w\in W_{ad}$ for all $\theta\in[0,\theta_0]$ with some $\theta_0>0$, one has*

$$\int\int_Q(B^*\psi_{1\epsilon}+\frac{\partial\varphi}{\partial w}(t,x,w_\epsilon)-2(w_\epsilon-\bar{w}))w\,dtdx\le 0. \tag{4.25}$$

Proof. Let $y_{1\epsilon w}$ be the strong solution of (4.4) corresponding to $w_\epsilon + \theta w$. Then one can easily check that $y_{1\epsilon w} \to y_{1\epsilon}$ strongly in $C([0,T];X)$ as $\theta \to 0$ and

$$\frac{y_{1\epsilon w}(t,x) - y_{1\epsilon}(t,x)}{\theta} = z_{1\epsilon}(t,x) \ \ \forall\theta > 0, \text{ a.e. } (t,x) \in Q \tag{4.26}$$

where $z_{1\epsilon}$ is a strong solution to

$$\begin{cases} \frac{\partial z_1}{\partial t} + Az_1 = Bw \ \text{ a.e. in } Q, \\ z_1(0,x) = 0, \ x \in \Omega, \\ z_1(t,\xi) = 0, \ (t,\xi) \in \Sigma. \end{cases}$$

Define the limits

$$\Delta_1 := \limsup_{\theta\to 0} \iint_Q \frac{g(t,x,y_{1\epsilon w}(t,x) + \bar{y}_2(t,x)) - g(t,x,y_{1\epsilon}(t,x) + \bar{y}_2(t,x))}{\theta} dtdx,$$
$$\Delta_2 := \limsup_{\theta\to 0} \iint_Q \frac{\epsilon\alpha_\epsilon^2(y_{1\epsilon w}(t,x) + \bar{y}_2(t,x)) - \epsilon\alpha_\epsilon^2(y_{1\epsilon}(t,x) + \bar{y}_2(t,x))}{\theta} dtdx.$$

Applying the classical mean value theorem to the integrands above, one gets

$$\Delta_1 = \limsup_{\theta\to 0} \iint_Q \frac{\partial g}{\partial y}(t,x,y_{1\epsilon} + \bar{y}_2 + \theta_1(y_{1\epsilon w} - y_{1\epsilon}))\frac{y_{1\epsilon w}(t,x) - y_{1\epsilon}(t,x)}{\theta} dtdx,$$
$$\Delta_2 = \epsilon \limsup_{\theta\to 0} \iint_Q (\alpha_\epsilon(y_{1\epsilon w} + \bar{y}_2) + \alpha_\epsilon(y_{1\epsilon} + \bar{y}_2)) \times$$
$$\alpha_\epsilon'(y_{1\epsilon} + \bar{y}_2 + \theta_2(y_{1\epsilon w} - y_{1\epsilon}))\frac{y_{1\epsilon w}(t,x) - y_{1\epsilon}(t,x)}{\theta} dtdx$$

where $\theta_1 = \theta_1(t,x)$, $\theta_2 = \theta_2(t,x) \in [0,1]$ a.e. in Q. Then using (4.26), (H4b), and the Lebesgue dominated convergence theorem, we obtain

$$|\iint_Q (\frac{\partial g}{\partial y}(t,x,y_{1\epsilon} + \bar{y}_2 + \theta_1(y_{1\epsilon w} - y_{1\epsilon}))\frac{y_{1\epsilon w} - y_{1\epsilon}}{\theta} - \frac{\partial g}{\partial y}(t,x,y_{1\epsilon} + \bar{y}_2)z_{1\epsilon})dtdx| \leq$$
$$\iint_Q |\frac{\partial g}{\partial y}(t,x,y_{1\epsilon} + \bar{y}_2 + \theta_1(y_{1\epsilon w} - y_{1\epsilon})) - \frac{\partial g}{\partial y}(t,x,y_{1\epsilon} + \bar{y}_2)| \cdot |z_{1\epsilon}| dtdx \to 0 \ \text{ as } \theta \to 0.$$

Thus one has

$$\Delta_1 = \iint_Q \frac{\partial g}{\partial y}(t,x,y_{1\epsilon}(t,x) + \bar{y}_2(t,x))z_{1\epsilon}(t,x)dtdx. \tag{4.27}$$

Note that $\alpha_\epsilon'(\cdot)$ is continuous by (4.3) with $|\epsilon\alpha_\epsilon'(\cdot)| \leq 1$ and $\alpha_\epsilon(y_{1\epsilon w} + \bar{y}_2) + \alpha_\epsilon(y_{1\epsilon} + \bar{y}_2) \in L^2(0,T;X)$ by (4.8). Then (4.26) infers

$$\Delta_2 = 2\epsilon \iint_Q \alpha_\epsilon'(y_{1\epsilon}(t,x) + \bar{y}_2(t,x))\alpha_\epsilon(y_{1\epsilon}(t,x) + \bar{y}_2(t,x))z_{1\epsilon}(t,x)dtdx. \tag{4.28}$$

Apparently $w_\epsilon + \theta w \to w_\epsilon$ strongly in $L^2(Q)$ as $\theta \to 0$ for all w satisfying the conditions of the theorem. Due to assumption (H5b) and the classical mean value theorem one has

$$|\frac{\varphi(t,x,w_\epsilon + \theta w) - \varphi(t,x,w_\epsilon)}{\theta}| = |\frac{\partial\varphi}{\partial w}(t,x,w_\epsilon + \theta_3\theta w)w| \leq \kappa_1(t,x)|w(t,x)|,$$
$$\frac{\partial\varphi}{\partial w}(t,x,w_\epsilon + \theta_3\theta w)w \to \frac{\partial\varphi}{\partial w}(t,x,w_\epsilon)w \ \text{ a.e. in } Q \text{ as } \theta \to 0$$

where $\theta_3 = \theta_3(t,x) \in [0,1]$ a.e. in Q. So the Lebesgue dominated convergence theorem yields

$$\iint_Q \frac{\partial \varphi}{\partial w}(t,x,w_\epsilon + \theta_3\theta w)wdtdx \to \iint_Q \frac{\partial \varphi}{\partial w}(t,x,w_\epsilon)wdtdx \ \text{ as } \theta \to 0. \tag{4.29}$$

Employing the optimality of $(w_\epsilon, y_{1\epsilon})$ in problem $(\mathrm{P}_{1\epsilon})$, we get

$$0 \geq \limsup_{\theta \to 0} \frac{J_{1\epsilon}(w_\epsilon + \theta w, y_{1\epsilon w}) - J_{1\epsilon}(w_\epsilon, y_{1\epsilon})}{\theta} \geq$$
$$\limsup_{\theta\to 0} \iint_Q [\frac{g(t,x,y_{1\epsilon w} + \bar{y}_2) - g(t,x,y_{1\epsilon} + \bar{y}_2)}{\theta} + \frac{\varphi(t,x,w_\epsilon + \theta w) - \varphi(t,x,w_\epsilon)}{\theta}]dtdx -$$
$$\limsup_{\theta\to 0} \iint_Q \frac{(w_\epsilon + \theta w - \bar{w})^2 - (w_\epsilon - \bar{w})^2}{\theta} dtdx -$$
$$\limsup_{\theta\to 0} \epsilon \iint_Q \frac{\alpha_\epsilon^2(y_{1\epsilon w} + \bar{y}_2) - \alpha_\epsilon^2(y_{1\epsilon} + \bar{y}_2)}{\theta} dtdx.$$

By virtue of (4.27)–(4.29) we arrive at the inequality

$$0 \geq \iint_Q (\frac{\partial g}{\partial y}(t,x,y_{1\epsilon} + \bar{y}_2) - 2\epsilon\alpha_\epsilon'(y_{1\epsilon} + \bar{y}_2)\alpha_\epsilon(y_{1\epsilon} + \bar{y}_2))z_{1\epsilon}dtdx +$$
$$\iint_Q (\frac{\partial \varphi}{\partial w}(t,x,w_\epsilon) - 2(w_\epsilon - \bar{w}))wdtdx. \tag{4.30}$$

Now substituting the solution $\psi_{1\epsilon}$ of (4.24) into (4.30) and integrating the latter inequality by parts, one obtains (4.25). This ends the proof of the theorem. □

4.5. Corollary. *For each $\epsilon > 0$ the maximal perturbation w_ϵ in problem $(\mathrm{P}_{1\epsilon})$ satisfies the following bang-bang relations:*

$$w_\epsilon(t,x) = c \ \ a.e. \ \ \{(t,x) \in Q \mid (B^*\psi_{1\epsilon})(t,x) + \frac{\partial \varphi}{\partial w}(t,x,w_\epsilon) - 2(w_\epsilon(t,x) - \bar{w}(t,x)) < 0\},$$
$$w_\epsilon(t,x) = d \ \ a.e. \ \ \{(t,x) \in Q \mid (B^*\psi_{1\epsilon})(t,x) + \frac{\partial \varphi}{\partial w}(t,x,w_\epsilon) - 2(w_\epsilon(t,x) - \bar{w}(t,x)) > 0\}$$

where $\psi_{1\epsilon}$ is the corresponding solution to the adjoint system (4.24).

