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APPLICATION OF DECISION ANALYSIS TO POLLUTION CONTROL: THE RHINE RIVER STUDY

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Abstract

As water resources are more intensively used and as water quality deteriorates there is an increasing need for improved decision making processes to manage them. Since both economic and social criteria, and several interest groups with often conflicting preferences are involved, multi-dimensional utility functions are employed in the analysis.

This paper presents a preliminary application of a model employing several forms of utility functions to the control of water quality on the Rhine River in which optimal treatment levels are found by simultaneously solving a system of non-linear equations. The applicability of additive and multiplicative forms of utility objective functions is studied, and the relation of this model to real world decision making is described.

Application of Decision Analysis to Pollution Control: The Rhine River Study*

Antony R. Ostrom** and Jacques G. Gros***

Introduction

Rational decision making affecting water quality generally involves aggregating incommensurate inputs from many sources. All too often, the decision making process is dominated by those criteria which can be easily quantified; putting little weight on subjective information. A need exists for techniques that can incorporate all the relevant inputs in one rational framework. This paper describes an approach that applies decision analysis and optimal control theory to resource management problems, and specifically to water quality management. In particular, concern centers on three broad topics: difficulty in defining the problem, difficulty in treating multiple objectives, and difficulty in handling uncertainty.

A precise definition of the problem to be studied is of critical importance. What is the complete set of technologically feasible alternatives, one of which will be found to be

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the optimal decision, that should be compared? Who are the decision makers: one person, a group, or several groups with overlapping jurisdictions? What are their objectives, how are they measured, and do they conflict? What is the decision making procedure that will be followed, and what sources of information does the decision maker use? What are the time scales of the decision, of the implementation of the decision, and of the impacts? Who is significantly affected by the decision, do their preferences differ from those of the decision makers, and should their feelings be included in the analysis?

Water resource and pollution control decisions have environmental, monetary, and social effects. As these are often incommensurate, it is natural to inquire how they can be included logically in one objective function. A related question is how some of these effects can be quantified, since there might exist no natural measures of some of them. Also, how does one put into commensurate units impacts over time -- especially where discounting may be inappropriate (see Meyer [12])?

Finally, how can uncertainty be handled? There is uncertainty in predicting the environmental impacts of a decision, in predicting costs, and in predicting actions of opposing and of friendly interest groups. Some of these uncertainties result from a lack of knowledge, others from incomplete information, others still from the unpredictability of nature.

These are some of the issues that should be considered in the analysis of water resource problems. We will present an approach that explicitly incorporates many of them. In particular, it includes trade-offs among objectives and among groups, and provides a structure for discussion of management alternatives. Objective and subjective aspects of the problem can be formally included, and uncertainty in the problem components is handled in a rigorous way.

Sensitivity analysis of variations in the preferences of the decision maker can be easily carried out; in fact we will present such a sensitivity analysis as part of our application of the approach to a pollution control problem of the Rhine River. The purpose of the application is to find operating rules for pollution control plants and to obtain quality levels for different reaches of the Rhine as it flows through the Federal Republic of Germany. The approach used is to maximize the expected value of utility under varying and uncertain ambient river conditions. This work should be viewed as one step towards finding analytic techniques for studying "...the next problem in public planning: the multiobjective multiple-decision-maker problem" (Cohon and Marks, [3]).

The Approach

There are too many impacts, interest groups, and considerations of equity for one person to integrate all the components of a water pollution control decision problem and come up with the "best" decision. Therefore, we propose a decision analytic approach to river basin pollution control. Decision analysis provides a framework for the systematic consideration of most of the relevant information and for assessing and including the preferences of the decision making group in the analysis. It allows the analyst to break the problem into its component parts, analyze them separately, and then recombine them into the original problem. This recombination is done in a systematic fashion, the decision maker providing the logic.

One of the first steps in the approach is to find an appropriate set of objectives and associated measures of effectiveness that indicate the degree to which they are achieved. (We refer to these measurable quantities as attributes.) This is done by the decision maker with the help of the analyst. For our pollution control problem on the Rhine, the objectives could be:

1. Minimize environmental degradation;
2. Minimize cost;
3. Maximize human health and safety.

These are far too abstract for actual use in the analysis. Objectives exist in hierarchies (Keeney and Raiffa [7]). The three we mentioned are upper level objectives; below them are a set of sub-objectives that describe them in a more useful fashion; for instance, "minimize water pollution." One usually associates "BOD," "COD," "DO" or "parts per million of some chemical contaminant" with the sub-objective "minimize water pollution." (See O'Conner [14] for some of the issues involved in the development of indices for water quality.) These are not direct measures of the objective; they are correlates used because the primary property is inherently unmeasurable or because the natural measure is analytically intractable. It is not altogether necessary to use such well-established measures; other subjective measures that make sense to the decision maker can also be used. If necessary, the analysis can be redone to determine how the choice of alternative measures biases the results.

It is desirable that the measurable quantities (attributes) be complete -- that is, that they include whatever could influence the decision; be of minimum size, for ease of analysis; and be non-redundant so that impacts are not double-counted.

Let us assume that, based on discussions with the decision maker, a set of n attributes has been found for our pollution control problem. Let X_i represent the i^{th} attribute, and x_i refer to a specific value of the attribute X_i. The consequence $\underline{x} = (x_1, x_2, \ldots, x_N)$ of implementing a particular alternative can be described in terms of the levels x_i of the attributes. But these consequences cannot be predicted with certainty. Therefore, associated with each set of consequence levels $\underline{x}$ is a probability density $p_v(\underline{x})$ for the v^{th} alternative. These densities can be obtained from historical records, from subjective feelings, or by combinations of the two (see Schlaifer [17]).

One of the most noteworthy aspects of decision analysis is the degree to which subjective information can be incorporated. We include it by quantifying the decision maker's preferences in terms of a utility function. This function (denoted $u(\underline{x})$) has two desirable properties:

a) $u(x_1', x_2', \ldots, x_N') \geq u(x_1'', x_2'', \ldots, x_N'')$ if and only if $(x_1', x_2', \ldots, x_N')$ is preferred to $(x_1'', x_2'', \ldots, x_N'')$; that is, the utility function is a monotonically increasing function of the decision maker's preferences; and b), in situations involving uncertainty, the expected value of $u(\underline{x})$ is the appropriate guide for decision making. (These two properties follow from some axioms specified in von Neumann and Morgenstern [13], and Pratt, Raiffa and Schlaifer [15].) In other words, the decision problem becomes finding the alternative "v" that maximizes

$$\int u(\underline{x})\ p_v(\underline{x})\ d\underline{x} \quad . \tag{1}$$

It would be convenient if the preference assessment could be divided into a number of small, simple tasks. Keeney [6] has presented a review of some simple multi-attribute utility function forms. He has shown that two independence properties are of critical importance in establishing the proper form, and unless these properties hold, certain simple utility function forms are theoretically inappropriate. He calls these properties value independence and utility independence. A set of attributes is said to be <u>value independent</u> if preferences among gambles depend only on the marginal probability density functions of the individual attributes. (In general, preferences among gambles would depend on the joint probability density function and not just on the marginal functions.) A set of N-1 attributes is said to be <u>utility independent</u> of the remaining attribute (the complement) if preferences among gambles over the set, with the complement held at a fixed level, do not depend on what the fixed level is.

If value independence holds, then the multi-attribute utility function $u(\underline{x})$ can be written in the additive form:

$$u(\underline{x}) = \sum_{i=1}^{N} k_i u_i(x_i) \quad , \tag{2}$$

where $u_i(x_i)$ is the single-attribute utility function for attribute i and the k_i's are scaling constants. On the other hand, if each N-1 attribute set is utility independent of its complement, then the multi-attribute utility function can be written either in the additive form, as above, or in the multiplicative form:

$$1 + k\, u(\underline{x}) = \prod_{i=1}^{N} \left(1 + k k_i u_i(x_i)\right) \quad , \tag{3}$$

where k and k_i are constants, with $\sum_{i=1}^{N} k_i = 1$ in the additive form and $\sum_{i=1}^{N} k_i \neq 1$ in the multiplicative form ($k_i \geq 0$). Usually, utility functions are scaled from 0 for the least preferred situation to 1 for the most preferred situation; hence k is a scaling constant so that $u(\underline{x})$ is scaled properly. As k approaches 0 ($\sum_{i=1}^{N} k_i$ approaches 1), the multiplicative form approaches the additive form.

It is worth mentioning once again that unless these independence properties hold, the additive or multiplicative forms of the multi-attribute utility function are not appropriate. Otherwise, much more complicated forms should be used, increasing the difficulty of assessment. Having said that, we must admit that these forms have been generally used in the literature because of their simplicity, because they can model many important preference patterns, and because they illustrate the essence of assessing and using utility functions. Since the information required in the assessment of the additive and the multiplicative form is similar, we can discuss them together. The next section describes the assessment process, and the following one shows how the various components of the problem are combined to describe an optimal strategy.

