Working Paper

Risk-Reshaping Contracts and Stochastic Optimization

Georg Ch. Pflug

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Risk-Reshaping Contracts and Stochastic Optimization

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Abstract

Insurance contracts and lotteries are just the opposite sides of the same coin: These are contracts, which allow to reshape an uncertain financial position by exchanging risks between two contractors. In this paper, we discuss some basic problems of operations research which are connected with such kind of contracts.

1 Introduction

Actions in economic planning are taken in an uncertain environment: The economic result of a decision may depend on the future, like on future prices, interest or exchange rates, but also on accidents, catastrophes and political decisions. One may say that uncertainity and risk are inevitable factors in economic decision making.

Developed economies offer instuments to buy and sell risks as if they were goods: Insurances and banks are ready to buy risks for some specific price. Typically, the seller's risk is not fully taken by the buyer, but his risk distribution is changed, it is reshaped.

Risk as part of the economic decision process has been studied by many economists starting in the 70ies. The typical research question was how to assess utilities to risky alternatives making observed human behavior rational.

Only in the 90ies the problem of how to assess prices and preferences to risk reshaping contracts became an important question in business administration. This area of research is part of the Stochastic Operations Research, in particular of Stochastic Optimization. In this paper, we review some basic research questions connected with risk-reshaping contracts.

We will concetrate here on one-stage decision problems (i.e. just one decision has to be made), bearing however in mind that most practical decisions are multi-stage (we have to decide now, but we know that there are future time moments allowing us to take corrective actions). For instance, asset-liability management problems of pension funds are always of the multi-stage type, since the fund is supposed to operate for a long period, in which inflows and outflows occur and investment or deinvestment decisions have to be taken in regular time intervals.

We may distinguish between binary decisions (where we may choose just between two alternatives), discrete decisions (where we have to select the decision from a finite set) or continuous decisions (where the decision space is a continuum).

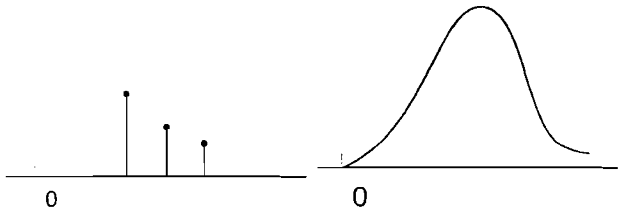
Let us consider a binary decision problem first.

Suppose that our future costs are described by a random variable Y. (Profits are considered as negative costs). If somebody offers a contract such that this random variable Y changes into another random variable Z_0 for the price of p_0, we are faced with the problem, whether to take this offer or not. We may and will include the price p_0 into the cost variable, and consider $Z = Z_0 + p_0$ as the new costs.

What is required for the decision process is a preference relation, which allows us to decide between Y and Z.

Let F_Y resp. F_Z be the distribution functions of Y resp. Z. These distribution functions may be decomposed into the absolutely continuous parts with densities f_Y resp. f_Z and the discrete parts.

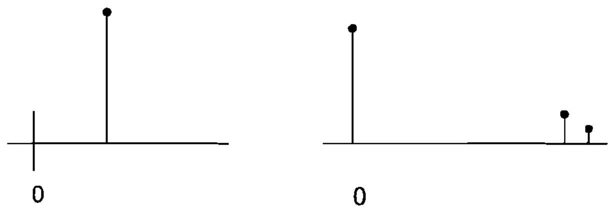
We visualize probability distributions on the real line in the following manner: The discrete part is shown by bars with a dot on top and the continuous part is represented by its density function.



The discrete part (left) and the continuous part (right) of a distribution.

