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ON CONVERGENCE OF THE SEQUENTIAL JOINT MAXIMIZATION METHOD FOR APPLIED EQUILIBRIUM PROBLEMS

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Abstract

The convergence of the sequential joint maximization method (Rutherford [10]) for searching economic equilibria is studied in the case of Cobb-Douglas utility functions. It is shown that convergence is closely related to the behavior of certain inhomogeneous Markov chains. In particular, convergence takes place if each good is either produced or available in the economy.

Key words: Applied equilibrium problem, joint maximization method, Cobb-Douglas utility.

Contents

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1 Introduction

The sequential joint maximization method was proposed by Rutherford [10] as a heuristic procedure for applied equilibrium problems. It turned out to be effective in applications to rather complex intertemporal equilibrium models for integrated assessment of international environmental policies (see Manne [6], Manne and Rutherford [7]). In the present paper we analyze some convergence properties of the method. We consider the case of Cobb-Douglas utility functions which allow to illustrate the main features of the procedure in the most simple manner. For example, it is shown that convergence of the joint maximization method is related to new problems for inhomogeneous Markov processes. We also illustrate the convergence of the method without requiring the gross substitutability assumptions.

2 General equilibrium problem

Let us introduce some necessary notations. Consider an economy consisting of m consumers and l producers. Each consumer k is characterized by a utility function $U(x_k)$, consumption vector $x_k \in Q_k \subset R^n$, initial endowment $w_k \in R^n_+$ and shares α_{ki} in profits of producer i, $\sum_{k=1}^m \alpha_{ki} = 1$. Producer i is characterized by the set of feasible activity vectors $y_i \in Y_i \subset R^n$ and a production vector-function $g_i(y_i) = (g_{i1}(y_i), \ldots, g_{in}(y_i))$. Let $p \in R^n_+$ denote a price vector of goods in the economy, $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_l)$, $Q = Q_1 \times \ldots \times Q_m$, $Y = Y_1 \times \ldots \times Y_l$.

Demand for goods in the economy is generated according to the principle of utility maximization: it is assumed that each consumer k chooses a vector y_k^+ of goods that maximizes his/her utility subject to a budget constraint (2) and others, for example, environmental constraints (3):

$$U_k(x_k) \longrightarrow \max_{x_k}, \tag{1}$$

$$px_k \leq I_k(y,p), \tag{2}$$

$$x_k \in Q_k \in R^n, \tag{3}$$

where income function $I_k(y,p)$ has the form:

$$I_k(y,p) = pw_k + \sum_{i=1}^{l} \alpha_{ki} p g_i(y_i), \quad \sum_{k=1}^{m} \alpha_{ki} = 1, \tag{4}$$

where $pg_i(y_i)$ denotes an inner product of vectors p and $g_i(y_i)$. This approach allows to generate an arbitrary number of demand functions $x_k(I_k, p)$ by choosing appropriate utility functions $U_k(x_k)$.

Producer i chooses the production levels y_i from the profit maximization:

$$pg_i(y_i) \longrightarrow \max_{y_i}, \tag{5}$$

$$y_i \in Y_i \subset R^n. \tag{6}$$

We also consider a "market player" (see Zangwill and Garcia [12]):

$$p(\sum_{k=1}^{m} x_k - \sum_{k=1}^{m} w_k - \sum_{i=1}^{l} g_i(y_i)) \longrightarrow \max_{p}, \tag{7}$$

$$p \geq 0, \quad \sum_{j=1}^{n} p_j = 1. \tag{8}$$

Vectors x^*, y^* and p^* constitute a general equilibrium if vectors x_k^* are solutions of (1)-(3) for fixed $p = p^*$, $y = y^*$, $k = 1, \ldots, m$; y_i^* is a solution of (5)-(6) for fixed $p = p^*$, $i = 1, \ldots, l$, and $p = p^*$ is a solution of (7)-(8) for fixed $x = x^*$, $y = y^*$, i.e. the following material and financial balances are fulfilled:

$$\sum_{k=1}^{m} x_k^* \leq W + G(y^*), \tag{9}$$

$$p^* \sum_{k=1}^{m} x_k^* = p^*(W + G(y^*)), \tag{10}$$

where $W = \sum_{k=1}^{m} w_k$, $G(y) = \sum_{i=1}^{l} g_i(y_i)$ (component-wise summation). Thus a general equilibrium x^*, y^*, p^* is in fact a Nash equilibrium of the appropriate game with $(m+l+1)$ players.

