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Dissipative Control Systems and Disturbance Attenuation for Nonlinear H^∞ - Problems

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Dissipative Control Systems and Disturbance Attenuation for Nonlinear H^{∞} – Problems

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Abstract

We characterize functions satisfying a dissipative inequality associated with a control problem. Such a characterization is provided in terms of epicontingent and viscosity supersolutions to a Partial Differential Equation called the Hamilton-Jacobi-Bellman-Isaacs equation. Links between viscosity and epicontingent supersolutions are studied. Finally, we derive (possibly discontinuous) disturbance attenuation feedback of the H^{∞}–problem from contingent formulation of the Isaacs' Equation.

1 Introduction

Consider the following control system with two independent controls

$$\begin{cases} x'(t) = f(t, x(t), u(t), w(t)), \quad x(t_0) = x_0 \\ u(t) \in \boldsymbol{R}^p \text{ and } w(t) \in \boldsymbol{R}^l \end{cases} \tag{1.1}$$

where the state x belongs to $\boldsymbol{R}^n$. One of questions of interest studied in H^{∞}-theory lies in finding a control $u(\cdot)$ insuring[1] that the following so-called L^2-gain[2]:

$$\frac{\int_0^T L(s, x(s), u(s), w(s))ds}{\|w(\cdot)\|^2_{L^2(0,T)}}$$

[1]When this control can be expressed by a feedback law, it is often called the disturbance attenuation feedback.

[2]Whose minimum in the linear case is the H^{∞}-norm of a suitable transfer function. See [13] or [5] for a detailed description of this fact.

is less or equal to some fixed constant γ^2. This leads to the H^∞-control problem described in [5] for instance. See also the bibliography of this book for further references. A reformulation of the above problem consists in studying the value-function of the optimal control problem described below.

The goal of the controller u is to minimize a cost function $\mathcal{J}_\gamma$ given by

$$\mathcal{J}_\gamma(t_0, x_0, u(\cdot), w(\cdot)) := \int_{t_0}^{T} (L(s, x(s), u(s), w(s)) - \gamma^2 \|w(s)\|^2) ds \tag{1.2}$$

against all possible choices of w which is the disturbance of the system. The result of this optimal action of the controller is a quantity, called the cost (or value), which depends on the initial conditions of the system

$$V(t_0, x_0) := \inf_{u(\cdot)\in \mathcal{U}} \sup_{w(\cdot)\in \mathcal{W}} \mathcal{J}_\gamma(t_0, x_0, u(\cdot), w(\cdot))$$

Here $\mathcal{U}$ and $\mathcal{W}$ are the sets of measurable functions from $[t_0, T]$ into $\boldsymbol{R}^p$ and $\boldsymbol{R}^l$ respectively, sometimes we shall denote these sets by $\mathcal{U}(t_0)$ and $\mathcal{W}(t_0)$.

The cost function V is a supersolution of the Partial Differential Equation

$$-V_t + H(t, x, -V_x) = 0$$

where H is the Hamiltonian of the control system defined by

$$H(t, x, p) = \sup_{u \in \boldsymbol{R}^p} \inf_{w \in \boldsymbol{R}^l} \left(\langle p, f(t, x, u, w) \rangle - L(t, x, u, w) + \gamma^2 \|w\|^2 \right)$$

for two suitable concepts of supersolutions. Next, we provide a characterization of sub/supersolutions of this Partial Differential Equation thanks to some monotonicity properties - called *dissipative inequalities* - of the cost-function $\mathcal{J}_\gamma$ along suitable trajectories of the system.

Let us recall the definition of dissipative inequality (see [14]) associated to some extended function $\Theta(\cdot,\cdot) : \boldsymbol{R}_+ \times \boldsymbol{R}^n \mapsto \boldsymbol{R} \cup \{\infty\}$. Fix a measurable control $\bar{u}(\cdot)$. If for any measurable disturbance $w(\cdot)$, we have

$$\left\{ \begin{array}{ll} t_2 \geq t_1 \geq t_0 & \Longrightarrow \Theta(t_2, x(t_2, \bar{u}(\cdot), w(\cdot))) - \Theta(t_1, x(t_1, \bar{u}(\cdot), w(\cdot))) \leq \\ & \leq \int_{t_1}^{t_2} (\gamma^2 \|w(s)\|^2 - L(s, x(s, \bar{u}(\cdot), w(\cdot)), \bar{u}(s), w(s))) ds \end{array} \right. \tag{1.3}$$

where $x(\cdot, \bar{u}(\cdot), w(\cdot)))$ denotes the solution to (1.1) corresponding to $\bar{u}(\cdot)$ and $w(\cdot)$, then Θ is called a *storage function* (associated to the control $\bar{u}(\cdot)$). When there exists a control $u(\cdot,\cdot)$ in the feedback form[3] such that (1.3) holds true with $\bar{u}(t) = u(t, x(t))$ for a function Θ, it is called the *disturbance attenuation*

[3] Namely $u : R_+ \times R^n \mapsto R^p$.

feedback. Notice that if Θ is nonnegative and $t_0 = 0$, $\Theta(0, x_0) = 0$, then (1.3) yields

$$\int_0^T L(s, x(s, \bar{u}(\cdot), w(\cdot)), \bar{u}(s), w(s)) ds \leq \gamma^2 \int_0^T \|w(s)\|^2 ds$$

for all $w \in L^2(0,T)$, which means that the L^2-gain is not greater than γ^2.

