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Central Paths and Selection of Equilibria

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Abstract

For two populations of players playing repeatedly a same bimatrix game, a dynamics associated with the method of analytic centers for linear programming is described. All populations' evolutions converge to static equilibria. All evolutions starting in a same connected set converge to a same equilibrium. If a starting time is sufficiently large, "almost all" evolutions end up at a single equilibrium representing all populations' pure strategy groups (phenotypes) with nonzero proportions. The dynamics is interpreted as populations' rule to learn best replying.

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1 Introduction

Game-evolutionary models of economic and biological macrosystems treat evolution as a process of multiple over-time repetition of a static game in large populations of players. The evolution is reflected in changes of fractions of players' groups playing same strategies; these fractions are naturally identified with time varying mixed strategies. Particular evolution laws are usually deduced from "physical" suppositions, or behavioral assumptions. "Physical" suppositions lie in the basis of a model of Darwinean evolution known as the replicator dynamics (see Hofbauer and Sigmund, 1988). Behavioral or learning assumptions justify game-evolutionary models for economics (see, e.g., Friedman, 1991; Samuelson and Zhang, 1992; Fudenberg and Kreps, 1993; Young, 1993; Kaniovski and Young, 1995).

Game dynamics driving the populations to an equilibrium ensure their stable coexistence in a faraway future. Therefore classification of equilibrium convergent game dynamics tends to be one of the principal tasks in evolutionary game theory. Another motive for treating equilibrium convergent game dynamics comes from the computational area, in connection with the problem of updating mixed strategies so as to reach an equilibrium starting from a nonequilibrium. Updating procedures were given in, e.g., Brown, 1951, Robinson, 1951 (fictitious play), and Garcia and Zangwill, 1981 (path following).

Generally, a static game admits many equilibria, each of them representing an admissible coexistence regime (convention) in the community of players. In the standard game-theoretical setting all equilibria are equivalent, whereas in real game situations particular equilibria (conventions) are usually selected. Thus, a question of equilibrium selection mechanisms arises. Such mechanisms were considered in Harsanyi and Selten, 1988. In Young, 1993, a dynamical equilibrium selection mechanism was proposed; it was shown that, under appropriate conditions, trajectories driven by a special game dynamics prefer to converge to a single equilibrium.

The equilibrium selection property seems to be quite rare in the world of game dynamics. Note that the replicator dynamics (Hofbauer and Sigmund, 1988), as well as some generalizations of the fictitious play dynamics (see Fudenberg and Kreps, 1993; Kaniovski and Young, 1995) are generally not equilibrium selection. The goal of the present paper is to give an example of an equilibrium convergent and "almost" equilibrium selection bimatrix game dynamics. All models' trajectories converge to equilibria, and, under an appropriate choice of parameters, "almost all" trajectories approach a single, interior, equilibrium; a corresponding convention represents all players' strategy groups with nonzero proportions.

The suggested model originates from the theory of analytic centers and interior point methods for linear programming (see Sonnevend, 1986; Sonnevend, et. al., 1991). Earlier, a dynamics of the kind was used in Harsanyi, 1982, for proving the existence of an odd number of equilibria.

The model is outlined in section 2. In section 3 we give a game-evolutionary interpretation by showing that the model corresponds to a certain learning pattern. Section 4 contains main definitions and an outline of the results. Section 5 is devoted to general convergence properties of the model. In section 6 an equilibrium selection statement is proved. The paper adjoins Kryazhimskii and Sonnevend, 1996.

2 Central path equations

We deal with a bimatrix game determined by $m \times n$ payoff matrices A_0 and B_0 ($n, m \geq 2$). Mixed strategies of the first and second players are as usual identified with points of S_n^0 and S_m^0, respectively. Here $S_k^0 = \{z \ : \ z_1 + \ldots + z_k = 1, \quad z_1, \ldots, z_k \geq 0\}$; z_i stands for the ith coordinate of a vector z; finite-dimensional vectors are treated as columns. A mixed strategy *(Nash) equilibrium*, $(\hat{x}^0, \hat{y}^0)$, is defined by

$$\hat{x}^0 \in \text{Argmax } \{p_{A^0}(x^0, \hat{y}^0) \ : \ x^0 \in S_n^0\}, \quad \hat{y}^0 \in \text{Argmax } \{p_{B^0}(\hat{x}^0, y^0) \ : \ x^0 \in S_m^0\},$$

where

$$p_{A^0}(x^0, y^0) = y^{0T} A^0 x^0, \quad p_{B^0}(x^0, y^0) = y^{0T} B^0 x^0$$

represent the payoffs to the first and second players, respectively, at a mixed strategy pair (x^0, y^0). In what follows, Argmax $\{r(z) \ : \ z \in E\}$ stands for the set of all maximizers of a scalar function $r(\cdot)$ on a set E; if the maximizer is unique, it is denoted argmax $\{r(z) \ : \ z \in E\}$.