We use some common assumptions:

(i) utility functions $U_k(x_k)$ are concave and continuous on Q_k;

(ii) sets Q_k are closed and convex, $0 \in Q_k \subseteq R^n_+$;

(iii) production functions $g_{ij}(y_i)$ are concave, $i = 1, \ldots, m$, $j = 1, \ldots, n$;

(iv) sets Y_i, $i = 1, \ldots, l$, are convex compacts, $Y_i \subset R^n_+$;

(v) for any product $j = 1, \ldots, n$ there exist activity vectors $y_i \in Y_i$ such that $W_j + G_j(y) > 0$.

Let us note that the case of nonlinear functions $g_i(y_i)$ (instead of traditional $g_i(y_i) = y_i$) is important when decomposition schemes are used (see, for example, [3]).

If utilities $U_k(\cdot)$, $k = 1, \ldots, m$, are positively homogeneous and income functions $I_k(y, p) := t_k$, $k = 1, \ldots, m$, are constant, then the general equilibrium problem is reduced to an optimization problem (see Eisenberg and Gale [2], Gale [4], Eisenberg [1], Polterovich [8], [9]).

Definition 2.1 *Function $U(x)$, $x \in Q$, is called positively homogeneous with degree β on a cone $Q \in R^n$ if for any $x \in Q$ and $r > 0$*

$$U(rx) = r^{\beta} U(x).$$

The following positively homogeneous utility functions are often used:

$U(x) = x_1^{\beta_1} \times \ldots \times x_n^{\beta_n}$, $\sum_{j=1}^n \beta_j = 1$, $0 \le \beta_j \le 1$ (Cobb-Douglas function);

$U(x) = \min_{1 \le i \le n} \{x_1/a_1, \ldots, x_n/a_n\}$, $a_j \ge 0$ (Leontief function);

$U(x) = \sum_{i=1}^n c_i x_i$, $c_i \ge 0$ (linear function).

Theorem 2.1 *Assume in addition to* **(i)**-**(v)** *that*

(vi) *function U_k is positively homogeneous with degree β_k and nonnegative on Q_k, set Q_k is a cone with the vertex at the origin and contains a vector $x'_k \in Q_k$ such that $U_k(x'_k) > 0$, $k = 1, \ldots, m$;*

(vii) *the income function $I_k(y, p) = t_k$ is constant, $k = 1, \ldots, m$.*

Then vectors x^*, y^* and p^* constitute an equilibrium iff vectors x^*_k, $k = 1, \ldots, m$, y^*_i, $i = 1, \ldots, l$, are solutions of the following optimization problem:

$$\sum_{k=1}^{m} \frac{t_k}{\beta_k} \ln U_k(x_k) \longrightarrow \max_{x,y}, \tag{11}$$

$$\sum_{k=1}^{m} x_k \le W + G(y), \tag{12}$$

$$x \in Q, \quad y \in Y, \tag{13}$$

and p^* is a Lagrange multiplier vector corresponding to inequalities (12).

This statement is a generalization of the results by Polterovich [8], [9] to the case of nonlinear production functions g_i. The following proof basically repeats the proof by Polterovich [8].

Lemma 2.1 *Assume that function $f(x)$ is concave and positively homogeneous with degree $\beta > 0$, set Q is a cone with $0 \in Q$, and $t > 0$. Then at the optimal solution of the optimization problem*

$$\frac{t}{\beta} \ln f(x) \longrightarrow \max_x, \tag{14}$$

$$qx \leq t, \tag{15}$$

$$x \in Q, \tag{16}$$

constraint (15) is fulfilled as equality (in the optimum) and the Lagrange multiplier corresponding to (budget) constraint (15) equals one.

Proof. Let x^* be the optimal solution of (14)-(16). Since $0 \in Q$ and $t > 0$, then in (15) Slater's condition is fulfilled. By Kuhn-Tucker theorem for any $x \in Q$

$$\frac{t}{\beta} \ln f(x^*) \geq \frac{t}{\beta} \ln f(x) + \lambda(t - qx), \tag{17}$$

where $\lambda \geq 0$. Note that

$$qx^* = t. \tag{18}$$

Otherwise, there exists a vector rx^*, $r > 1$, satisfying constraints (15) and (16), thereby contradicting the optimality of x^*:

$$f(x^*) < r^\beta f(x^*) = f(rx^*).$$

Putting in (17) $x = rx^*$, $r > 0$, and using (18) and homogeneity of f we obtain

$$\lambda(r - 1) \geq \ln r. \tag{19}$$

If $r < 1$ then $\lambda \leq \ln(r/(r-1))$, and passing to the limit $t \to 1 - 0$ we obtain $\lambda \leq 1$. Passing in (19) to the limit $t \to 1 + 0$ we obtain the opposite inequality $\lambda \geq 1$. □