The problem will be reduced to the statement of a criterion allowing to determine storage functions as sub/supersolutions of a Hamilton-Jacobi-Bellman Partial Differential Equation. This result is related to those of James [12] who proved - in the continuous case - that storage functions are viscosity subsolutions to some PDE and that any continuous viscosity subsolution of this PDE is a storage function. In the present work, we provide relations between storage functions, viscosity and epicontingent supersolutions in the lower semicontinuous case.

Another aim of our paper is to use epicontingent supersolutions to derive a disturbance attenuation law (possibly set-valued) in the context of H^∞-problem.

2 Preliminaries

In this section we recall some basic definitions of set-valued analysis. The subdifferential of a function $\phi : \boldsymbol{R}^n \mapsto \overline{\boldsymbol{R}}$ at $x_0 \in \mathrm{Dom}(\phi)$ is defined by

$$\partial_-\phi(x_0) := \left\{ p \in \boldsymbol{R}^n \mid \liminf_{x \longrightarrow x_0} \frac{\phi(x) - \phi(x_0) - \langle p, x - x_0 \rangle}{\|x - x_0\|} \geq 0 \right\}$$

and the superdifferential of ϕ at x_0 by $\partial_+\phi(x_0) := -\partial_-(-\phi)(x_0)$ (see [6]). The contingent epiderivative of ϕ at x in the direction v is given by:

$$D_\uparrow \phi(x)(v) := \liminf_{(h,v') \longrightarrow (0+,v)} \frac{\phi(x + hv') - \phi(x)}{h}$$

or equivalently by

$$Epi(D_\uparrow \phi(x)) = T_{Epi(\phi)}(x, \phi(x))$$

where Epi stands for the epigraph and the contingent cone[4] to a set A at a point $x \in A$ is defined in the following way:

$$T_A(x) := \{ v \mid \liminf_{h \longrightarrow 0+} d(x + hv, A)/h = 0 \}$$

Here d denotes the distance. We define in a symmetric way the contingent hypoderivative:

$$D_\downarrow \phi(x)(v) := -D_\uparrow(-\phi)(x)(v)$$

These definitions are related by a result from [3, Proposition 6.4.8]:

Lemma 2.1 *Consider a function $\phi : \boldsymbol{R}^n \mapsto \overline{\boldsymbol{R}}$. Then*

$$\partial_-\phi(x) = \{ p_x \in \boldsymbol{R}^n \mid \forall\, u \in \boldsymbol{R}^n,\ D_\uparrow \phi(x)(u) \geq \langle p_x, u \rangle \} \tag{2.1}$$

[4] or Bouligand's cone, see [2] and [3].

3 Dissipative Systems and Hamilton-Jacobi Equations

Consider a subset U of a Banach space and continuous functions

$$f : \boldsymbol{R}_+ \times \boldsymbol{R}^n \times U \mapsto \boldsymbol{R}^n \ \ \& \ \ L : \boldsymbol{R}_+ \times \boldsymbol{R}^n \times U \mapsto \boldsymbol{R}$$

We associate to these data the control system

$$\begin{cases} x'(s) & = & f(s, x(s), u(s)) \\ u(s) & \in & U \end{cases} \tag{3.1}$$

and assume that

$$\begin{cases} \mathbf{H_1)} & \forall\, (t_0, x_0) \in \boldsymbol{R}_+ \times \boldsymbol{R}^n,\ \exists\, \varepsilon > 0,\ \exists\, k \in L^1(t_0 - \varepsilon, t_0 + \varepsilon) \\ & \text{such that for almost all } t \in [t_0 - \varepsilon, t_0 + \varepsilon],\ \forall\, u \in U, \\ & f(t, \cdot, u) \ \text{ is } k(t) - \text{ Lipschitz on } B(x_0, \varepsilon) \\ & \\ \mathbf{H_2)} & \forall\, (t_0, x_0) \in \boldsymbol{R}_+ \times \boldsymbol{R}^n,\ \exists\, \varepsilon > 0,\ \exists\, c > 0 \text{ satisfying :} \\ & |t - t_0| + \|x - x_0\| \le \varepsilon \implies \\ & \|f(t, x, u)\| + |L(t, x, u)| \le c(1 + \|u\|^2) \end{cases} \tag{3.2}$$

Under these assumptions for every $u \in L^2_{loc}(\boldsymbol{R}_+, U)$ and $(t_0, x_0) \in \boldsymbol{R}_+ \times \boldsymbol{R}^n$ the control system (3.1) with the initial condition

$$x(t_0) = x_0 \tag{3.3}$$

has a solution on a neighborhood of t_0 in $\boldsymbol{R}_+$.

Definition 3.1 *The control system (3.1) is called dissipative if there exists a function* $V : \boldsymbol{R}_+ \times \boldsymbol{R}^n \mapsto \boldsymbol{R} \cup \{\pm\infty\}$ *(called a storage function) such that for every* $u \in L^2_{loc}(\boldsymbol{R}_+, U)$ *and any solution* $x(\cdot)$ *to (3.1) corresponding to* $u(\cdot)$ *and defined on some time interval* $[T_1, T_2]$ *we have*

$$\forall\, T_1 \le t_1 \le t_2 \le T_2,\ \ V(t_2, x(t_2)) \le V(t_1, x(t_1)) + \int_{t_1}^{t_2} L(s, x(s), u(s))ds$$

In the above, $\int_{t_1}^{t_2} L(s, x(s), u(s))ds = +\infty$ whenever $s \mapsto L(s, x(s), u(s))$ is not integrable on $[t_1, t_2]$. However, assumption $\mathbf{H_2)}$ yields integrability of this function for t_1 and t_2 sufficiently close to t_0.