We associate the above game with a family of *penalized* games parametrized by a nonnegative parameter t (time). In the penalized game corresponding to time t, the payoffs to the first and second players are given by

$$p_{A^0}(t, x^0, y^0) = t p_{A^0}(x^0, y^0) + \sum_{i=1}^{n} \log x_i^0 + \phi_1^T x^0,$$

$$p_{B^0}(t, x^0, y^0) = t p_{B^0}(x^0, y^0) + \sum_{i=1}^{m} \log y_i^0 + \phi_1^T x^0,$$

where ϕ_1, ϕ_2 are fixed vectors in R^n, R^m respectively; players' strategy spaces are again S_n^0 and S_m^0. The logarithmic terms penalize the players for the approach to the "boundary" of the mixed strategy space $S_n^0 \times S_m^0$ (as some pure strategy fractions x_i^0, y_j^0 go to zero, the penalty terms go to $-\infty$). As time t is small, the penalty terms dominate; players' major care is keeping all their pure strategy fractions far from zero. As time t is large, the matrix payoffs dominate; players' major care is playing the game. At time t an equilibrium $(x^0(t), y^0(t))$ in the penalized game is defined by

$$x^0(t) = \text{argmax } \{p_{A^0}(t, x^0, y^0(t)) \ : \ x^0 \in \text{int } S_n^0\}, \tag{2.1}$$

$$y^0(t) = \text{argmax } \{p_{B^0}(t, x^0(t), y^0) \ : \ y^0 \in \text{int } S_m^0\}; \tag{2.2}$$

here int S_k^0 stands for the set of all $z \in S_k^0$ such that $z_i > 0$ ($i = 1, \ldots, k$). Note that due to the strict concavity of the maps $x^0 \mapsto p_{A^0}(t, x^0, y^0(t))$ and $y^0 \mapsto p_{B^0}(t, x^0(t), y^0)$, their maximizers on int S_n^0 and, respectively, int S_m^0 are unique; therefore in (2.1), (2.2) the usage of argmax instead of Argmax is correct.

We are concerned with the question if the equilibria $(x^0(t), y^0(t))$ in the penalized games converge to those in the initial game as t goes to infinity. The fact that $x^0(t)$, $y^0(t))$ have only nonzero coordinates, that is, $(x^0(t), y^0(t))$ never touches the "boundary" of the strategy space $S_n^0 \times S_m^0$, allows us to call paths $t \mapsto (x^0(t), y^0(t))$ *central* (avoiding the boundary).

The existence of an equilibrium $(x^0(t), y^0(t))$ in the penalized games is stated in Lemma 2.2. To prove it we shall use the next simple observation. Below int $S_k^0(\kappa)$ denotes the set of all $z \in S_k^0$ such that $z_i \geq \kappa$ $(i = 1, \ldots, k)$.

Lemma 2.1 *Let $t_* \geq 0$. There is a $\kappa > 0$ such that for every $t \in [0, t_*]$, every $x_*^0 \in$ int S_n^0 and every $y_*^0 \in$ int S_m^0 it holds that*

$$\text{argmax } \{p_{A^0}(t, x^0, y_*^0) \ : \ x^0 \in \text{int } S_n^0\} \in S_n^0(\kappa),$$

$$\text{argmax } \{p_{B^0}(t, x_*^0, y^0) \ : \ y^0 \in \text{int } S_m^0\} \in S_m^0(\kappa).$$

This statement follows easily from the fact that $p_{A^0}(t, x^0, y^0)$ and $p_{B^0}(t, x^0, y^0)$ tend to $-\infty$ as one of the coordinates of x^0, respectively, y^0 goes to zero.