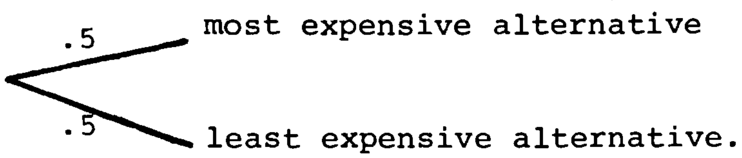
Utility Function Assessment

A problem encountered in applying utility functions to water resource problems is in their assessment and verification. This is a long process requiring familiarity with the assessment technique by the individual whose utility function is being assessed. Hence, one generally begins the assessment process with an explanation of the underlying theory and a description of the technique to be followed. During assessment, an initial series of questions is asked to establish whether or not the independence properties hold, and if so, among which attributes. If some independence properties do hold, this makes the form of the utility function simpler; in addition, the mathematical interrelationships of the function may be used to define the entire function on the basis of a reduced data set. Questioning proceeds by asking the subject to state preferences in terms of simple hypothetical gambles involving various attributes. Let us illustrate the technique by considering the questioning about pollution abatement cost for one reach of the Rhine. Let this be attribute 1 (denoted X_1).

We have found that the various pollution control alternatives on the Rhine have waste water treatment costs in the range 60DM/hr to 2270DM/hr, depending upon degree of treatment and size of treatment plant. (These estimates are based on work by Stehfest [20].) As mentioned previously, utility functions are generally scaled from 0 for the least preferred situation to 1 for the most preferred. Hence, we set

$$u_1(2270) = 0 \quad \text{and} \quad u_1(60) = 1 \quad . \tag{4}$$

The decision maker is then offered the choice between two options. Option 1 is a lottery* in which there is an equal chance of having the most expensive alternative (costing 2270) and the least expensive alternative (costing 60). This lottery can be illustrated as follows:



Option 2 is that the certain outcome which costs 1200 will be the one used. Is option 2 preferred by the decision maker over option 1? Of course, stated in this fashion, the choice is

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*See Luce and Raiffa [9] for a discussion of the "lottery" concept in decision analysis.

hypothetical and does not correspond to a real life situation; or does it? The local pollution control authority may be negotiating with the federal government about the pollution control equipment to be used. The local authority perceives that after it has presented its most preferred alternative (the least costly one), there is a 50% chance that the federal authorities will accept it, and, if they do not, will require the least preferred alternative (the most costly one). Perhaps during the negotiation process the local authority would be offered the alternative, costing 1200, as a compromise. Would it accept the compromise, or force the issue between the least and the most expensive alternatives? The authority may well accept the compromise, even though its value is less preferred than the expected value of the lottery (1165), so as to avoid the risk of ending up with the least preferred alternative. Well, if 1200 is preferred, then by the nature of assessing $u_1(x_1)$, $u_1(1200)$ must be greater than the expected utility value of the lottery. That value is:

$$.5*u_1(60) + .5*u_1(2270) = .5*1. + .5*0. = .5 \quad ;$$

thus $u_1(1200) > 0.5$. Now, the same lottery could be compared with an alternative costing 1500 for sure; and let us suppose that the lottery was preferred. Then $u_1(1500)$ must be less than the .5 expected utility value of the lottery. Finally, suppose that the decision maker says that he is indifferent to the choice between the alternative costing

1345 for sure and the lottery. Then the utility value assigned to 1345, which we call the indifference value, must equal that of the lottery; so $u_1(1345) = .5$.

The process then repeats itself. A certain amount, 1900, is compared to the lottery yielding an equal chance of obtaining 2270 (the least preferred value) and 1345 (the indifference value of the previous lottery). Suppose that the lottery is preferred, but that the decision maker is indifferent to the choice between 1840 and the second lottery. Then $u_1(1840)$ must be assigned a value equal to the expected utility value of the lottery, or

$$u_1(1840) = .5u_1(1345) + .5u_1(2270)$$

$$= .5*.5 + .5*0. = .25 \quad .$$

Continuing in a like manner with the value 1345 (the indifference point for the first lottery) and 60 (the most preferred value) we can find the cost with a utility of .75. These five points (the points with utility value 0, .25, .5, .75, and 1) define a utility curve for a single consequence. (See Figure 1.) Techniques are available for making consistency checks and for fitting piecewise continuous functions through these five points (Schlaifer [18]; Keeney and Sicherman [8]). Often, an exponential function

$$u_1(x_1) = c_1\left(1-e^{b_1(x_1 - x_1^*)}\right) \tag{5}$$

is fit to the five points. In our example $c_1 = 2.063$,

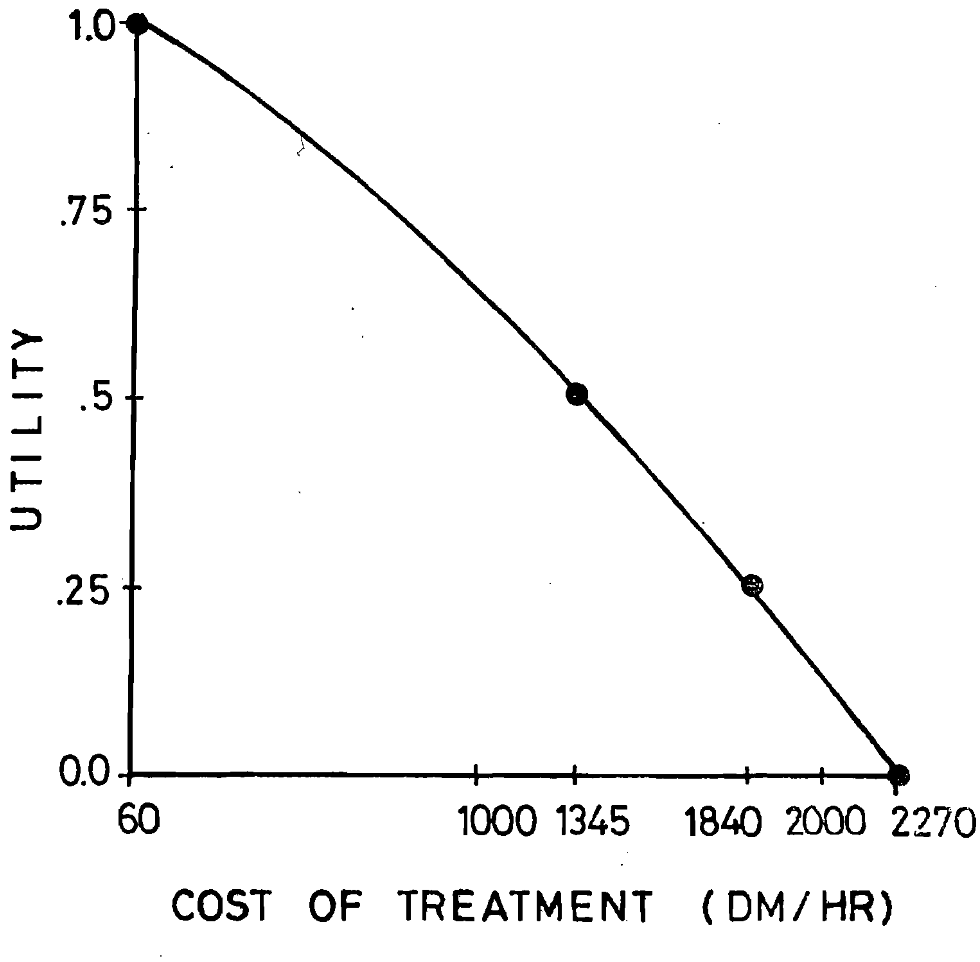

Figure 1. Utility for waste water treatment costs.

b_1 = .0003 and $x_1{}^*$, the maximum value of x_1, = 2270. In the Rhine River example to follow, we will use single-attribute utility functions of this form.

It is worth mentioning that the resulting utility function is concave; that is, it satisfies the following property:

$$u_1(px_1' + (1 - p)x_1'') \geq pu_1(x_1') + (1 - p)\, u_1(x_1'') \quad , \qquad (6)$$

where $0 < p < 1$. In other words, in the choice between a lottery and having the expected value of the lottery for certain, the expected value is preferred. One advantage of the utility function approach is that this preference (referred to as being risk averse) can be explicitly taken into account.

We have described a method of obtaining single-attribute utility functions (see Schlaifer [17] for other methods). But water resource decision problems deal with more than just one attribute; and the additive and multiplicative forms reflect this. We will now consider trade-offs among attributes. The goal is to assess values for the k_i's ($i = 1,2,\ldots,N$) of the additive and multiplicative forms. To simplify the discussion, we will consider a three-attribute assessment problem. (For any other value of N, the approach is analogous.) For these three attributes, the following limits are appropriate:

$$2270 \geq x_1 \geq 60$$

$$x_{2*} \leq x_2 \leq x_{2*}$$

$$x_{3*} \leq x_3 \leq x_{3*}$$

where the limit to the left is the least preferred and the limit to the right the most preferred value of that attribute. As before, we will scale the single-attribute utility functions from 0 for the least preferred value of the attribute to 1 for the most preferred:

$$u_1(2270) = 0 \qquad u_1(60) = 1$$

$$u_2(x_{2*}) = 0 \qquad u_2(x_2{}^*) = 1$$

$$u_3(x_{3*}) = 0 \qquad u_3(x_3{}^*) = 1 \quad . \qquad (7)$$

Similarly, the multi-attribute utility function is scaled from 0 for the least preferred consequence (all three attributes at their least preferred value) to 1 for the most preferred (all three attributes at their most preferred value):

$$u(2270, x_{2*}, x_{3*}) = 0 \quad \text{and} \quad u(60, x_2{}^*, x_3{}^*) = 1 \quad . \qquad (8)$$