Inequality of Definition 3.1 is called the dissipative inequality. In general the storage function is neither unique nor continuous. Semicontinuous storage functions can be studied as sub/supersolutions to Hamilton-Jacobi equations. But first we recall that lower and upper semicontinuous envelopes of a storage function are again storage functions.

Proposition 3.2 *Assume that for every* $u \in L^2_{loc}(\mathbf{R}_+, U)$ *the solution* $x(\cdot)$ *to (3.1), (3.3) is defined on* $\mathbf{R}_+$ *and there exist* $\varepsilon > 0$, $k \in L^1_{loc}$ *such that for any* u, $f(t,\cdot,u)$ *is* $k(t)$*−Lipschitz on* $B(x(t),\varepsilon)$. *Further assume that for a constant* $c > 0$ *and a function* $\varphi : \mathbf{R}_+ \times \mathbf{R}^n \mapsto \mathbf{R}_+$ *which is bounded on bounded sets the following inequalities hold true*

$$\begin{cases} \|f(t,x,u)\| \leq c(1+\|u\|^2)(1+\|x\|) \\ |L(t,x,u))| \leq c(1+\|u\|^2 + \varphi(t,x)) \end{cases}$$

If V *is a storage function, then so do its lower and upper*[5] *envelopes* $V_\star$, $V^\star$.

The proof proceeds by classical arguments and is omitted. Because of the above result, below, we shall only study lower semicontinuous storage functions.

Theorem 3.3 *Assume that (3.2) holds true. If* V *is a storage function, then for all* $(t,x) \in \mathrm{Dom}(V)$

$$\begin{cases} \qquad\qquad\quad \sup_{u\in U} D_\uparrow V(t,x)(1, f(t,x,u)) - L(t,x,u) \leq 0 \\ t > 0 \implies \sup_{u\in U} D_\uparrow(-V)(t,x)(-1,-f(t,x,u)) - L(t,x,u) \leq 0 \end{cases}$$

Proof — Let $u \in U$ be a constant control. Consider an associated solution to (3.1) satisfying $x(t) = x$, defined on $[t-\delta, t+\delta]$. Then for $h > 0$,

$$V(t-h, x(t-h)) \geq V(t,x) - \int_{t-h}^{t} L(s,x(s),u)ds$$

and therefore

$$\frac{-V(t-h,x(t-h)) - (-V(t,x))}{h} \leq \frac{1}{h}\int_{t-h}^{t} L(s,x(s),u)ds$$

We can pass to the lower limit and apply Lebesgue's theorem, using (3.2) and that $x(t-h) = x - hf(t,x,u) + o(h)$. In this way we get

$$D_\uparrow(-V)(t,x)(-1,-f(t,x,u)) \leq L(t,x,u)$$

Since u is arbitrary, we derived the second inequality. The proof of the first one is similar by applying the dissipative inequality between t and $t+h$. □

Define Hamiltonians

$$H^\natural(t,x,p) = \sup_{u\in U}(\langle p, f(t,x,u)\rangle - L(t,x,u))$$

[5]The lower envelope $V_\star$ of V is the largest lower semicontinuous function which is smaller than V. In a shorter way, $Epi(V_\star)$ is the closure of the epigraph of V. The upper envelope $V^\star$ is defined by considering hypographs: $Hypo(V^\star) := cl(Hypo(V))$.

and

$$H^\flat(t,x,p) = \inf_{u\in U} \left(\langle p, f(t,x,u)\rangle + L(t,x,u)\right)$$

Let us check, like in [12], that any storage function is a viscosity sub/supersolution[6] defined thanks to subdifferentials:

Corollary 3.4 *If V is a storage function, then it is a viscosity subsolution to*

$$V_t + H^\sharp(t,x,V_x) = 0 \tag{3.4}$$

and a viscosity supersolution to

$$-V_t + H^\flat(t,x,-V_x) = 0 \tag{3.5}$$

The proof follows from Lemma 2.1 and Theorem 3.3.

If V is differentiable, then both statements boil down to :

$$\forall\,(t,x)\in \boldsymbol{R}_+ \times \boldsymbol{R}^n,\;\; V_t + H^\sharp(t,x,V_x) \le 0$$

We observe that $\partial_+ V(t,x)$ is adapted to deal with upper semicontinuous storage functions and $\partial_- V(t,x)$ is adapted to lower semicontinuous storage functions. Indeed, a statement converse to Corollary 3.4 holds true.