Lemma 2.2 *For every $t \geq 0$ there exists an $(x^0(t), y^0(t))$ satisfying (2.1), (2.2).*

Proof. Consider the maps

$$X^0(t, \cdot) : y^0 \mapsto X^0(t, y^0) = \text{argmax } \{p_{A^0}(t, x^0, y^0) \ : \ x^0 \in \text{int } S_n^0\},$$

$$Y(t, \cdot) : x^0 \mapsto Y^0(t, x^0) = \text{argmax } \{p_{A^0}(t, x^0, y^0) \ : \ y^0 \in \text{int } S_m^0\}$$

on int S_m^0 and int S_n^0 respectively. The maps are well defined and continuous. By Lemma 2.1 $X^0(t, \cdot)$ and $Y^0(t, \cdot)$ take values, respectively, in $S_n^0(\kappa)$ and $S_m^0(\kappa)$ for some $\kappa > 0$. Therefore the composite map $(x^0, y^0) \mapsto (X^0(t, Y^0(t, x^0)), Y^0(t, X^0(t, y^0)))$ is continuous and carries $S_n^0(\kappa) \times S_m^0(\kappa)$ into itself. By Brower's theorem it has a fixed point $(x^0(t), y^0(t))$ in $S_n^0(\kappa) \times S_m^0(\kappa)$. By the definition of $X^0(t, \cdot)$ and $Y^0(t, \cdot)$ the point $(x^0(t), y^0(t))$ satisfies (2.2). □

In our analysis we shall use the independent variables $x = (x_1, \ldots, x_{n-1}) = (x_1^0, \ldots, x_{n-1}^0)$, $y = (y_1, \ldots, y_{n-1}) = (y_1^0, \ldots, y_{m-1}^0)$ replacing x_n^0, y_m^0 by $1 - \sum x_i$, and, respectively, $1 - \sum y_j$. Thus, from now on we identify mixed strategies of the first and second players with elements of S_{n-1} and S_{m-1}, respectively, where $S_k = \{z \ : \ z_1 + \ldots + z_k \leq 1, \quad z_1, \ldots, z_k \geq 0\}$. Using the partitions

$$A^0 = \begin{pmatrix} A^{00} & b_1 \\ c_1^T & d_1 \end{pmatrix}, \quad B^0 = \begin{pmatrix} B^{00} & b_2 \\ c_2^T & d_2 \end{pmatrix},$$

where A^{00} and B^{00} are $(m-1) \times (n-1)$ matrices, $b_1, b_2 \in R^{n-1}$, $c_1, c_2 \in R^{m-1}$, $d_1, d_2 \in R^1$, we represent the payoffs $p_{A^0}(x^0, y^0)$, $p_{B^0}(x^0, y^0)$, respectively, in the form

$$p_A(x, y) = \begin{pmatrix} y \\ 1 - \sum_{j=1}^{m-1} y_j \end{pmatrix}^T A^0 \begin{pmatrix} x \\ 1 - \sum_{i=1}^{n-1} x_i \end{pmatrix} = y^T A x + g_1^T x + h_1^T y + d_1, \qquad (2.3)$$

$$p_B(x, y) = \begin{pmatrix} y \\ 1 - \sum_{j=1}^{m-1} y_j \end{pmatrix}^T B^0 \begin{pmatrix} x \\ 1 - \sum_{i=1}^{n-1} x_i \end{pmatrix} = y^T B x + g_2^T x + h_2^T y + d_2; \qquad (2.4)$$

here

$$A = A^{00} - C_1 - B_1 + D_1, \qquad g_1 = c_1 - \bar{d}_1, \quad h_1 = b_1 - \bar{d}_1,$$

$$B = B^{00} - C_2 - B_2 + D_2, \quad g_2 = c_2 - \bar{d}_2, \quad h_2 = b_2 - \bar{d}_2,$$

$$B_k = (b_k, \ldots, b_k) \quad C_k = \begin{pmatrix} c_k^T \\ \ldots \\ c_k^T \end{pmatrix}, \quad \bar{d}_k = \begin{pmatrix} d_k & \ldots & d_k \\ & \ldots & \\ d_k & \ldots & d_k \end{pmatrix} \quad (k = 1, 2).$$

The equilibria in the bimatrix game are expressed through

$$\bar{x} \in \text{Argmax } \{p_A(x, \bar{y}) \ : \ x \in S_{n-1}\}, \tag{2.5}$$

$$\bar{y} \in \text{Argmax } \{p_B(\bar{x}, y) \ : \ x \in S_{m-1}\}. \tag{2.6}$$

Namely $(\hat{x}, \hat{y}) \in S_n^0 \times S_m^0$ is an equilibrium in the bimatrix game if and only if $\hat{x}_i = \bar{x}_i$ for all $i = 1, ..., n-1$ and $\hat{y}_j = \bar{y}_j$ for all $j = 1, ..., m-1$, where $(\bar{x}, \bar{y})$ satisfies (2.5), (2.6). We shall identify every above $(\bar{x}, \bar{y})$ with a corresponding $(\hat{x}, \hat{y})$ and also call it a (Nash) equilibrium.