Proof of Theorem 2.1. Consider an equilibrium x_k^*, $k = 1, \dots, m$, y_i^*, $i = 1, \dots, l$, and p^*. Obviously, vector x_k^* is a solution of the problem

$$\frac{t_k}{\beta_k} \ln U_k(x_k) \longrightarrow \max_{x_k}, \tag{20}$$

$$p^* x_k \leq t_k, \tag{21}$$

$$x_k \in Q_k, \tag{22}$$

By Lemma 2.1

$$p^* x_k^* = t_k, \quad \lambda_k = 1.$$

Using Kuhn-Tucker theorem for any $x_k \in Q_k$ we have

$$\frac{t_k}{\beta_k} \ln U_k(x_k^*) \geq \frac{t_k}{\beta_k} \ln U_k(x_k) + (t_k - p^* x_k). \tag{23}$$

Summing (23) over k and taking into account that $p^* x_k^* = t_k$ we obtain

$$\sum_{k=1}^{m} \frac{t_k}{\beta_k} \ln U_k(x_k^*) \geq \sum_{k=1}^{m} \frac{t_k}{\beta_k} \ln U_k(x_k) + p^*(\sum_{k=1}^{m} x_k^* - \sum_{k=1}^{m} x_k). \tag{24}$$

For producers at equilibrium we have

$$p^* g_i(y_i^*) \geq p^* g_i(y_i), \quad y_i \in Y_i, \quad i = 1, \dots, l, \tag{25}$$

and hence

$$p^* G(y^*) \geq p^* G(y), \quad y = (y_1, \dots, y_l) \in Y. \tag{26}$$

By definition of the equilibrium

$$p^* \sum_{k=1}^{m} x_k^* = p^*(W + G(y^*)), \quad p^* \geq 0. \tag{27}$$

From (24), (26), (27) for any $x_k \in Q_k$ and y we obtain

$$\sum_{k=1}^{m} \frac{t_k}{\beta_k} \ln U_k(x_k^*) \geq \sum_{k=1}^{m} \frac{t_k}{\beta_k} \ln U_k(x_k) + p^*(W + G(y) - \sum_{k=1}^{m} x_k). \tag{28}$$

Vectors x_k^*, $k = 1, \dots, m$, and y_i^*, $i = 1, \dots, l$, satisfy conditions (13). From (27), (28) follows that these vectors form a solution to problem (11)-(13) and p^* is an optimal Lagrange multiplier vector to constraint (12).

The proof of the inverse statement proceeds as follows. Let x_k^*, $k = 1, \dots, m$, and y_i^*, $i = 1, \dots, l$, be a solution of (11)-(13) and p^* be an optimal Lagrange multiplier corresponding to constraint (12). This means that relations (27), (28) hold true. From

(27), (28) for $x_k = x_k^*$ we obtain (26) and hence (25). Then, for $y_i = y_i^*$ we obtain (24). Therefore

$$\frac{t_k}{\beta_k} \ln U_k(x_k^*) \geq \frac{t_k}{\beta_k} \ln U_k(x_k) + (p^* x_k^* - p^* x_k). \tag{29}$$

Taking $x_k = r x_k^*$ after simple transformations for all $r > 0$

$$(r-1) p^* x_k^* \geq t_k \ln r. \tag{30}$$

Hence

$$p^* x_k^* = t_k. \tag{31}$$

Substituting $p^* x_k^*$ in (29) by t_k we obtain (23), which jointly with (31) shows that x_k^* is a solution of problem (1)-(3) of k-th consumer. Thus consumption and production vectors x_k^*, $k = 1, \ldots, m$, y_i^*, $i = 1, \ldots, l$, as well as p^* satisfy equilibrium conditions (12) and (27). □

Parameter $\gamma_k = \frac{t_k}{\beta_k}$ in (11) is called Negishi's weight of utility U_k in the aggregated utility

$$U(x_1, \ldots, x_m) = \sum_{k=1}^{m} \gamma_k \ln U_k(x_k).$$

Consider a parametric optimization problem (11)-(13), denote its solution sets $X(t)$, $Y(t)$ and optimal Lagrange multiplier set $P(t)$ (corresponding to (12)). Now construct the following set valued mapping:

$$\begin{aligned} I(t) \quad = \quad & \{z \in R^m | z_k = p(w_k + \sum_{i=1}^{l} \alpha_{ki} g_i(y_i)), \quad k = 1, \ldots, m; \\ & \quad p \in P(t), \quad (y_1, \ldots, y_l) \in Y(t)\}. \end{aligned} \tag{32}$$

The next lemma connects equilibriums of model (1)-(6) with fixed points of $I(t)$.