Here are some examples of cost-reshaping contracts:

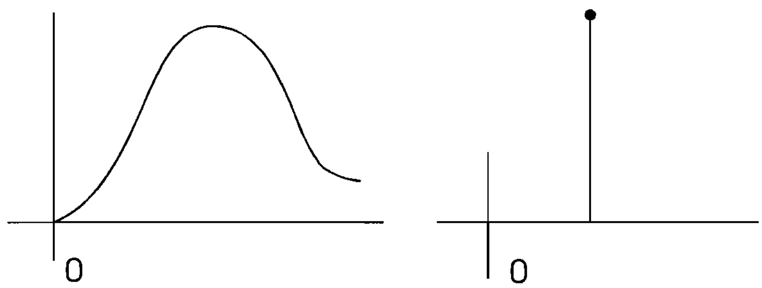
(i) **A lottery**
For a fixed price $Y = p_0$, one may buy a ticket which offers a random win Z, distributed according to some discrete probability distribution.



The price of the ticket (left) and the distribution of wins (right)

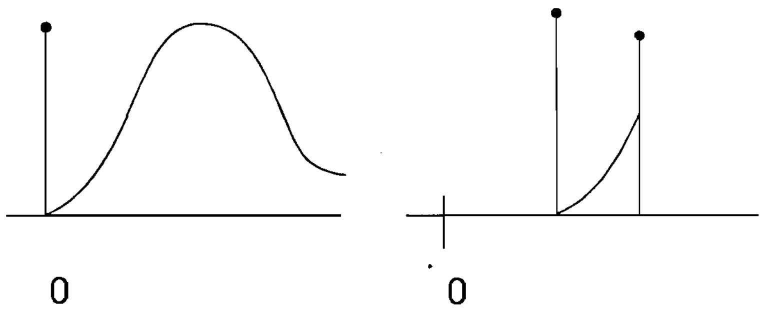
Typically, the expectation $\mathbb{E}(Z)$ is smaller than the price p_0. Although Z is riskier, it has a smaller expectation, Thus the lottery has a negative risk premium $\mathbb{E}(Z) - p_0 < 0$. It seems completely irrational to buy lottery ticket. In fact, firms should never play lotteries. since it is irrational to do. For individuals, there is a psychological argument, namely the regret principle (see section 3), which may be adopted as excuse for irrationality.

(ii) **An insurance contract**
Random costs Y are taken over by the insurer for a fixed price $Z = p_0$. Typically the insurer gets a risk premium (safety loading) $p_0 - \mathbb{E}(Y) > 0$. From the insurer's side, the contract looks like a lottery with positive risk premium, a lottery which is worthwile to play.



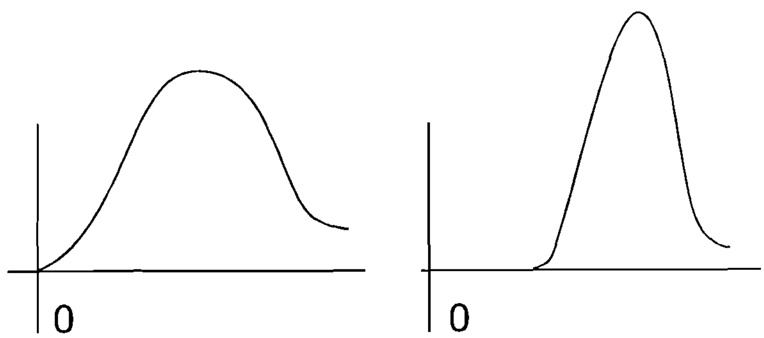
The distribution of the random claim Y (left) and the (deterministic) insurance premium (right)

(iii) **An insurance with deductible**
For a premium of p_0, random costs Y are taken over by the insurer, if they exceed a prespecified amount z_0 (the deductible). Thus by this contract, the customer changes his cost distribution from Y to $Z = \min(Y, z_0) + p_0$.



The distribution of the total damage Y (left) and the cost distribution Z of the insured customer (right)

(iv) **Reinsurance**
Let Y be the total outpayments of an insurance company in one period. Suppose that the company has a reinsurance contract, which allows to claim $\gamma(Y - m)^+$ from the reinsurer (a^+ equals a, if a is positive and 0 otherwise). This contract changes Y to $Z = Y - \gamma(Y - m)^+ + p_0$. Proportional reinsurance ($m = 0$) and stop-loss reinsurance ($\gamma = 1$) are special cases.



The cost distribution without (Y :left) and with (Z :right) a reinsurance contract.