Proof. Taking $\tilde{w} = w - w_\epsilon$ for any $w \in W_{ad}$, one has $w_\epsilon + \theta\tilde{w} = (1-\theta)w_\epsilon + \theta w \in W_{ad}$ for each $\theta \in [0,1]$. Due to (4.25) with $w = \tilde{w}$ we obtain

$$\iint_Q (B^*\psi_{1\epsilon} + \frac{\partial \varphi}{\partial w}(t,x,w_\epsilon) - 2(w_\epsilon - \bar{w}))(w - w_\epsilon)dtdx \leq 0 \ \ \forall w \in W_{ad} \tag{4.31}$$

that implies the bang-bang relations. □

5 Suboptimal Control under Worst Perturbations

In this section we study the boundary optimal control problem (P_2) stated in Section 2. According to the splitting procedure, the optimal solutions of (P_2) allow us to find optimal boundary controls to the original minimax problem (P) under the worst perturbations.

One can see that problem (P_2) considered is a boundary optimal control problem for parabolic systems with hard control constraints acting in the Dirichlet boundary conditions and moving state constraints generated by the splitting procedure. To remove (approximate) the latter constraints we are going to develop a penalization technique that provides a useful suboptimality information for the original minimax problem.

Let $\alpha(\cdot)$ be the maximal monotone operator defined in (4.1) and $\alpha_\epsilon(\cdot)$ be a smooth approximation of $\alpha(\cdot)$ of the form (4.2). For each $\epsilon > 0$ we consider a parametric family of approximation problems for (P_2) formulated as follows:

$$(P_{2\epsilon}) \quad \text{minimize } J_{2\epsilon}(u, y_2) := \iint_Q g(t, x, \bar{y}_1(t, x) + y_2(t, x))dtdx + \iint_\Sigma h(t, \xi, u(t, \xi))dtd\xi + \|u - \bar{u}\|^p_{L^p(0,T;U)} + \epsilon\|\alpha_\epsilon(\bar{y}_1 + y_2)\|^2_{L^2(0,T;X)}$$

over $u \in U_{ad}$ subject to the system

$$\begin{cases} \dfrac{\partial y_2}{\partial t} + Ay_2 = 0 \text{ a.e. in } Q, \\ y_2(0, x) = 0, \ x \in \Omega, \\ y_2(t, \xi) = u(t, \xi), \ (t, \xi) \in \Sigma. \end{cases} \tag{5.1}$$

Remember that solutions to (5.1) are considered in the *mild* sense, i.e., there exists a (unique) function $y_2 \in C([0, T]; X)$ satisfying

$$y_2(t) = \mathcal{L}u := -A\int_0^t S(t - \tau)Du(\tau)d\tau \ \ \forall t \in [0, T]. \tag{5.2}$$

The next result justifies the existence of optimal solutions to the approximating minimization problem $(P_{2\epsilon})$ for all ϵ.

5.1. Proposition. *For each $\epsilon > 0$ problem $(P_{2\epsilon})$ has at least one optimal solution pair $(u_\epsilon, y_{2\epsilon}) \in U_{ad} \times C([0, T]; X)$.*

Proof. First we note that for any $\epsilon > 0$ problem $(P_{2\epsilon})$ has a feasible pair $(\bar{u}, \bar{y}_2)$ that generated by the given optimal solution $(\bar{u}, \bar{w})$ to the original minimax problem (P). Let (u, y_2) be an arbitrary feasible pair to $(P_{2\epsilon})$. It follows from assumptions (H4a) and (H6a) that

$$\iint_Q g(t, x, \bar{y}_1(t, x) + y_2(t, x))dtdx + \iint_\Sigma h(t, \xi, u(t, \xi))dtd\xi > -\infty. \tag{5.3}$$

To estimate the given trajectory $\bar{y}_1$ for (2.10) we use (2.5) and (2.7) that easily infer

$$\|\bar{y}_1(t)\|_X \leq M_1(e^{-\omega t}\|y_0\|_X + \frac{(\|B\| \max\{|c|, d\} + \|f\|_\infty)\sqrt{\text{mes}(\Omega)}}{\omega}(1 - e^{-\omega t})).$$

Then employing arguments similar to the proof of Proposition 4.1, we conclude that

$$\|\alpha_\epsilon(\bar{y}_1 + y_2)\|_{L^2(0,T;X)} < \infty \ \text{ for each } \epsilon > 0. \tag{5.4}$$

Thus it follows from (5.3), (5.4) and the boundedness of U_{ad} in $L^p(0, T; U)$ that

$$J_{2\epsilon}(u, y_2) > -\infty \ \ \forall u \in U_{ad}, \ \epsilon > 0,$$

i.e., the cost functional $J_{2\epsilon}$ is *proper* in the minimization problem $(P_{2\epsilon})$ for any $\epsilon > 0$.

Fixed $\epsilon > 0$ and taking into account the uniqueness of mild solutions (5.2) corresponding to controls u, we consider the cost functional in $(P_{2\epsilon})$ on the admissible control set U_{ad} equipped with the weak topology of $L^p(0, T; U)$. Now using Corollary 3.2, Propositions 3.4 and 3.5 as well as the convexity of h in u and the continuity of the operator $\alpha_\epsilon(\cdot)$ in (4.2), we conclude that for big p the cost functional in $(P_{2\epsilon})$ is weakly semicontinuous in $L^p(0,T;U)$ on the weakly compact set U_{ad}; cf. the proof of Theorem 3.6. Thus the existence of optimal solutions in $(\mathrm{P}_{2\epsilon})$ follows from the classical Weierstrass theorem. □

Next let us prove the *strong convergence* of optimal solutions for the approximating problems $(\mathrm{P}_{2\epsilon})$ to the given optimal solution $(\bar{u},\bar{y}_2)$ of the state-constrained problem (P_2).

5.2. Theorem. *Let $(\bar{u},\bar{y}_2)$ be the given optimal solution to problem (P_2) and let $\{(u_\epsilon,y_{2\epsilon})\}$ be a sequence of optimal solutions to the approximating problems $(\mathrm{P}_{2\epsilon})$. Then there is a subsequence of $\{\epsilon\}$ along which*

$$u_\epsilon \to \bar{u} \text{ strongly in } L^p(0,T;U),\ y_{2\epsilon} \to \bar{y}_2 \text{ strongly in } C([0,T];X), \text{ and}$$
$$J_{2\epsilon}(u_\epsilon,y_{2\epsilon}) \to J_2(\bar{u},\bar{y}_2) \text{ as } \epsilon \to 0.$$

Proof. From the optimality of $(u_\epsilon,y_{2\epsilon})$ in $(\mathrm{P}_{2\epsilon})$ and the feasibility of $(\bar{u},\bar{y}_2)$ in this problem one has

$$J_{2\epsilon}(u_\epsilon,y_{2\epsilon}) \le J_{2\epsilon}(\bar{u},\bar{y}_2) = J_2(\bar{u},\bar{y}_2) \ \ \forall \epsilon > 0. \tag{5.5}$$

This implies, in particular, that there is a constant $M > 0$ independent of ϵ such that

$$\epsilon\|\alpha_\epsilon(\bar{y}_1 + y_{2\epsilon})\|^2_{L^2(0,T;X)} \le M, \ \text{ i.e.,}$$

$$\epsilon\|\alpha_\epsilon(\bar{y}_1 + y_{2\epsilon})\|_{L^2(0,T;X)} \to 0 \ \text{ as } \epsilon \to 0. \tag{5.6}$$

Due to the weak compactness of U_{ad} in $L^p(0,T;U)$ we can find a function $\tilde{u} \in U_{ad}$ and a subsequence of $\{u_\epsilon\}$ along which

$$u_\epsilon \to \tilde{u} \ \text{ weakly in } L^p(0,T;U) \text{ as } \epsilon \to 0. \tag{5.7}$$

