RM-75-34

POLYHEDRAL DYNAMICS - II: GEOMETRICAL STRUCTURE AS A BASIS FOR DECISION MAKING IN COMPLEX SYSTEMS

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July 1975

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Abstract

The tools of polyhedral dynamics and dynamic programming are combined through the medium of cross-impact analysis to attack problems of organizational structure. It is argued that the standard cross-impact approaches to such problems are deficient in that they ignore the true multi-dimensional nature of such systems, as well as providing no systematic mechanism for rational decision making. The results of the analysis are illustrated by applications to the structuring of a large scientific organization and by the analysis of a simplified version of a problem arising in the energy field.

1. Introduction

Over the past few decades it has become increasingly apparent, even to the casual observer, that modern technology and communication facilities have led to an almost unbelievable degree of specialization on the part of the labor force. A corollary of this process has been the ever-increasing size, complexity, and compartmentalization on the part of virtually all societal organizations, at least in the industrial and post-industrial countries of the world. A cursory glance at any government directory, university catalogue, or industrial organization chart will quickly confirm the above trend.

The movement toward specialization poses a serious problem for modern managers in that they must somehow arrange an organizational structure that integrates the diverse talents at their disposal into a smooth-functioning unit (or units) working toward the overall organizational goal. In addition, the manager's job is complicated by the dramatically reduced system time-constraints forced upon him by the developmental rate of modern technology and social change. No longer can a manager afford

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the luxury of choosing an organizational structure and expecting it to effectively serve its function for any appreciable length of time. The organization "obsolete before its conception" is almost a cliché of our times, due to the inability of many (most?) decision makers to adapt to the new order of things.

The basic question is how can any mere mortal hope to successfully juggle the multiplicity of goals, constraints, and resources, judiciously discounting future imponderables, and come out with any type of even feasible structure, let alone one which would be flexible enough to bend with the prevailing political, economic, and social breezes. As Mark Twain put it, "it wouldn't be a cinch for an angel." The answer to this question, of course, is that no one really knows how to successfully create such organizational structures. As a result, a general factotum of analysis is applied: when you do not know what to do, apply what you do know which, in this case, generally means that various structures that have served reasonably well in the past are resurrected with greater or lesser, but basically random, success.

One of the procedures that so-called "modern" managers have employed in their quest for the Holy Grail of perfect organization is cross-impact analysis. In its simplest form, this technique consists in identifying two sets of objects, say men and tasks. A cross-impact matrix is then constructed having an entry in the (i,j) position if the talent of man i is necessary for completion of task j. Often this is purely a 0-1 situation, although refinements are possible, as we shall point out later. Using the cross-impact matrix, the last step of the analysis usually consists in forming a directed <u>planar</u> graph and attempting to draw conclusions about the structure under study by various graph-theoretic techniques. Such cross-impact analyses are also employed in many areas outside organizational structure, particularly in cases where large numbers of independent input variables (decisions) affect many output variables, precise causal effects being poorly understood. We shall examine a case of this type arising in the energy field later.

The thesis which we propose to argue in this paper is that the traditional cross-impact approach to system structure is deficient in two essential ways:

Dimensionality - the reduction of the cross-impact matrix to a planar directed graph projects what is inherently a highly multidimensional object, namely the organization or system under study, into a two-dimensional world. It seems intuitively clear, by geometrical reasoning or otherwise, that artificially constraining any object to "live" in a world smaller than its natural dimension will result in a loss of information concerning the basic structure of that object. Elementary projective geometry shows us that infinitely many objects (curves, vectors, etc.) may project onto the same object, and there is no reason to suspect that large organizations, which are at least as complicated as elementary lines and planes, can be satisfactorily understood by projecting the essence of their being into a two-dimensional world. When put in such bald terms, it seems quite astonishing that any useful information could be gleaned from such approaches.