Theorem 3.5 *Assume (3.2). If V is an upper semicontinuous viscosity subsolution to (3.4), then V is storage function. If V is a lower semicontinuous viscosity supersolution to (3.5), then V is storage function.*

Proof — Fix $0 \le t_1 \le t_2$, an L^2_{loc}-control $u(\cdot)$ and let $x(\cdot)$ be an associated solution to (3.1), (3.3). Consider a sequence $u_n(\cdot)$ of piecewise continuous controls converging to $u(\cdot)$ in L^2-norm on $[t_1, t_2]$. Hence

$$\forall\,(p_t,p_x)\in \partial_+ V(t,x),\;\; p_t + \langle p_x, f(t,x,u_n(t))\rangle - L(t,x,u_n(t)) \le 0$$

Consequently,

$$\forall\,(p_t,p_x)\in \partial_-(-V)(t,x),\;\; -p_t + \langle -p_x, f(t,x,u_n(t))\rangle - L(t,x,u_n(t)) \le 0$$

[6]Consider an Hamiltonian $H : [0,T]\times R^n \times R^n \mapsto R$. A function $\Theta : [0,T]\times R^n \mapsto \overline{R}$ is a viscosity supersolution (cf [6]) to the following PDE

$$-\Theta_t + H(t,x,-\Theta_x) = 0$$

if and only if

$$\forall\,(t,x)\in \mathrm{Dom}(\Theta),\; \forall\,(p_t,p_x)\in \partial_-\Theta(t,x),\;\; -p_t + H(t,x,-p_x) \ge 0$$

and is a viscosity subsolution to the above PDE if and only if

$$\forall\,(t,x)\in \mathrm{Dom}(\Theta),\; \forall\,(p_t,p_x)\in \partial_+\Theta(t,x),\;\; -p_t + H(t,x,-p_x) \le 0$$

Using the same arguments as in [7] we deduce that

$$D_{\uparrow}(-V)(t,x)(-1,-f(t,x,u_n(t))) \leq L(t,x,u_n(t))$$

This is equivalent to

$$(-1,-f(t,x,u_n(t)),L(t,x,u_n(t))) \in T_{Epi(-V)}(t,x,-V(t,x)) \subset T_{Epi(-V)}(t,x,y)$$

for any $y \geq -V(t,x)$. Thanks to the Viability Theorem (cf [2]) applied to the map

$$(t,x) \mapsto -1 \times -f(t,x,u_n(t)) \times L(t,x,u_n(t))$$

on the epigraph of $-V$, we know that there exist some $y_n(\cdot), z_n(\cdot)$ solving

$$(y'(s),z'(s)) = (-f(s,y(s),u_n(s)),L(s,y(s),u_n(s)))$$

such that $(y_n(0),z_n(0)) = (x(t_2),-V(t_2,x(t_2)))$ which is viable in the set $\text{Epi}(-V)$. This implies that for any $t \leq t_2$,

$$-V(t_2,x(t_2)) \geq -V(t_2-t,y_n(t)) - \int_{t_2-t}^{t_2} L(s,y_n(t_2-s),u_n(s))ds \qquad (3.6)$$

Consider the solution $(y(\cdot),z(\cdot))$ to

$$\begin{cases} (y'(s),z'(s)) &= (-f(s,y(s),u(s)),L(s,y(s),u(s))) \\ (y(0),z(0)) &= (x(t_2),-V(t_2,x(t_2))) \end{cases}$$

By the Gronwall inequality, $(y_n(\cdot),z_n(\cdot))$ converge to $(y(\cdot),z(\cdot))$ when $n \to \infty$. Since $-V$ is lower semicontinuous, taking the lower limit in (3.6), this yields for $t := t_2 - t_1$ and the solution $x(t) := y(t_2-t)$ to (3.1)

$$-V(t_2,x(t_2)) \geq -V(t_1,x(t_1)) - \int_{t_1}^{t_2} L(s,x(s),u(s))ds$$

The second statement is proved similarly. □

Remark — In the difference with [12] we do not have to assume that V is locally bounded. So our framework satisfies the requirements of [14]. □

4 Solutions of Partial Differential Equations

As it was shown in [12], the dissipative inequality leads to a first order partial differential equation, called Isaacs' equation. In the previous section, we already explained the meaning of viscosity sub/supersolutions, here let us recall another approach through contingent solutions (cf [3]) of Hamilton-Jacobi-Bellman-Isaacs' equations. We shall see further that the value function V from the introduction satisfies in a suitable sense these partial differential equations. But now, we only want to define and compare two definitions of solutions of Isaacs' equation.

Denote by $\overline{\boldsymbol{R}}$ the extended real line $\boldsymbol{R} \cup \{\pm\infty\}$ and define the Hamiltonian of the H^∞-control problem:

$$H(t,x,p) := \sup_{u \in \boldsymbol{R}^p} \inf_{w \in \boldsymbol{R}^l} \{ \langle p, f(t,x,u,w) \rangle - L(t,x,u,w) + \gamma^2 \|w\|^2 \}$$

Consider the associated Hamilton-Jacobi-Isaacs' equation:

$$-\Theta_t + H(t,x,-\Theta_x) = 0 \tag{4.1}$$

Definition 4.1 *A function $\Theta : [0,T] \times \boldsymbol{R}^n \mapsto \overline{\boldsymbol{R}}$ is an epicontingent supersolution to (4.1) if and only if for all $(t,x) \in \mathrm{Dom}(\Theta)$ with $t < T$*

$$\sup_{u \in \boldsymbol{R}^p} \inf_{w \in \boldsymbol{R}^l} \{ -D_\uparrow \Theta(t,x)(1, f(t,x,u,w)) - L(t,x,u,w) + \gamma^2 w^2 \} \geq 0 \tag{4.2}$$

