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How Set-Valued Maps Pop Up in Control Theory

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Abstract

We describe four instances where set-valued maps intervene either as a tool to state the results or as a technical tool of the proof. The paper is composed of four rather independent sections:

1. Set-Valued Optimal Synthesis and Differential Inclusions
2. Viability Kernel
3. Nonsmooth Solutions to Hamilton-Jacobi-Bellman Equations
4. Interior and Boundary of Reachable Sets

1 Optimal Synthesis

We define optimal synthesis in two cases: for the Mayer problem with locally Lipschitz value function and for the time optimal control problem with lower semicontinuous time optimal function.

1.1 Mayer Problem with Lipschitz Value Function

Consider a complete separable metric space U, a continuous $f : R^n \times U \mapsto R^n$, a locally Lipschitz $\varphi : R^n \mapsto R$ and the minimization problem

$$\min \{\varphi(x(1)) \mid x \text{ is solves } (1),\ x(0) = \xi_0\}$$

(1) $$x'(t) = f(x(t), u(t)),\ \ u(t) \in U$$

The value function of this problem is defined by

$$V(t_0, x_0) = \inf \{\varphi(x(1)) \mid x \text{ solves } (1),\ x(t_0) = x_0\}$$

The value function generates the optimal synthesis since it is constant along optimal trajectories. It is well known that in general it is nonsmooth. If

$$(2)\quad \begin{cases} i) & \forall\, R>0,\ \exists\, c_R>0,\ \forall\, u,\ f(\cdot,u) \text{ is } c_R-\text{Lipschitz on } B_R \\ ii) & \exists\, k>0,\ \forall\, x,\ \sup_{u\in U} \|f(x,u)\| \le k(1+\|x\|) \end{cases}$$

where B_R denotes the closed ball of center zero and radius R, then the value function is locally Lipschitz (see for instance [25, FLEMING & RISHEL]). The optimal feedback set-valued map is given by

$$U(t,x) = \left\{ u \in U \mid \frac{\partial V}{\partial(1,v)}(t,x) = 0,\ v = f(x,u) \right\}$$

where $\frac{\partial V}{\partial(1,v)}$ denotes the directional derivative of V in the direction $(1,v)$. The sets $U(t,x)$ may be empty at points where V is not differentiable. The "optimal" control system can be described then in the following way

$$x'(t) = f(x(t),u(t)),\ \ u(t) \in U(t,x(t)),\ \ x(t_0) = x_0$$

A natural question arises : What are the solutions of the above closed loop system? A possible answer comes from the theory of differential inclusions: Solutions are absolutely continuous functions such that

$$x'(t) \in f(x(t),U(t,x(t))) \ \text{ a.e. } \ \&\ \ x(t_0) = x_0$$

Let us introduce the set-valued map of "optimal dynamics"

$$G(t,x) := f(x,U(t,x)) = \bigcup_{u\in U(t,x)} \{f(x,u)\}$$

Theorem 1.1 ([30, H.F.]) *Assume that V is locally Lipschitz. Then the following two statements are equivalent:*

i) x solves the differential inclusion

$$(3)\qquad x'(t) \in G(t,x(t)),\ \ x(t_0) = x_0$$

ii) x is optimal: $V(t_0,x_0) = \varphi(x(1))$

Proof — The proof is extremely simple. Fix a trajectory x of (3) and set $\psi(t) = V(t,x(t))$. Then ψ is absolutely continuous and for almost all t

$$\psi'(t) = \frac{\partial V}{\partial(1,x'(t))}(t,x(t))$$

If $i)$ holds true, then $\psi'(t) = 0$ a.e. and thus ψ is constant equal to $\varphi(x(1))$. If $ii)$ is satisfied, then, $\psi' = 0$ and thus $\frac{\partial V}{\partial(1,x'(t))}(t,x(t)) = 0$ a.e.. □

It was proved in [12, CANNARSA & FRANKOWSKA] that for smooth problems G is upper semicontinuous but its values are not convex. We recall next the definition of upper semicontinuous maps.