Dynamism and Control - cross-impact studies are essentially static in their approach to structural analysis. The output of such an analysis is a picture, albeit a distorted one, of the system structure at a particular moment in time. Furthermore, the standard techniques give no information as to what should be done to effect changes in the system in order to modify its structure in some purposeful fashion. The above objections to cross-impact analysis are well known and most likely would be heartily seconded by any analyst or manager who has had occasion to employ them. The basic question that arises is what, if anything, can one do to overcome the obstacles? Is there any methodology which is capable of dealing with the multidimensional nature of large organizations and translate it into understandable terms? And finally, can such a methodology be readily taught, understood, and used by real-world decision makers? In the remainder of this paper, we shall attempt to provide affirmative answers to all of these questions.

The fundamental tools to be used in our development are the theories of polyhedral dynamics and dynamic programming. Since the basic problem naturally separates into the two components of structural understanding and effective action, we find it necessary to employ tools specifically designed for each task. Polyhedral dynamics will enable us to cope with the multidimensional nature of large organizations by providing a systematic procedure for determining the way in which the structure is put together, the nature of the system components, and the effect of various local changes upon the global organizational structure. With the understanding of the inherent geometry of the system given by the methods of polyhedral dynamics, we may then apply the well known recursive techniques of dynamic programming in order to effect feedback decision making policies in an optimal manner.

Since the ideas of polyhedral dynamics are not well known, we shall begin our discussions with a brief review of the basic ideas. More details and examples may be found in [1,4]. Following the introductory section, we consider the basic cross-impact techniques and the use of polyhedral dynamics to upgrade their utility. Dynamic programming methods are then introduced to facilitate decisionmaking and the paper closes with some hypothetical examples and a discussion of further extensions.

2. Polyhedral Dynamics

To set the mathematical stage for what follows, we briefly sketch the main ideas surrounding the conceptual tool which we have chosen to call polyhedral dynamics [4]. The basic idea is to recognize that every relation between two sets of objects can be given a geometrical interpretation as a simplicial complex and that the methods of modern algebraic topology may be employed to analyze the structure and connectivity patterns associated with the complex. Of special importance for applications will be the manner in which the polyhedra of the complex are connected to each other through chains of varying dimensions. It will be through such chains of connection that we will be able to account for the inherently multidimensional world in which all activities naturally take place.

Our basic set-up is the following: we have two finite sets of objects, say X and Y and a relation λ between them. Thus, $\lambda \subset X \times Y$. For example, suppose $X = \{1,2,3,4\}$, $Y = \{-1,0\}$, and λ is the relation $<$, i.e. if $x \varepsilon X$, $y \varepsilon Y$, then $(x,y) \varepsilon \lambda$ if and only if $x < y$. In this example, the relation λ is empty since there is no pair (x,y) such that $x < y$. A less trivial example is obtained if we let X = {supermarket, post office, bank, bakery}, Y = {milk, eggs, cakes, envelopes}. If the relation λ is defined to mean that facility x offers item y for sale, then we see that x_1, supermarket, is λ-related to each y, x_2, post office, is λ-related to y_4, envelopes, etc. Thus, $\lambda = \{(x_1,y_1), (x_1,y_2), (x_1,y_3), (x_1,y_4), (x_2,y_4), (x_4,y_2)\} \subset X \times Y$. One of the great advantages to the polyhedral dynamics approach is that the above framework is extremely general, allowing for an almost unlimited variety of applications.