The first question is designed to get the relative magnitudes of the k_i's in Equations 2 and 3. The decision maker is asked to rank the attributes he would like to change from the worst level to the best level, given that he started with all of them at the worst level $(2270, x_{2*}, x_{3*})$. The response might be that he would prefer the change in X_3 from x_{3*} to $x_3{}^*$; next, the change in X_1; and last, the change in X_2. This implies

$$k_3 > k_1 > k_2 \quad .$$

Now let us assess their actual values. The first question to the decision maker might be: "Consider having the worst level of each attribute, and changing the third attribute from its least preferred value to some intermediate value x_3'; or, alternatively, changing the first attribute over its complete range, from its least preferred to its most preferred value. Do you prefer the first change or the second change?" This response might be that the second change is preferred and his preference would be written as follows:

$$(60,x_{2*},x_{3*}) \succ (2270,x_{2*},x_3') \quad .$$

Further, let us suppose that if the intermediate value for the third attribute is changed to x_3'', he becomes indifferent to the choice between the two changes; hence their utility values are equal:

$$u(2270,x_{2*},x_3'') = u(60,x_{2*},x_{3*}) \quad . \qquad (9)$$

Evaluating both sides using the additive or multiplicative forms, we find that

$$k_3 u_3(x_3'') = k_1 \quad . \qquad (10)$$

In a similar fashion, trade-offs between X_2 and X_3 are considered. Suppose the decision maker agrees that, starting once again at the worst consequence, he is indifferent to the choice between changing X_3 from x_{3*} to x_3''', and changing X_2 from x_{2*} to x_2^*; then

$$k_3 u_3(x_3''') = k_2 \quad . \tag{11}$$

If the proper multi-attribute utility function form is additive, then the assessment process is finished (except for verification and consistency checks). One more piece of information is needed, however. Considering the most preferred value of each attribute, we find

$$u(60, x_2^*, x_3^*) = k_1 u_1(60) + k_2 u_2(x_2^*) + k_3 u_3(x_3^*) \quad , \tag{12}$$

or

$$1 = k_1 + k_2 = k_3 \quad . \tag{13}$$

Because $u_3(x_3)$ has been independently assessed, we have three independent equations (Equations 10, 11, and 13) which can be solved for the three unknowns (k_1, k_2, and k_3).

If the multiplicative form is the proper one to use, then a further assessment question needs answering. One way of stating it is as follows: "For what value of p is the decision maker indifferent between

1. the certain outcome with the first two attributes at their least preferred values and the third at its most preferred value;
2. the lottery with a chance p of obtaining the most

preferred value of all attributes, and a chance (1-p) of obtaining the least preferred value of all attributes.

This can be diagrammed as follows:

$$(2270,\ x_{2*},\ x_3{}^*) \sim \begin{cases} \xrightarrow{\ p\ } (60, x_2{}^*, x_3{}^*) \\ \xrightarrow{1-p} (2270, x_{2*}, x_{3*}) \end{cases} .$$

It can be shown quite easily, using the multiplicative form, that $k_3 = p$, and this completes the assessment. One more constant of Equation (3) needs determination. Considering the most preferred value of each attribute, we find:

$$1 + k\, u(60, x_2{}^*, x_3{}^*) = \prod_{i=1}^{3} (1 + k k_i u_i(x_i{}^*)) \qquad (14)$$

or

$$1 + k = \prod_{i=1}^{3} (1 + k k_i) \quad . \qquad (15)$$

When this equation is solved for k, the multiplicative form is completely determined.

There are alternative assessment techniques for obtaining values for the k_i's (see Raiffa [16]). In general, a real-life assessment involves adapting a variety of methods to fit the occasion. It should not be expected on the first try that the decision maker be completely comfortable with his answers. Generally, assessment is an iterative process, with several attempts at verifying and then improving the utility functions.

After a few iterations, the utility function should closely represent the decision maker's preferences.

The additive and multiplicative forms exhibit one major difference with respect to the decision maker's preference pattern. It can best be explained in terms of a simple example. Consider two lotteries in which values of X_2 and X_3 vary. Let there be a lottery, lottery 1, in which there is a .5 chance of obtaining (x_2^*, x_{3*}) and a .5 chance of obtaining (x_{2*}, x_3^*). Let there be a second lottery, lottery 2, in which there is a .5 chance of obtaining (x_2^*, x_3^*) and a .5 chance of obtaining (x_{2*}, x_{3*}). (The marginal probability of getting the most preferred and least preferred value of each attribute is the same in both lotteries.) If, faced with the choice between these two lotteries, the decision maker finds himself indifferent, then the additive form is proper (this behavior is sometimes referred to as being multiattribute risk neutral); otherwise the multiplicative form is proper.

In summary, we have shown how it is possible to assess a utility function for attributes of water resource problems. Next we consider how one can solve a problem involving maximization of the expected value of a utility function, given certain forms of technological relations relating river conditions at different points along the river. Then, we will apply the theory to a study of water quality management on the Rhine River.

Solution Technique

As mentioned previously (Equation 1), the problem we wish to solve is

$$\max_{v} \int u(\underline{x})\ p_v(\underline{x})\ d\underline{x}\ , \tag{16}$$

where $p_v(\underline{x})$ describes in probabilistic terms the impact on the attribute set $\underline{X}$ due to choosing decision alternative "v." It is computationally convenient to transform this problem by taking expected values with respect to the probability of occurrence of varying ambient environmental conditions along the river (e.g., streamflow rate or waste loads). Then technological constraints relate the attribute set $\underline{X}$ to the decision alternative chosen and to different ambient conditions. Let q be the indicator of ambient environmental conditions and p(q) its probability density. We are interested in attributes on different reaches of the Rhine; let there be n attributes describing each reach. Let at least one of these attributes in each reach refer to the degree of pollution control in that reach.

Further, let us number the attributes so that the first n are those describing reach 1 (let us denote this n attribute set $\underline{x}(1,n)$); the next n are those describing reach 2 (denoted $\underline{x}(n + 1,2n)$), etc. Assume that there are R reaches, numbered from upstream to downstream. Further, let us assume that we can relate attributes in reach r to those in reaches r and r-1 and to ambient river conditions by the following equation:

$$\underline{x}((r - 1)n + 1, rn) = f_r(\underline{x}((r - 1)n + 1, rn),$$
$$\underline{x}((r - 2)n + 1, (r - 1)n), q).$$

This equation needs some explanation. Suppose one attribute in each reach refers to DO level, another to BOD, and a third to some control mechanism such as aeration or sewage treatment. Now, we can relate DO in reach r to the DO, BOD, and COD in the next upstream reach (reach r-1), and to whatever waste treatment has been carried out in reaches r and r-1, by using the Streeter-Phelps [22] or other similar equations. For later analytic convenience, let us rewrite the last relationship by substracting $\underline{x}((r - 1)n + 1, rn)$ from both sides, so that the relation can be expressed in a form that equals zero:

$$F_r = F_r(\underline{x}((r - 1)n + 1, rn), \underline{x}((r - 2)n + 1, (r - 1)n), q)$$
$$= f_r(\underline{x}((r - 1)n + 1, rn), x((r - 2)n + 1, (r - 1)n), q)$$
$$- \underline{x}((r - 1)n + 1, rn) = 0 \quad . \tag{18}$$

Now, we can restate our problem as follows:

$$\max \int u(\underline{x})\ p(q)\ dq \ , \tag{19}$$

where

$$\underline{x} = \{\underline{x}(1, n),\ x(n + 1, 2n), \ldots, x((R - 1)n + 1, Rn)\}\ ,$$

and, subject to

$$F_r = 0 \tag{20}$$

for r = 1,...,R. We are assuming that if the ambient river conditons are known, the values of the attributes in reach 1 can be found.

To find a solution, optimal control theory can be used. A Hamiltonian can be defined as follows:

$$H = u(\underline{x})\ p(q) + \sum_{r=1}^{R} \lambda_r F_r \quad , \tag{21}$$

where the λ_r's are multipliers. The necessary conditions for the maximum (Bryson and Ho [1]) are:

$$\frac{\partial H}{\partial x_i} = 0 \quad , \quad i = 1,2,\ldots,Rn \quad , \tag{22}$$

subject to

$$F_r = 0 \quad , \quad r = 1,2,\ldots,R \quad . \tag{23}$$

The sufficient conditions, assuming that H is a convex function of the x_i's, are

$$\frac{\partial^2 H}{\partial x_i^2} < 0, \quad i = 1,\ldots,Rn \quad . \tag{24}$$

Let us now turn to the application of this approach to the Rhine.

Rhine River Application

We have two goals in this application. First, using the approach presented above, we will generate a set of optional waste treatment policies that maximize expected utility; second, we will determine the effect of changing the parameters of the utility function on the optimal decisions. At the same time we would like to keep everything simple enough so that what is done is readily apparent to the reader. The portion of the Rhine River in the Federal Republic of Germany between Mannheim and the Netherlands border was chosen for analysis. (See Figure 2.) Along this stretch are the heavily industrialized Ruhr and four large cities -- Mainz, Cologne, Koblenz and Mannheim. Chemical oxygen demand (COD) loads are large, particularly near the industrialized areas (see Figure 3a).