Let X, Y be metric spaces and $F : X \rightsquigarrow Y$ be a set-valued map, i.e., $\forall\, x \in X,\ F(x) \subset Y$. The (Painlevé–Kuratowski) upper limit is defined by

$$\mathrm{Limsup}_{x\to x_0} F(x) := \left\{ \lim_{n\to\infty} y_n \mid x_n \to x_0,\ y_n \in F(x_n) \right\}$$

If Y is compact, then F is upper semicontinuous on X if and only if

$$\forall\, x_0 \in X,\ \ \mathrm{Limsup}_{x\to x_0} F(x) \subset F(x_0)$$

When the data f, φ are smooth enough, then the value function V has "regular" directional derivatives and therefore the map G inherits upper semicontinuity, but in the same time the function

$$a \mapsto \frac{\partial V}{\partial a}(t, x(t))$$

is concave. If it is both concave and convex, then V is differentiable at $(t, x(t))$. So the values of G may be nonconvex at points where V is not differentiable. We would like to underline that qualitative theory of differential inclusions is build for upper semicontinuous set-valued maps with convex values. Most of its results are not valid without convexity assumptions. Because of that one should not expect optimal trajectories to have a nice structure when V is nonsmooth.

1.2 Time Optimal Feedback

We describe next the problem for which the value function is in general discontinuous. Consider a complete separable metric space U and a continuous map $f : R^n \times U \mapsto R^n$. Let $y(\cdot; x, u)$ denote the solution to

$$(4) \qquad y'(t) = f(y(t), u(t)) \qquad t \geq 0,\ \ y(0) = x$$

where $u(t) \in U$ almost everywhere and let $K \subset R^n$ be a closed set. Consider the *time optimal control problem for system (4) with target* K:

$$T(x) = \inf_u\ \inf\{t \geq 0 \mid y(t; x, u) \in K\}.$$

By the usual convention $T(x) = +\infty$ when no trajectory starting at x reaches K. A vector $p \in R^n$ is called a (proximal) *normal* to $S \subset R^n$ at a point $x \in \bar{S}$ if

$$dist_S(x + p) = \|p\|$$

Proximal normals were introduced in [9, BONY]. The Hamiltonian associated to the above control system is defined by $H(x,p) = \sup_{u \in U} \langle p, f(x,u) \rangle$.

To define time optimal synthesis we need the following extension of directional derivative. For $\varphi : R^n \mapsto R \cup \{+\infty\}$ the upper contingent derivative of φ at x_0 in the direction v is defined by

$$D_{\downarrow}\varphi(x_0)(v) \;=\; \limsup_{h \to 0+,\, v' \to v} \frac{\varphi(x_0 + hv') - \varphi(x_0)}{h}$$

See [3, AUBIN & FRANKOWSKA] for properties of contingent derivatives. In the two results below we impose assumptions (2) and that for all $x \in R^n$, $f(x, U)$ is closed and convex.

Theorem 1.2 ([13, CANNARSA, H.F. & SINESTRARI]) *Let $\bar{u}(\cdot)$ be a fixed control such that the corresponding trajectory $\bar{y}(\cdot) = y(\cdot; x_0, \bar{u})$ satisfies*

$$\bar{y}(t) \notin K, \;\; \forall\, t \in [0, T_0[\,;\;\; \bar{y}(T_0) \in K\,;\;\; H(\bar{y}(T_0), \nu) > 0$$

for some normal ν to $R^n \backslash K$ at $\bar{y}(T)$. Then $\bar{u}$ is time optimal if and only if, for every $t \in [0, T_0[$,

$$D_{\downarrow}T(\bar{y}(t))(v) = -1, \qquad \forall\, v \in Dy(t) := \mathrm{Limsup}_{s \downarrow t} \frac{\bar{y}(s) - \bar{y}(t)}{s - t}$$