We may associate a matrix Λ to each relation $\lambda \subset X \times Y$. The entries in Λ are either 0 or 1 and are defined by the rule

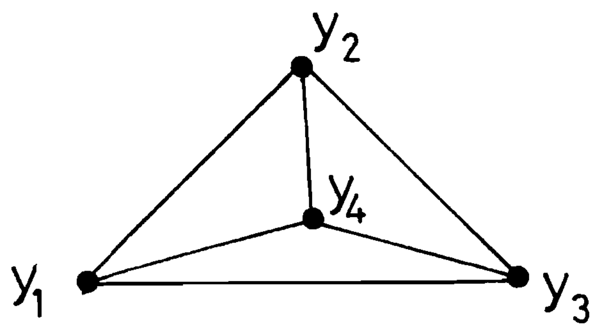
$$[\Lambda]_{ij} = \begin{cases} 1, & (x_i,y_j) \varepsilon \lambda \\ 0, & (x_i,y_j) \not\varepsilon \lambda \end{cases} .$$

If card X = n, card Y = m, then Λ is an $n \times m$ matrix. By transposing the matrix Λ, we obtain the conjugate relation $\lambda^{-1} \subset Y \times X$ defined in a manner analagous to that above. It is clear that knowledge of λ defines λ^{-1}, and conversely.

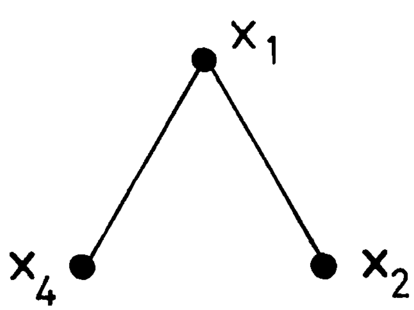
For analytical purposes, we shall work with the incidence matrices Λ and Λ'. However, it is often useful to employ a geometrical picture of the simplicial complex induced by a relation λ on the sets X and Y. If we regard $x_i \varepsilon X$ as defining row i of Λ, while $y_j \varepsilon Y$ defines column j, then we may define the simplicial complex $K_X(Y;\lambda)$ having vertices $\{y_1,...,y_m\}$ and simplices named $\{x_1,\ldots,x_n\}$. Thus, the simplex x_i will consist of the polyhedron defined by all vertices in Y which are λ-related to x_i. For instance, in the second example above, the matrix Λ is

Λ = λ	y_1 (=milk)	y_2 (=eggs)	y_3 (=cakes)	y_4 (=envelopes)
x_1 (= supermarket)	1	1	1	1
x_2 (= post office)	0	0	0	1
x_3 (= bank)	0	0	0	0
x_4 (= bakery)	0	0	1	1

The complex $K_X(Y;\lambda)$ is



consisting of the 3-simplex (tetrahedron) x_1, and the two 0-simplices x_2 and x_4. The conjugate complex $K_Y(X;\lambda^{-1})$, corresponding to the incidence matrix Λ', has the geometrical form



consisting of the 0-simplices y_1 and y_2 (which are identical), and the 1-simplices y_3 and y_4.

A major objective of polyhedral dynamics is to analyze the way in which the complex $K_X(Y;\lambda)$ (or $K_Y(X;\lambda^{-1})$) is connected through its component simplices. The first step in this direction is to perform what is called a Q-analysis [4] of $K_X(Y;\lambda)$. We say that a simplex σ_p is said a face of the simplex σ_r if σ_p and σ_r share at least one vertex and the dimension of $\sigma_p(=p)$ is less than that of σ_r (recall that a simplex σ_n is of dimension n if it consists of n + 1 vertices). We now have the important

Definition 1. Two simplices σ_p and σ_r are said to be connected by a q-chain if there exists a sequence of simplices σ^i, i = 1, 2,...,m such that

i) σ_p is a face of σ^1,

ii) σ^j is a face of σ^{j+1}, j = 1,...,m-1 ,

iii) σ_r is a face of σ^m,

iv) $\min_{1 \leq i \leq m} \{\dim \sigma^i\} = q$.

Thus, intuitively speaking, q is the strength of the "weakest" link in any chain connecting σ_p and σ_r.