The payoffs $p_{A^0}(t, x^0, y^0)$, $p_{B^0}(t, x^0, y^0)$ in the penalized games are rewritten into

$$p_A(t, x, y) = tp_A(x, y) + \sum_{i=1}^{n} \log x_i + \log \left(1 - \sum_{i=1}^{n} x_i\right) + \psi_1^T x + \phi_{1n}, \tag{2.7}$$

$$p_B(t, x, y) = tp_B(x, y) + \sum_{i=j}^{m} \log y_j + \log \left(1 - \sum_{j=1}^{n} y_j\right) + \psi_1^T y + \phi_{2n}, \tag{2.8}$$

where $\psi_{1i} = \phi_{1i} - \phi_{1n}$ $(i = 1, \ldots, n-1)$, $\psi_{2i} = \phi_{2i} - \phi_{2m}$ $(i = 1, \ldots, m-1)$. An equilibrium in the penalized game at time t is expressed through

$$x(t) = \text{argmax } \{p_A(x, y(t)) \ : \ x \in S_{n-1}\}, \tag{2.9}$$

$$y(t) = \text{argmax } \{p_B(x(t), y) \ : \ y \in S_{m-1}\}; \tag{2.10}$$

Namely, $(x^0(t), y^0(t))$ satisfies (2.1), (2.2) if and only if $x^0(t)_i = x(t)_i$ for all $i = 1, ..., n-1$ and $y^0(t)_j = y(t)_j$ for all $j = 1, ..., m-1$, where $(x(t), y(t))$ satisfies (2.9), (2.10). We identify every above $(x(t), y(t))$ with a corresponding $(x^0(t), y^0(t))$.

By Lemma 2.2 for every $t \geq 0$ there exists a $(x(t), y(t))$ satisfying (2.9) and (2.10); by Lemma 2.1 $(x(t), y(t))$ lies in the interior of $S_{n-1} \times S_{m-1}$. This together with the strong concavity of the functions maiximized in (2.9) and (2.10), yield that $(x(t), y(t)) \in$ int $S_{n-1}\times$ int S_{m-1} is an equilibrium in the penalized game at time t, that is, satisfies (2.9), (2.10), if and only if the derivatives of the above functions at $x(t)$ and, respectively, $y(t)$ vanish; more specifically, $(x(t), y(t))$ solves the algebraic equation

$$\Lambda_1(t, x, y) = 0, \tag{2.11}$$

$$\Lambda_2(t, x, y) = 0, \tag{2.12}$$

where

$$\Lambda_1(t, x, y) = t(A^T y + g_1) + \sum_{i=1}^{n-1} \frac{e_i}{x_i} - \frac{\sum_{i=1}^{n-1} e_i}{1 - \sum_{i=1}^{n-1} x_i} + \psi_1,$$

$$\Lambda_2(t, x, y) = t(Bx + h_2) + \sum_{j=1}^{m-1} \frac{f_j}{y_j} - \frac{\sum_{i=1}^{m-1} f_j}{1 - \sum_{j=1}^{m-1} y_j} + \psi_2,$$

$e_i \in R^{n-1}$, $f_j \in R^{m-1}$, $e_{i,i} = 1$, $e_{i,k} = 0$ for $k \neq i$, $f_{j,j} = 1$, $f_{j,k} = 0$ for $k \neq j$. We call (2.11), (2.12) the *central path* equations.

The formal differentiation of the algebraic equations (2.11), (2.12) in t results in the differential equation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = -H^{-1}(t,x,y) \begin{pmatrix} A^T y + g_1 \\ Bx + h_2 \end{pmatrix}. \tag{2.13}$$

Here

$$H(t,x,y) = \begin{pmatrix} -D(x) & tA^T \\ tB & -D(y) \end{pmatrix}, \tag{2.14}$$

$$D(z) = \mathrm{Diag}\left(\frac{1}{z_1^2} + \frac{1}{(1 - \sum_{i=1}^k z_i)^2}, \ldots, \frac{1}{z_k^2} + \frac{1}{(1 - \sum_{i=1}^k z_i)^2}\right) \quad (z \in R^k); \tag{2.15}$$

$\mathrm{Diag}(\zeta_1, \ldots, \zeta_k)$ stands for the diagonal $k \times k$ matrix with $\zeta_1, \ldots, \zeta_k$ on the diagonal. One can treat (2.14) as a dynamical system generating the central paths. Below we will couple (2.14) with an initial condition

$$x(t_0) = x^0, \quad y(t_0) = y^0 \tag{2.16}$$

thus determining a Cauchy problem. A solution $(x(\cdot), y(\cdot))$ of this problem will always be understood as that defined on $[0, \infty)$ and satisfying the constraint $(t, x(t), y(t)) \in \mathcal{H}$. Here and in what follows $\mathcal{H}$ is the set of all $(t, x^0, y^0) \in [0, \infty) \times \mathrm{int}\, S_{n-1} \times \mathrm{int}\, S_{m-1}$ such that the matrice $H(t,x,y)$ (2.14) is invertible.