The proof is not as straightforward as in the previous section, since $T(\cdot)$ may be merely lower semicontinuous. It is shown first that the co-state $p(\cdot)$ of Pontryagin's maximum principle verifies an adjoint inclusion. Then the Viability Theorem from [2, AUBIN] is applied to show that $t \mapsto V(t, \bar{y}(t))$ is Lipschitz even when V is discontinuous. The above result suggests to define the **time optimal synthesis** in the following way:

$$U(x) = \{u \in U \mid D_{\downarrow}T(x)(f(x,u)) = -1\}$$

The associated set-valued map of "optimal dynamics" is

$$G(x) = f(x, U(x))$$

In view of Theorem 1.2 it is natural to expect optimal trajectories to solve the following closed loop system

$$y'(t) = f(y(t), u(t)), \;\; u(t) \in U(y(t)), \;\; y(0) = x$$

Consider the differential inclusion

$$y'(t) \in G(y(t)), \;\; y(0) = x \tag{5}$$

The time optimal function $T(\cdot)$ being in general discontinuous, the arguments from the proof of Theorem 1.1 are not valid any longer. For this reason we have to change the notion of solution.

Definition 1.3 (**[37, MARCHAUD]**) *A continuous map* $y : [0, T_0] \to R^n$ *is a contingent solution of* (5) *if*

$$Dy(t) \cap G(y(t)) \neq \emptyset\,, \quad \forall\, t \in [0, T_0[\ \ \&\ \ y(0) = x$$

We already know that every time optimal solution is a contingent solution of (5) under all assumptions of Theorem 1.2. Conversely,

Theorem 1.4 (**[13, CANNARSA, H.F. & SINESTRARI]**) *Suppose that* $y(\cdot)$ *is a contingent solution of* (5) *in* $[0, T_0]$ *satisfying*

$$y(t) \notin K,\ \forall\, t \in [0, T_0[\,;\ \ y(T_0) \in K\,.$$

Then y *is time optimal.*

2 Viability Kernel

We provide next three examples leading to the notion of viability kernel.

Example 1: Implicit Control System

$$f(x, x', u) \ = \ 0, \quad u \in U$$

The way to make it "explicit" is to define the set-valued map

$$F(x) \ = \ \{v \mid \exists\, u \in U,\ \ f(x, v, u) = 0\}$$

and to study the differential inclusion

$$x'(t) \in F(x(t)) \quad \text{a.e.}$$

But in general F is not defined on the whole space but only on a subset $\mathrm{Dom}(F) := \{x \mid F(x) \neq \emptyset\}$. Furthermore there are $x_0 \in \mathrm{Dom}(F)$ from where no trajectory defined over R_+ of the control system starts. □

Example 2: Control System with State Constraints

$$x'(t) \ = \ f(x(t), u(t)),\ \ h(x(t), u(t)) \leq 0,\ \ u(t) \in U$$

To "get rid" from the constraints let us introduce the set-valued map

$$U(x) = \{u \in U \mid h(x, u) \leq 0\}$$

The new control system is

$$x' \ = \ f(x, u),\ \ u \in U(x)$$

Again U may be defined only over a subset $K = \{x \mid U(x) \neq \emptyset\}$. Furthermore, there are $x_0 \in K$ from where no trajectory of the control system satisfying state constraints starts. □

Example 3: Bounded Chattering

The problem is to find solutions to the control system

$$x'(t) = f(x(t), u(t)), \quad x(0) = x_0, \quad u(t) \in U(x(t)), \quad \|u'(t)\| \leq M$$

That is u has to be absolutely continuous and, in particular, it can not have jumps. Define a new dynamical system

$$\begin{cases} x'(t) &= f(x(t), u(t)), \quad x(0) = x_0, \quad u(t) \in U(x(t)) \\ u'(t) &\in B_M \end{cases}$$