Stehfest [21] has proposed two water quality models for the Rhine. The first is a complex ecological model; the second, an empirical model, essentially the Streeter-Phelps [22] equations applied to degradable COD, non-degradable COD and oxygen deficit. Stehfest has applied the latter model to data for eighteen reaches; his results for the case of 1970 treatment levels (approximately 80% of population and industry receiving mechanical treatment, and 40% receiving additional biological treatment) are reproduced in Figure 3b. Stehfest then estimated 1985 COD discharges and relative population levels for each reach. (The relative population indices are needed so that the treatment cost functions which follow include economies of scale.) We shall use Stehfest's [21]

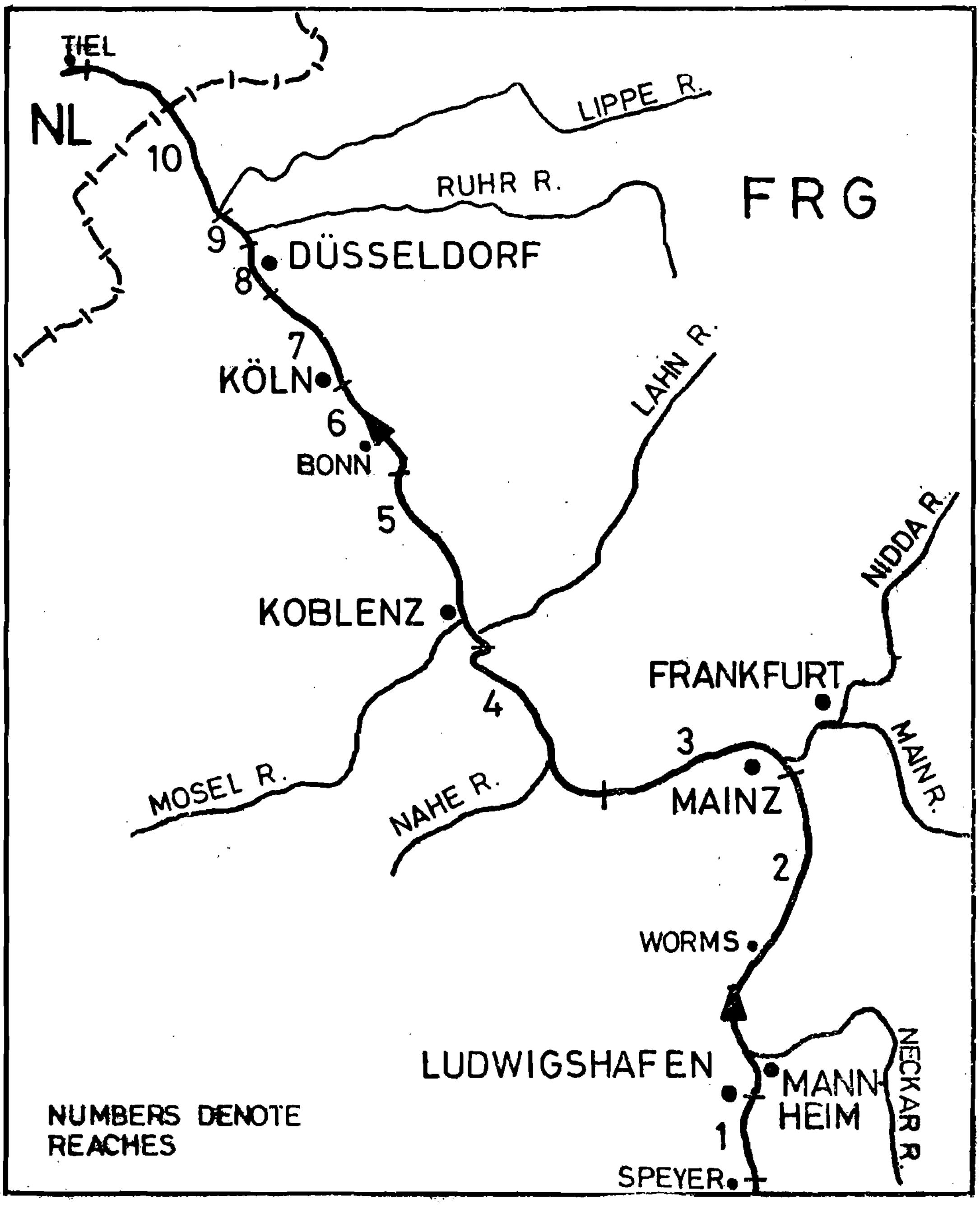

Figure 2. Region of the Rhine Basin chosen for study, divided into ten reaches.

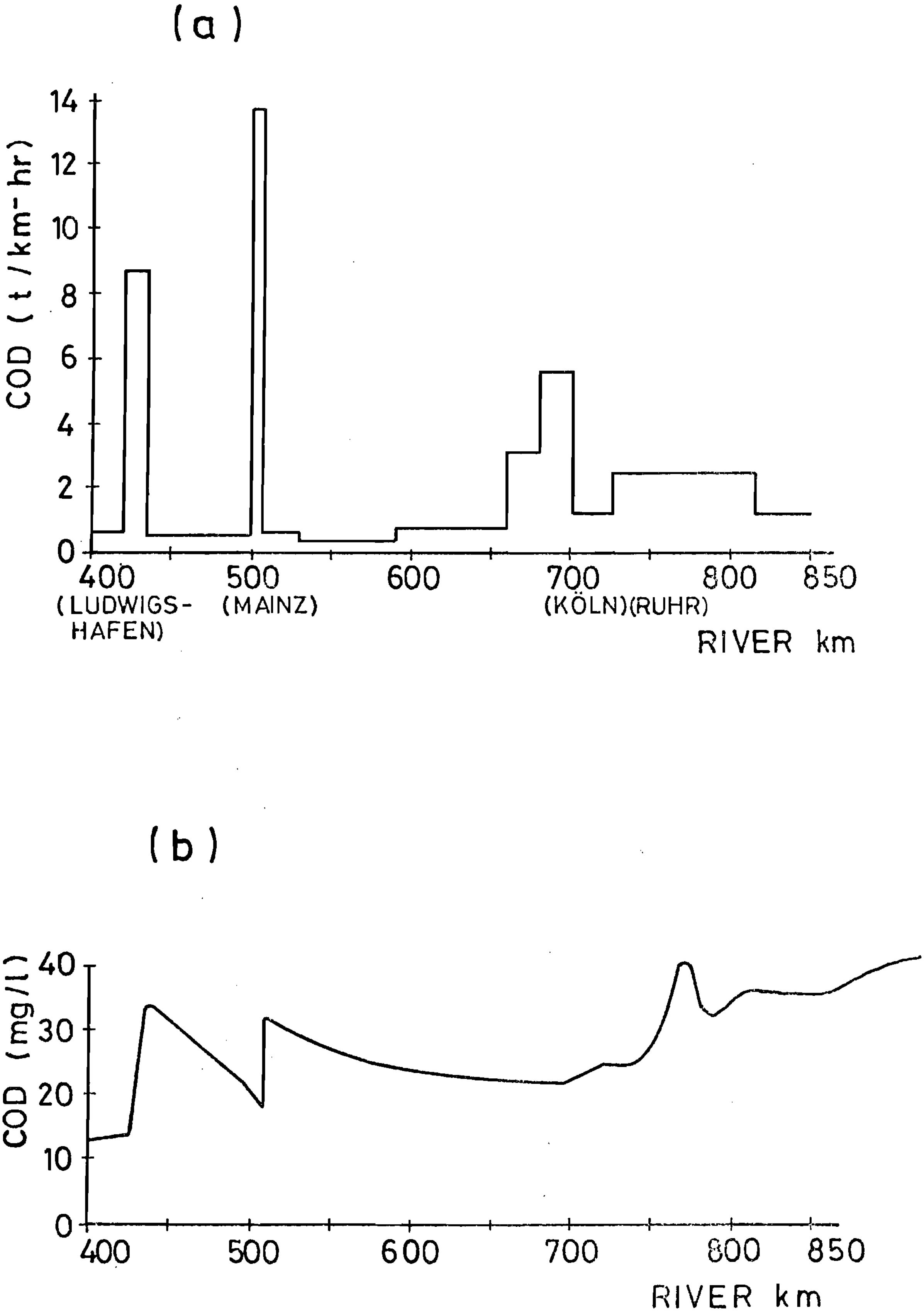

Figure 3. COD on the Rhine River.

(a) 1970 loads.

(b) Mean daily levels with 1970 treatment strategy.
Source: Stehfest [21]

data as a point of departure for our preliminary analysis.

Figure 4 presents the thirty-six-year streamflow record. We shall use this record as a cumulative probability density, and discretize it into five segments.

The degradable COD level at the beginning of a reach will be one attribute*, and efficiency of treatment of waste water the second. For this application, these will be the only attributes for each reach; the cost of waste treatment will be expressed as a function of removal efficiency, streamflow, and relative population. For ease of discussion, we will change our notation from that of the previous section, where all attributes were denoted x_i, to one where different symbols represent different physical entities. Let:

w_r = concentration of COD at start of reach r [mg/l]

W_r = degradable COD inflow before treatment [kg/hr]

y_r = degradable COD removal efficiency in reach r [%]

$W_r(1 - y_r)$ = degradable COD added to reach r [kg/hr]

Z_r = COD treatment cost in reach r [DM/hr]

Q_r = discharge in reach r [m^3/hr]

q_r = inflow at start of reach r [m^3/hr]

d_r = length of reach r [m]

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*Stehfest [19] has proposed a variant of the form of this attribute. He suggests weighting the COD level by the population on each reach. Another alternative or addition might be water quality weighted by volume. Degradable COD was chosen as the primary water quality indicator of interest. The inclusion of a DO equation would complicate the problem without adding to the reader's insights; the implications of extending the model to include DO will be discussed in extensions.