(v) **Options**
The (European put) option contract allows the owner of a share to sell this share at some future time instant T for the price of c, if he wants. Suppose that Y denotes the price of this share at time T and let p_0 be the price of the option. Then the decision maker has to decide between Y (not to buy the option) and $Z = \max(Y, c) - p_0$ (to buy the option). Notice that Y and Z are profits and not costs in this example.

(vi) **Swaps, Caps, Floors and Collars**
The swap contract allows to make and exchange between a fixed-interest loan and a variable-interest loan. It is a kind of insurance contract. Caps, floors and collars are contracts, which put limits to the interests of a variable-interest loan. They are kinds of reinsurance contracts.

The basic research questions for risk-reshaping contracts are:

(i) **The estimation problem:** How can we estimate the distributions of Y and Z from data?

(ii) **The decision making problem:** How should an individual agent calculate his preferences for Y or Z and make his decisions?

(iii) **The pricing problem:** How should a company fix the price of a contract, which offers to reshape Y to Z?

We discuss these questions in the following sections.

2 Modeling and estimation

The basic problem of decision making under uncertainity is the statistical problem of estimating the distribution of the random costs Y (and likewise of Z). Without information about the distributions (risk assessment) no decision is possible.

The quality of the estimates depends on the available data, their quality and the accuracy of the model. In the simplest case like in property insurance, independent, identically distributed observations $Y_1, Y_2, \ldots, Y_n$ of the individual claims can be observed and the unknown claim distribution F_Y can be estimated by the empirical d.f.

$$\widehat{F}_Y(t) = \frac{1}{n} \sum_{i=1}^{n} I(Y_i \leq t)$$

or smoothed variants of it. It is important to stress that $\widehat{F}_Y$ is only an approximation of F_Y and there is always an estimation error present. This error is often ignored and the decisions are made as if $\widehat{F}_Y$ would be the true F_Y. The estimation error can be quantified by formulas for confidence regions, like the Dvoretzky-Kiefer-Wolfowitz inequality

$$\mathbb{P}\{\sup_t |\widehat{F}_Y(t) - F_Y(t)| \geq \epsilon\} \leq 58 \exp(-n\epsilon^2)$$

or other exponential inequalities (see Shorack and Wellner [7]).

Let $\mathcal{F}_{Y,\epsilon}$ be a the confidence region for F_Y

$$\mathcal{F}_{Y,\epsilon} = \{F_Y : \sup_t |F_Y(t) - \widehat{F}_Y(t)| \leq \epsilon\}.$$

A decision to prefer Z over Y (i.e. to prefer $\widehat{F}_Z$ over $\widehat{F}_Y$) is *robust of level* ϵ, if all elements of $\mathcal{F}_{Y,\epsilon}$ are preferrable over all elements of $\mathcal{F}_{Z,\epsilon}$.

Very often, i.i.d. data for Y are not available, but the distribution of Y has to be inferred in an indirect manner. Consider e.g. the problem of deciding whether or not to swap a variable interest loan with a collar (interest rates are variable only within prespecified limits). To determine the distribution, we have to have a stochastic model of the interest rates. Let (η_t) be the stochastic process describing the interest rates. Z is some (complicated) function of the whole trajectory $(\eta_t)_{0 \leq t \leq T}$. We estimate the distribution of (η_t) first by fitting a parametric model to the past observed data. Assuming stationarity of the process or at least of its trend, we get a stochastic model for the future interest process. Finally, the distribution of the derived quantity Z is estimated – typically by simulation, very rarely by analytical considerations.

As before, the estimation error is not negligible and must be quantified by confidence regions to get robust decisions.

In cases of extreme lack of information, estimation by statistical methods is replaced by expert opinions. This very subjective method requires that the expert lists the set of possible scenarios together with an assignment of probabilities to each of them.

3 Decision making

In this section, we discuss principles of assigning preferences to decision alternatives under uncertainity.

3.1 Risk functions

Deterministic values are comparable, since there is only one reasonable way of ordering the real line: If one can get the same good for less money, everybody will take the better offer. But how to compare random distributions?