Theorem 2.2 *Suppose assumptions* **(i)-(vi)** *are fulfilled.*

If x^*, y^* p^* constitute an equilibrium of (1)-(6) then

$$t^* = \{t_k^* = p^*(w_k + \sum_{i=1}^{l} \alpha_{ki} g_i(y_k^*)), \quad k = 1, \ldots, m\} \tag{33}$$

is a fixed point of $I(t)$.

If t^* is a fixed point of $I(t)$, i.e. $t^* \in I(t^*)$, then there exist $x^* \in X(t^*)$, $y^* \in Y(t^*)$ and $p^* \in P(t^*)$ constituting an equilibrium of the original model (1)-(6).

Proof. Let x^*, y^*, p^* be an equilibrium of (1)-(6). Construct t^* by (33). Now consider optimization problem (11)-(13) with $t = t^*$. By Theorem 2.1 x^*, y^*, p^* belong to solutions of (11)-(13), i.e. $x^* \in X(t^*)$, $y^* \in Y(t^*)$ and $p^* \in P(t^*)$. Hence

$$t^* \in I(t^*) = \{z| \quad z_k = p(w_k + \sum_{i=1}^{l} \alpha_{ki} g_i(y_k)), \quad k = 1, \ldots, m, \\ p \in P(t^*), \; y \in Y(t^*)\}.$$

Now prove the reverse statement. From $t^* \in I(t^*)$ and the definition of $I(t)$ it follows that there exist $p^* \in P(t^*)$ and $(y_1^*, \ldots, y_l^*) \in Y(t^*)$ such that

$$t_k^* = p^*(w_k + \sum_{i=1}^{l} \alpha_{ki} g_i(y_i^*)), \quad k = 1, \ldots, m. \tag{34}$$

By Theorem 2.1 x^*, y^*, p^* constitute an equilibrium of the original model (1)-(6), where t_k^* stands for $I_k(y,p)$, $k = 1, \ldots, m$. But due to (34) budget constraint (2) can be rewritten in the form

$$px_k \le t_k^* = p^*(w_k + \sum_{i=1}^{l} \alpha_{ki} g_i(y_i^*)).$$

It means that x_k^* provides a solution of consumer k's problem (1)-(3) under fixed $p = p^*$ and $y = y^*$. This completes the proof. □

3 Cobb-Douglas utilities

Notice that the aggregated utility function (11) in Theorem 2.1 is in fact a logarithm of the following Cobb-Douglas type function

$$U(x) = \prod_{k=1}^{m} U^{t_k/\beta_k}(x_k).$$

So it is natural to analyze possibilities of computational procedures first of all in the following case.

(viii) *Assume that consumer's utility functions have Cobb-Douglas form:*

$$U_k(x_k) = x_{k1}^{\beta_{k1}} \times x_{k2}^{\beta_{k2}} \times \ldots \times x_{kn}^{\beta_{kn}}, \tag{35}$$

$$x_k = (x_{k1}, \ldots, x_{kn}) \ge 0,$$

where

$$0 \le \beta_{ki} \le 1, \quad \sum_{i=1}^{n} \beta_{ki} = 1, \quad k = 1, \ldots, m.$$

This utility functions are positively homogeneous of degree 1.

Consider optimization problem (11)-(13) in the case of Cobb-Douglas utilities:

$$U^*(t) = \max_{x,y} \sum_{k=1}^{m} t_k \ln(x_{k1}^{\beta_{k1}} \cdot \ldots \cdot x_{km}^{\beta_{km}}) \tag{36}$$

$$\sum_{k=1}^{m} x_k \le W + G(y), \tag{37}$$

$$x \ge 0, \quad y \in Y. \tag{38}$$

Lemma 3.1 *In (36)-(38) an optimal production vector y^* is a solution of the problem:*

$$\max_{y\in Y} \sum_{j=1}^{n} \left(\sum_{k=1}^{m} t_k \beta_{kj} \right) \ln(W_j + G_j(y)). \tag{39}$$

An optimal Lagrange multiplier vector p^* has the form:

$$p_j^* = \frac{1}{W_j + G_j(y^*)} \sum_{k=1}^{m} t_k \beta_{kj}, \quad j = 1, \ldots, n. \tag{40}$$