Denote by $\tilde{y}_2$ the mild solution of (5.1) corresponding to $\tilde{u}$. Now using Proposition 3.4, we can find a subsequence of $\{\epsilon\}$ such that

$$y_{2\epsilon}(t,x) \to \tilde{y}_2(t,x) \ \text{ a.e. in } Q \tag{5.8}$$

along this subsequence for p sufficiently big. Then following the procedure in the proof of Theorem 4.3 with the usage of (5.6) and Lemma 4.2, we conclude that

$$a \le \bar{y}_1(t,x) + \tilde{y}_2(t,x) \le b \ \text{ a.e. in } Q,$$

i.e., $(\tilde{u},\tilde{y}_2)$ is a feasible solution to the state-constrained problem (P_2). Therefore,

$$J_2(\tilde{u},\tilde{y}_2) \ge J_2(\bar{u},\bar{y}_2). \tag{5.9}$$

Now let us pass to the limit in (5.5) as $\epsilon \to 0$ taking into account (5.7)–(5.9) and the lower semicontinuity of the functional (2.14) on U_{ad} in the weak topology of $L^p(0,T;U)$ (see the proof of Theorem 3.6). This yields

$$\lim_{\epsilon\to 0}\|u_\epsilon - \bar{u}\|^p_{L^p(0,T;U)} = 0 \ \text{ and } \ \lim_{\epsilon\to 0}\epsilon\|\alpha_\epsilon(\bar{y}_1 + y_{2\epsilon})\|^2_{L^2(0,T;X)} = 0. \tag{5.10}$$

The first equality in (5.10) means that $u_\epsilon \to \bar{u}$ as $\epsilon \to 0$ *strongly* in $L^p(0,T;U)$. For big p the latter implies the *strong* convergence $y_{2\epsilon} \to \bar{y}_2$ in $C([0,T];X)$ by virtue of

Proposition 3.1. The value convergence in Theorem 5.2 follows from the solution convergence obtained and the second equality in (5.10). This ends the proof of the theorem. □

Next let us establish necessary optimality conditions for the approximating problems ($P_{2\epsilon}$) under the following additional assumptions on the integrand h:

(H6b) $h(t,\xi,u)$ is continuously differentiable in u with the derivative measurable in (t,ξ). Moreover, there is a nonnegative function $\gamma_1 \in L^q(\Sigma)$ such that $|\frac{\partial h}{\partial u}(t,\xi,u)| \leq \gamma_1(t,\xi)$ a.e. $(t,\xi) \in \Sigma$, $\forall u \in [\mu,\nu]$ where $1/p + 1/q = 1$.

Let $(u_\epsilon, y_{2\epsilon})$ be an optimal solution to problem ($P_{2\epsilon}$) for any fixed $\epsilon > 0$. Consider feasible variations of u_ϵ of the form $u_\epsilon + \theta u \in U_{ad}$ with $u \in L^p(0,T;U)$ where $\theta \in [0,\theta_0]$ for some $\theta_0 > 0$. Denote by $y_{2\epsilon u}$ the mild solution of (5.1) corresponding to $u_\epsilon + \theta u$ and consider a function $\phi : [0,\theta_0] \to \mathbf{R}$ defined by

$$\phi(\theta) := J_{2\epsilon}(u_\epsilon + \theta u, y_{2\epsilon u}). \tag{5.11}$$

It follows from the definition that ϕ attains its minimum at $\theta = 0$. Moreover, it is easy to see from (5.2) and (3.3) for big p that

$$\begin{gathered} y_{2\epsilon u} \to y_{2\epsilon} \text{ strongly in } C([0,T]; H^{1/2-\epsilon}(\Omega)) \text{ as } \theta \to 0 \text{ and} \\ \frac{y_{2\epsilon u}(t,x) - y_{2\epsilon}(t,x)}{\theta} = \mathcal{L}u \quad \forall \theta > 0, \text{ a.e. } (t,x) \in Q. \end{gathered} \tag{5.12}$$

The following results provide necessary conditions for optimality of $(u_\epsilon, y_{2\epsilon})$ in ($P_{2\epsilon}$) ensuring, due to Theorem 5.2, suboptimality conditions to the state-constrained problem (P_2).

5.3. Theorem. *Let $(u_\epsilon, y_{2\epsilon})$ be an optimal solution to problem ($P_{2\epsilon}$) and let $\mathcal{L}^* : (C([0,T];X))^*$
$\to L^q(0,T;U)$ be the adjoint operator to the operator $\mathcal{L}$ in (3.2). Then one has*

$$\begin{aligned} 0 \leq & \iint_\Sigma [\mathcal{L}^*(\frac{\partial g}{\partial y}(t,x,\bar{y}_1 + y_{2\epsilon}) + 2\epsilon\alpha'_\epsilon(\bar{y}_1 + y_{2\epsilon})\alpha_\epsilon(\bar{y}_1 + y_{2\epsilon})) + \frac{\partial h}{\partial u}(t,\xi,u_\epsilon)]u dt d\xi + \\ & 2p\int_0^T \|u_\epsilon - \bar{u}\|_U^{p-2}(\int_\Gamma (u_\epsilon - \bar{u})u d\xi)dt \end{aligned} \tag{5.13}$$

where $u \in L^p(0,T;U)$ such that $u_\epsilon + \theta u \in U_{ad}$ for all $\theta \in [0,\theta_0]$ with some $\theta_0 > 0$.

Proof. Taking into account that the function $\phi(\cdot)$ in (5.11) has its minimum at $\theta = 0$ and using the classical mean value theorem, we get

$$\begin{aligned} & 0 \leq \liminf_{\theta\to 0} \frac{\phi(\theta) - \phi(0)}{\theta} = \\ & \liminf_{\theta\to 0} \frac{1}{\theta}[\iint_Q (g(t,x,\bar{y}_1 + y_{2\epsilon u}) - g(t,x,\bar{y}_1 + y_{2\epsilon}))dtdx + \\ & \iint_\Sigma (h(t,\xi,u_\epsilon + \theta u) - h(t,\xi,u_\epsilon))dtd\xi + (\|u_\epsilon + \theta u - \bar{u}\|^p_{L^p(0,T;U)} - \|u_\epsilon - \bar{u}\|^p_{L^p(0,T;U)}) + \\ & \epsilon(\|\alpha_\epsilon(\bar{y}_1 + y_{2\epsilon u})\|^2_{L^2(0,T;X)} - \|\alpha_\epsilon(\bar{y}_1 + y_{2\epsilon})\|^2_{L^2(0,T;X)})] = \end{aligned}$$

$$\liminf_{\theta\to 0}\frac{1}{\theta}[\iint_Q \frac{\partial g}{\partial y}(t,x,\bar{y}_1+y_{2\epsilon}+\theta_1(y_{2\epsilon u}-y_{2\epsilon}))(y_{2\epsilon u}-y_{2\epsilon})dtdx +$$
$$\iint_\Sigma \frac{\partial h}{\partial u}(t,\xi,u_\epsilon+\theta_2\theta u)\theta u dtd\xi +$$
$$\int_0^T (\|u_\epsilon+\theta u-\bar{u}\|_U^{p-2}+\cdots+\|u_\epsilon-\bar{u}\|_U^{p-2})(\int_\Gamma \theta u(2u_\epsilon-2\bar{u}+\theta u)d\xi)dt +$$
$$\epsilon\iint_Q (\alpha_\epsilon(\bar{y}_1+y_{2\epsilon u})+\alpha_\epsilon(\bar{y}_1+y_{2\epsilon}))\alpha'_\epsilon(\bar{y}_1+y_{2\epsilon}+\theta_3(y_{2\epsilon u}-y_{2\epsilon}))(y_{2\epsilon u}-y_{2\epsilon})dtdx]$$

where $\theta_i=\theta_i(t,x)\in[0,1]$ a.e. in Q for $i=1,2,3$. Observe that $\theta_i(y_{2\epsilon u}-y_{2\epsilon})\to 0$ strongly in $L^2(Q)$ as $\theta\to 0$ for $i=1,2,3$ and that $\alpha_\epsilon(\bar{y}_1+y_{2\epsilon u})+\alpha_\epsilon(\bar{y}_1+y_{2\epsilon})\in L^2(0,T;X)$. Then similarly to Section 4, by using assumptions (H4b) and (H6b), Proposition 3.5, and the Lebesgue dominated convergence theorem, we obtain

$$0\le \iint_Q (\frac{\partial g}{\partial y}(t,x,\bar{y}_1+y_{2\epsilon})+2\epsilon\alpha'_\epsilon(\bar{y}_1+y_{2\epsilon})\alpha_\epsilon(\bar{y}_1+y_{2\epsilon}))\mathcal{L}u dtdx +$$
$$\iint_\Sigma \frac{\partial h}{\partial u}(t,\xi,u_\epsilon)u dtd\xi + 2p\int_0^T \|u_\epsilon-\bar{u}\|_U^{p-2}(\int_\Gamma (u_\epsilon-\bar{u})u d\xi)dt. \tag{5.14}$$