Naturally there may exist $x_0 \in \mathrm{Dom}(U)$ from where no trajectory of the above system starts. □

In all three cases we reduced the control system under investigation to the following so called viability problem

$$\text{(6)} \qquad \begin{cases} x'(t) &\in F(x(t)), \text{ for almost all } t \geq 0, \\ x(t) &\in K, \ \forall\, t \geq 0, \\ x(0) &= x_0 \in K \end{cases}$$

The viability kernel $Viab(K)$ of K (under F) is the set of all initial conditions $x_0 \in K$ from which starts at least one solution (defined over R_+) of the differential inclusion (6). The notion of viability kernel was introduced in [1, AUBIN]. If

$$\begin{cases} i) & F \text{ is upper semicontinuous with closed convex images} \\ ii) & \exists\, k > 0, \ \sup_{v \in F(x)} \|v\| \leq k(\|x\| + 1) \end{cases}$$

then the viability kernel Viab(K) is closed and enjoys some stability properties. Algorithms were obtained to compute the viability kernel which for low dimensions run on PC's. See [36, FRANKOWSKA & QUINCAMPOIX], [38, SAINT-PIERRE], [15,16,17, CARDALIAGUET, QUINCAMPOIX & SAINT-PIERRE]. These (global) algorithms were inspired by "zero-dynamics" of [10, BYRNES & ISIDORI].

Since the notion of viability kernel revealed to be very useful in "computing" Lyapunov functions, time-optimal function, solving the target problem and in some applied problems (see [8, BONNEUIL & MULLERS], [19, CARTELIER & MULLERS], [22, DOYEN & GABAY], [23, DOYEN, GABAY & HOURCADE] and also [2, AUBIN] and its bibliography) the research is carried out in Université Paris-Dauphine by P. Cardaliaguet, L. Doyen, M. Quincampoix, P. Saint-Pierre and N. Seube to perfection algorithms for computing the viability kernel.

3 Solutions to HJB Equations

We address here the Hamilton-Jacobi-Bellman equation of optimal control. Consider again the Mayer Problem from Section 1.1. As we already observed for locally Lipschitz data the value function is locally Lipschitz. It is easy to understand how the generalized (bilateral) solutions arise in the Lipschitz case.

Definition 3.1 ([21]) *Let $\psi : X \mapsto R \cup \{+\infty\}$ be an extended function and $x_0 \in X$ be such that $\psi(x_0) \neq \infty$. The* subdifferential *of ψ at x_0 is :*

$$\partial_- \psi(x_0) = \left\{ p \mid \liminf_{x \to x_0} \frac{\psi(x) - \psi(x_0) - \langle p, x - x_0 \rangle}{\|x - x_0\|} \geq 0 \right\}$$

The value function V is nondecreasing along solutions of the control system and is constant along optimal solutions. For this reason the following statement follows easily by classical arguments (see for instance [25, FLEMING & RISHEL]): If V is differentiable at (t, x), then

$$-\frac{\partial V}{\partial t}(t,x) + H\left(x, -\frac{\partial V}{\partial x}(t,x)\right) = 0$$

$$\forall\, (p_t, p_x) \in \text{Limsup}_{(t',x') \to (t,x)} \frac{\partial V}{\partial x}(t', x'), \quad -p_t + H(x, -p_x) = 0$$

The Hamiltonian $H(x, \cdot)$ being convex, we have

$$\forall\, (p_t, p_x) \in \partial V(t,x), \ -p_t + H(x, -p_x) \leq 0$$

where $\partial V(t,x)$ denotes Clarke's generalized gradient.