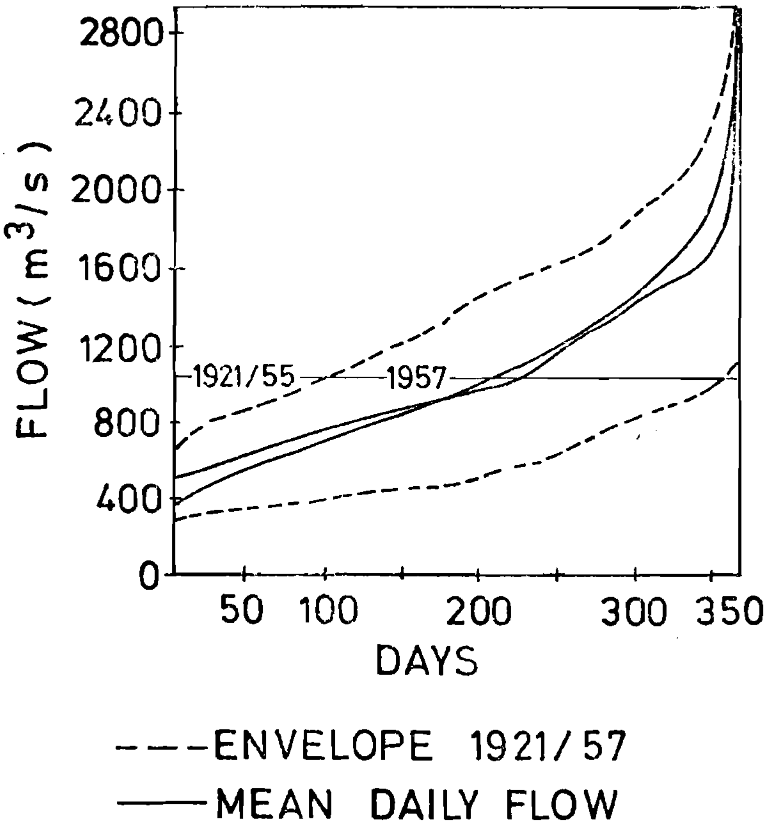

Figure 4. Distribution of flows on the Rhine River at Rheinfelden, 1921-1957. Source: Ref. [2]

t_r = time taken for a unit volume of water to pass through reach r [hr]

k_1 = rate constant for COD decay [hr^{-1}]

A_r = average cross-sectional area of reach r [m^2] .

The equations that relate these variables from one reach to the next are:

$$Q_{r+1} = Q_r + q_{r+1} \tag{25}$$

$$w_{r+1} = \left[w_r + \frac{(1 - y_r)W_r}{Q_r} \right] \exp\,(- k_1 d_r A_r/Q_r) \tag{26}$$

for r = 0,..., R-1. For convenience, we will rewrite this last relation as a function that equals zero:

$$F_{r+1} = \left[w_r + \frac{(1 - y_r)W_r}{Q_r} \right] \exp\,(- k_1 d_r A_r/Q_r) - w_{r+1} = 0 \quad . \tag{27}$$

We also need a function that relates treatment cost to COD removal efficiency. A continuous function approximation to the costs that Stehfest [20] uses for the Rhine is

$$Z_r = \left(\gamma_r e^{B_r y_r} \right) W_r y_r \quad , \tag{28}$$

where γ_r = .000053 and B_r = 5.11 when relative population equals 1; γ_r = .00011 and B_r = 4.44 when it equals 2; and γ_r = .000086 and B_r = 4.44 when it equals 3. Cost is expressed in DM/hr.

We next specify the form of the utility functions for each reach for the two attributes cost and water quality.

As mentioned previously, exponential functions are often used for single-attribute utility functions; they can be written

$$u_{Z_r}(Z_r) = c_r\left(1 - e^{b_r(Z_r - Z_r*)}\right), \quad r = 1, 2, \ldots, R \qquad (29)$$

$$u_{w_r}(w_r) = d_r\left(1 - e^{h_r(w_r - w_r*)}\right), \quad r = 1, 2, \ldots, R \quad , \qquad (30)$$

where u_{Z_r} and u_{w_r} are single-attribute utility functions for costs and COD levels of reach r, respectively. (In our examples $Z_r* = 2270$ and $w_r* = 30$.) Constants b_r and h_r are obtained from the assessment process; constants c_r and d_r are needed so that these single-attribute utility functions are scaled from 0 to 1. Once b_r is obtained, c_r is fixed; similarly, d_r is fixed when h_r is obtained. We can thus specify these single-attribute utility functions by specifying one constant for each of them. Later on, as part of our analysis, we will vary the parameters b_r and h_r (known as risk aversion coefficients) to see how the optimal decision changes.

For equity among reaches we will use the same form of utility function for each reach (the minimum and maximum costs in each reach's utility function depend upon relative population density in the reach). Further, we will attach equal weights to each reach when calculating the utility function for the whole basin.*

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*The implications of doing otherwise are discussed in Gros [4] and Gros and Ostrom [5].

Consider using the additive multi-attribute utility function form for one reach of the Rhine:

$$u_r(Z_r, w_r) = a_r u_{Z_r}(Z_r) + g_r u_{w_r}(w_r) \quad . \tag{31}$$

Specifying either a_r or g_r is sufficient since the other is given by $1 = a_r + g_r$. Thus to specify the additive utility function for one reach, we must specify b_r, h_r and either a_r or g_r. Alternatively, consider using the multiplicative multi-attribute utility function form for one reach of the Rhine:

$$1 + k\, u_r(Z_r, w_r) = \left[1 + k\, a_r u_{Z_r}(Z_r)\right]\left[1 + k\, g_r\, u_{w_r}(w_r)\right] \quad . \tag{32}$$

Here we must specify both a_r and g_r to define the multiplicative form for one reach. Then k can be obtained from

$$1 + k = \left[1 + k\, a_r\right]\left[1 + k\, g_r\right] \quad . \tag{33}$$

If the decision maker's multi-reach utility function is the additive form of the single-reach utility functions with equal weights, then the multi-reach function is specified:

$$u(\underline{x}) = \frac{1}{R} \sum_{r=1}^{R} u_r(Z_r, w_r) \quad . \tag{34}$$

If the multi-reach utility function is multiplicative, then one more constant is needed--the multiplier m for each reach:

$$1 + k'\, u(\underline{x}) = \prod_{r=1}^{R} \left(1 + k' m\, u_r(Z_r, w_r)\right) \quad , \tag{35}$$

where k' corresponds to k in Equation (3) with $Rm \neq 1$.

What we propose to do is vary the parameters b_r, h_r, a_r, g_r, and m, and see how the decision changes. As we mentioned in relation to Equation (3), the additive form is just a special case of the multiplicative form, with k (or k' in the multi-reach problem) approaching 0. Therefore, one might say that we are doing a sensitivity analysis on the multiplicative form, the additive form being one variation.

Computational Approach

An R-reach problem requires the solution of 3R non-linear equations simultaneously for 3R unknowns (w_r, y_r and λ_r, $r = 1,2,\ldots,R$). For the purpose of keeping computation times within reasonable limits, the eighteen reaches of Stehfest [20] were aggregated into ten longer ones. Although this implies a certain simplification of the characteristics of the inflow and treatment processes, our first objective was to demonstrate the general applicability of the model, which could later, if necessary, be extended to larger problems. Data for the aggregated system is illustrated in Table 1. Note that for simplicity Stehfest's 1985 estimates of the inflows and waste discharges have been treated as point sources at the start of each reach.

The second reach for example, represents the Rhine basin area of Mannheim, including effluents from paper mills and chemical plants at Ludwigshafen and Neckar River. Reach 3 starts at Mainz, at the confluence of the Main River and the Rhine; reach 5 at Koblenz where the Mosel joins the Rhine;

Table 1. Inflow, COD Discharge and Relative Population Density for the Ten-Reach Aggregation.

Reach	KM	Inflow ($10^5 m^3/hr$)	COD ($10^5 kg/hr$)	Relative Density	Approximate Location
	400	0.000	.203	1	Speyer
1					
	420	3.600	2.741	2	Ludwigshafen
2					
	500	7.200	1.640	3	Mainz
3					
	530	2.884	.366	1	Rüdesheim
4					
	590	10.640	.958	1	Oberlandstein
5					
	660	1.648	1.016	2	Bad Godesberg
6					
	680	0.000	2.339	3	Köln
7					
	725	0.000	1.224	3	Düsseldorf
8					
	755	0.000	2.448	3	Wittlaer
9					
	815	2.884	1.321	1	Wesel
10					
	900	-	-	-	Tiel (Netherlands)

and reach 7 with the region around Cologne. The last reach extends across the Dutch border to the town of Tiel.

The ten-reach example was solved using the rapidly convergent Newton-Raphson algorithm on a CDC 6600 computer, requiring from five to eight "Newton Steps" until the solution converged. CPU time (execution and output) was approximately 16.7 seconds for the sequential solution of five problems, one for each value of the uncertain streamflow, q.

The result for each of the five discrete values of q was an optimal control law $y_r(q)$. A typical example of the dependence of the control strategy in reach 4 on the value of q is illustrated in Figure 5, where an additive form of utility was employed. Not all reaches had the same shape of control strategy. The optimal policy for reaches 1 and 6, for example, was to decrease treatment efficiency for an increase in flow rate, since dilution had more impact than self-purification (which is related to flow time). From the set of $y_r(q)$ and the probability distribution of q, the expected values of the cost and quality attributes and the expected utilities for cost and quality were computed.

The above procedure was repeated for each of the two forms of utility function discussed previously:

1) Additive form over attributes; additive form over reaches (the "sum-sum" or "$\Sigma\Sigma$" form);
2) Product form over attributes; additive form over reaches (the "sum-product" or "$\Sigma\Pi$" form).