Optimal consumption x_k^*, $k = 1, \ldots, m$, is calculated as follows:

$$x_{kj}^* = \frac{t_k \beta_{kj}}{p_j^*}, \quad j = 1, \ldots, n. \tag{41}$$

Proof. Denote $p = (p_1, \ldots, p_n) \ge 0$ vector of Lagrange multipliers corresponding to inequality (37). The required follows from the following assertions:

$$U^*(t) = \max_{y\in Y, x\ge 0} \min_{p\ge 0} \left(\sum_{k=1}^{m} t_k \ln U_k(x_k) - p(\sum_{k=1}^{m} x_k - W - G(y)) \right) =$$

$$\max_{y\in Y} \min_{p\ge 0} \left(\sum_{k=1}^{m} \sum_{j=1}^{n} \max_{x_{kj}\ge 0} (t_k \beta_{kj} \ln(x_{kj}) - p_j x_{kj}) + p(W + G(y)) \right) =$$

$$\max_{y\in Y} \min_{p\ge 0} \left(\sum_{k=1}^{m} \sum_{j=1}^{n} (\beta_{kj} t_k \ln \frac{t_k \beta_{kj}}{p_j} - t_k \beta_{kj}) + p(W + G(y)) \right) =$$

$$\max_{y\in Y} \left(\sum_{j=1}^{n} \min_{p_j\ge 0} \left((\sum_{k=1} (t_k \beta_{kj}) \ln \frac{1}{p_j} + p_j (W_j + G_j(y)) \right) \right) +$$

$$\sum_{k=1}^{m} \sum_{j=1}^{n} t_k \beta_{kj} \ln \beta_{kj} + \sum_{k=1}^{m} (t_k \ln t_k - t_k) =$$

$$\max_{y\in Y} \sum_{j=1}^{n} \left(\sum_{k=1}^{m} t_k \beta_{kj} \right) \ln(W_j + G_j(y)) -$$

$$\sum_{j=1}^{n} \left(\sum_{k=1}^{m} t_k \beta_{kj} \right) \ln \left(\sum_{k=1}^{m} t_k \beta_{kj} \right) + \sum_{k=1}^{m} \sum_{j=1}^{n} t_k \beta_{kj} \ln \beta_{kj} + \sum_{k=1}^{m} t_k \ln t_k.$$

□

Consider the set valued mapping $I(t)$ in the case of Cobb-Douglas utilities.

Lemma 3.2 *In the case of Cobb-Douglas utility functions (35) the set valued mapping* $I(t)$ *has the form:*

$$I(t) = \{A(y)t|\ y \in Y(t)\} \tag{42}$$

where $t = (t_1, \ldots, t_m)^T$*,* $Y(t)$ *is a solution set of (39) and matrix* $A(t) = \{a_{pq}\}_{p,q=1}^m$ *has elements*

$$a_{pq}(t) = \sum_{j=1}^n \frac{w_{pj} + \sum_{i=1}^l \alpha_{pi} g_{ij}(y_i)}{W_j + \sum_{i=1}^l g_{ij}(y_i)} \beta_{qj}. \tag{43}$$

Proof. By definition

$$\begin{aligned} I(t) = \{z \in R^m| \quad & z_k = p(w_k + \textstyle\sum_{i=1}^l \alpha_{ki} g_i(y_i)), \quad k = 1, \ldots, m, \\ & p \in P(t), \quad y \in Y(t)\}, \end{aligned}$$

where $Y(t)$ and $P(t)$ are solutions of (36)-(38). But by Lemma 3.1 $Y(t)$ is a solution set for (39) and

$$P(t) = \{p \in R^n|\ p_j = \frac{1}{W_j + G_j(y)} \sum_{q=1}^m t_q \beta_{qj}, \quad j = 1, \ldots, n, \quad y \in Y(t)\}.$$

Then for $z = (z_1, \ldots, z_p, \ldots, z_n) \in I(t)$ we have

$$\begin{aligned} z_p &= \textstyle\sum_{j=1}^n p_j (w_{pj} + \sum_{i=1}^l \alpha_{pi} g_{ij}(y_i)) \\ &= \textstyle\sum_{j=1}^n \left(\sum_{q=1}^m \frac{\beta_{qj}}{W_j + G_j(y)} t_q \right) \left(w_{pj} + \sum_{i=1}^l \alpha_{pi} g_{ij}(y_i) \right) \\ &= \textstyle\sum_{q=1}^m \left(\sum_{j=1}^n \frac{w_{pj} + \sum_{i=1}^l \alpha_{pi} g_{ij}(y_i)}{W_j + G_j(y)} \beta_{qj} \right) t_q \\ &= \textstyle\sum_{q=1}^n a_{pq} t_q. \end{aligned}$$