Using Definition 1, it is not difficult to see that the property of q-connectivity induces an equivalence relation upon the simplices of K, i.e. two simplices σ_p and σ_r are equivalent if and only if they are connected by a q-chain, q = 0, 1,.... The process of performing a Q-analysis consists of determining the cardinality of the equivalence classes under the relation of q-connectivity. In other words, we are interested in finding the number of distinct q-connected components in $K_X(Y;\lambda)$. We write the results of the Q-analysis in a vector

$$Q = (Q_n, Q_{n-1}, \ldots, Q_1, Q_0) \quad ,$$

whose components Q_i represent the number of distinct i-connected components in the complex. A simple algorithm for performing the Q-analysis directly from the incidence matrix Λ is given in the Appendix.

To illustrate the technique, consider the earlier example of services and goods. The relevant incidence matrix for $K_X(Y;\lambda)$ was

$$\Lambda = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} .$$

Employing the algorithm of the Appendix, we form the matrix $\Lambda\Lambda' - \Omega$ (where $[\Omega]_{ij} = 1$ for all i,j). This gives

$$\Lambda\Lambda' - \Omega = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 3 & 0 & - & 0 \\ & 0 & - & - \\ & & - & - \\ & & & 0 \end{bmatrix} \end{array} ,$$

where we write only the upper triangular part and use (-) to denote (-1). The above matrix gives us the connectivity pattern of the simplices x_1, x_2, x_3, x_4. Performing the q-analysis, we have

$$\begin{aligned} \text{at } q = 3,\ Q_3 &= 1,\ \{x_1\} , \\ q = 2,\ Q_2 &= 1,\ \{x_1\} , \\ q = 1,\ Q_1 &= 1,\ \{x_1\} , \\ q = 0,\ Q_0 &= 1,\ \{x_1, x_2, x_4\} . \end{aligned}$$

The Q vector is thus

$$Q = \overset{3}{(1} \quad 1 \quad 1 \quad \overset{0}{1)} \quad .$$

Note that the simplex x_3 (bank) plays no role whatsoever in $K_X(Y;\lambda)$ since it has no relation to any good in Y. For the purposes of analysis, it should be entirely eliminated from consideration. Note, however, that this is not the same thing as having a simplex which is totally disconnected from the rest of the complex. The latter situation would show up as $Q_0 > 1$.

At this juncture one might object that the 0-1 incidence pattern necessary for Q-analysis is often unrealistic for practical problems in the sense that it is purely qualitative, failing to distinguish varying degrees of connective strength between the elements of X and Y. To deal with this situation, we allow for γ to be a weighted relation. Thus, the elements of the incidence matrix may now be any real number. A 0-1 incidence matrix Λ is then induced from Γ by means of slicing parameters, θ_{ij} which are specified by the analyst. If we call the weighted incidence matrix $\Gamma = [\gamma_{ij}]$, then the elements of Λ are determined by the rule

$$[\Lambda]_{ij} = \begin{cases} 0, & \gamma_{ij} < \theta_{ij} \\ 1, & \gamma_{ij} \geq \theta_{ij} \end{cases} , \qquad \begin{matrix} i = 1, \ldots, n \\ j = 1, \ldots, m \end{matrix} \quad .$$

Hence, we see that our structural view of the complex will be greatly influenced by the level at which we choose to observe it, the level being specified by the parameters θ_{ij}.

3. Dynamical Change

Since the vector Q gives an unambiguous, multidimensional description of how the complex K_X Y;λ) is connected, we are in position to describe a mechanism for dynamical changes in the structure. The possibilities for effecting structural changes consist of

1) changing the elements of the sets X and Y either by addition or deletion of vertices and simplices, and/or

2) modifying the relation λ by either

a) redefinition or

b) in the case of a weighted relation, changing the threshold levels θ_{ij}.

Of course, there is no uniform rule as to which combination of possibilities should be used, since the choice will be highly dependent upon the situation under study and the operating constraints.