The latter implies (5.13) end ends the proof of the theorem. □

5.4. Corollary. *For each $\epsilon>0$ the optimal control u_ϵ to* ($P_{2\epsilon}$) *satisfies the following bang-bang relations*:

$$u_\epsilon(t,\xi) = \mu \text{ a.e. } \{(t,\xi)\in\Sigma \mid \mathcal{L}^*(\frac{\partial g}{\partial y}(t,x,\bar{y}_1+y_{2\epsilon})+2\epsilon\alpha'_\epsilon(\bar{y}_1+y_{2\epsilon})\alpha_\epsilon(\bar{y}_1+y_{2\epsilon}))+ \frac{\partial h}{\partial u}(t,\xi,u_\epsilon)+2p\|u_\epsilon-\bar{u}\|_U^{p-2}(u_\epsilon-\bar{u})<0\},$$
$$u_\epsilon(t,\xi) = \nu \text{ a.e. } \{(t,\xi)\in\Sigma \mid \mathcal{L}^*(\frac{\partial g}{\partial y}(t,x,\bar{y}_1+y_{2\epsilon})+2\epsilon\alpha'_\epsilon(\bar{y}_1+y_{2\epsilon})\alpha_\epsilon(\bar{y}_1+y_{2\epsilon}))+ \frac{\partial h}{\partial u}(t,\xi,u_\epsilon)+2p\|u_\epsilon-\bar{u}\|_U^{p-2}(u_\epsilon-\bar{u})>0\}.$$

Proof. Let $u=\hat{u}-\bar{u}$ in (5.13) for any $\hat{u}\in U_{ad}$. Taking $\theta_0=1$, one has $u_\epsilon+\theta u=(1-\theta)u_\epsilon+\theta\hat{u}\in U_{ad}$ for all $\theta\in[0,\theta_0]$. Then the bang-bang relations follow directly from (5.13). □

6 Necessary Optimality Conditions with State Constraints

In the last part of this paper we furnish the limiting processes to derive necessary optimality conditions for the original minimax control problem (P). They are based on passing to the limit in the necessary optimality conditions for the approximating problems ($P_{1\epsilon}$) and ($P_{2\epsilon}$) by taking into account the splitting procedure and the strong convergence results proved in the previous sections. First let us summarize the approximation and suboptimality results obtained for the given optimal solution $(\bar{u},\bar{w})$ to the original problem (P).

6.1. Theorem *Let $(\bar{u},\bar{w})$ be an optimal solution to the minimax control problem* (P) *and let $\bar{y}$ be the corresponding mild trajectory of system* (2.1). *Assume that all the hypotheses*

(H1)–(H6) *are fulfilled and that p is sufficiently big. Then for each* $\epsilon > 0$ *there exist optimal solutions* $\{(w_\epsilon, y_{1\epsilon})\}$ *and* $\{(u_\epsilon, y_{2\epsilon})\}$ *to problems* $(\mathrm{P}_{1\epsilon})$ *and* $(\mathrm{P}_{1\epsilon})$, *respectively, such that*

$$(u_\epsilon, w_\epsilon, y_\epsilon) = (u_\epsilon, w_\epsilon, y_{1\epsilon} + y_{2\epsilon}) \to (\bar{u}, \bar{w}, \bar{y})$$
$$\textit{strongly in } L^p(0,T;U) \times L^2(0,T;W) \times C([0,T];X) \ \textit{as} \ \epsilon \to \infty$$

and the necessary optimality conditions in Theorems 4.4 *and* 5.3 *hold.*

Analysing the necessary conditions of Theorems 4.4 and 5.3, we can observe that to pass to the limit therein one needs to get a uniform bound for the term $\epsilon\alpha'_\epsilon(\cdot)\alpha_\epsilon(\cdot)$. Such a bound does not follow from the previous consideration without additional assumptions. To furnish this, let us impose the following *constrained qualification conditions* for the state constraints in the minimax problem (P) that take into account the nature of this problem through the splitting procedure of Section 2. It what follows $\|\cdot\|_\infty$ denotes the norm in $L^\infty(Q)$.

(CQ1) *There exist* $\tilde{w} \in W_{ad}$ *and* $\eta_1 > 0$ *such that for all* $\zeta \in L^\infty(Q)$ *with* $\|\zeta\|_\infty \le 1$ *and the strong solution* $\tilde{y}_1$ *of* (2.10) *corresponding to* $\tilde{w}$ *one has*

$$a \le \tilde{y}_1(t,x) + \bar{y}_2(t,x) + \eta_1\zeta(t,x) \le b \ \ \textit{a.e. in } Q.$$

(CQ2) *There exist* $\tilde{u} \in U_{ad}$ *and* $\eta_2 > 0$ *such that for all* $\zeta \in L^\infty(Q)$ *with* $\|\zeta\|_\infty \le 1$ *and the mild solution* $\tilde{y}_2$ *of* (2.11) *corresponding to* $\tilde{u}$ *one has*

$$a \le \bar{y}_1(t,x) + \tilde{y}_2(t,x) + \eta_2\zeta(t,x) \le b \ \ \textit{a.e. in } Q.$$

Note that the qualification conditions imposed are different from the classical Slater interiority condition in the corresponding spaces. In particular, they do *not* imply that the sets of feasible trajectories y_1 and y_2 have nonempty interiority in the spaces $W^{1,2}([0,T];X)$ and $C([0,T];X)$, respectively. We refer the reader to [4] and [25] for more discussions on the related qualification conditions for the case of parabolic systems with distributed controls.

The next lemma provides desired uniform estimates that turn out to be crucial in the limiting procedures developed below.

6.2. Lemma. *Let* $(\bar{u}, \bar{w}, \bar{y})$, $(w_\epsilon, y_{1\epsilon})$, *and* $(u_\epsilon, y_{2\epsilon})$ *satisfy the conditions in Theorem* 6.1. *Assume, in addition, that the qualification conditions* (CQ1) *and* (CQ2) *hold. Then there exists a constant* $C > 0$ *independent of* ϵ *such that for any* $\epsilon > 0$ *one has the estimates*

$$\|\epsilon\alpha'_\epsilon(y_{1\epsilon} + \bar{y}_2)\alpha_\epsilon(y_{1\epsilon} + \bar{y}_2)\|_1 \le C, \tag{6.1}$$
$$\|\epsilon\alpha'_\epsilon(\bar{y}_1 + y_{2\epsilon})\alpha_\epsilon(\bar{y}_1 + y_{2\epsilon})\|_1 \le C \tag{6.2}$$

where $\|\cdot\|_1$ *denotes the norm in* $L^1(Q)$.

Proof. First let us consider inequality (4.3) and put there $w = \tilde{w} - w_\epsilon$ where $\tilde{w}$ satisfies the qualification condition (CQ1). Employing the latter condition and the monotonicity of $\alpha_\epsilon(\cdot)$ in (4.2), we have

$$0 \ge \iint_Q \frac{\partial g}{\partial y}(t,x,y_{1\epsilon} + \bar{y}_2)(\tilde{y}_1 - y_{1\epsilon})dtdx +$$

$$\iint_Q (\frac{\partial \varphi}{\partial w}(t,x,w_\epsilon) - 2(w_\epsilon - \bar{w}))(\tilde{w} - w_\epsilon)dtdx -$$
$$2\iint_Q \epsilon\alpha_\epsilon'(y_{1\epsilon} + \bar{y}_2)\alpha_\epsilon(y_{1\epsilon} + \bar{y}_2))(\tilde{y}_1 - y_{1\epsilon})dtdx =$$
$$\iint_Q \frac{\partial g}{\partial y}(t,x,y_{1\epsilon} + \bar{y}_2)(\tilde{y}_1 - y_{1\epsilon})dtdx + \iint_Q (\frac{\partial \varphi}{\partial w}(t,x,w_\epsilon) - 2(w_\epsilon - \bar{w}))(\tilde{w} - w_\epsilon)dtdx +$$
$$2\iint_Q \epsilon\alpha_\epsilon'(y_{1\epsilon} + \bar{y}_2)(\alpha_\epsilon(y_{1\epsilon} + \bar{y}_2) - \alpha_\epsilon(\tilde{y}_1 + \bar{y}_2 + \eta_1\zeta))(y_{1\epsilon} + \bar{y}_2 - \tilde{y}_1 - \bar{y}_2 - \eta_1\zeta)dtdx +$$
$$2\iint_Q \epsilon\alpha_\epsilon'(y_{1\epsilon} + \bar{y}_2)\alpha_\epsilon(y_{1\epsilon} + \bar{y}_2)\eta_1\zeta dtdx \geq$$
$$\iint_Q \frac{\partial g}{\partial y}(t,x,y_{1\epsilon} + \bar{y}_2)(\tilde{y}_1 - y_{1\epsilon})dtdx + \iint_Q (\frac{\partial \varphi}{\partial w}(t,x,w_\epsilon) - 2(w_\epsilon - \bar{w}))(\tilde{w} - w_\epsilon)dtdx +$$
$$2\iint_Q \epsilon\alpha_\epsilon'(y_{1\epsilon} + \bar{y}_2)\alpha_\epsilon(y_{1\epsilon} + \bar{y}_2)\eta_1\zeta dtdx \quad \forall \zeta \in L^\infty(Q) \text{ with } \|\zeta\|_\infty \leq 1.$$

Now it follows from assumptions (H4b), (H5b) and Theorem 4.3 that there is a constant $C > 0$ independent of ϵ such that

$$\iint_Q \epsilon\alpha_\epsilon'(y_{1\epsilon} + \bar{y}_2)\alpha_\epsilon(y_{1\epsilon} + \bar{y}_2)\zeta dtdx \leq C \quad \forall \epsilon > 0 \text{ and } \zeta \in L^\infty(Q) \text{ with } \|\zeta\|_\infty \leq 1.$$

The latter obviously implies (6.1).