3 A learning interpretation

In this section, for simplicity we restrict ourselves to the 2×2 matrices,

$$A_0 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B_0 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}. \tag{3.1}$$

The central path equations (2.11), (2.12) have the form

$$t(Ay + g_1) + \frac{1}{x} - \frac{1}{1-x} + \psi_1 = 0, \tag{3.2}$$

$$t(Bx + h_2) + \frac{1}{y} - \frac{1}{1-y} + \psi_2 = 0, \tag{3.3}$$

where

$$A = a_{11} - a_{12} - a_{21} + a_{22}, \quad g_1 = a_{21} - a_{22}, \tag{3.4}$$

$$B = b_{11} - b_{12} - b_{21} + b_{22}, \quad h_2 = b_{12} - b_{22} \tag{3.5}$$

(see [Vorobyov, p. 106]).

Imagine two large populations, $\mathcal{A}$ and $\mathcal{B}$, of players. The players from $\mathcal{A}$ and $\mathcal{B}$ are equipped with the payoff matrices A_0 and B_0 respectively. Each player adopts one of the two admissible strategies, 1 or 2. At every instant t a pair of players from $\mathcal{A}$ and $\mathcal{B}$ is picked up at random to play the bimatrix game with the above payoff matrixes. The groups of players adopting 1 and 2 in both populations vary over time. Fractions $x(t)$ and $y(t)$ of players' groups in $\mathcal{A}$ and $\mathcal{B}$ adopting strategy 1 at time t characterize a state of the populations.

We assume that $x(t)$ and $y(t)$ evolve in accordance with a rule of *learning to reply best.* The avarage payoffs to players of $\mathcal{A}$ adopting, respectively, strategies 1 and 2 at time t are given by

$$\alpha_1(t) = y(t)a_{11} + (1 - y(t))a_{12}, \quad \alpha_2(t) = y(t)a_{21} + (1 - y(t))a_{22}. \tag{3.6}$$

Symmetrically, the avarage payoffs to players of $\mathcal{B}$ adopting, respectively, strategies 1 and 2 are

$$\beta_1(t) = x(t)b_{11} + (1 - x(t))b_{21}, \quad \beta_2(t) = x(t)b_{12} + (1 - x(t))b_{22}.$$

The *best reply strategy* $s^+_{\mathcal{A}}(t)$ for the population $\mathcal{A}$ at time t is that of the group of players whose avarage payoff is larger,

$$s^+_{\mathcal{A}}(t) = \begin{cases} 1, & \alpha_1(t) > \alpha_2(t) \\ 2, & \alpha_1(t) < \alpha_2(t) \\ \text{arbitrary}, & \alpha_1(t) = \alpha_2(t) \end{cases}. \tag{3.7}$$

Let $x^+(t)$ be the fraction of the best repliers in $\mathcal{A}$ at time t,

$$x^+(t) = \begin{cases} x(t), & s^+_{\mathcal{A}}(t) = 1 \\ 1 - x(t), & s^+_{\mathcal{A}}(t) = 2 \\ 1, & s^+_{\mathcal{A}}(t) \text{ is arbitrary} \end{cases}.$$

Consider the difference between the maximum and minimum avarage payoffs,

$$\Delta^+_{\mathcal{A}}(t) = \max\{\alpha_1(t), \alpha_2(t)\} - \min\{\alpha_1(t), \alpha_2(t)\}.$$

For the population $\mathcal{A}$ a rule to learn best replying is expressed in the following three conditions:

(i) (*learning on payoff differences*) the greater avarage payoff difference $\Delta^+_{\mathcal{A}}(t)$ occurs at time t (the clearer is which strategy is better at time t), the greater part of the population $\mathcal{A}$ replies best at t;

(ii) (*learning in time*) if the population $\mathcal{A}$ registers a same avarage payoff difference at different times ($\Delta^+_{\mathcal{A}}(t_1) = \Delta^+_{\mathcal{A}}(t_2)$, $t_1 < t_2$), then at a greater time (t_2) a greater part of the population replies best;

(iii) (*ability to complete learning*) if times corresponding to a same avarage payoff difference go to infinity, the fractions of the best repliers at these times approach 1.