If we assume that at any time t, there is a certain set of admissible decisions D_t, then the result of a decision will, in general, change the structure vector

$$Q(t) \rightarrow Q(t + 1) = T(Q(t)),$$

where T is the transformation taking Q(t) into Q(t + 1). Since most decision making is carried out over some decision-horizon, we can employ dynamic programming techniques to determine optimal policies, assuming some measure of utility can be attached to any given system structure. For example, assume that we agree to measure the system structure by the two vectors

$$Q(t) \text{ and } Q^{-1}(t) \quad ,$$

corresponding to the conjugate complexes $K_X(Y;\lambda)$ and $K_Y(X;\lambda^{-1})$. Furthermore, let us agree that our measure of utility is given by

$$g_t(Q(t), Q^{-1}(t), d(t)) \quad , \qquad d(t) \varepsilon D_t \quad ,$$

and that we desire to maximize the quantity

$$J = \sum_{t=1}^{N} g_t(Q(t), Q^{-1}(t), d(t)) \quad .$$

Introducing the optimal value function

$f_a(Q, Q^{-1})$ = value of J when the system structure is (Q, Q^{-1}), the process begins at time a and an optimal policy is used,

the Principle of Optimality [3] immediately yields the recurrence formula

$$f_a(Q, Q^{-1}) = \max_{d \varepsilon D(a)} [g_a(Q(a), Q^{-1}(a), d) + f_{a+1}(T(Q, Q^{-1}))] \quad ,$$

$$a = N - 1, N - 2, \ldots, 1 \quad ,$$

$$f_N(Q, Q^{-1}) = \max_{d \varepsilon D(N)} g_N(Q, Q^{-1}, d) \quad .$$

As is well known, these equations may be used to recursively determine the optimal value and policies associated with the system under study.

4. Cross-Impact Matrices and Applications

We now turn our attention to the main point of this report--the application of polyhedral dynamics to the management of large organizations. Before presenting our main examples, we briefly review the standard cross-impact set-up.

Classically, cross-impact analyses are concerned with two sets of objects, rather unimaginatively called A and B, and the causal relationships between their elements. We may form a cross-impact matrix $\mathscr{C}$ from A and B by defining the elements as

$$[\mathscr{C}]_{ij} = \begin{cases} 1, \text{ if element } a_i \varepsilon A \text{ influences element } b_j \varepsilon B \\ 0, \text{ otherwise} \end{cases}$$

Thus, $\mathscr{C}$ may actually be considered as an incidence matrix. The departure from our discussion of the previous section takes place when, rather than forming a true multidimensional simplicial complex from $\mathscr{C}$, standard approaches project the whole of $\mathscr{C}$ onto a directed planar graph by identifying the elements of A and B as nodes of the graph, and letting an arc pass from a_i to b_j if and only if element $[\mathscr{C}]_{ij} = 1$. The futility of such an approach should now be fairly evident since almost all of the inherent structure present in $\mathscr{C}$ is destroyed by such a projection. In very few cases will it be possible for a two-dimensional object, the digraph, to accurately reflect the true multidimensional nature of $\mathscr{C}$.

Example 1: Scientific Organization

As a static example of the use of polyhedral dynamics, consider the organization of a group of scholars encompassing many disciplines into a cohesive, interdisciplinary research institute. The basic problem here is how to organize the available talent to simultaneously promote interdisciplinary contact while still retaining the professional stimulus of group specialization, i.e. we must resolve the problem of organizing the "minds and bodies" into a single organizational scheme. One obvious extreme is to have everyone in a single group, thereby precluding group direction in any special area. The other end of this spectrum is to organize solely along project (or specialization) lines, which destroys the interdisciplinary aspect. Most likely, the best route is some sort of compromise between the two.