Next let us consider inequality (5.14) and put there $u = \tilde{u} - u_\epsilon$ where $\tilde{u}$ satisfies the qualification condition (CQ2). Using this condition and the monotonicity of $\alpha_\epsilon(\cdot)$, we obtain

$$0 \leq \iint_Q \frac{\partial g}{\partial y}(t,x,\bar{y}_1 + y_{2\epsilon})\mathcal{L}(\tilde{u} - u_\epsilon)dtdx + \iint_\Sigma \frac{\partial h}{\partial u}(t,\xi,u_\epsilon)(\tilde{u} - u_\epsilon)dtd\xi +$$
$$2p\int_0^T \|u_\epsilon(t) - \bar{u}(t)\|_U^{p-2}(\int_\Gamma (u_\epsilon - \bar{u})(\tilde{u} - u_\epsilon)d\xi)dt +$$
$$2\iint_Q \epsilon\alpha_\epsilon'(\bar{y}_1 + y_{2\epsilon})\alpha_\epsilon(\bar{y}_1 + y_{2\epsilon})(\mathcal{L}\tilde{u} - \mathcal{L}u_\epsilon)dtdx \leq$$
$$\iint_Q \frac{\partial g}{\partial y}(t,x,\bar{y}_1 + y_{2\epsilon})(\tilde{y}_2 - y_{2\epsilon})dtdx + \iint_\Sigma \frac{\partial h}{\partial u}(t,\xi,u_\epsilon)(\tilde{u} - u_\epsilon)dtd\xi +$$
$$2p\int_0^T \|u_\epsilon(t) - \bar{u}(t)\|_U^{p-2}(\int_\Gamma (u_\epsilon - \bar{u})(\tilde{u} - u_\epsilon)d\xi)dt -$$
$$2\iint_Q \epsilon\alpha_\epsilon'(\bar{y}_1 + y_{2\epsilon})(\alpha_\epsilon(\bar{y}_1 + y_{2\epsilon}) - \alpha_\epsilon(\bar{y}_1 + \tilde{y}_2 + \eta_2\zeta))(\bar{y}_1 + y_{2\epsilon} - \bar{y}_1 - \tilde{y}_2 - \eta_2\zeta)dtdx -$$
$$2\eta_2\iint_Q \epsilon\alpha_\epsilon'(\bar{y}_1 + y_{2\epsilon})\alpha_\epsilon(\bar{y}_1 + y_{2\epsilon})\zeta dtdx \leq$$
$$\iint_Q \frac{\partial g}{\partial y}(t,x,\bar{y}_1 + y_{2\epsilon})(\tilde{y}_2 - y_{2\epsilon})dtdx + \iint_\Sigma \frac{\partial h}{\partial u}(t,\xi,u_\epsilon)(\tilde{u} - u_\epsilon)dtd\xi +$$
$$2p\int_0^T \|u_\epsilon(t) - \bar{u}(t)\|_U^{p-2}(\int_\Gamma (u_\epsilon - \bar{u})(\tilde{u} - u_\epsilon)d\xi)dt -$$
$$2\eta_2\iint_Q \epsilon\alpha_\epsilon'(\bar{y}_1 + y_{2\epsilon})\alpha_\epsilon(\bar{y}_1 + y_{2\epsilon})\zeta dtdx \quad \forall \zeta \in L^\infty(Q) \text{ with } \|\zeta\|_\infty \leq 1.$$

It follows from (H4b), (H6b) and Theorem 5.2 that there is a constant $C > 0$ independent of $\epsilon > 0$ such that

$$\iint_Q \epsilon\alpha_\epsilon'(\bar{y}_1 + y_{2\epsilon})\alpha_\epsilon(\bar{y}_1 + y_{2\epsilon})\zeta dtdx \leq C \quad \forall \epsilon > 0 \text{ and } \zeta \in L^\infty(Q) \text{ with } \|\zeta\|_\infty \leq 1.$$

The latter estimate yields (6.2) and ends the proof of the lemma. □

Along the optimal trajectory $\bar{y}(t,x)$ to the minimax problem (P) we define the set

$$Q_{ab} := \{(t,x) \in Q \mid \bar{y}(t,x) = a \text{ or } \bar{y}(t,x) = b\}.$$

where the state constraints (2.2) are *active.* This set plays an essential role to characterize limits of the functions in (6.1) and (6.2) that can be considered as elements of the space $(L^\infty(Q))^*$.

Recall [8, Theorem 16 on p. 298] that the space $(L^\infty(Q))^*$ can be identified with the space $ba(Q)$ of those bounded additive functions (*generalized measures*) on subsets of Q that vanish on sets of the Lebesgue measure zero. This means that for any $\Lambda \in (L^\infty(Q))^*$ there is a unique $\lambda \in ba(Q)$ such that

$$\Lambda(\beta) = \iint_Q \beta\lambda(dtdx) \ \ \forall \beta \in L^\infty(Q). \tag{6.3}$$

It what follows we will not distinguish between $(L^\infty(Q))^*$ and $ba(Q)$, i.e., we identify Λ and λ in (6.3). For any $\lambda \in L^\infty(Q))^*$ we consider its *support set* supp λ where λ is not zero. Recall that such a support set is defined to within subsets of the Lebesgue measure zero on Q.

In what follows, convergence along a *generalized sequence* means the convergence of a *net* in the weak* topology of the space $(L^\infty(Q))^*$ where the topological and sequential limits are different.

6.3. Lemma. *Let all the assumptions of Lemma 6.2 hold. Then there exist* $\lambda_i \in (L^\infty(Q))^*$ *with* supp $\lambda_i \subset Q_{ab}$, $i = 1,2$, *and a generalized sequence of* $\{\epsilon\}$ *along which*

$$\begin{aligned} &2\epsilon\alpha'_\epsilon(y_{1\epsilon} + \bar{y}_2)\alpha_\epsilon(y_{1\epsilon} + \bar{y}_2) \to \lambda_1 \ \ weakly^* \ \ in \ \ (L^\infty(Q))^* \ \ and \\ &2\epsilon\alpha'_\epsilon(\bar{y}_1 + y_{2\epsilon})\alpha_\epsilon(\bar{y}_1 + y_{2\epsilon}) \to \lambda_2 \ \ weakly^* \ \ in \ \ (L^\infty(Q))^* \ \ as \ \ \epsilon \to 0. \end{aligned} \tag{6.4}$$

Proof. We consider only the first relationship (6.4); the proof of the second one is similar. For any $\epsilon > 0$ define a linear functional on $L^\infty(Q)$ by the formula

$$\Lambda_{1\epsilon}(\beta) := 2\iint_Q \epsilon\alpha'_\epsilon(y_{1\epsilon} + \bar{y}_2)\alpha_\epsilon(y_{1\epsilon} + \bar{y}_2)\beta dtdx \ \ \forall \beta \in L^\infty(Q).$$

Due to Lemma 6.2 one has

$$|\Lambda_{1\epsilon}(\beta)| \le C\|\beta\|_\infty \ \ \forall \beta \in L^\infty(Q)$$

that yields the boundedness of the set $\{\Lambda_{1\epsilon}\}$ in the space $(L^\infty(Q))^*$. Employing the well-known result on the weak* compactness of the unit ball in a dual space, we find an element $\Lambda_1 \in (L^\infty(Q))^*$ and a generalized sequence of $\{\epsilon\}$ along which

$$\lim_{\epsilon \to 0} \Lambda_{1\epsilon}(\beta) = \lim_{\epsilon \to 0} 2\iint_Q \epsilon\alpha'_\epsilon(y_{1\epsilon} + \bar{y}_2)\alpha_\epsilon(y_{1\epsilon} + \bar{y}_2)\beta dtdx = \Lambda_1(\beta) \ \ \forall \beta \in L^\infty(Q). \tag{6.5}$$

Now considering $\lambda_1 \in ba(Q)$ corresponding to Λ_1 by virtue of (6.3), we conclude from (6.5) and the definition of the weak* convergence that (6.4) holds.