□

Remark. Notice that matrix $A(y)$ in (42) has a remarkable feature: the sum of elements in each column of $A(y)$ equals to 1. Indeed,

$$\sum_{p=1}^m a_{pq} = \sum_{p=1}^m \sum_{j=1}^n \frac{w_{pj} + \sum_{i=1}^l \alpha_{pi} g_{ij}(y_i)}{W_j + G_j(y)} \beta_{qj} =$$

$$\sum_{j=1}^n \beta_{qj} \frac{\sum_{p=1}^m w_{pj} + \sum_{i=1}^l g_{ij}(y_i) \sum_{p=1}^m \alpha_{pi}}{W_j + G_j(y)} = \sum_{j=1}^n \beta_{qj} = 1.$$

4 The lack of gross substitutability

Let us now calculate the excess demand function in the case of Cobb-Douglas utilities and for a fixed (possibly zero) feasible production plan $y \in Y$.

Let p be a given price vector. Each consumer k solves the problem:

$$x_{k1}^{\beta_{k1}} \times \ldots \times x_{kn}^{\beta_{kn}} \longrightarrow \max_{x_k},$$

$$px_k \le p(w_k + \sum_{i=1}^{l} \alpha_{ki} g_i(y_i)) = p\overline{w}_k, \quad x_k \ge 0,$$

where $\overline{w}_k = w_k + \sum_{i=1}^{l} \alpha_{ki} g_i(y_i)$.

By Lemma 2.1 this problem is equivalent to:

$$(p\overline{w}_k) \sum_{j=1}^{n} \beta_{kj} \ln x_{kj} - \sum_{j=1}^{n} p_j x_{kj} + \sum_{j=1}^{n} p_j \overline{w}_{kj} \longrightarrow \max_{x_{k1},\ldots,x_{kn} \ge 0}.$$

Its solution is

$$x_{kj} = \frac{1}{p_j} (p\overline{w}_k) \beta_{kj}, \quad j = 1, \ldots, n.$$

Thus excess demand function $f(p) = \{f_j(p)\}$ has the following components:

$$f_j(p) = \frac{1}{p_j} \sum_{k=1}^{m} (p\overline{w}_k) \beta_{kj} - W_j - G_j(y).$$

Let us check the gross substitutability condition. We have

$$\frac{\partial f_j(p)}{\partial p_i} = \frac{1}{p_j} \sum_{k=1}^{m} \overline{w}_{ki} \beta_{kj} \ge 0.$$

If, for instance, $\overline{w}_k > 0$ and $\beta_k = (\beta_{k1}, \ldots, \beta_{kn}) > 0$ for all k, then

$$\frac{\partial f_j(p)}{\partial p_i} > 0 \ \text{ for all } \ i, j, \ i \ne j,$$

and, hence, the gross substitutability condition is satisfied. In this case an equilibrium in the (exchange) economy can be found by a Walrasian tâtonment process. But if for some pair (i, j) it happens that $\sum_{k=1}^{m} \overline{w}_{ki} \beta_{kj} = 0$ then $\partial f_j(p)/\partial p_i = 0$ and the convergence of this tâtonment process is not guaranteed. An advantage of the sequential joint optimization method, as will follow from the next section, is its convergence in the absence of gross substitutability.

Let us consider a simple numerical example.

Example. *Consider an exchange economy with only two consumers and two types of goods.*

The first consumer has utility function $U_1(x_1) = x_{12}$ *and endowment vector* $w_1 = (1, 1)$, *i.e. he solves the problem*

$$x_{12} \longrightarrow \max_{x_{11}, x_{12}},$$

$$p_1 x_{11} + p_2 x_{12} \le p_1 + p_2, \quad x_{11}, x_{12} \ge 0.$$

The second consumer has utility function $U_2(x_2) = \sqrt{x_{21} x_{22}}$ *and endowment vector* $w_2 = (1, 0)$, *i.e. he solves the problem*

$$\sqrt{x_{21} x_{22}} \longrightarrow \max_{x_{21}, x_{22}},$$

$$p_1 x_{21} + p_2 x_{22} \le p_1, \quad x_{21}, x_{22} \ge 0.$$

The economy has the following equilibrium solutions:

$$p^* = (0, 1), \quad x_1^* = (x_{11}, 1), \quad x_2^* = (x_{21}, 0),$$

where x_{11}, x_{21} are arbitrary, but $0 \le x_{11} + x_{21} \le 2$.