Individuals have different perceptions of risk, they exhibit differences in their risk aversion. Economists have accounted for this by introducing different types of preference relations, most based on axiomatic priciples, like the widely used utility indices introduced by Arrow and Pratt.

We adopt here a pragmatic way. We call any function, which maps distribution functions to the real line a *risk function* and allow to make comparisons and preference relations on the basis of these functions. A large collection of risk functions has been proposed, the decision maker has to choose one of them or to invent a new one.

Let $\mathcal{R}(Y)$ denote a risk function associated the random cost variable Y. We suppose that $\mathcal{R}(Y)$ depends only on the distribution F_Y of Y.

Here are some examples for risk functions:

- **Linear risk functions**

  These are characterized by the fact that they linear in the distribution function: If Y is a cost distribution which satisfies

  $$Y = \begin{cases} Y_1 & \text{with probability } \alpha \\ Y_2 & \text{with probability } 1-\alpha \end{cases}$$

  then $\mathcal{R}(Y) = \alpha\mathcal{R}(Y_1)+(1-\alpha)\mathcal{R}(Y_2)$ for linear risk functions. Examples are the *Arrow-Pratt type* risk functions

  $$\mathcal{R}(Y) = \mathbb{E}(U(Y)) = \int U(v)\, dF_Y(v),$$

  where U is some utility function; the *value at risk*

  $$\mathcal{R}(Y) = \mathbb{E}(Y\, I\{Y \geq t\}) = \int_t^{\infty} v\, dF_Y(v)$$

  where $I\{\,\}$ is the indicator function, i.e.

  $$I\{Y \geq t\} = \begin{cases} 1 & \text{if } Y \geq t \\ 0 & \text{if } Y < t \end{cases}$$

and t be some fixed threshold amount (for instance three times the expectation of Y).
Related risk functions are the *excess ratio*

$$\mathcal{R}(Y) = \frac{\mathbb{E}(Y I\{Y \geq t\})}{\mathbb{E}(Y)}$$

and the *exceedance probability*

$$\mathcal{R}(Y) = \mathbb{P}\{Y \geq t\} = 1 - F_Y(t).$$

Notice that for the same threshold value t, the exceedance probability is always smaller than the excess ratio, since by Stephensons inequality

$$\frac{\mathbb{E}(Y I\{Y \geq t\})}{\mathbb{E}(I\{Y \geq t\})} \geq \mathbb{E}(Y).$$

- **Quadratic risk functions**

  These are quadratic in the distribution function F_Y, e.g. the *Markowitz value*

$$\mathcal{R}(Y) = \mathbb{E}(Y) + \delta \text{ Var } (Y) = \int v \, dF_Y(v) + \frac{\delta}{2} \int \int (v-u)^2 \, dF_Y(u) \, dF_Y(v).$$

  Here and in the following, δ denotes the factor of risk aversion.

- **Expectation/dispersion risk functions**

$$\mathcal{R}(Y) = \mathbb{E}(Y) + \lambda \mathbb{E}[g(Y - \mathbb{E}(Y))]$$

  where g is some convex function with $g(0) = 0$. Examples are:
  The *upper semivariance*

$$\text{Var}^+(Y) = \mathbb{E}[(Y - \mathbb{E}(Y))^+]^2$$

  and the *lower semivariance*

$$\text{Var}^-(Y) = \mathbb{E}[(Y - \mathbb{E}(Y))^-]^2.$$

- **Quantile function based risk functions:**
  These are linear in the quantile function $Q_Y(t) = F_Y^{-1}(t)$, e.g. *Yaari's index*

$$\mathcal{R}(Y) = \int_0^\infty g(F_Y(u)) \, du = \int_0^1 g(u) \, dQ_Y(u)$$

  for some monotone function g.

Preference relations can be built on the basis of risk functions: Let $\mathcal{R} = (\mathcal{R}_0, \dots, \mathcal{R}_k)$ be a set of r isk functions. For two random variables Y and Z we may define the preference relation

$$Y \overset{\mathcal{R}}{\prec} Z$$

iff

$$\mathcal{R}_1(Y) \le \mathcal{R}_1(Z)$$

$$\dots$$

$$\mathcal{R}_k(Y) \le \mathcal{R}_k(Z)$$

Risk functions and preference structures reflect the individual situation of the decision maker, his risk aversion and objectives of behavior.