It remains to show that supp $\lambda_1 \subset Q_{ab}$. Note that due to the state constraints (2.2) the set

$$\{(t,x) \in Q \mid \bar{y}(t,x) < a \ \text{ or } \ \bar{y}(t,x) > b\}$$

has measure zero. Thus assuming that supp $\lambda_1 \not\subset Q_{ab}$, we find a set $\tilde{Q}$ such that

$$\begin{aligned} &\text{mes}(\tilde{Q}) > 0, \ \ \lambda_1(\tilde{Q}) \neq 0, \ \ \text{and} \\ &\tilde{Q} \subset \{(t,x) \in Q |\ a < \bar{y}_1(t,x) + \bar{y}_2(t,x) < b\}. \end{aligned} \tag{6.6}$$

The latter implies that

$$\tilde{Q} \subset \cup_{r>0} Q_r \ \text{ where } \ Q_r := \{(t,x) \in Q \mid a + r \leq \bar{y}_1(t,x) + \bar{y}_2(t,x) \leq b - r\}.$$

Noting that $Q_{r_1} \subset Q_{r_2}$ if $r_1 > r_2$, we get

$$\text{mes}(\tilde{Q} \cap Q_r) \neq 0 \ \text{ for all small } r > 0. \tag{6.7}$$

Moreover, for any $\delta > 0$ one can find $\tilde{r} > 0$ such that

$$\text{mes}(\tilde{Q} \setminus Q_{\tilde{r}}) \leq \text{mes}(\cup_{r>0} Q_r \setminus Q_{\tilde{r}}) < \delta. \tag{6.8}$$

Now employing the convergence $y_{1\epsilon} \to \bar{y}$ in Theorem 4.3 and then using the Egorov theorem, we find a set $Q_\rho \subset Q_{\tilde{r}} \cap \tilde{Q}$ with $\text{mes}((Q_{\tilde{r}} \cap \tilde{Q}) \setminus Q_\rho) < \rho$ and a subsequence of $\{y_{1\epsilon}(t,x)\}$ that converges to $\bar{y}_1(t,x)$ uniformly in Q_ρ. When $\rho > 0$ is sufficiently small, we have from (6.7) that $\text{mes}(Q_\rho) \neq 0$ and

$$a < y_{1\epsilon}(t,x) + \bar{y}_2(t,x) < b \ \text{ in } Q_\rho \text{ for small } \epsilon.$$

Due to (4.2) the latter yields

$$\epsilon \alpha'_\epsilon(y_{1\epsilon}(t,x) + \bar{y}_2(t,x))\alpha_\epsilon(y_{1\epsilon}(t,x) + \bar{y}_2(t,x)) = 0 \ \text{ in } Q_\rho \text{ for small } \epsilon. \tag{6.9}$$

Observe in addition that

$$\tilde{Q} = (\tilde{Q} \cap Q_{\tilde{r}}) \cup (\tilde{Q} \setminus Q_{\tilde{r}}) = Q_\rho \cup ((\tilde{Q} \cap Q_{\tilde{r}}) \setminus Q_\rho) \cup (\tilde{Q} \setminus Q_{\tilde{r}}).$$

Now let us consider any $\beta \in L^\infty(Q)$ with supp $\beta \subset \tilde{Q}$. Denoting

$$\gamma_\epsilon(t,x) := 2\epsilon\alpha'_\epsilon(y_{1\epsilon}(t,x) + \bar{y}_2(t,x))\alpha_\epsilon(y_{1\epsilon}(t,x) + \bar{y}_2(t,x))\beta(t,x), \tag{6.10}$$

we have

$$\Lambda_{1\epsilon}(\beta) = \iint_{Q_\rho} \gamma_\epsilon(t,x)dtdx + \iint_{(\tilde{Q}\cap Q_{\tilde{r}})\setminus Q_\rho} \gamma_\epsilon(t,x)dtdx + \iint_{\tilde{Q}\setminus Q_{\tilde{r}}} \gamma_\epsilon(t,x)dtdx. \tag{6.11}$$

Since $\gamma_\epsilon \in L^1(Q)$ and δ is sufficiently small in (6.8), one gets

$$|\iint_{\tilde{Q}\setminus Q_{\tilde{r}}} \gamma_\epsilon(t,x)dtdx| < \varepsilon \ \ \forall \varepsilon > 0.$$

Taking into account this fact, relationships (6.5), (6.9)-(6.11), $\text{mes}((\tilde{Q} \cap Q_{\tilde{r}}) \setminus Q_\rho) < \rho$, $\beta \in L^\infty(\tilde{Q})$, and estimate (6.1), we conclude that there is a nonnegative function $c(\rho)$ such that $c(\rho) \to 0$ as $\rho \to 0$ and

$$|\Lambda_1(\beta)| \leq c(\rho) \ \text{ for any } \ \beta \in L^\infty(Q) \text{ with supp } \beta \subset \tilde{Q} \tag{6.12}$$

when ρ is sufficiently small. Therefore, $\Lambda_1(\beta) = 0$ for all β in (6.12). This contradicts assumption (6.6) and ends the proof of the lemma. □

Now we are ready to prove necessary optimality conditions for the original minimax problem (P) with state constraints. First let us obtain results that characterize the worst perturbations in (P). Given $\bar{y} \in C([0,T];X)$ and $\lambda_1 \in (L^\infty(Q))^*$, we consider the *adjoint system*

$$\begin{cases} \dfrac{\partial \psi_1}{\partial t} - A\psi_1 = -\dfrac{\partial g}{\partial y}(t,x,\bar{y}) + \lambda_1 \text{ a.e. in } Q, \\ \psi_1(T,x) = 0, \ x \in \operatorname{cl}\Omega, \\ \psi_1(t,\xi) = 0, \ (t,\xi) \in \Sigma \end{cases} \tag{6.13}$$

and define its solution $\psi_1(t,x)$ in the following sense:

$$\iint_Q \psi_1(t,x)(\frac{\partial v}{\partial t} + Av)dtdx = \iint_Q \frac{\partial g}{\partial y}(t,x,\bar{y}(t,x))vdtdx - \iint_Q v\lambda_1(dtdx) \tag{6.14}$$
$$\forall v \in W_0^{2,1,\infty}(Q)$$

The next theorem shows that, along optimal processes to (P), there is a solution to (6.13) belonging to the space $BV(0,T;H^{-1}(\Omega))$ of $H^{-1}(\Omega)$-valued finctions with bounded variation on $[0,T]$ and satisfying some additional conditions.

6.4. Theorem. *Let* $(\bar{u},\bar{w},\bar{y})$ *be an optimal triple in problem* (P) *under assumptions* (H1)–(H5) *and let the qualification condition* (CQ1) *hold. Then there exist a measure* $\lambda_1 \in (L^\infty(Q))^*$ *with* $\operatorname{supp}\lambda_1 \subset Q_{ab}$ *and a trajectory* $\psi \in BV(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;X)$ *of the adjoint system* (6.13) *such that*

$$\iint_Q (B^*\psi_1 + \frac{\partial \varphi}{\partial w}(t,x,\bar{w}))(w - \bar{w})dtdx \le 0 \ \ \forall w \in W_{ad}. \tag{6.15}$$

Proof. We prove this theorem by passing to the limit in the necessary optimality conditions of Theorem 4.4 for the approximating problems $(P_{1\epsilon})$. Let $\psi_{1\epsilon}$ be the strong solution to the adjoint system (4.24) corresponding to $(u_\epsilon, y_{1\epsilon})$ in Theorem 4.4. Multiplying (4.24) by $v \in W_0^{2,1,\infty}(Q)$ and integrating the latter by parts, we get

$$\iint_Q \psi_{1\epsilon}(\frac{\partial v}{\partial t} + Av)dtdx = \iint_Q \frac{\partial g}{\partial y}(t,x,y_{1\epsilon} + \bar{y}_2)vdtdx - $$
$$\iint_Q 2\epsilon\alpha'_\epsilon(y_{1\epsilon} + \bar{y}_2)\alpha_\epsilon(y_{1\epsilon} + \bar{y}_2)vdtdx \ \ \forall v \in W_0^{2,1,\infty}(Q). \tag{6.16}$$