Excess demand functions here have the form:

$$f_1(p) = -\frac{1}{2},$$

$$f_2(p) = \frac{1}{p_2}(\frac{3}{2} p_1 + p_2) - 1.$$

Thus $\partial f_1(p)/\partial p_2 = 0$ and the gross substitutability condition is not satisfied. The classical Walrasian tâtonment process $dp/d\tau = f(p)$ does not converge here in the sense that its first component goes to $-\infty$. Let us show that the sequential joint maximization method can overcome this difficulty.

5 Sequential joint maximization method

Rutherford's [10], [11] sequential joint maximization method can be viewed as an attempt to solve the inclusion $t \in I(t)$ by the following sequence of vectors $t^s = (t_1^s, \ldots, t_m^s)$, $s = 0, 1, \ldots$:

t^0 is an arbitrary nonnegative vector, $\sum_{k=1}^m t_k^0 = 1$;

$$\bar{t}^{s+1} \in I(t^s), \tag{44}$$

$$t^{s+1} = (1 - \sigma_s) t^s + \sigma_s \bar{t}^{s+1}, \tag{45}$$

where $I(t)$ is defined by (32), parameters $\sigma_s > 0$ play a role of step multipliers. If $\sigma_s = 1$ then the (full step) process has the form

$$t^{s+1} \in I(t^s). \tag{46}$$

An empirical result is that sequence t^s (with some $0 < \sigma \leq \sigma_s \leq 1$) converges to fixed points of $I(t)$ (equilibrium incomes) (see Rutherford [10], [11], Manne [6], Manne and Rutherford [7]). The corresponding equilibria of model (1)-(6) can be found as solutions $X(t)$, $Y(t)$ and $P(t)$ of optimization problem (11)-(13).

Let us analyze some convergence properties of this method in the case of Cobb-Douglas utilities. In this case method has the form:

$$t^0 \geq 0, \quad \sum_{k=1}^{m} t_k^0 = 1; \tag{47}$$

$$t^{s+1} = ((1-\sigma_s)E + \sigma_s A(y^s))t^s, \quad y^s \in Y(t^s), \quad s = 0, 1, \ldots. \tag{48}$$

Note that when starting in a simplex, i.e. $\sum_{k=1}^{m} t_k^0 = 1$, the method always remains within a simplex, i.e. $\sum_{k=1}^{m} t_k^s = 1$, due to the fact that the column sums of $A(y)$ equal 1.

Let us note that if the set $Y(t)$ is a singleton then (48) is reduced to the process

$$t^{s+1} = \overline{A}(t^s)t^s,; \quad s = 0, 1, \ldots, \quad \overline{A}(t^s) = (1-\sigma_s)E + \sigma_s A(Y(t^s)), \tag{49}$$

which generates a sequence of inhomogeneous nonnegative matrices

$$\overline{A}(t^0), \overline{A}(t^1), \ldots, \overline{A}(t^s), \ldots$$

They are stochastic matrices, therefore the convergence of (48) is connected with the convergence of the backward products

$$\overline{A}(t^s)\overline{A}(t^{s-1}) \times \ldots \times \overline{A}(t^0).$$

The main complexity here is concerned with endogenously generated inhomogeneity of such products by the sequence $t^0, t^1, \ldots$. It leads to new challenging problems of Markov processes. In this article we mention only some straightforward results.

Proposition 5.1 *If functions $G_j(y)$ are strictly concave and monotonously increasing, $\sigma_s \geq \sigma > 0$, then subsequences $\{t^{s_l}\}$ such that*

$$\lim_{l \to \infty} \|t^{s_l+1} - t^{s_l}\| = 0,$$

converge to an equilibrium.

Proof. Notice that solution $Y(t)$ of problem (39) with strictly concave and increasing functions $G_j(y)$ is unique and continuously depends on t, the same holds for $I(t)$. Suppose $t^{s_l} \longrightarrow t^*$ and $\|t^{s_l+1} - t^{s_l}\| \longrightarrow 0$, $s \longrightarrow \infty$. Then $t^{s_l+1} \longrightarrow t^*$ and from

$$t^{s_l+1} = (1-\sigma_{s_l})t^{s_l} + \sigma_{s_l} I(t^{s_l})$$

it follows

$$t^* = I(t^*).$$

By Theorem 2.2 t^* is the equilibrium income vector of the original model (1)-(6). □

The proposition provides a tool to select a subsequence of points converging to an equilibrium. But in general, there may be no such subsequences t^{s_l} satisfying the conditions of this proposition.

In the following three cases $\sigma_s = \sigma > 0$ and matrices $A(y),\ y \in Y(t)$ do not depend on y. Then process (47)-(48) becomes a standard homogeneous Markov chain with well known conditions of convergence to a stable distribution (see Gantmaher [5]).