3.2 Decision problems

Let $Z_1, \dots, Z_k$ finitely many alternatives. If we single out one appropriate risk function $\mathcal{R}_0$, the decision is to take the alternative with minimal $\mathcal{R}_0$.

Sometimes the decision maker has to decide about a parameter (or parameter vector) x. For instance, he may decide, which part of his loan he wants to swap to variable interest and which part to keep fixed interest.

Such a continuum of alternatives leads to a (nonlinear, constrained) optimization problem:

Let the set of alternatives be $(Z_x; x \in \mathbb{X})$. Let $\mathcal{R}_0, \mathcal{R}_1, \dots, \mathcal{R}_k$ be a set of risk functions. One of them, namely $\mathcal{R}_0$ serves as objective function, the others are constraint functions.

The decision problem under uncertainity reads

$$\left\| \begin{array}{l} \text{Minimize } \mathcal{R}_0(f(x, \xi)) \\ \text{subject to} \\ \mathcal{R}_1((f(x, \xi)) \le b_1 \\ \dots \\ \mathcal{R}_k((f(x, \xi)) \le b_k \\ x \in \mathbb{X}. \end{array} \right.$$

Example. Portfolio optimization.

Suppose that we can buy any mix of k different assets, each with random return Z_i. Suppose that we measure the risk with the Markowitz function, but want to keep the probability of an extreme loss bounded by some prespecified quantity. The problem reads then

$$
\left\|
\begin{array}{ll}
\text{Maximise } \mathbb{E}(x_1 Z_1 + \ldots + x_k Z_k) - \delta \text{ Var } (x_1 Z_1 + \ldots + x_k Z_k) & \\
\text{subject to} & \\
x_1 + \ldots + x_k = B & \text{the budget} \\
\mathbb{P}\{x_1 Z_1 + \ldots + x_k Z_k \leq t\} \leq \alpha & \text{the bound for extreme losses} \\
x_i \geq 0. &
\end{array}
\right.
$$

This is a nonlinear optimization problem, with quadratic objective. The first constraint is linear, the second is typically highly nonlinear. The solution of this problem is done by standard nonlinear optimization techniques.

3.3 Regret

Suppose that the random costs are of the form $Z_x = f(x, \xi)$, where ξ is some random variable, which describes the uncertain future. If we would be clearvoyant and know the future ξ in advance, we could choose the decision x in dependence of ξ and obtain the minimally possible costs $\min_{x \in \mathbb{X}} f(x, \xi)$. The difference to $f(x, \xi)$ is the regret function

$$\tilde{f}(x, \xi) = f(x, \xi) - \min_{x \in \mathbb{X}} f(x, \xi).$$

If we replace the original costs Z_x be the regret values $\tilde{Z}_x = \tilde{f}(x, \xi)$ and solve the decision problem for $\tilde{Z}_x$ we obtain a different solution (the regret solution) in general. Notice that the regret values are psychological values rather than costs and that the regret solution is a more emotional than rational decision.

Example. Suppose that a lottery offers a ticket for the price of 1 Euro, offering a 1% chance of winning 80 Euro. Let x_1 be the action "buy a ticket" and x_2 "do not buy a ticket". Let $\xi = 1$ mean that our ticket wins and $\xi = 2$ that it looses. We have $f(x_1, 1) = -79, f(x_1, 2) = 1, f(x_2, 1) = 0, f(x_2, 2) = 0$. The regret values are $\tilde{f}(x_1, 1) = 0, \tilde{f}(x_1, 2) = 1, \tilde{f}(x_2, 1) = 79, \tilde{f}(x_2, 2) = 0$. Notice that $\mathbb{E}(\tilde{Z}_{x_1}) = 0.99$ and $\mathbb{E}(\tilde{Z}_{x_2}) = 0.79$, but $\text{Var}(\tilde{Z}_{x_1}) = 0.0099$ whereas $\text{Var}(\tilde{Z}_{x_2}) = 61.79$ and for small risk aversion factors, the regret solution is to buy the ticket.