The strong solution $\psi_{1\epsilon}$ to (4.24) can be represented in the form

$$\psi_{1\epsilon}(t) = -\int_t^T S(\tau - t)(\frac{\partial g}{\partial y}(\tau,x,y_{1\epsilon} + \bar{y}_2) - 2\epsilon\alpha'_\epsilon(y_{1\epsilon} + \bar{y}_2)\alpha_\epsilon(y_{1\epsilon} + \bar{y}_2))d\tau \ \ \forall t \in [0,T] \tag{6.17}$$

where the semigroup $S(\cdot)$ generated by the operator $-A$ has the *contraction property* in $L^1(\Omega)$; see [6]. Employing in (6.17) the latter property and estimate (6.1), we find a constant $M > 0$ independent of ϵ and t such that

$$\|\psi_{1\epsilon}(t)\|_{L^1(\Omega)} \le$$
$$\int_t^T \|S(\tau - t)(\frac{\partial g}{\partial y}(\tau,x,y_{1\epsilon} + \bar{y}_2) - 2\epsilon\alpha'_\epsilon(y_{1\epsilon} + \bar{y}_2)\alpha_\epsilon(y_{1\epsilon} + \bar{y}_2))\|_{L^1(\Omega)}d\tau \le$$
$$\int_t^T \|S(\tau - t)\| \cdot \|\frac{\partial g}{\partial y}(\tau,x,y_{1\epsilon} + \bar{y}_2) - 2\epsilon\alpha'_\epsilon(y_{1\epsilon} + \bar{y}_2)\alpha_\epsilon(y_{1\epsilon} + \bar{y}_2))\|_{L^1(\Omega)}d\tau \le$$
$$\|\frac{\partial g}{\partial y}\|_1 + \|2\epsilon\alpha'_\epsilon(y_{1\epsilon} + \bar{y}_2)\alpha_\epsilon(y_{1\epsilon} + \bar{y}_2))\|_1 \le M < \infty \ \ \forall t \in [0,T], \ \epsilon > 0.$$

This means that $\{\psi_{1\epsilon}\}$ is bounded in $C([0,T];L^1(\Omega))$. Moreover, it follows from (4.24) and (6.1) that the sequences $\{\frac{\partial\psi_{1\epsilon}}{\partial t} - A\psi_{1\epsilon}\}$ is bounded in $L^1(Q)$. Employing the Sobolev imbedding theorem, we conclude that $\frac{\partial\psi_{1\epsilon}}{\partial t}$ is bounded in $L^1(0,T;H^{-1}(\Omega))$. Furthermore, based on (4.24) and the previous estimates, one gets the boundedness of $\{\psi_{1\epsilon}\}$ in $L^2(0,T;H_0^1(\Omega))$ and $L^\infty(0,T;X)$ similarly to [2, Section 5.1.2] and [25, Section 4.2.1]. Now involving standard compactness arguments, we find a function $\psi_1 \in BV(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;X)$ and a subsequence of $\{\psi_{1\epsilon}\}$ (without relabelling) such that

$$\begin{gathered}\psi_{1\epsilon}(t) \to \psi_1(t) \text{ strongly in } H^{-1}(\Omega),\\ \psi_{1\epsilon} \to \psi_1 \text{ strongly in } L^2(0,T;H_0^1(\Omega)), \text{ and}\\ \psi_{1\epsilon} \to \psi_1 \text{ weakly}^* \text{ in } L^\infty(0,T;X) \text{ as } \epsilon \to 0.\end{gathered}$$

Passing to the limit in (6.16) as $\epsilon \to 0$ and taking Lemma 6.3 into account, we conclude that ψ_1 satisfies the adjoint system (6.13) in the sense of (6.14). Finally, to obtain (6.15) we pass to the limit in condition (4.31) that immediately follows from (4.25). Using Theorem 4.3, (H3), and the convergence $\psi_{1\epsilon} \to \psi_1$ in $L^2(0,T;X)$, we arrive at (6.15) and end the proof of the theorem. □

6.5. Corollary. *Under the assumptions of Theorem* 6.4 *the maximal perturbation* $\bar{w}$ *in problem* (P) *satisfies the following bang-bang principle*:

$$\bar{w}(t,x) = c \text{ a.e. } \{(t,x) \in Q \mid (B^*\psi_1)(t,x) + \frac{\partial\varphi}{\partial w}(t,x,\bar{w}(t,x)) < 0\},$$

$$\bar{w}(t,x) = d \text{ a.e. } \{(t,x) \in Q \mid (B^*\psi_1)(t,x) + \frac{\partial\varphi}{\partial w}(t,x,\bar{w}(t,x)) > 0\} \tag{6.18}$$

where $\psi_1(t,x)$ *is the corresponding solution to the adjoint system* (6.13).

Proof. This follows directly from (6.15). □

Next we are going to obtain necessary optimality conditions for optimal boundary controllers in the minimax problem (P) by passing to the limit in the necessary optimality conditions for the approximating problems ($P_{2\epsilon}$). To furnish this we need to show that the Dirichlet operator $\mathcal{L}$ in (3.2) is *continuous* from $L^\infty(\Sigma)$ into $L^\infty(\Omega)$. The following theorem contains this property and provides the desired optimality conditions for the original problem .

6.6. Theorem. *Let* $(\bar{u},\bar{w},\bar{y})$ *be an optimal triple in problem* (P) *under assumptions* (H1)–(H4), (H6) *and let the qualification condition condition* (CQ2) *hold. Then there is a measure* $\lambda_2 \in (L^\infty(Q))^*$ *with support* supp $\lambda_2 \subset Q_{ab}$ *such that for any* $u \in U_{ad}$ *one has*

$$0 \le \iint_\Sigma [\mathcal{L}^*(\frac{\partial g}{\partial y}(t,x,\bar{y})) + \frac{\partial h}{\partial u}(t,\xi,\bar{u})](u-\bar{u})dtd\xi + \iint_\Sigma (u-\bar{u})(\mathcal{L}^*\lambda_2)(dtd\xi) \tag{6.19}$$

Proof. Let $\{(u_\epsilon, y_{2\epsilon})\}$ be a sequence of optimal solutions to (P_ϵ) that strongly converges to $(\bar{u},\bar{y}_2)$ due to Theorem 5.2 and satisfies necessary optimality conditions in Theorem 5.3. It directly follows from (5.13) that

$$0 \le \iint_\Sigma [\mathcal{L}^*(\frac{\partial g}{\partial y}(t,x,\bar{y}_1+y_{2\epsilon}) + 2\epsilon\alpha_\epsilon'(\bar{y}_1+y_{2\epsilon})\alpha_\epsilon(\bar{y}_1+y_{2\epsilon})) + \frac{\partial h}{\partial u}(t,\xi,u_\epsilon)](u-u_\epsilon)dtd\xi +$$
$$2p\int_0^T \|u_\epsilon - \bar{u}\|_U^{p-2}(\int_\Gamma (u_\epsilon-\bar{u})(u-u_\epsilon)d\xi)dt \quad \forall u \in U_{ad}; \tag{6.20}$$

cf. the proof of Corollary 4.5. We need to pass to the limit in (6.20) as $\epsilon \to 0$ (along a generalized sequence, without relabelling). Due to Theorem 5.2 and the well-known continuity of the operator $\mathcal{L}^* : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Gamma))$ (see, e.g., [14]) one has

$$\iint_\Sigma [\mathcal{L}^*(\frac{\partial g}{\partial y}(t,x,\bar{y}_1+y_{2\epsilon}) + \frac{\partial h}{\partial u}(t,\xi,u_\epsilon)](u-u_\epsilon)dtd\xi \to$$
$$\iint_\Sigma [\mathcal{L}^*(\frac{\partial g}{\partial y}(t,x,\bar{y})) + \frac{\partial h}{\partial u}(t,\xi,\bar{u})](u-\bar{u})dtd\xi \;\; \forall u \in U_{ad}$$

and the last term in (6.20) converges to 0 as $\epsilon \to 0$. To get (6.19) from (6.20) it remains to show that for any $u \in U_{ad}$ one has

$$\iint_\Sigma (u-u_\epsilon)\mathcal{L}^*(2\epsilon\alpha'_\epsilon(\bar{y}_1+y_{2\epsilon})\alpha_\epsilon(\bar{y}_1+y_{2\epsilon}))dtd\xi \to \iint_\Sigma (u-\bar{u})(\mathcal{L}^*\lambda_2)(dtd\xi) \quad (6.21)$$

as $\epsilon \to 0$ along a generalized subsequence. Taking Lemma 6.3 into account, one can conclude that (6.21) follows from the continuity of the operator $\mathcal{L}^* : (L^\infty(Q))^* \to (L^\infty(\Sigma))^*$ in the weak* topologies of the spaces. This weak* continuity of the adjoint operator is a direct consequence of the strong continuity of the operator $\mathcal{L}$ in (3.2) considered from $L^\infty(\Sigma)$ into $L^\infty(Q)$. To justify the latter continuity we involve some results from the theory of *generalized solutions* to parabolic equations along with previous considerations.