Case 1. Consider an exchange economy, i.e. $g_i(y_i) = 0,\ i = 1, \ldots, l$. Then matrix $A(y),\ y \in Y(t^s)$ is constant and has the form

$$A = \begin{pmatrix} \sum_{j=1}^n \frac{w_{1j}}{W_i}\beta_{1j} & \sum_{j=1}^n \frac{w_{1j}}{W_j}\beta_{2j} & \ldots & \sum_{j=1}^n \frac{w_{1j}}{W_j}\beta_{mj} \\ \sum_{j=1}^n \frac{w_{2j}}{W_j}\beta_{1j} & \sum_{j=1}^n \frac{w_{2j}}{W_j}\beta_{2j} & \ldots & \sum_{j=1}^n \frac{w_{2j}}{W_j}\beta_{mj} \\ \ldots & \ldots & \ldots & \ldots \\ \sum_{j=1}^n \frac{w_{mj}}{W_j}\beta_{1j} & \sum_{j=1}^n \frac{w_{mj}}{W_j}\beta_{2j} & \ldots & \sum_{j=1}^n \frac{w_{mj}}{W_j}\beta_{mj} \end{pmatrix}.$$

Case 2. If levels of productions are fixed, i.e. Y consists of a single point, then $A(y^s),\ y^s \in Y$ is also constant and has the form (43).

Case 3. Suppose that

$$\alpha_{pi} = \alpha_p, \quad i = 1, \ldots, l,$$

$$W_j > 0 \text{ and } G_j(y) = 0 \text{ for } j = 1, \ldots, n';$$

$$W_j = 0 \text{ and } G_j(y') > 0 \text{ for } j = n'+1, \ldots, n \text{ and some } y' \in Y$$

(in particular we may have $W_j = 0$ for all $j = 1, \ldots, n$), i.e. each good is either produced (but not available as endowment) or not produced (but available as endowment) in the economy. Then matrix $A(y)$ is also constant and has the form

$$A = \begin{pmatrix} \sum_{j=1}^{n'} \frac{w_{1j}}{W_j}\beta_{1j} + \alpha_1 \sum_{j=n'+1}^n \beta_{1j} & \ldots & \sum_{j=1}^{n'} \frac{w_{1j}}{W_j}\beta_{mj} + \alpha_1 \sum_{j=n'+1}^n \beta_{mj} \\ \ldots & \ldots & \ldots \\ \sum_{j=1}^{n'} \frac{w_{mj}}{W_j}\beta_{1j} + \alpha_m \sum_{j=n'+1}^n \beta_{1j} & \ldots & \sum_{j=1}^{n'} \frac{w_{mj}}{W_j}\beta_{mj} + \alpha_m \sum_{j=n'+1}^n \beta_{mj} \end{pmatrix}.$$

Lemma 5.1 *Let either of Cases 1, 2 or 3 apply, $\sigma_s = \sigma > 0$, and thus matrix $A(y^s) = A$ be constant. If A has a positive row, then $\overline{A} = (1-\sigma)E + \sigma A$ is stable with maximum eigenvalue $\lambda_A = 1$, so*

$$\lim_{s\to\infty} t^{s+1} = \lim_{s\to\infty} \overline{A}^s t^0 = t_A,$$

where t_A is a single eigenvector of A corresponding $\lambda_A = 1$:

$$\overline{A} t_A = t_A.$$

Then $At_A = t_A$ and by Theorem 2.2 t_A is an equilibrium income vector.

Example (continued, from section 4). In this example matrix A is constant (as in Case 1) and equals to

$$A = \begin{pmatrix} 1 & 3/4 \\ 0 & 1/4 \end{pmatrix}.$$

It has a unique eigenvector $t_A = (1,0)^T$, $\sum_{k=1}^{m}(t_A)_k = 1$, corresponding to the maximal eigenvalue 1. Sequence $t^{s+1} = At^s$, starting from any initial point t^0, $\sum_{k=1}^{m} t_k^0 = 1$, very quickly converges to t_A.

6 Concluding remarks

In this article we have indicated only some convergence properties of the joint maximization method and related issues. In particular we demonstrate that even the case of Cobb-Douglas utility functions leads to a new type of problems for inhomogeneous Markov processes, where the time dependence of the transition matrix is endogenously generated by the probability distribution of its current states. It is worth mentioning that the convergence of the joint maximization method does not require the gross substitutability assumptions to be met. Further convergence analysis requires more in-depth study of the mapping $I(t)$ and matrix $A(y)$ in (42), (43).

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