We formalize (i) – (iii) by setting

$$x^+(t) = \omega_{\mathcal{A}}(t, \Delta^+_{\mathcal{A}}(t)); \tag{3.8}$$

here $\omega_{\mathcal{A}}(\cdot,\cdot) : (t, \Delta) \mapsto \omega_{\mathcal{A}}(t, \Delta)$ is a continuous function from $[0, \infty) \times [0, \infty)$ to $(0, 1)$ increasing in both arguments and such that $\lim_{t\to\infty} \omega_{\mathcal{A}}(t, \Delta) = 1$ for every $\Delta > 0$. Let us call $\omega_{\mathcal{A}}(\cdot,\cdot)$ a *learning function* for the population $\mathcal{A}$. Introducing a similar *learning function* for the population $\mathcal{B}$, $\omega_{\mathcal{B}}(\cdot,\cdot) : (t, \Delta) \mapsto \omega_{\mathcal{B}}(t, \Delta)$, and putting

$$\Delta^+_{\mathcal{B}}(t) = \max\{\beta_1(t), \beta_2(t)\} - \min\{\beta_1(t), \beta_2(t)\},$$

we define a learning rule for the population $\mathcal{B}$. Namely, for the fraction $y^+(t)$ of the best repliers in the population $\mathcal{B}$, we set

$$y^+(t) = \omega_{\mathcal{B}}(t, \Delta^+_{\mathcal{B}}(t)). \tag{3.9}$$

Now we shall show that the central path equations (3.2), (3.3) are equivalent to the learning dynamics (3.8), (3.9) for particular learning functions $\omega_{\mathcal{A}}(\cdot,\cdot)$ and $\omega_{\mathcal{B}}(\cdot,\cdot)$. We set

$$\psi_1 = \frac{1}{\omega_{\mathcal{A}}(0,\Delta_{\mathcal{A}}^{+}(0))} - \frac{1}{1-\omega_{\mathcal{A}}(0,\Delta_{\mathcal{A}}^{+}(0))}, \tag{3.10}$$

$$\tau_{\mathcal{A}}(t,\Delta) = \left(\frac{1}{1-\omega_{\mathcal{A}}(t,\Delta)} - \frac{1}{\omega_{\mathcal{A}}(t,\Delta)} - \psi_1\right)\frac{1}{\Delta}. \tag{3.11}$$

Let $\omega_{\mathcal{A}}(\cdot,\cdot))$ be continuously differentiable, and $(\partial\omega_{\mathcal{A}}(t,\Delta)/\partial\Delta)_{\Delta=0} > 0$. Then $\tau_{\mathcal{A}}(\cdot,\cdot)$ is continuous on $[0,\infty)\times[0,\infty)$, with

$$\tau_{\mathcal{A}}(t,0) = \tau_{\mathcal{A}}^{0}(t) > 0, \tag{3.12}$$

$$\begin{aligned}
\tau_{\mathcal{A}}^{0}(t) &= \frac{\partial}{\partial\Delta}\left(\frac{1}{1-\omega_{\mathcal{A}}(t,\Delta)} - \frac{1}{\omega_{\mathcal{A}}(t,\Delta)} - \psi_1\right)_{\Delta=0} \\
&= \left(\frac{1}{(1-\omega_{\mathcal{A}}(t,\Delta))^2} + \frac{1}{\omega_{\mathcal{A}}^2(t,\Delta)} - \frac{\partial\omega_{\mathcal{A}}^2(t,\Delta)}{\partial\Delta}\right)_{\Delta=0}
\end{aligned}$$

(here we have used Lopital's rule) and

$$\frac{\partial\tau_{\mathcal{A}}(t,\Delta)}{\partial\Delta} = \frac{1}{\Delta}\left(\frac{1}{(1-\omega_{\mathcal{A}}(t,\Delta))^2} + \frac{1}{\omega_{\mathcal{A}}^2(t,\Delta)}\right)\frac{\partial\omega_{\mathcal{A}}(t,\Delta)}{\partial\Delta} - \frac{1}{\Delta^2}\tau_{\mathcal{A}}(t,\Delta). \tag{3.13}$$

For a fixed t (3.13) determines a linear differential equation for $\Delta \mapsto \tau_{\mathcal{A}}(t,\Delta)$. The first term on its right hand side is nonnegative (by the assumption on $\omega_{\mathcal{A}}(\cdot,\cdot)$). This, together with the positiveness of the initial condition (3.12), imply that $\tau_{\mathcal{A}}(t,\Delta)$ is positive for all $t,\Delta \geq 0$. Besides, from (3.11) and the assumption that $\lim_{t\to\infty}\omega_{\mathcal{A}}(t,\Delta) = 1$, we have that

$$\lim_{t\to\infty}\tau_{\mathcal{A}}(t,\Delta) = \infty$$

for all $\Delta \geq 0$.