Let us consider a function $v \in L^2(\Sigma)$ in the Dirichlet boundary condition for (2.11). Employing [15, Theorem 9.1], we know that there is a unique $y(v) \in L^2(Q)$, called a generalized solution to (2.11), such that

$$\iint_Q y(v)(-\frac{\partial z}{\partial t} + Az)dtdx = -\iint_\Sigma v\frac{\partial v}{\partial \nu_A}dtd\xi \quad (6.22)$$
$$\forall z \in \{z \in H^{2,1}(Q) \mid z(t,\xi)=0,\ (t,\xi)\in\Sigma,\ z(T,x)=0\}.$$

Now let $v \in L^\infty(\Sigma)$ and let $y = \mathcal{L}v$ be the corresponding mild solution to (2.11). We are going to show that such y satisfies (6.22), i.e., coincides with the generalized solution to (2.11) in this case. Let us consider the given controller v as an element of the space $L^p(0,T;U)$ for big p and then use the fact that the space $\mathcal{D}(\Sigma)$ is dense in $L^p(0,T;U)$, i.e., there is a sequence $\{v_n\} \subset \mathcal{D}(\Sigma)$ with

$$v_n \to v \text{ strongly in } L^p(0,T;U) \text{ as } n \to \infty.$$

It is well known that for each $v_n \in \mathcal{D}(\Sigma)$ system (2.11) has a unique classical solution y_n that automatically is a mild solution and a generalized solution to (2.11). Therefore, $y_n = \mathcal{L}(v_n)$ and y_n satisfies (6.22) for all $n = 1,2,\ldots$. Moreover, it follows from Proposition 3.1 that

$$\|\mathcal{L}v - y_n\|_{C([0,T];X)} = \|\mathcal{L}v - \mathcal{L}v_n\|_{C([0,T];X)} \to 0 \text{ as } n \to \infty.$$

Employing all these facts, one has

$$|\iint_Q \mathcal{L}v(-\frac{\partial z}{\partial t} + Az)dtdx + \iint_\Sigma v\frac{\partial z}{\partial \nu_A}dtd\xi| \le$$
$$|\iint_Q (\mathcal{L}v - y_n)(-\frac{\partial z}{\partial t} + Az)dtdx| + |\iint_\Sigma (v-v_n)\frac{\partial z}{\partial \nu_A}dtd\xi| \le$$
$$\|\mathcal{L}v - y_n\|_{C([0,T];X)} \cdot \| - \frac{\partial z}{\partial t} + Az\|_{L^2(0,T;X)}T^{1/2} +$$
$$\|v - v_n\|_{L^p(0,T;U)}\|\frac{\partial z}{\partial \nu_A}\|_{L^2(0,T;U)}T^{1/\hat{q}} \to 0 \text{ as } n \to \infty$$

where $\tilde{q} := \frac{2(p-1)}{p-2}$. Thus we obtain

$$\iint_Q \mathcal{L}v(-\frac{\partial z}{\partial t} + Az)dtdx = -\iint_\Sigma v\frac{\partial z}{\partial \nu_A}dtd\xi$$
$$\forall z \in \{z \in H^{2,1}(Q) \mid z(t,\xi) = 0,\ (t,\xi) \in \Sigma,\ z(T,x) = 0\}.$$

The latter means that the mild solution $y = \mathcal{L}v$ is a generalized solution to (2.11) for any $v \in L^\infty(\Sigma)$. Using the uniqueness of generalized solutions and the fact that the generalized solution operator is a continuous map of $L^\infty(\Sigma) \to L^\infty(Q)$ (see [15, pp. 205–206]), we conclude that the linear operator $\mathcal{L}$ is continuous from $L^\infty(\Sigma)$ into $L^\infty(Q)$. This allows us to pass to the limit in (6.21) and finish the proof of the theorem. □

Summarizing the results obtained, we come up to the following theorem that contains necessary optimality consitions for both worst disturbances and optimal controllers in the original minimax problem.

6.7. Theorem. *Let* $(\bar{u}, \bar{w})$ *be an optimal solution to the minimax problem* (P) *and let* $\bar{y}$ *be the corresponding trajectory of system* (2.1). *Assume that all the hypotheses* (H1)–(H6) *and the constraint qualification conditions* (CQ1), (CQ2) *hold. Then there are measures* $\lambda_i \in (L^\infty(Q))^*$ *with* supp $\lambda_i \subset Q_{ab}$, $i = 1,2$, *an adjoint trajectory* $\psi_1 \in BV(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;X)$ *satisfying* (6.14) *such that conditions* (6.15) *and* (6.19) *are fulfilled and the worst disturbance* $\bar{w}$ *obeys the bang-bang relations* (6.18).

REFERENCES

1. Balakrishnan AV (1981) Applied Functional Analysis, 2nd edition. Springer-Verlag, New York

2. Barbu V (1993) Analysis and Control of Nonlinear Infinite Dimensional Systems. Academic Press, Boston

3. Barbu V , Precupanu T (1986) Convexity and Optimization in Banach Spaces. Reidel Publishing, Dordrecht–Boston–Lancaster

4. Bergounioux M, Tiba D (1995) General optimality conditions for constrained convex control problems. SIAM J Control Optim, to appear

5. Berkovitz LD (1985) The existence of value and saddle point in games of fixed duration. SIAM J Control Optim 23: 172–196

6. Brézis H, Strauss W (1973) Semi-linear second-order elliptic equations in L^1. J Math Soc Jap 25: 565–590

7. Bonnans JF, Tiba D (1987) Equivalent control problems and applications. Lecture Notes in Control and Information Sciences 97: 154–161, Springer-Verlag, Berlin

8. Dunford N, Schwartz JT (1958) Linear Operators. Part I. Intersciense, New York

9. Fattorini HO, Murphy T (1994) Optimal boundary control of nonlinear parabolic equations. Lecture Notes in Pure and Applied Mathematics 160: 91–109. Marcel Dekker, New York

10. Fattorini HO, Murphy T (1994) Optimal problems for nonlinear parabolic boundary control systems. SIAM J Control Optim 32: 1577–1596

11. Friedman A (1987) Optimal control for parabolic variational inequalities. SIAM J Control Optim 25: 482–497

12. He ZX (1987) State constrained control problems governed by variational inequalities. SIAM J Control Optim 25: 119–144

13. Lasiecka I, Triggiani R (1983) Dirichlet boundary control problem for parabolic equations with quadratic cost: analyticity and Riccati's feedback synthesis. SIAM J Control Optim 21: 41–67

14. Lasiecka I, Triggiani R (1987) The regulator problem for parabolic equations with Dirichlet boundary control. Part I: Riccati's feedback synthesis and regularity of optimal solutions. Appl Math Optim 16: 147–168

15. Lions JL (1971) Optimal Control of Systems Governed by Partial Differential Equations. Springer-Verlag, Berlin

16. Lions JL, Magenes E (1972) Non–Homogeneous Boundary Value Problems and Applications. Part I. Springer-Verlag, New York

17. Mackenroth U (1982) Convex parabolic boundary control problems with pointwise state constraints. J Math Anal Appl 87: 256–277

18. Mordukhovich BS (1990) Minimax design for a class of distributed control systems. Autom Remote Control 50: 1333–1340

19. Mordukhovich BS, Zhang K (1993) Robust optimal control for a class of parabolic distributed systems. Proc 1993 Amer Cont Conf: 466–467

20. Mordukhovich BS, Zhang K (1994) Approximation results for robust control of heat-diffusion equations. Proc 1994 Amer Cont Conf: 2648–2649

21. Mordukhovich BS, Zhang K (1994) Minimax boundary control problems for parabolic systems with state constraints. Proc 1994 Amer Cont Conf: 3085–3089

22. Mordukhovich BS, Zhang K (1994) Feedback control for state-constrained heat equations with uncertain disturbances. Proc 33rd Conf Dec Contr: 1774–1775

23. Mordukhovich BS, Zhang K (1994) Existence, approximation, and suboptimality conditions for minimax control of heat transfer systems with state constraints. Lecture Notes in Pure and Applied Mathematics 160: 251–270. Marcel Dekker, New York

24. Neittaanmäki P, Tiba D (1988) A variational inequality approach to constrained control problems for parabolic equations. Appl Math Optim 17: 185–201

25. Neittaanmäki P, Tiba D (1994) Optimal Control of Nonlinear Parabolic Systems. Marcel Dekker, New York

26. Pazy N (1983) Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer-Verlag, New York

27. Poljak BT (1969) Semicontinuity of integral functionals and existence theorems on extremal problems. Math. USSR Sb. 7: 59–77

28. Tröltzsch F (1984) Optimality Conditions for Parabolic Control Problems and Applications. Teubner Texte, Leipzig

29. Washburn D (1979) A bound on the boundary input map for parabolic equations with applications to time optimal control. SIAM J Control Optim 17: 652–671

30. Wouk A (1979) A Course of Applied Functional Analysis. John Wiley & Sons, New York.