On the other hand V is a viscosity solution to the HJB equation (see [21, CRANDALL, EVANS & LIONS]). In particular,

$$\forall\, (p_t, p_x) \in \partial_- V(t,x), \ -p_t + H(x, -p_x) \geq 0$$

But $\partial_- V(t,x) \subset \partial V(t,x)$ and therefore V is a bilateral solution:

$$\forall\, (p_t, p_x) \in \partial_- V(t,x), \ -p_t + H(x, -p_x) = 0$$

This notion of solution is valid as well for lower semicontinuous functions [7, BARRON & JENSEN], [32,33, FRANKOWSKA]. In [7] the authors extended the maximum principle of PDE's to lower semicontinuous functions. The alternative approach proposed in [33] is based on viability theory.

The "geometry" behind the method of proving uniqueness of nonsmooth solutions is the following one (details can be found in [4, AUBIN & FRANKOWSKA]): Consider the reachable set $R(t)$ at time t of

$$\begin{cases} x' = -f(x, u(t)) & x(1) = x_1 \quad u(t) \in U \\ z' = 0 & z(1) = \varphi(x_1) \end{cases}$$

for all possible choices of x_1. Then $R(t)$ is the epigraph of $V(t,\cdot)$:

$$R(t) = \{(x,r) \mid r \geq V(t,x)\}$$

The semigroup properties of reachable sets are used to investigate tangents to the epigraph of V. Namely consider the differential inclusion

$$x'(t) \in F(x(t)), \quad x(0) = x_0$$

where F is a locally Lipschitz set-valued map with convex compact images, and define its reachable map

$$R(t,x_0) = \{x(t) \mid x \text{ solves the above inclusion}\}$$

Then by well known results from [24, FILIPPOV]

$$R(t,x_0) = x_0 + tF(x_0) + o(t)B$$

where B denotes the closed unit ball. The fact that V is a bilateral solution follows from monotonicity properties of the value function. Proofs of the converse are based on Viability Theory [2, AUBIN].

In conclusion, we have to underline that for problems with lower semicontinuous cost function, it is natural to use **subdifferentials** rather than **upper directional derivatives**, because subdifferentials are related to tangents to the epigraph of V: $Epi(V) = \{(t,x,r) \mid r \geq V(t,x)\}$ which is closed. On the other hand to construct optimal synthesis via subdifferentials one needs extra assumptions which may be difficult to check. We used here upper directional derivatives, related to tangents to the hypograph of V: $\{(t,x,r) \mid r \leq V(t,x)\}$ not closed in general.

Next we discuss briefly the method of **characteristics of HJB equations**. Since the Hamiltonian of Mayer's problem is not differentiable at $(x,0)$, we consider the Bolza problem:

$$(P) \qquad \text{minimize} \int_{t_0}^{T} L(x(t),u(t))dt \; + \; \varphi(x(T))$$

over solution-control pairs (x,u) of control system

$$(7) \qquad \begin{cases} x'(t) = f(x(t)) + g(x(t))u(t), & u \in L^1 \\ x(t_0) = x_0 \end{cases}$$

The *Hamiltonian* H in this case is defined by

$$H(x,p) = \sup_u \left(\langle p, f(x) + g(x)u \rangle - L(x,u) \right)$$

and the value function is given by

$$V(t_0, x_0) = \inf_u \int_{t_0}^{T} L(x(t; t_0, x_0, u), u(t))dt + \varphi(x(T; t_0, x_0, u))$$

where $x(\cdot; t_0, x_0, u)$ denotes the solution to (7) corresponding to the control u. The HJB equation is

$$-V_t + H(x, -V_x) = 0, \quad V(T, \cdot) = \varphi(\cdot)$$

If H is smooth, then the characteristic system of this equation is the following Hamiltonian system

$$\begin{cases} x'(t) &= H'_p(x(t)p(t)) \qquad x(T) = x_T \\ -p'(t) &= H'_x(x(t), p(t)) \qquad p(T) = -\nabla\varphi(x_T) \end{cases}$$

It is well known that for nonconvex problems it is natural to expect shocks for such system:

$$\exists\ x_1(T) \neq x_2(T),\ \exists\ t_0 < T \text{ such that } x_1(t_0) = x_2(t_0)$$

This implies that for some initial conditions and some initial time t_0 we have multiple optimal trajectories or equivalently

$$\exists\ x_0 \quad \text{such that } \operatorname{Limsup}_{x \to x_0} \left\{ \frac{\partial V}{\partial x}(t_0, x) \right\}$$

is not a singleton.