To study this problem by polyhedral dynamics, we define the two sets

$$X = \{\text{all disciplines}\} \quad ,$$

$$Y = \{\text{application areas}\} \quad .$$

For our example, we let

$$X = \left\{\begin{array}{l}\text{engineer, biologist, physicist, social scientist,} \\ \text{mathematician, computer programmer, economist}\end{array}\right\},$$

$$Y = \left\{\begin{array}{l}\text{water resources, energy, urban, ecology, bio-medical,} \\ \text{organizations, methodology, food, industrial}\end{array}\right\}.$$

For the sake of argument, let us assume that we have the weighted relation λ giving rise to the following incidence matrix Λ

	λ	Water	Ene.	Urb.	Ecol.	Bio-med.	Org.	Meth.	Food	Ind.
					Y					
	ENG	30	15	5	5	10	10	15	0	35
	BIO	20	15	5	50	40	0	0	20	5
	PHY	10	30	5	10	20	10	5	15	5
X	SSCI	5	5	40	10	10	20	0	15	10
	MATH	15	10	10	15	10	40	50	10	10
	COMP PROG	10	10	10	10	5	5	20	10	15
	ECON	10	15	25	0	5	15	10	30	5

The numbers in this incidence matrix are chosen to represent an estimate of the per cent of total project resources which should be devoted to personnel from the corresponding disciplines.

We now perform a Q-analysis at different slicing levels in an attempt to understand the structure inherent in the above organizations. First of all, we slice uniformly at the level $\theta = 1$. This induces the incidence matrix

$\mathcal{O} = 1$

$K_X(Y;\lambda)$	Water	Ene.	Urb.	Ecol.	Bio-med.	Org.	Meth.	Food	Ind.
ENG	1	1	1	1	1	1	1	0	1
BIO	1	1	1	1	1	0	0	1	1
PHY	1	1	1	1	1	1	1	1	1
SSCI	1	1	1	1	1	1	0	1	1
MATH	1	1	1	1	1	1	1	1	1
COM PRG	1	1	1	1	1	1	1	1	1
ECON	1	1	1	0	1	1	1	1	1

$\Lambda =$ (matrix above)

The Q-analysis gives for the complex $K_X(Y;\lambda)$ at

$q = 8$: $Q_8 = 1$ {PHY, MATH, COMP PROG}

$q = 7$: $Q_7 = 1$ {PHY, ENG, SSCI, MATH, COMP PROG, ECON}

$q \leq 6$: $Q_6 = 1$ {all}

Thus, the structure vector is

$$Q = \overset{8}{(}1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad \overset{0}{1}) ,$$

and we see that the complex $K_X(Y;\lambda)$ is strongly connected at the slicing level $\mathcal{O} = 1$, there being only a single component at all connectivity levels.

Suppose now we feel that a resource level less than 15% will be too low for effective work. To test the effect of this change on the organizational structure, we increase our slicing level to $\mathcal{O} = 15$. This yields the new incidence matrix

$\theta > 15$:

		Water	Ene.	Urb.	Ecol.	Bio-med.	Org.	Meth.	Food	Ind.
	ENG	1	0	0	0	0	0	0	0	1
	BIO	1	0	0	1	1	0	0	1	0
	PHY	0	1	0	0	1	0	0	0	1
Λ =	SSCI	0	0	1	0	0	1	0	0	0
	MATH	0	0	0	0	0	1	1	0	0
	COMP PROG	0	0	0	0	0	0	1	0	0
	ECON	0	0	1	0	0	0	0	1	0

The Q-analysis gives

$q = 3,\ Q_3 = 1,\ \{BIO\}$

$q = 2,\ Q_2 = 2,\ \{BIO\},\ \{PHY\}$

$q = 1,\ Q_1 = 6,\ \{BIO\},\ \{PHY\},\ \{SSCI\},\ \{MATH\},\ \{ECON\},\ \{ENG\}$

$q = 0,\ Q_0 = 1,\ \{all\}$.