The above properties of $\tau_{\mathcal{A}}(\cdot,\cdot)$ (3.11) have been deduced from the assumptions upon the learning function $\omega_{\mathcal{A}}(\cdot,\cdot)$. Inversely, starting from a $\tau_{\mathcal{A}}(\cdot,\cdot)$ having the above properties, one can arrive at a learning function $\omega_{\mathcal{A}}(\cdot,\cdot)$ satisfying (3.11). Let us perform a particular exercise of the kind. Assume that

$$\tau_{\mathcal{A}}(t,\Delta) = \tau(t),$$

where $\tau(\cdot) : [0,\infty) \mapsto [0,\infty)$ is an increasing function such that

$$\lim_{t\to\infty}\tau(t) = \infty.$$

Then (3.11) defines the value $\omega_{\mathcal{A}}(t,\Delta)$ of a learning function to be a positive solution of the quadratic equation

$$1 - 2\omega - \omega(1-\omega)(\psi_1 + \Delta\tau(t)) = 0;$$

explicitly,

$$\omega_{\mathcal{A}}(t,\Delta) = \begin{cases} \frac{2+\psi_1(t,\Delta)+(4+\psi_1^2(t,\Delta))^{1/2}}{2\psi_1(t,\Delta)}, & 2\psi_1(t,\Delta) \neq 0 \\ 1/2, & 2\psi_1(t,\Delta) = 0 \end{cases}, \tag{3.14}$$

where

$$\psi_1(t,\Delta) = \psi_1 + \Delta\tau(t). \tag{3.15}$$

An elementary analysis shows that $\omega_{\mathcal{A}}(\cdot,\cdot)$ defined by (3.14), (3.15) satisfies the assumptions characterizing a learning function for the population $\mathcal{A}$.

Consider a simplest case, $\tau(t) = t$. The corresponding learning function $\omega_{\mathcal{A}}(\cdot,\cdot)$ is defined by (3.14), (3.15). Take a t, for which the best reply strategy, $s_{\mathcal{A}}^+(t)$, of the population $\mathcal{A}$ is 1. By (3.8) we have

$$x(t) = x^+(t) = \omega_{\mathcal{A}}(t, \alpha_1(t) - \alpha_2(t)).$$

As noticed above, (3.11) holds. For $\Delta = \alpha_1(t) - \alpha_2(t)$ it turns into

$$t(\alpha_1(t) - \alpha_2(t)) + \frac{1}{x(t)} - \frac{1}{1 - x(t)} + \psi_1 = 0.$$

Noticing that by (3.6), (3.4)

$$\alpha_1(t) - \alpha_2(t) = Ay(t) + g_1,$$

we easily obtain that $x = x(t)$ and $y = y(t)$ satisfy the central path equation (3.2). Similarly, this equation is verified for the two complementary cases, $s_{\mathcal{A}}^+(t) = 2$ and $s_{\mathcal{A}}^+(t)$ arbitrary. Identical arguments show that $x = x(t)$ and $y = y(t)$ satisfy another central path equation, (3.4); the latter follows from the learning rule (3.9) for the population $\mathcal{B}$, where the learning function $\omega_{\mathcal{B}}(\cdot,\cdot)$ is defined symmetrically to (3.14), (3.15) (with $\tau(t) = t$).

Remark 3.1 The equivalency between the learning and central path dynamics holds also for learning functions more general than those specified above (that is, corresponding to $\tau(t) = t$). This equivalency requires however more general central path equations. Let us outline three steps of generalization.

Define, first, the learning functions $\omega_{\mathcal{A}}(\cdot,\cdot)$, $\omega_{\mathcal{B}}(\cdot,\cdot)$ by (3.14), (3.15), with an arbitrary $\tau(\cdot)$ increasing and satisfying $\lim_{t\to\infty}\tau(t) = \infty$. Then the learning dynamics (3.8), (3.9) is equivalent to the central path dynamics (3.2) (3.3) with t replaced by $\tau(t)$; we have time rescaling.