A function $\Theta : [0,T] \times \boldsymbol{R}^n \mapsto \overline{\boldsymbol{R}}$ is an hypocontingent subsolution to (4.1) if and only if for all $(t,x) \in \mathrm{Dom}(\Theta)$ with $t < T$

$$\sup_{u \in \boldsymbol{R}^p} \inf_{w \in \boldsymbol{R}^l} \{ -D_\downarrow \Theta(t,x)(1, f(t,x,u,w)) - L(t,x,u,w) + \gamma^2 w^2 \} \leq 0$$

Lemma 2.1 yields

Proposition 4.2 *Consider a function $\Theta : [0,T] \times \boldsymbol{R}^n \mapsto \overline{\boldsymbol{R}}$.*

- *If $\Theta(\cdot,\cdot)$ is an epicontingent supersolution to (4.1), then it is a viscosity supersolution to (4.1).*
- *If $\Theta(\cdot,\cdot)$ is an hypocontingent subsolution to (4.1), then it is a viscosity subsolution to (4.1).*

Let us underline that the converse implication is not true, in general.

Counter Example — Consider the case when the dynamics depend only on w. For any $(x,y) \in \boldsymbol{R}^2$ define

$$f(x,y,w) := \begin{cases} 0 \text{ if } (x,y) \notin \boldsymbol{R}_+(1,1) \\ w \text{ where } w \in W := co\{(0,0);(1,1)\} \text{ elsewhere} \end{cases}$$

The set-valued map $f(\cdot,\cdot,W)$ is obviously upper semicontinuous with convex nonempty values. We shall exhibit a Lipschitz function $\phi : \boldsymbol{R}_+ \times \boldsymbol{R}^2 \mapsto \boldsymbol{R}$ which satisfies for any x

$$\forall\, (p_x, p_y) \in \partial_-\phi(x,y), \;\; \max_{w \in W} \langle (p_x,p_y), f(x,y,w) \rangle \leq 0 \tag{4.3}$$

but does not satisfy

$$\sup_{w\in W} D_{\uparrow}\phi(x,y)(f(x,y,w)) \leq 0 \tag{4.4}$$

To accomplish this task, define ϕ as follows:

$$\begin{cases} \phi(x,y) = 0 & \text{if } x \leq 0 \text{ or } y \leq 0 \\ \phi(x,y) = y & \text{if } 0 \leq y \leq x \\ \phi(x,y) = x & \text{if } 0 \leq x \leq y \end{cases}$$

Inequality (4.4) means that for any $w \in W$,

$$(f(x,y,w),0) \in T_{Epi(\phi)}(x,y,\phi(x,y))$$

which is obviously false for instance if $(x,y) = (1,1)$.

Inequalities (4.3) are fulfilled because

$$(x,y) \notin \boldsymbol{R}_{+}(1,1) \implies \max_{w} \langle (p_x,p_y), f(x,y,w)\rangle = 0,\ \forall\,(p_x,p_y) \in \partial_{-}\phi(x,y)$$

$$\begin{cases} (x,y) = (0,0) & \implies \partial_{-}\phi(0,0) = \{0\} \\ (x,y) \in \boldsymbol{R}_{+}(1,1)\setminus\{(0,0)\} & \implies \partial_{-}\phi(0,0) = \emptyset \end{cases}$$

5 Regularity of the Value-Function

The value function[7]

$$V(t_0,x_0) := \inf_{u(\cdot)\in\mathcal{U}} \sup_{w(\cdot)\in\mathcal{W}} \mathcal{J}_{\gamma}(t_0,x_0,u(\cdot),w(\cdot))$$

is Lipschitz under suitable regularity of data. We shall need the following assumptions:

The growth of the dynamic is bounded in the following way

$$\exists\, E > 0,\ \|f(t,x,u,w)\| \leq E(1+\|u\|^2)(1+\|x\|) \tag{5.1}$$

The dynamic is Lipschitz continuous as follows

$$\begin{cases} f \text{ is continuous and } \exists\,\alpha(\cdot) \in L^1(0,T;\boldsymbol{R}_{+}), \text{ such that} \\ \forall\, w \in \boldsymbol{R}^l,\ f(t,\cdot,\cdot,w) \text{ is } \alpha(t) - \text{Lipschitz} \end{cases} \tag{5.2}$$

The integral cost satisfies assumptions insuring that the value is finite :

$$\begin{cases} & L \text{ is continuous and } \exists\,(a,b) \in \boldsymbol{R}_{+}^2,\ \forall\,(t,x,u,w), \\ i) & L(t,x,u,w) \leq \gamma^2\|w\|^2 + a(1+\|u\|^2) \\ ii) & L(t,x,u,w) \geq \gamma^2\|w\|^2 - b \\ iii) & \forall\, R > 0,\ \exists\, k_R > 0,\ \forall\, w \\ & L(\cdot,\cdot,\cdot,w) \text{ is } k_R - \text{Lipschitz on } \boldsymbol{R}_{+} \times B(0,R) \times \boldsymbol{R}^p \end{cases} \tag{5.3}$$

[7] where $\mathcal{J}_{\gamma}$ is defined by (1.2).

$$\left\{ \begin{array}{l} \forall\,(t,x,u) \in \boldsymbol{R}_+ \times \boldsymbol{R}^n \times \boldsymbol{R}^p, \text{ the set} \\ \{\,(L(t,x,u,w) - \gamma^2\|w\|^2 + \boldsymbol{R}_+, f(t,x,u,w))\,|\,w \in \boldsymbol{R}^l\,\} \\ \text{is closed and convex.} \end{array} \right. \tag{5.4}$$

Proposition 5.1 *Let assumptions (5.1) - (5.3) hold true. Then $V(\cdot,\cdot)$ is continuous and Lipschitz in x with a constant independent from t.*

The proof is very technical and can be found in [9].