4 Pricing

In the foregoing sections we have discussed the reshaping problem from the viewpoint of the customer: Should he, or to what extent should he sign a contract for risk reshaping. The problem turned out to be a stochastic optimization problem, which is in the form of a nonlinear constrained optimization problem. In this section we will adopt the viewpoint of the offering side and present some basic principles of setting up prices for such reshaping contracts.

In principle, the situation between both sides is symmetric and the same considerations, which apply to the customer apply also to the offerer. However, since the latter deals with many of such contracts, his situation is different. Because of repeated reshaping offers, the offering side should come up with a *pricing strategy*, rather than individual decisions.

To begin with, assume that Y represents random costs, an insurance is going to buy. What is a reasonable price for this contract, from the insurers point of view? It seems clear that the price p for "buying" the costs Y should not be lower than the expected costs

$$p(Y) \geq \mathbb{E}(Y),$$

or, introducing the difference $s(Y) = p(Y) - \mathbb{E}(Y)$ as *safety loading*,

$$s(Y) \geq 0.$$

Rather primitive, but widely used pricing strategies for safety loadings are introduce

- the *constant safety loading*

$$s(Y) = C$$

or

- the *proportional safety loading*

$$s(Y) = \gamma\, \mathbb{E}(Y)$$

where γ is some factor. Both proposals suffer from the shortcoming that other characteristics than the expectation of Y do not enter the price. It is natural to include at least the dispersion of Y: The higher the dispersion, the higher should be the price.

- *standard deviation safety loading*

$$s(Y) = \gamma \ \mathrm{Std}(Y)$$

or

- *variance safety loading*

$$s(Y) = \gamma \ \mathrm{Var}(Y)$$

Another pricing principle is based on utility functions: Let $U(x)$ be a strictly monotone convex utility function and U^{-1} its inverse.

- *Utility safety loading*

$$s(Y) = U^{-1}(\mathbb{E}[U(Y)]) - \mathbb{E}(Y)$$

$s(Y)$ is always nonnegative, since by Jensen's inequality for all convex integrable U

$$\mathbb{E}[U(Y)] \geq U(\mathbb{E}[Y])$$

and since U is strictly monotone. Notice that in the case of a deterministic loss variable $Y \equiv const$, the safety loading is zero. A good example for such a pricing strategy is to take $U(x) = \exp(\alpha x)$, which results in

$$p(Y) = \frac{1}{\alpha} \log \mathbb{E}[\exp(\alpha Y)].$$

Other examples are $U(x) = x^2$ resulting in

$$p(Y) = \sqrt{\mathbb{E}(Y^2)}$$

or $U(x) = [(x - \mathbb{E}(Y))^+]^2$ resulting in

$$s(Y) = \sqrt{\mathbb{E}[(Y - \mathbb{E}(Y))^+]^2},$$

the upper semi standard deviation.

The just discussed pricing stategies do not at all take into account, how many contracts are to be issued. Suppose for simplicity that we know that N similar contracts are issued for the price $p(Y)$ each and that the random claims connected with these contracts are i.i.d. random variables $Y_i; i = 1, \ldots N$. (We will later touch the point that especially for insurance against catastrophic risks, the independence assumption is not justified and dangerous. Dependency has to be assumed.)

Given the distribution of Y_i, we may calculate the probability, that the business resulting out of all the N contracts will be a loss:

$$\mathbb{P}\{Np(Y) - \sum_{i=1}^{N} Y_i < 0\}. \tag{1}$$

Introduce the sum $S_N = \sum_{i=1}^{N} Y_i$ and the Laplace transform $L_Y(t) = \mathbb{E}[\exp(tY)]$. By the well known exponential inequality we have for all t

$$\mathbb{P}\{S_N > Np(Y)\} \leq \mathbb{E}[\exp(tS_N)]e^{-tNp(Y)} = [L_Y(t)e^{-tp(Y)}]^N.$$

Let

$$a = \inf_{t \geq 0} L_Y(t) e^{-tp(Y)}. \tag{2}$$

Then

$$\mathbb{P}\{S_N > Np(Y)\} \leq a^N.$$

We see that the loss probability decreases geometrically with the number of contracts N. To put it differently, if the loss probability is our objective, the optimal price $p(Y)$ to be asked for each contract decreases with N.