In the two results below we impose the following assumptions:

$\mathbf{H_1}$) f, g are locally Lipschitz; f, g, $L(\cdot, u)$ are differentiable, $\varphi \in C^1$
$\mathbf{H_2}$) $\forall\ (t_0, x_0) \in [0, T] \times \mathbf{R}^n$ an optimal solution of (P) does exist and $V : [0, T] \times \mathbf{R}^n \mapsto \mathbf{R}$ is locally Lipschitz
$\mathbf{H_3}$) $L(x, \cdot)$ is continuous, convex and $\exists\ c > 0$ such that $L(x, u) \geq c\,\|u\|^2$
$\mathbf{H_4}$) For all $r > 0$, there exists $k_r \geq 0$ such that

$$\forall\ u \in \mathbf{R}^m, \quad L(\cdot, u) \text{ is } k_r - \text{Lipschitz on } B_r(0)$$

$\mathbf{H_5}$) H' is locally Lipschitz and the Hamiltonian system is complete.

Theorem 3.2 *Every solution (x, p) to the Hamiltonian system*

$$\begin{cases} x'(t) &= H'_p(x(t), p(t)) \quad x(t_0) = x_0 \\ -p'(t) &= H'_x(x(t), p(t)) \quad p(t_0) \in -\operatorname{Limsup}_{x \to x_0} V'_x(t_0, x) \end{cases}$$

is so that x is optimal.

Theorem 3.3 (BYRNES & H.F.) *Assume that $H(x,\cdot)$ is strictly convex and let $(\overline{x},\overline{u})$ be a trajectory-control pair. If $\overline{x}$ is an optimal trajectory of the Bolza problem, then for all $t\in]t_0,T]$, V is differentiable at $(t,\overline{x}(t))$.*

The above extends earlier results of [14, CANNARSA & SONER]) of calculus of variations. Further study of shocks is continued in [18, CAROFF & FRANKOWSKA].

4 Interior and Boundary of Reachable Sets

4.1 Local Controllability

Consider the control system

$$x'(t) = f(x(t),u(t)), \quad u(t)\in U, \; x(0) = x_0 \tag{8}$$

where f verifies (2) and $f(x,U)$ are closed and convex. Its reachable set at time $t\geq 0$ is given by

$$R(t) = \{\, x(t) \mid x \text{ is solves } (8)\}$$

We address the following question: When $x_0 \in \mathrm{Int}(R(t))$ for all $t>0$?

Let us first recall the Graves theorem (1947): if $f : X \longmapsto Y$ is C^1 and $f'(x_0)$ is surjective, then $\forall\, \varepsilon>0,\; f(x_0)\in \mathrm{Int}(f(B_\varepsilon(x_0)))$, where $B_\varepsilon(x_0)$ denotes the closed ball of center x_0 and radius ε.

A very similar result holds true also for set-valued maps. Here we apply it to the reachable map $R(\cdot)$. But in order to get such extension of Graves' theorem, one needs to differentiate set-valued maps on metric spaces. Recall first the notion of Painlevé–Kuratowski lower limit of sets. Let $F : X \leadsto Y$ be a set-valued map. The lower limit is given by

$$\mathrm{Liminf}_{x\to x_0} F(x) := \left\{ \lim_{x\to x_0} y_x \mid y_x \in F(x)\right\}$$

We introduce k-order variations of reachable sets:

$$R^k(0) := \mathrm{Liminf}_{t\to 0+} \frac{R(t)-x_0}{t^k}$$

Notice that for all $k\geq 1$, $R^k(x_0)\subset R^{k+1}(x_0)$.