6 Disturbance Attenuation Feedback

6.1 Existence of continuous feedback and epicontingent supersolutions

Let $\Theta : [0,T] \times \boldsymbol{R}^n \mapsto \boldsymbol{R} \cup \{+\infty\}$ be an extended function.

Proposition 6.1 *Suppose that there exists a continuous feedback $u : [0,T] \times \boldsymbol{R}^n \mapsto \boldsymbol{R}^p$ such that for any measurable control $w(\cdot)$ an associated solution $x(\cdot)$ to (1.1) satisfies (1.3) with $\bar{u}(t) = u(t,x(t))$. Then $\Theta(\cdot,\cdot)$ is an epicontingent supersolution to (4.1) and consequently also its viscosity supersolution.*

Proof — Fix $w \in \boldsymbol{R}^l$ and let $x(\cdot) = x(\cdot, t_0, x_0, \bar{u}, w)$ be an associated solution to (1.1). Applying inequality (1.3) with $t_1 := t_0 \leq t_2 := t_0 + h$ we obtain for all $h > 0$

$$\begin{array}{l} \frac{1}{h}(\Theta(t_0+h, x(t_0+h)) - \Theta(t_0,x_0)) \leq \\ \leq \frac{1}{h}\int_{t_0}^{t_0+h}(\gamma^2\|w(s)\|^2 - L(s,x(s),\bar{u}(s),w))ds \end{array}$$

We can pass to lower limit in the last inequality when $h \to 0+$ to get

$$D_\uparrow\Theta(t_0,x_0)(1, f(t_0,x_0,u(t_0,x_0),w)) \leq \gamma^2\|w\|^2 - L(t_0,x_0,u(t_0,x_0),w)$$

Since it occurs for any w we obtain (4.2). □

Consider next an epicontingent supersolution Θ to (4.1). The question arises whether there exists a feedback $u(\cdot,\cdot)$ such that Θ is a storage function.

Let us give a necessary condition for Θ to satisfy (1.3) in the case of existence of a continuous feedback $u(t,x)$:

Theorem 6.2 *Assume (5.1) - (5.4). Let $\Theta(\cdot,\cdot) : [0,T] \times \boldsymbol{R}^n \mapsto \boldsymbol{R}_+ \cup \{\infty\}$ be a lower semicontinuous function, Lipschitz with respect to the second variable[8] with a constant independent from t, which satisfies the contingent inequality*

$$\sup_w\{D_\uparrow\Theta(t,x)(1, f(t,x,u(t,x),w)) - \gamma^2\|w\|^2 + L(t,x,u(t,x),w)\,\} \leq 0 \tag{6.1}$$

[8] This is the case for instance, of the value-function studied in the previous section.

for some continuous $u(\cdot,\cdot)$ and all $t < T$. If for some $w(\cdot) \in L^2(0,T)$, a solution $x(\cdot)$ to

$$x'(t) = f(t, x(t), u(t, x(t)), w(t)), \quad x(0) = x_0 \tag{6.2}$$

is defined on the whole[9] interval $[0,T]$, then the following dissipative inequality holds true: $\forall\, 0 \leq t_1 \leq t_2 \leq T$,

$$\Theta(t_2, x(t_2)) \leq \Theta(t_1, x(t_1)) + \int_{t_1}^{t_2} (\gamma^2 \|w(s)\|^2 - L(s, x(s), u(s, x(s)), w(s)))ds$$

Proof — The proof is inspired by the proof of a Viability Theorem for Tubes (see [10] and [11]). We need the following result.

Lemma 6.3 ([9]) *Let $\alpha(\cdot) : [0,T] \mapsto \boldsymbol{R}$ be a lower semicontinuous function and $\beta : [0,T] \mapsto \boldsymbol{R}$ be an integrable function. If $\sup_{s\in[0,T[} D_\uparrow \alpha(s)(1) < +\infty$ and*

$$\text{for almost all } \ s \in [0,T], \ D_\uparrow \alpha(s)(1) \leq \beta(s)$$

then for any $0 \leq t_1 \leq t_2 \leq T$,

$$\alpha(t_2) \leq \alpha(t_1) + \int_{t_1}^{t_2} \beta(\sigma) d\sigma$$

Proof of Theorem 6.2 — Let us fix $w(\cdot) \in L^2(0,T)$ and consider a solution $x(\cdot)$ to (6.2) defined on $[0,T]$. We claim that

$$D_\uparrow \alpha(t)(1) \leq b \ \text{ for every } t \in [0,T[$$

and

$$D_\uparrow \alpha(t)(1) \leq \beta(t) \ \text{ for almost every } t \in [0,T] \tag{6.3}$$

where $\alpha(t) := \Theta(t, x(t))$ and $\beta(t) := \gamma^2 \|w(t)\|^2 - L(t, x(t), u(t, x(t)), w(t))$. Indeed, fix $t \in [0,T[$. Let $h_n \to 0+$ be such that

$$D_\uparrow \alpha(t)(1) = \lim_{n\to\infty} \frac{\Theta(t + h_n, x(t + h_n)) - \Theta(t, x(t))}{h_n} =$$