Example. Suppose that the distribution of Y is exponential with mean 1, i.e. $L_Y(t) = \frac{1}{t-1}\theta(1-t)$, where

$$\theta(u) = \begin{cases} 1 & \text{if } u < 0 \\ \infty & \text{if } u \geq 0 \end{cases}.$$

We find for a given by (2) $a = p(Y)e^{1-p(Y)}$. From a ruin condition, like $a^N = [p(Y)e^{1-p(Y)}]^N \leq \alpha$, the price $p(Y)$ can be determined.

A more detailed pricing strategy takes the timing aspect of claims into account. Suppose that again N contracts are issued, each for an independent replication of Y, which stands for the yearly claim distribution. The claims appear in random moments of time and in random height. Suppose that the claim moments follow a Poisson process $\Pi(t)$ with intensity λN and suppose that the instream of premium is $Np(Y)$ per year. Define the risk process as

$$X(t) = Np(Y)t - \sum_{i=1}^{\Pi(t)} Y_i,$$

where Y_i are i.i.d. replicates of Y, each with expectation 1. Denote by $\Psi(u)$ the ruin probability

$$\Psi(u) = \mathbb{P}\{u + X(t) < 0 \quad \text{for some } t > 0\}.$$

The ruin probability is determined by $N, P(Y), \lambda$ and the distribution of Y, however explicit formulas are known only in simple cases (such as exponentially distributed Y). A good approximation is given by the Cramer-Lundberg formula

$$\Psi(u) \sim \frac{e^{-Ru}\rho}{L'_Y(R) - p(Y)/\lambda}$$

where $\rho = \frac{p(Y)}{\lambda} - 1$ and R is the solution of $\frac{p(Y)}{\lambda} = \frac{L_Y(R)-1}{R}$. Various other approximations have been proposed, see Grandell (1992). Notice that in this setup, the number of contracts N does not enter the ruin probability simply because the whole model is only valid for large N.

4.1 Pricing by trading at stock exchange

If a good is traded at the stock exchange, the price is determined by the aggregate offers and demands of many economic agents. Individual constraints and preferences are no more visible. Only divisible goods can be traded. If the goods are of individualistic quality (like shares of individual companies) they are considered as separate. If the are just determined by few characteristics (like oil of specific quality class) all offers are pooled to get a unique market price for this good.

The only characteristic of zero-coupon bonds is the time to maturity. All zero-coupon bonds with same maturity can be pooled together. Such a bond promises for instance to pay the fixed sum x in one month from now. Its price p, found at stock exchange, allows to calculate the one-month interest rate r by the equation $p = x(1+r)^{-1}$. It is clear that all contracts with the same one-month maturity must lead to the same rate r, otherwise arbitrage (free lunch) would be possible.

4.2 Pricing of derivatives by no-arbitrage law

Suppose now that the return X, which a contract offers after one month time is not determined now, but depends on the unknown future. Two such contracts X_1 and X_2 are only similar if the distributions of X_1 and X_2 coincide. If such contracts were be traded at stock exchange, they would necessarily have the same price due to the no-free-lunch argument.

For instance, suppose that the rights emerging from life insurance contracts would be traded, then the life insurance contract of a 50 year old male person for the sum of 1000 Euro would get a certain price, which is independent of the name of this person, although the contracts of two different persons have different random pays, but they coincide in distribution.

At present, there is no mechanism at stock exchange for equalizing prices for nonidentical, but stochastically identical risks. If – in some future time – stock exchanges would start to trade such contracts, market risk aversions would appear, which replace the todays more individualistic views.