Thus, the Q vector for this situation is

$$Q = \overset{3}{(1}\ \ 2\ \ 6\ \ \overset{0}{1)} \ .$$

Now the situation has changed from the earlier highly connected structure to a situation which is only totally connected at the 0-level. As soon as we step up to the 1-level, the organizational complex splits into six disjoint pieces. Notice, also, that at the $\theta = 1$ level, the simplex BIO seemed to be a weak component in the structure, being only a 6-simplex. However, at the $\theta > 15$ level, BIO becomes the highest dimensional simplex in the complex. We conclude that the group BIO is far more important to the organization than would appear at first glance.

In this way, we can study the relative integration of all the professions into the organization by choosing different slicing levels. Similarly, if we return to the original weighted relation λ and perform a Q-analysis for the inverse relation λ^{-1}, we may look at the organizational structure from the point of view of project integration rather than personnel. A combination of these two relations, studied at different slicing levels, will finally yield a global picture of the entire organization and will pinpoint the disciplines and/or projects acting as obstructions to a well structured organization.

Example 2: Energy Policy Making

We turn now to a dynamic example of policy making in the energy area. Suppose we are concerned with environmental impact of various types of primary energy sources. The basic task might be to plan an energy program which shifts emphasis from one source to another as technological advances take place, while at the same time paying attention to the effect on the environment of various sources.

For the sake of illustration, assume the primary sources are given by the elements of the set

$$X = \left\{ \begin{array}{l} \text{fossil fuels, nuclear reactors, solar energy, thermal} \\ \text{energy, fusion reactors} \end{array} \right\} ,$$

while the environmental pollutants form the set

$$Y = \{\text{heat, particulate matter, radiation, CO2}\} .$$

Further, we postulate the relation $\lambda \subset X \times Y$ to be $(x_i, y_j) \varepsilon \lambda$ if and only if source x_i gives rise to pollutant y_j. A reasonable incidence matrix for this situation might be

$\Lambda =$

λ	x_1	x_2	x_3	x_4	x_5
y_1	1	1	1	1	1
y_2	1	1	0	0	0
y_3	0	1	0	0	1
y_4	1	0	0	0	0

Carrying out the Q-analysis for Λ, we find the initial structure vector is

$$Q = \overset{4}{(}1 \quad 1 \quad 1 \quad 1 \quad \overset{0}{1)}$$

indicating a well-connected complex. However, our interest is in temporal decision making which will change the structure in some way.

Suppose we introduce costs associated with taking various decisions and that the allowable decisions are to focus attention on some subset of sources in X by means of deletion of certain vertices from consideration as energy sources during the given time period. Since X has five elements, our decision set has thirty-two elements at each time. If we assign a cost

$$g_i(\hat{Q},\hat{X})$$

to the decision $\hat{X}$ and structure vector $\hat{Q}$ at time period i, then we may use dynamic programming as sketched in the last section to determine our optimal decision policy. For instance, if the decision were made to eliminate fossil fuels from consideration in a given period, then the above structure vector would change to

$$Q = \overset{3}{(}1 \quad 1 \quad 1 \quad \overset{0}{2)}$$

since y_4, carbon dioxide pollution, would be eliminated from the complex. Of course, one would have to weigh the benefits of this structure vector against other possibilities to assess the optimal policy but, once the functions $\{g_i\}$ were assigned, it would be a simple task to compute the best policies.

5. Extensions

In addition to the foregoing Q-analysis, there are several other concepts that one might introduce to study the connectivity properties of the complex $K_Y(X;\lambda)$ (or $K_X(Y;\lambda^{-1})$. We mention two possiblities:

a) Eccentricity - since the individual simplices form the basic building blocks of $K_Y(X;\lambda)$, it is of some interest to have some measure of how well integrated each individual simplex is into the total structure. Our structure vector Q is not sufficient for this purpose since it only gives a gross measure of how many distinct q-connected components are in $K_Y(X;\lambda)$, but says nothing about the individual simplices of these components.