By the mean value theorem and assumptions (5.1), (5.4) there exist $\bar{w} \in \boldsymbol{R}^l$ and a subsequence h_{n_k} such that

$$\frac{x(t + h_{n_k}) - x(t)}{h_{n_k}} \longrightarrow f(t, x(t), u(t, x(t)), \bar{w})$$

From assumptions (5.3) *ii)* and (6.1), for all $(t, x, w) \in [0,T[\times \boldsymbol{R}^n \times \boldsymbol{R}^l$,

$$D_\uparrow \Theta(t, x)(1, f(t, x, u(t, x), w)) \leq \gamma^2 \|w\|^2 - L(t, x, u(t, x), w) \ \leq b$$

[9] One way to insure this statement is to assume that $u(\cdot,\cdot)$ is bounded.

Since $\Theta(t,\cdot)$ is Lipschitz

$$D_\uparrow \alpha(t)(1) = D_\uparrow \Theta(t,x(t))(1, f(t,x(t),u(t,x(t)),\bar{w})) \leq b$$

Consider a point t such that $x'(t) = f(t,x(t),u(t,x(t)),w(t))$. Since $\Theta(t,\cdot)$ is Lipschitz, using (6.1) we obtain

$$D_\uparrow \alpha(t)(1) = \liminf_{h\longrightarrow 0+} \frac{\Theta(t+h,x(t+h)) - \Theta(t,x(t))}{h} =$$

$$\liminf_{h\longrightarrow 0+} \frac{\Theta(t+h,x(t)+\int_t^{t+h} f(s,x(s),u(s,x(s)),w(s))ds) - \Theta(t,x(t))}{h} =$$

$$= D_\uparrow \Theta(t,x(t))(1, f(t,x(t),u(t,x(t)),w(t))) \leq \beta(t)$$

Applying Lemma 6.3, we derive

$$\Theta(t_2,x(t_2)) \leq \Theta(t_1,x(t_1)) + \int_{t_1}^{t_2} (\gamma^2\|w(s)\|^2 - L(s,x(s),u(s,x(s)),w(s)))ds \quad \square$$

6.2 Value function and disturbance attenuation feedback

We next prove that the value function V satisfies the dissipative inequality (1.3) when there exists an "optimal feedback".

Proposition 6.4 *We assume (5.1) - (5.4) and that*

$$\begin{cases} \text{there exists a feedback } u : [0,T]\times \boldsymbol{R}^n \mapsto \boldsymbol{R}^p, \ \forall\,(t,x) \\ V(t,x) = \sup_{w(\cdot)\in \mathcal{W}(t)} \mathcal{J}_\gamma(t,x,u(\cdot,\cdot),w(\cdot)) \end{cases} \tag{6.4}$$

Then for any measurable disturbance $w(\cdot)$ the solution $x(\cdot)$ to (1.1) satisfies the dissipative inequality (1.3) with $\bar{u}(t) = u(t,x(t))$.

Proof — Consider $0 \leq t_1 \leq t_2 \leq T$ and a measurable disturbance $w(\cdot)$. Denote by $\mathcal{W}(t_1)[t_2]$ the set of all measurable $\tilde{w} : [t_1,T] \longrightarrow \boldsymbol{R}^l$ whose restriction to $[t_1,t_2]$ is equal to $w(\cdot)$. Then

$$\begin{cases} V(t_1,x(t_1)) & = \sup_{\bar{w}(\cdot)\in \mathcal{W}(t_1)} \mathcal{J}_\gamma(t_1,x(t_1),u(\cdot,\cdot),\bar{w}(\cdot)) \geq \\ & \geq \sup_{\tilde{w}(\cdot)\in \mathcal{W}(t_1)[t_2]} \mathcal{J}_\gamma(t_1,x(t_1),u(\cdot,\cdot),\tilde{w}(\cdot)) \end{cases}$$

which, by the very definition of $\mathcal{J}_\gamma$, is equal to

$$\int_{t_1}^{t_2} \left(L(s,x(s),u(s,x(s)),w(s)) - \gamma^2\|w(s)\|^2\right) ds + \sup_{\widehat{w}(\cdot)\in \mathcal{W}(t_2)} \mathcal{J}_\gamma(t_2,x(t_2),u,\widehat{w}(\cdot))$$

Hence

$$V(t_1,x(t_1)) \geq V(t_2,x(t_2)) + \int_{t_1}^{t_2} \left(L(s,x(s),u(s,x(s)),w(s)) - \gamma^2\|w(s)\|^2\right) ds \quad \square$$

Thanks to Proposition 6.1 we can state the following

Corollary 6.5 *We impose assumptions of Proposition 6.4. Assume furthermore that the feedback $u(\cdot,\cdot)$ satisfying (6.4) is continuous. Then the value function V is an epicontingent supersolution to (4.1).*

6.3 Disturbance attenuation feedback and supersolutions of Isaacs' equation

This section is devoted to the construction of an *attenuation* feedback $u(\cdot,\cdot)$ such that for any measurable $w(\cdot)$, the associated solution satisfies the dissipative inequality.