In some very specific situations, pricing of random contracts must be even today based on the principle of no-arbitrage. Suppose that p_X is the known price of a contract, which promises to the holder to get the random sum X in one month and suppose that X has a two point distribution

$$\mathbb{P}(X = x_1) = \pi_1; \qquad \mathbb{P}(X = x_2) = \pi_2.$$

Suppose further that Y is the value of a derivative contract which promises the sum of y_1 if $X = x_1$ and the sum of y_2 if $X = x_2$. We claim that today's price p_Y of the derivative contract is uniquely determined by p_X, if we assume that every contract is divisible. The argument goes as follows: Suppose that we design a portfolio consising of $y_2 - y_1$ parts of the first contract and $x_1 - x_2$ parts of the derivative contract. The price of this portfolio today is $(y_2 - y_1)p_X + (x_1 - x_2)p_Y$. If $X = x_1$, the value of the portfolio is $(y_2 - y_1)x_1 + (x_1 - x_2)y_2 = y_2x_1 - x_2y_1$ and if $X = x_2$ then the value is the same, namely $(y_2 - y_1)x_2 + (x_1 - x_2)y_1 = y_2x_1 - x_2y_1$. Thus this portfolio has no risk and therefore its price has to be the value at maturity divided by $(1 + r)$, where r is the one-month interest rate for a deterministic contract, i.e.

$$(y_2 - y_1)p_X + (x_1 - x_2)p_Y = (1 + r)(y_2x_1 - x_2y_1).$$

This last equation determines the price p_Y in a unique manner:

$$p_Y = (1 + r)\frac{y_2x_1 - x_2y_1}{x_1 - x_2} - p_X\frac{y_2 - y_1}{x_1 - x_2}.$$

We notice that the determination of p_Y is independent of the probabilities π_1 and π_2. We notice further that no unique determination of p_Y is possible, if X may take three or more different values.

5 Dependency

Independence of stochastic effects is often assumed for simplicity. However, realistic models must incorporate dependency structures. One important example is the claim structure for insurance companies. Both the time instants and the heights of the claims may be dependent, since they may be consequences of the same cause. Dependencies may drastically change the values of the risk functions and the ignorance may lead to wrong decisions.

Example. We come back to the insurance example of section 3. We suppose that N contracts are issued for the price $p(Y)$ each and that the costs connected with each contract are Y_i. We consider the value at risk for the threshold $t = Np(Y)$, i.e. $\mathbb{E}(\sum_{i=1}^N Y_i I\{\sum_{i=1}^N Y_i > Np(Y)\})$. Suppose that Y_i are exponentially distributed with mean 1. We consider two cases:
(1) the Y_i are independent; (2) the Y_i are identical.

(1) We calculate the value at risk as

$$\int_{Np(Y)}^{\infty} x\frac{N^{N-1}e^{-x}}{(N-1)!}\,dx = \int_{p(Y)}^{\infty} \frac{N^{N+1}u^N e^{-Nu}}{(N-1)!}\,du$$

which is an incomplete Γ-function which goes to zero as N tend to infinity. We conclude: Every new contract decreases the risk for the company. (Recall that this was already stated at the end of section 3).

(2) In the complete dependent situation the value at risk is

$$\mathbb{E}(NYI\{NY > Np(Y)\}) = N \int_{p(Y)}^{\infty} xe^{-x}\, dx$$

which increases with increasing N. We conclude that for highly dependent risks (for instance risks emerging from natural catastrophes), every new contract increases the risk of the insurer.

6 Conclusion

Modern instruments of financial engineering allow to reshape uncertain financial positions. Both parties, the one which is offering a reshaping contract and the other which is accepting the offer have to consider a highly complex stochastic optimization problem. This problem has several aspects, a probability aspect for modelling of stochastic processes, a statistical aspect for the estimation of parameters and distributions from data, a modelling aspect, since the appropriate risk functions must be chosen and an optimization aspect, namely the solution of the underlying nonlinear optimization problem for determinig the optimal action.

For all these parts there exist well developed methods, but a more integrative view is necessary. Integrative research questions are:

(i) How does the estimation error influence the quality of the decision?

(ii) How does the misspeficication error influence the quality of the decision?

(iii) What is a good compromise between realistic model and computable decision problem?

(iv) What numerical optimization method is appropriate for what problem?

We hope that further research will give some insights into these questions.

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