For the sum-sum and sum-product forms the results of the optimization are shown in Table 2. Expected values of

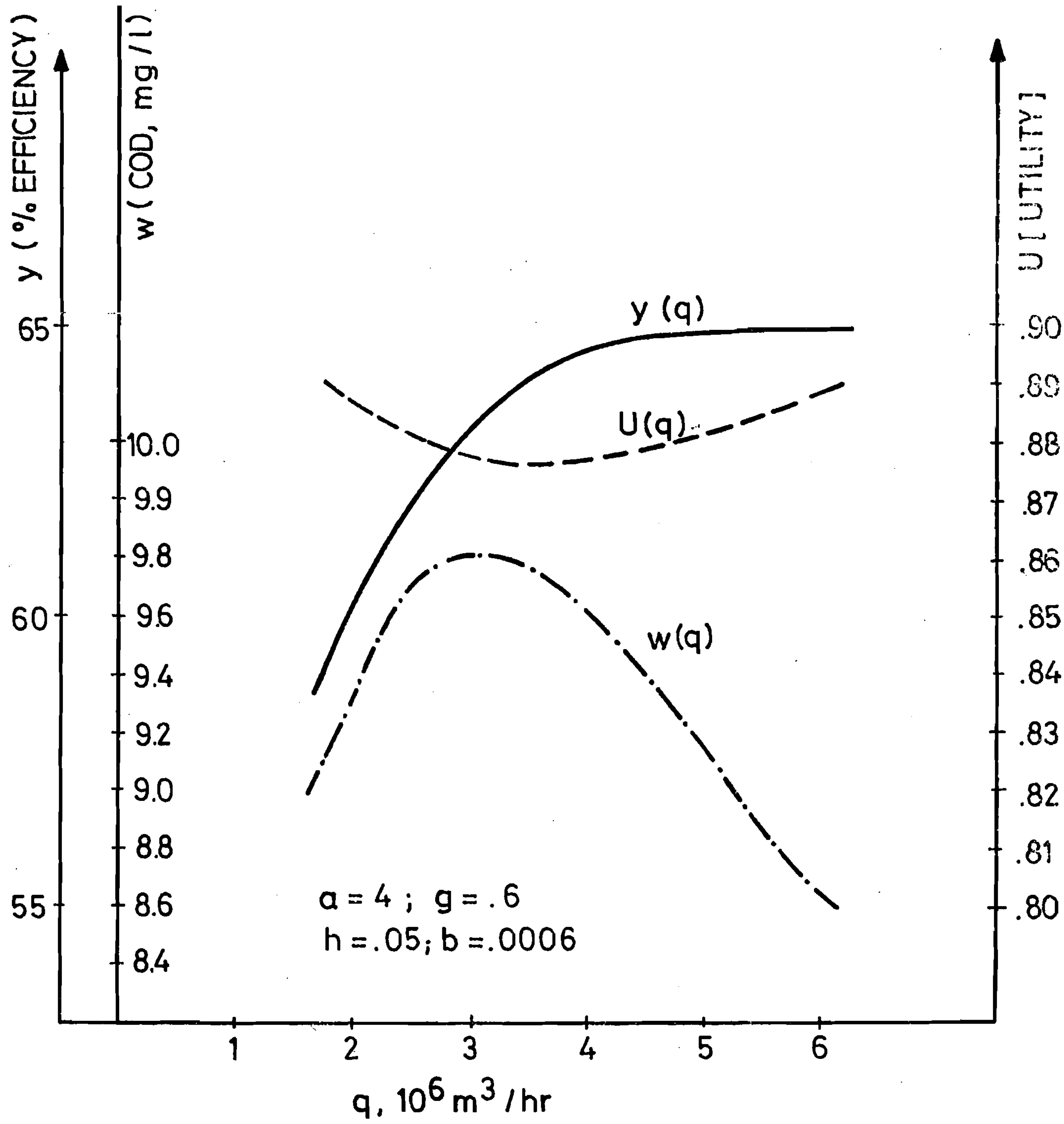

Figure 5. Control strategy y(q), COD w(q), and total utility U(q) for reach 4, using ΣΣ objective function.

Table 2. Comparison of Expected Values for Quality, Efficiency, Cost, and Utility Using ΣΣ and ΣΠ Utility Objective Functions.

Reach	Quality (COD, mg/h)		Efficiency %		Cost (DM/hr)		Utility	
	ΣΣ	ΣΠ	ΣΣ	ΣΠ	ΣΣ	ΣΠ	ΣΣ	ΣΠ
1	16.7	16.7	64	70	19	28	.73	.81
2	14.2	13.9	85	92	890	1318	.70	.76
3	9.0	7.0	83	89	463	676	.81	.89
4	9.5	6.7	63	74	31	64	.88	.94
5	7.2	5.0	72	83	197	381	.89	.93
6	6.1	4.0	77	87	276	483	.88	.93
7	8.4	5.2	83	91	677	1069	.82	.90
8	9.1	5.2	75	85	298	508	.85	.93
9	10.3	6.1	77	88	512	904	.82	.91
10	9.9	5.7	65	82	172	482	.87	.93
Border	7.4	4.0	-	-	-	-	.87	.94

$h = .05$, $h/b = 10.56$, $g/a = 1.5$

ΣΣ: $g = .6$
ΣΠ: $g = .9$

COD ($\mathcal{E} w_r$), treatment efficiency ($\mathcal{E} y_r$), treatment costs ($\mathcal{E} z_r$) and total utility ($\mathcal{E} u_r$) are compared for each reach. It is interesting to note that for the same risk aversion parameters (h_r and b_r), and the same ratio of weights (g_r/a_r), the sum-product form tends to result in generally higher quality levels on all reaches with correspondingly higher treatment efficiencies.

The results illustrate operating strategies for hypothetical treatment plants that are assumed to exist in 1985. To some extent, they are based on the definition of equity used in the analysis. It should be remembered that we used similar utility functions in each reach, and then weighted them equally to obtain the multi-reach utility function. This results in different operating levels in each reach, but with comparable utility values (see last two columns of Table 2). If a different concept of equity were used, such as one in which all treatment plants operate at the same level, or utility is based on per capita costs, different results would be expected.

Sensitivity Analysis

The optimal trajectories $y_r(q)$ for both objective function forms were subjected to sensitivity analysis on the single-attribute risk aversion parameters h and a. Holding the attribute weights fixed ($g/a = 1.5$), the ratio of risk aversion coefficients h/b was varied to test the effect on the expected quality in the river. Figures 6 and 7 show the relationship between expected COD levels on reach 4 for variations in

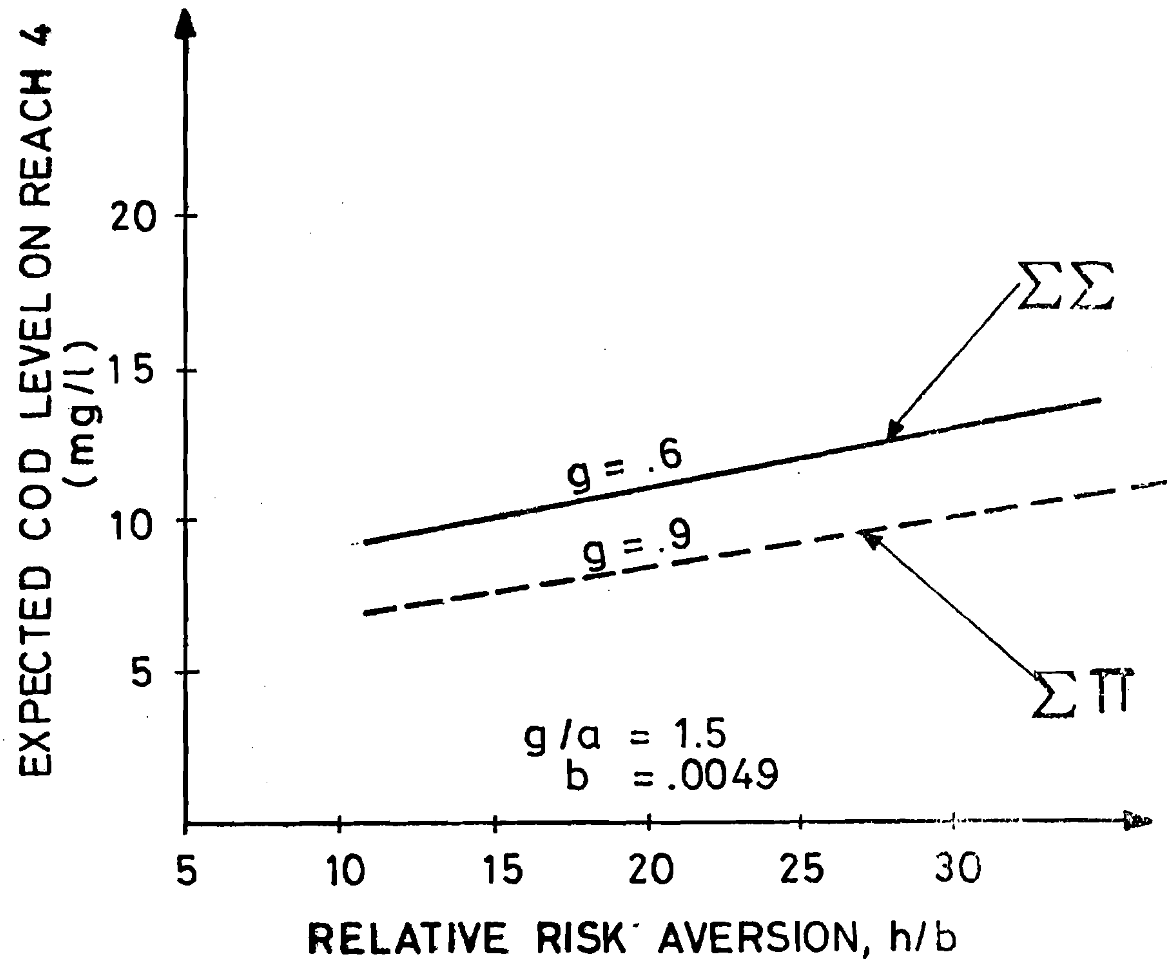

Figure 6. Sensitivity to risk aversion parameters: constant b with varying h.