To remedy this situation, for each simplex $\sigma \varepsilon K_Y(X;\lambda)$ we define the two numbers

$\hat{q}$ = dimension of σ as a simplex (= 1 + number of vertices comprising σ);

$\check{q}$ = the largest value of q for which σ is q-connected to another distinct simplex in the complex, i.e. the largest value of q for which σ appears in the q-analysis in a component not consisting of itself alone.

Using $\hat{q}$ and $\check{q}$, we define the eccentricity of σ as

$$\mathrm{ecc}(\sigma) = \frac{\hat{q} - \check{q}}{\check{q} + 1} \quad .$$

Since $\check{q} = -1$ if and only if σ is totally disconnected from the remainder of the complex, an eccentricity of ∞ is consistent with our intuitive feeling of how well integrated a particular simplex is into the complex. After all, it is hardly possible to be less "antisocial" than to be totally disconnected from the remainder of $K_Y(X;\lambda)$. Similarly, a large difference $\hat{q} - \check{q}$ implies that σ does not integrate well at high-dimensional levels, but this disunity must be normalized by the first q-level at which integration does occur, since it is intuitively clear that the lower this critical level is, the more disjoint σ is from the remainder of $K_Y(X;\lambda)$.

b) Patterns - the second notion we introduce is that of a pattern on $K_Y(X;\lambda)$. Here the basic idea is that $K_Y(X;\lambda)$ should be regarded as a dynamic structure in which temporal changes take place. The type of dynamics can be represented as a flow of numbers from one simplex to another, in other words, a pattern. More precisely, a pattern Π_i is a mapping

$$\Pi_i: \ \{\text{i-dimensional simplices}\} \rightarrow \{\text{number field } k\} \quad ,$$

i.e. Π_i is a rule which assigns a number from k to each i-dimensional simplex in $K_Y(X;\lambda)$. The total pattern Π on K may then be regarded as the direct sum of all individual patterns,

$$\Pi = \oplus \sum_{i=0}^{K} \Pi_i \quad ,$$

where $K = \dim K_Y(X;\lambda)$. Thus, Π is a graded pattern.

An important point to note is that the individual components of Π are defined only on i-dimensional objects. Hence, a change in Π_i from one moment to the next implies that in some way there has been a redistribution of numbers from one i-dimensional simplex to another. Thus, the particular numbers have meaning only when restricted to their appropriate dimension and redistribution is possible in only two ways: i) two i-dimensional simplices belong to the same i-connected component or ii) a mechanism from outside the complex is at work. This idea gives rise to the notion of an obstruction vector $\hat{Q}$ formed from Q as

$$\hat{Q} = Q - U \quad ,$$

where U is the unit point $U = (1,1,\ldots,1)$. $\hat{Q}$ is a measure of the geometrical obstructions to a free flow of patterns in $K_Y(X;\lambda)$ since it measures the ability of a pattern to change by method i). Further properties and uses of patterns, obstruction vectors, graded algebras, etc. are given in [2,4].

APPENDIX

Algorithm for Q-analysis

If the cardinalities of the sets Y and X are m and n, respectively, the incidence matrix Λ is an (m x n) matrix with entries 0 or 1. In the product $\Lambda\Lambda'$, the number in position (i,j) is the result of the inner product of row i with row j of Λ. This number equals the number of 1's common to rows i and j in Λ. Therefore, it is equal to the value (q + 1), where q is the dimension of the shared face of the simplices σ_p, σ_r represented by rows i and j. Thus, the algorithm is

1) form $\Lambda\Lambda'$ (an m x m matrix),
2) evaluate $\Lambda\Lambda' - \Omega$, where Ω is an m x m matrix all of whose entries are 1,
3) retain only the upper triangular part (including the diagonal) of the symmetric matrix $\Lambda\Lambda' - \Omega$. The integers on the diagonal are the dimensions of the Y_i as simplices. The Q-analysis then follows by inspection.

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