Theorem 4.1 ([31, H.F.]) *If $0\in f(x_0,U)$ and for some $v_1,...v_p \in R^k(0)$*

$$0 \in \mathrm{Int}\;\mathrm{co}\,\{v_1,...,v_p\}$$

then $x_0\in \mathrm{Int}(R(t))$ for all $t>0$. Furthermore there exist $L>0,\; \varepsilon>0$ such that for all small $t>0$, all $y_1\in B_\varepsilon(x_0)$ and $y\in R(t)$ there exists t_1 such that

$$y_1\in R(t_1) \;\;\&\;\; |t_1-t| \;\leq\; L\,\sqrt[k]{\|y_1-y\|}$$

4.2 Lipschitz Behavior of Controls

Consider again the control system (8) and let $(z,\overline{u})$ be its trajectory control pair. We impose assumptions (2) and that $f(\cdot,u) \in C^1$ for all u. The linearized control system is given by

$$(9)\begin{cases} w'(t) &= f'_x(z(t),\overline{u}(t))w(t) + y(t), \quad y(t) \in \overline{co}\, f(z(t),U) - z'(t) \\ w(0) &= 0 \end{cases}$$

and the corresponding reachable set by $R^L(T) = \{w(T) \mid w \text{ solves } (9)\}$.

Theorem 4.2 (**[31, H.F.]**) *Assume that* $0 \in \text{Int}\,(R^L(T))$. *Then* $z(T) \in \text{Int}(R(T))$ *and there exist* $\varepsilon > 0$, $L > 0$ *such that for all* $b \in B_\varepsilon(z(T))$ *we can find a control* $u(\cdot)$ *satisfying*

$$x_u(T) = b, \quad \mu(\{t \in [0,T] \mid u(t) \neq \overline{u}(t)\}) \leq L\, \|b - z(T)\|$$

4.3 Nonsmooth Maximum Principle

Consider the control system (1) and assume (2). Let $g : R^n \to R^k$ be a locally Lipschitz function and K_0, $K_1 \subset R^n$ be closed. We impose the following end-point constraints:

$$(10) \qquad x(0) \in K_0, \quad x(1) \in K_1$$

Define the reachable set at time one : $R(1) = \{x(1) \mid x \text{ solves } (1), (10)\}$. Let z be a trajectory of (1),(10). It is well known (see for instance [20, CLARKE], [39, WARGA] etc.) that if $g(z(1))$ is a boundary point of $g(R(1))$, then a maximum principle holds true. The aim of this section is to make evident that behind there is an "alternative" inverse mapping theorem, which is much more than the characterization of boundary of reachable sets. Recall that generalized Jacobian of a locally Lipschitz function $\varphi : R^n \mapsto R^m$ (see [20, CLARKE]) is defined by:

$$\partial\varphi(x_0) \;=\; \overline{co}\;(\text{Limsup}_{x\to x_0}\, \varphi'(x))$$

Theorem 4.3 (**[31, H.F.]**) *Let* $(z,\overline{u})$ *be a trajectory-control pair of (1), (10). Then at least one of the following two statements holds true:*
i) $\exists\, \lambda \in R^k$ *and an absolutely continuous* $p : [0,1] \to R^n$ *not both equal to zero, satisfying the maximum principle*

$$-p'(t) \;\in\; \partial_x f(z(t),\overline{u}(t))^\star p(t) \quad \text{a.e. in } [0,1]$$

$$\max_{u\in U}\, \langle p(t),\, f(z(t),u)\rangle \;=\; \langle p(t),\, f(z(t),\overline{u}(t))\rangle \quad \text{a.e.}$$

$$p(0) \in N_{K_0}(z(0)), \quad -p(1)) \in \partial g(z(1))^\star\lambda + N_{K_1}(z(1))$$