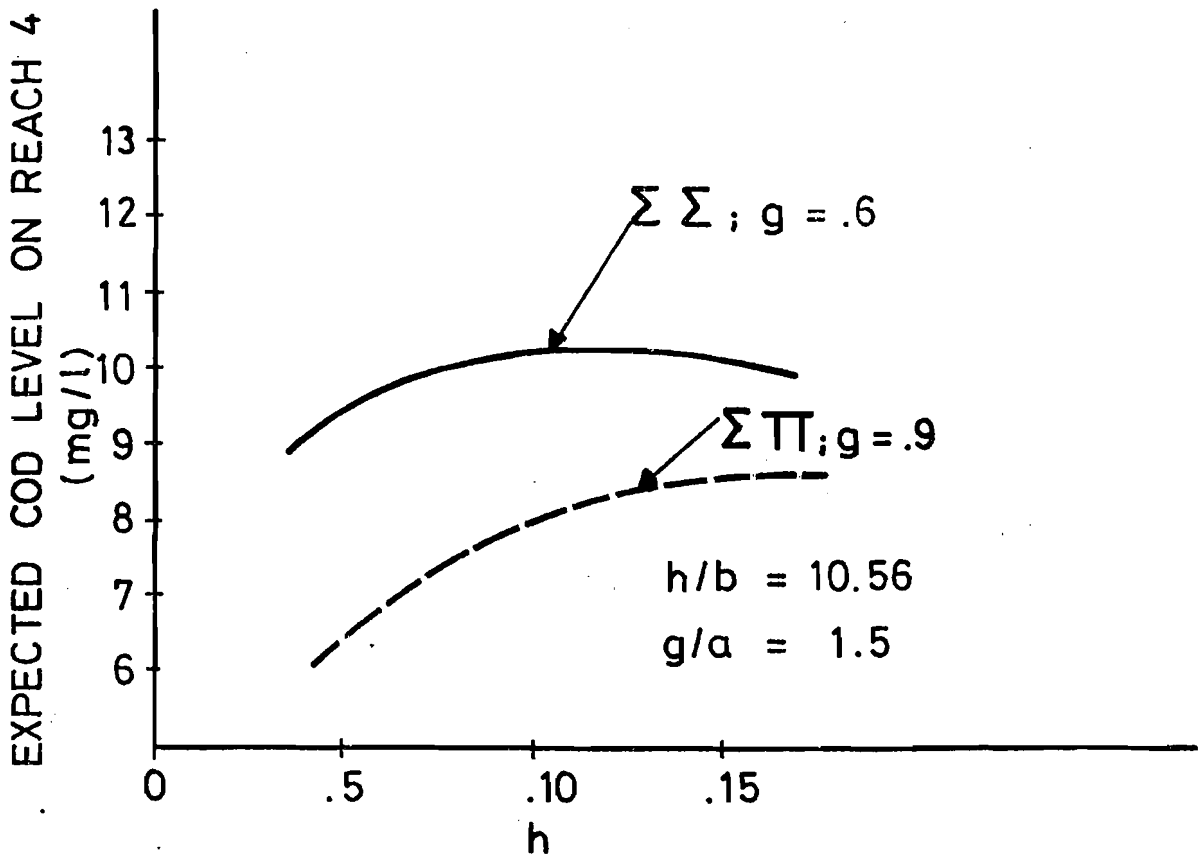

Figure 7. Sensitivity to risk aversion parameters: constant h/b.

the ratio of such aversions h/b. In Figure 6, b was kept constant and h was varied; in Figure 7 the ratio h/b was maintained constant while h was varied. The graphs show two important features: first, they demonstrate how the expected quality changes when we shift to a product utility function; second, they illustrate that expected quality is relatively insensitive to changes in the degree of risk aversion. One of the criticisms of decision theory (see Maass, [10]) is that is is difficult to obtain an accurate description of the shape of the utility function. Figures 6 and 7, however, show us that (in the range of this analysis) we need not worry too much about this problem since the optimal solution is relatively insensitive to the risk aversion coefficients.

A second and perhaps more interesting result is shown in Figures 8 and 9. Here risk aversion coefficients were kept constant and the ratio of attribute weights g/a was varied. Expected COD levels on reach 4 were examined for sensitivity to the value of g/a. In Figure 8, the expected COD drops off sharply for the $\Sigma\Sigma$ form of utility, as g/a approaches 1 and levels out at a value of approximately 6.25 mg/l. The $\Sigma\Pi$ form, however, is extremely sensitive to changes in the value of g/a and should be carefully examined in this light.

Figure 9 indicates that no matter how low a weight g is chosen for water quality, the COD level will not exceed some upper level (in this example, 10.1 mg/1).

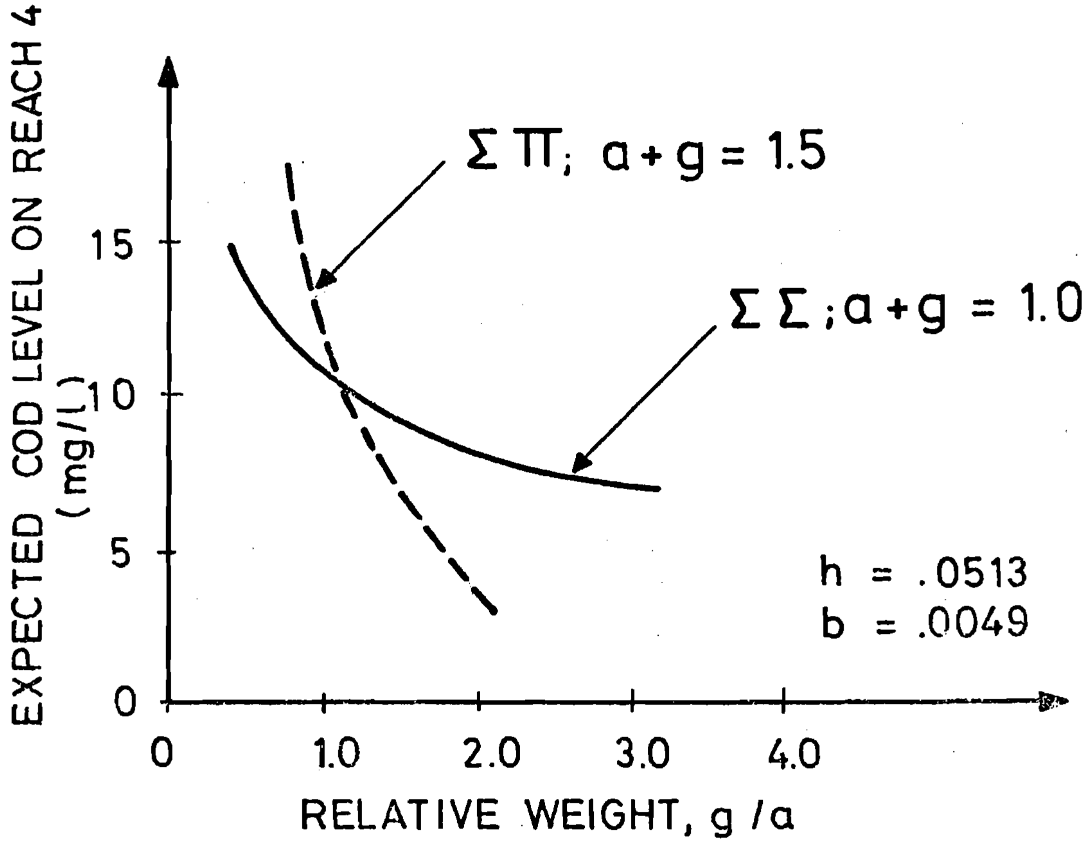

Figure 8. Sensitivity to relative scaling weights, g/a.