Consider the contingent inequality

$$\inf_{u\in R^p}\sup_{w\in R^l}\left(D_\uparrow\Theta(t,x)(1,f(t,x,u,w))-\gamma^2\|w\|^2+L(t,x,u,w)\right)\le 0 \qquad (6.5)$$

By Section 6.2 we know that if there exists a continuous feedback $u(\cdot,\cdot)$ as in (6.4), then the value function V verifies the above inequality. We also know that when the data are regular enough, then V is continuous and Lipschitz with respect to x with a constant independent from t.

We first show that a solution Θ to (6.5) allows to construct a discontinuous attenuation feedback. The feedback we propose is set-valued, i.e. $U(t,x)\subset \boldsymbol{R}^p$, but any - possibly discontinuous - selection $u(t,x)\in U(t,x)$ is a candidate for the single-valued feedback, provided it enjoys the following property :

$$\begin{cases}\forall\, w\in\mathcal{W},\ \forall\,(t_0,x_0)\in[0,T]\times\boldsymbol{R}^n,\ \exists\, x(\cdot)\ \text{ solving the system}\\ x'(t)=f(t,x(t),u(t,x(t)),w(t)),\ \ x(t_0)=x_0\end{cases}$$

Let Θ solve (6.5) on $[0,T[\times\boldsymbol{R}^n$ and define the set-valued feedback $U:[0,T[\times\boldsymbol{R}^n\mapsto\boldsymbol{R}^p$ by

$$U(t,x):=\{\,u\in U\mid \forall\, w,\ D_\uparrow\Theta(t,x)(1,f(t,x,u,w))-\gamma^2\|w\|^2+L(t,x,u,w)\le 0\,\}$$

The sets $U(t,x)$ are closed whenever f and L are continuous.

Theorem 6.6 *We impose assumptions (5.1) - (5.4). Further assume that Θ is lower semicontinuous and Lipschitz with respect to the second variable with a constant independent from t. Then for every $w\in L^2(0,T)$ and any solution x to the differential inclusion*

$$x'(t)\in f(t,x(t),U(t,x(t)),w(t))),\quad x(t_0)=x_0$$

we have an analogous of (1.3) : for all $t_0\le t_1\le t_2\le T$,

$$\Theta(t_2,x(t_2))\le\Theta(t_1,x(t_1))+\int_{t_1}^{t_2}(\gamma^2\|w(t)\|^2-L(t,x(t),u(t),w(t)))dt$$

where $u(t)$ is so that $x'(t)=f(t,x(t),u(t),w(t))$. In the other words Θ is a storage function.

The proof is similar to the proof of Theorem 6.2 and is omitted.

References

[1] AUBIN J.-P. & CELLINA A. (1984) DIFFERENTIAL INCLUSIONS, Springer-Verlag.

[2] AUBIN J.-P. (1991) VIABILITY THEORY, Birkhäuser, Boston, Basel, Berlin.

[3] AUBIN J.-P. & FRANKOWSKA H. (1990) SET-VALUED ANALYSIS, Birkhäuser, Boston, Basel, Berlin.

[4] BARRON E.N., EVANS L.C. & JENSEN R. (1984) *Viscosity solutions of Isaacs' equations and differential games with Lipschitz controls,* J. Diff. Eqs., No.53, pp. 213-233.

[5] BASAR T. & BERNHARD P. (1991) H^∞-OPTIMAL CONTROL AND MINIMAX DESIGN PROBLEMS, Birkhäuser, Boston, Basel, Berlin.

[6] CRANDALL M.G. & EVANS L.C. & LIONS P.L. (1984) *Some properties of viscosity solutions of Hamilton-Jacobi Equations*, Transactions of A.M.S., 282, pp. 487-502.

[7] FRANKOWSKA H. (1993) *Lower semicontinuous solutions of Hamilton-Jacobi-Bellman equation*, SIAM J. on Control and Optimization, 31, pp. 257-272.

[8] FRANKOWSKA H. & QUINCAMPOIX M. (1992) *Value functions of differential games,* in PROCEEDINGS OF PREMIER COLLOQUE FRANCO-ROUMAIN SUR L'OPTIMISATION, CONTRÔLE OPTIMAL, EQUATIONS AUX DÉRIVÉES PARTIELLES, 7-11 Septembre, 1992, International Series of Numerical Mathematics, vol. 107, Birkhäuser Verlag, Basel, pp. 193-206.

[9] FRANKOWSKA H. & QUINCAMPOIX M. (1996) *Dissipative control systems and disturbance attenuation for nonlinear H^∞ - problems*, Cahiers de CEREMADE.

[10] FRANKOWSKA H. & PLASKACZ S. (1996) *A measurable upper semicontinuous viability theorem for tubes*, J. of Nonlinear Analysis, TMA, 26, pp. 565-582.

[11] FRANKOWSKA H. & PLASKACZ S. & RZEZUCHOWSKI T. (1995) *Measurable viability theorems and Hamilton-Jacobi-Bellman equations* J. Diff. Eqs., Vol. 116, N.2, pp. 265-305.

[12] JAMES M.R. (1993) *A partial differential inequality for dissipative nonlinear systems,* Systems & Control Letters, 21, pp. 315-320.

[13] KNOBLOCH H. & FLOCKERZI D. (1995) TOPICS IN CONTROL THEORY, Birkhaüser.

[14] WILLEMS J.C. (1972) *Dissipative dynamical systems,* Arch. Rat. Mech. Anal., 45, pp. 321-351.