where $N_K(x)$ *denotes the Clarke normal cone to* K *at* x *and* $\partial_x f$ *the generalized Jacobian with respect to* x.
ii) $\exists\, L > 0,\ \varepsilon > 0$ *such that for all* $(a, b, c) \in R^k \times R^n \times R^n$ *satisfying*

$$\|a - g(z(1))\| + \|b\| + \|c\| \leq \varepsilon$$

there exists a trajectory-control pair (x_u, u) *such that*

$$g(x_u(1)) = a, \ \ x_u(0) \in b + K_0, \ \ x_u(1) \in c + K_1$$

and $\mu(\{t \mid u(t) \neq \bar{u}(t)\}) \leq L\,(\|a - g(x_{\bar{u}}(1))\| + \|b\| + \|c\|)$.

In particular, if $g(z(1))$ *is a boundary point of* $g(R(1))$, *then the statement i) holds true.*

The above results from the set-valued inverse mapping theorem on metric spaces. Denote by $\mathcal{U}$ the set of all measurable functions $u : [0, 1] \to U$. Let $x(\cdot; u, x_0)$ be the solution of (8) corresponding to the control u and define the set-valued map $G : R^n \times \mathcal{U} \rightsquigarrow R \times R^n \times R^n$ by

$$G(x_0, u) = (g(x(1; u, x_0)), x_0, x(1; u, x_0)) - \{0\} \times K_0 \times K_1$$

The "strategy" of the proof is the following one:

1. Approximate G via "smooth" maps by regularizing f and g.
2. Use the inverse mapping theorem on approximations.
3. Go to the limit.

Regularization technics implying nonsmooth maximum principle go back to [39, WARGA]. In [26, FRANKOWSKA] it was shown that Warga's scheme may be refined to get smaller objects than the derivatives containers. The inverse mapping theorem used on approximations is Theorem 4.4 below. Finally Stability Theorem 4.5 is applied to take limits.

Consider $G : X \rightsquigarrow Y$, where X is a complete separable metric space and Y is a Banach space with the norm Gâteaux differentiable away from zero. Let $y_0 \in G(x_0)$. The graph of G is defined by

$$\mathrm{Graph}(G) = \{(x, y) \mid y \in G(x)\}$$

The first order "contingent" variation is defined by

$$G^{(1)}(x_0, y_0) \ = \ \mathrm{Limsup}_{h \to 0+} \frac{G(B_h(x_0)) - y_0}{h}$$

Theorem 4.4 **([31, H.F.])** *If for some* $\varepsilon > 0,\ \rho > 0,\ M > 0$

$$\text{(11)} \qquad \rho B \subset \bigcap_{\substack{(x,y) \in \mathrm{Graph}(G) \\ (x,y) \in B_\varepsilon(x_0, y_0)}} \overline{co}\,(G^{(1)}(x, y) \cap MB)$$

then for all $(x_1, y_1, y_2) \in \text{Graph}(G) \times Y$ *near* (x_0, y_0, y_0)

$$\text{dist}\,(x_1,\ G^{-1}(y_2)) \ \leq\ \frac{1}{\rho}\ \|y_1 - y_2\|, \quad \text{where } G^{-1}(y) = \{x \mid y \in G(x)\}$$

Theorem 4.5 (**[31, H.F.]**) *Consider set-valued maps* $\{G_i\}_{i\geq 0}$ *from a complete metric space* X *to a Banach space* Y *having closed graphs. Let* $y_0 \in G_0(x_0)$. *We assume that for some* $\delta > 0$ *and for every* $\lambda > 0$ *there exists an integer* I_λ *such that for all* $i \geq I_\lambda$ *and all* $x \in B_\delta(x_0)$

$$G_i(x) \ \subset\ G_0(x) \ +\ \lambda B$$

If G_i *have "a Lipschitz inverse" on a neighborhood of* (x_0, y_0) *with the same Lipschitz constant, then so does* G.

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