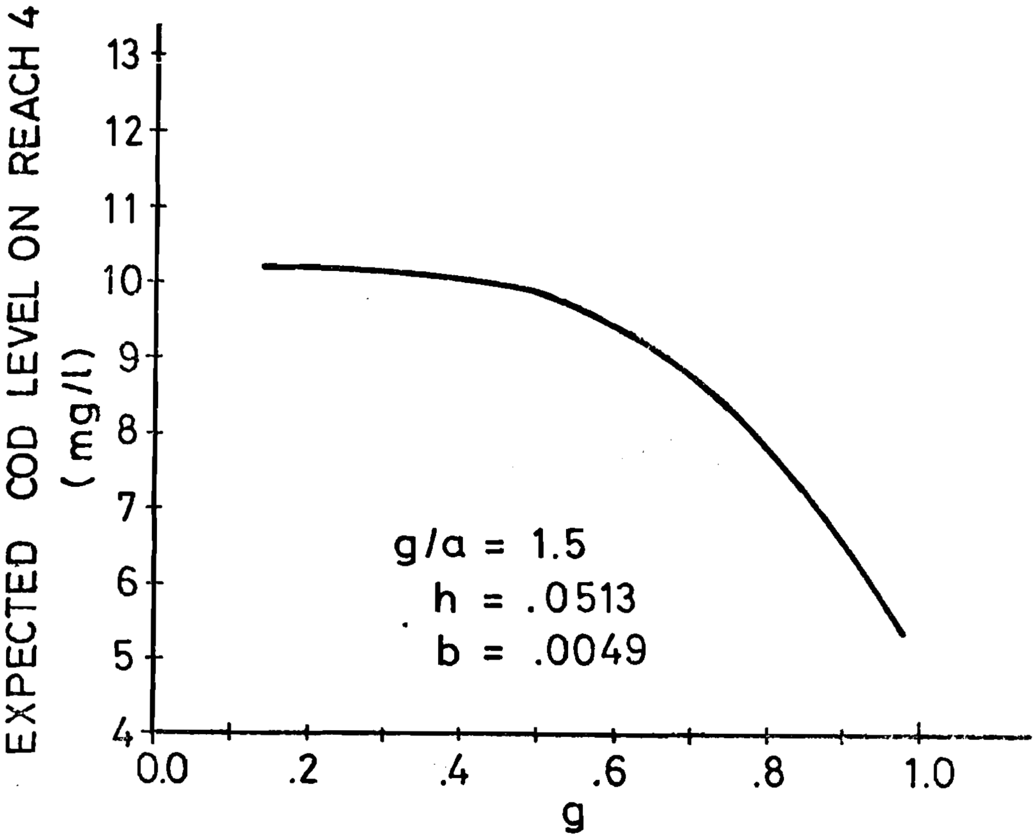

Figure 9. Sensitivity to scaling weights: constant g/a with varying g.

The sensitivity analysis presented has given us an important insight: if a similar anlysis is carried out in practice, more emphasis and greater care should be devoted to the assessment of those parameters which have a great effect on the optimal decision (the scaling weights); comparatively less time can be spent on parameters which have little effect (risk aversion parameters). It also gives us a good idea of the way in which optimal treatment strategies vary with changes in these parameters.

Discussion

Recent applications of control theory (see Young and Beck, [23]) have been concerned with optimal control trajectories that satisfy quality constraints determined *a priori*. The approach in this paper, however, is to treat preferences for variable water quality levels through the use of multiattribute utility functions. Rather than minimizing costs to achieve some predetermined quality levels, the model establishes a trade-off function between quality and costs of waste-water treatment.

Although the results for the Rhine application are preliminary, they show a marked difference between additive and multiplicative utility functions. If we are to take advantage of some of the appealing features of the product form, then caution must be used in selecting the attribute weights a and g. For our ten-reach problem, computational times were reasonable, and it appears that even larger and more complex problems can be handled. The model could be easily extended to include more reaches, and the Streeter-Phelps equation could be improved in a straightforward manner. Adding treatment plants on the tributaries requires a few more equations to relate flows at the confluences, and a correlation structure between those flows. Some of the implications of handling capital costs in this type of model are presented in Gros and Ostrom [5].

Extensions

It is clear that the inclusion of a dissolved oxygen demand (DOD) equation would enhance the model. This could be handled in a straightforward fashion. DOD would replace COD in the single-attribute utility function of equation (29), which would be rewritten as:

$$u(D_r) = d_r\left(1 - e^{h_r(D_r - D)}\right) \qquad r = 1,2,\ldots,R \quad , \qquad (36)$$

where

D_r is the DO deficit on reach r (mg/l)

D is the saturation concentration of oxygen (mg/l);

and the constraints in equation (26) would be replaced by one relating DOD in one reach to DOD and COD in the next upstream reach.

$$F_{r+1} = \frac{k_1}{k_2 - k_1}\left[w_r + \frac{(1 - y_r)\,W_r}{Q_r}\right]\left[\exp\left(-k_1 d_r A_r/Q_r\right) - \exp\left(-k_2 d_r A_r/Q_r\right)\right] + D_r \exp\left(-k_2 d_r A_r/Q_r\right) - D_{r+1} = 0 \quad , \qquad (37)$$

where

$$w_{r+1} = \left[w_r + \frac{(1 - y_r)\, W_r}{Q_r} \right] \exp\,(-k_1 d_r A_r / Q_r) \quad . \qquad (38)$$

The above expression for COD, however, can be written in a more general form as:

$$w_r = w_1 \prod_{i=1}^{r-1} e^{-k_1 t_i} + \sum_{i=1}^{r-1} \frac{(1-y_i)W_i}{Q_i} \left\{ \prod_{j=i}^{r-1} e^{-k_1 t_j} \right\} , \qquad (39)$$

where $t_r = d_r A_r / Q_r$, and then substituted into Equation 37.

Although there are still 3R equations, D_r, y_r, F_r, $r = 1,2,\ldots,R$, the $\frac{\partial H}{\partial y_r} = 0$ equations now contain extra derivative terms which can add considerably to the computing time. Alternatively, Equation 38 could be written as a separate constraint, denoted G_r. This implies that the Hamiltonian in Equation 39 would be rewritten as

$$H = u(\underline{x})\ p(q) + \sum_{r=1}^{R} \lambda_r F_r + \sum_{r=1}^{R} \mu_r G_r \quad . \qquad (40)$$

This, however, implies that we now need to solve 5R simultaneous non-linear equations for y_r, W_r, D_r, λ_r, and μ_r, $r = 1,2,\ldots,R$, instead of the 3R equations the COD model required.

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References

[1] Bryson, A.E. and Ho, Y. Applied Optimal Control. Ginn and Co., Waltham, Massachusetts (1969).

[2] Bundesanstalt fuer Gewaesserkunde, Deutsches Gewaesserkundliches Jahrbuch: Abflussgebiet Rhein, Abflussjahn (1968), Koblenz (1970).

[3] Cohon, Jared L. and Marks, David H. "A Review and Evaluation of Multiobjective Programming Techniques." Water Resources Research , II, No. 2, (April 1975), 208-220.

[4] Gros, Jacques G. "A Paretian Approach to Power Plant Setting in New England." Ph.D. thesis, Harvard University, May 1974.

[5] Gros, Jacques G. and Ostrom, Tony R. "A Decision Analytic Approach to River Basin Pollution Control." International Institute for Applied Systems Analysis Research Memorandum 75-9. Laxenburg, Austria, 1975.

[6] Keeney, Ralph L. "Concepts of Independence in Multiattribute Utility Theory", James L. Cochrane and Milan Zeleny, (eds.), Multiple Criteria Decision Making, University of South Carolina Press, Columbia, South Carolina, 1973, pp. 62-71.

[7] Keeney, Ralph L. and Raiffa, Howard. Decision Analysis with Multiple Objectives (forthcoming, 1976).

[8] Keeney, Ralph L. and Sicherman, Alan. "An Interactive Computer Program for Assessing and Analyzing Preferences Concerning Multiple Objectives." International Institute for Applied Systems Analysis Research Memorandum 75-12, Laxenburg, Austria, 1975.

[9] Luce, R.D. and Raiffa, H. "Gry i Decyzje." Panstwowe Wydawnictwo Naukowe, Warsaw, 1964, pp. 31-38.

[10] Maass, A., et al. Design of Water Resource Systems. Harvard University, Cambridge, Massachusetts, 1962.

[11] Materialienbang zum Umweltprogramm der Bundesregierung, Deutscher Bundestag, 6. Wahlperiode, 1971, pp. 121-172.

[12] Meyer, R.F. "On the Relationship Among the Utility of Assets, the Utility of Consumption and Investment Strategy in an Uncertain, but Time-Invariant World." J. Lawrence (ed.), OR 69; Proceedings of the Fifth International Conference on Operational Research, Travistock, London, 1970.

[13] von Neumann, J. and Morgenstern, O. Theory of Games and Economic Behavior. Princeton University Press, Princeton, New Jersey, 1947.

[14] O'Connor, Michael F. "The Application of Multi-attribute Scaling Procedures to the Development of Indices for Water Quality." University of Chicago Center for Mathematical Studies in Business Economics, Report 7339, August, 1973.

[15] Pratt, J.W., Raiffa, H. and Schlaifer, R.O. Introduction to Statistical Decision Theory. McGraw Hill, New York, 1965.

[16] Raiffa, Howard. "Preferences for Multiattributed Alternatives." RM-5868-DOT/RC, The Rand Corporation, April, 1969.

[17] Schlaifer, R.O. Analysis of Decisions under Uncertainty. McGraw Hill, New York, 1969.

[18] Schlaifer, R.O. Computer Programs for Elementary Decision Analysis. Harvard University, Cambridge, Massachusetts, 1971.

[19] Stehfest, H. Personal Communication, July 1975.

[20] Stehfest, H. "Basic Data for Optimal Control of Rhine River Quality." IIASA internal memorandum, March 1975.

[21] Stehfest, H. "Modelltheoretische Untersuchungen zur Selbstreinigung von Fliessgewaessern." Kernforschungszentrum, Karlsruhe, August, 1973.

[22] Streeter, H.W. and Phelps, E.B. "A Study of Pollution and Natural Purification of the Ohio River." U.S. Public Health Service Bulletin, 146 (1925).

[23] Young, P. and Beck, B. "The Modelling and Control of Water Quality in a River System," Automatica, 10 (1974), 455-468.