If, more generally, we define the learning functions $\omega_{\mathcal{A}}(\cdot,\cdot)$, $\omega_{\mathcal{B}}(\cdot,\cdot)$ through (3.14), (3.15) with different $\tau(\cdot)$, $\tau(\cdot) = \tau_{\mathcal{A}}(\cdot)$ and $\tau(\cdot) = \tau_{\mathcal{B}}(\cdot)$, the learning dynamics (3.8), (3.9) appears to be equivalent to the central path dynamics with different time scales; namely, in (3.2) and (3.3) we have, respectively, $\tau_{\mathcal{A}}(\cdot)$ and $\tau_{\mathcal{B}}(\cdot)$ instead of t.

Let, finally, the learning dynamics (3.8), (3.9) correspond to arbitrary learning functions $\omega_{\mathcal{A}}(\cdot,\cdot)$, $\omega_{\mathcal{B}}(\cdot,\cdot)$. Then for $x = x(t)$, $y = y(t)$ we have the central path equalities (3.2), (3.3), where t is replaced, respectively, by $\tau_{\mathcal{A}}(t, \Delta_{\mathcal{A}}^+(t))$ and $\tau_{\mathcal{B}}(t, \Delta_{\mathcal{B}}^+(t))$; the function $\tau_{\mathcal{A}}(\cdot,\cdot)$ is defined by (3.11), and $\tau_{\mathcal{B}}(\cdot,\cdot)$ is defined by an identical formula, with obvious changes.

Similar equivalencies between the learning and central path dynamics can also be established for the general $n \times m$ case.

4 Definitions. Outline of results

Let $Z(t \mid \psi_1, \psi_2)$ be the set of all pairs $(x, y) \in S_{n-1} \times S_{m-1}$ satisfying the central path equations (2.11), (2.12) ($t \geq 0$). We noticed in section 2 that this set is nonempty. (Due to the polynomial character of the equation (2.6) $Z(t \mid \psi_1, \psi_2)$ is, generically, finite; see Harsanyi, 1982).

Central paths will be understood as smooth single-valued branches of the multi-valued map $t \mapsto Z(t \mid \psi_1, \psi_2)$. Namely, we define a *central path starting from* $(x^0, y^0) \in \text{int } S_{n-1} \times \text{int } S_{m-1}$ *at time* t_0 to be a solution of the Cauchy problem (2.13), (2.16). A central path *starting in* $Z^0 \subset \text{int } S_{n-1} \times \text{int } S_{m-1}$ *at time* t_0 will be undestood as that starting at t_0 from some $(x^0, y^0) \in Z^0$. We shall call $Z^0 \subset \text{int } S_{n-1} \times \text{int } S_{m-1}$ a *set of uniqueness* for initial time t_0 ($t_0 \geq 0$) if for every $(x^0, y^0) \in Z^0$ there exists a unique central path starting from $(x^0, y^0) \in Z^0$ at t_0.

Note that $H^{-1}(\cdot)$ is Lipschitz on every closed subset of $\mathcal{H}$ (see section 2). Taking this into account we easily arrive at the following.

Lemma 4.1 *Let* $Z^0 \in \text{int } S_{n-1} \times \text{int } S_{m-1}$ *and for every* $(x^0, y^0) \in Z^0$ *there exists a solution of the Cauchy problem (2.13), (2.16). Then*

(a) Z^0 *is a set of uniqueness for initial time* t_0,

(b) for every $t \geq t_0$ *the central path* $(x(\cdot), y(\cdot))$ *starting from* $(x^0, y^0) \in Z^0$ *at* t_0 *satisfies* $(x(t), y(t)) \in Z(t \mid \psi_1, \psi_2)$ *where* ψ_1, ψ_2 *are such that* (x^0, y^0) *is the single solution of (2.11), (2.12) with* $t = t_0$ *(*ψ_1 *and* ψ_2 *obviously exist),*

(c) the function associating with every $(t, x^0, y^0) \in [t_0, \infty) \times Z^0$ *the value* $(x(t), y(t))$ *of the central path* $(x(\cdot), y(\cdot))$ *starting from* $(x^0, y^0) \in Z^0$ *at* t_0 *is continuous.*

Remark 4.1 Noticing that $H(t, x, y)$ is invertible for t close to zero and (x, y) lying in a neighborhood of an interior point of $S_{n-1} \times S_{m-1}$, one can easily prove the following local uniqueness statement. For every $(x^0, y^0) \in \text{int } S_{n-1} \times \text{int } S_{m-1}$ there exists an $\epsilon > 0$ such that for all $t \in [0, \epsilon)$ the set $Z[t \mid x^0, y^0]$ contains a single element $(x(t), y(t))$. The function $(x(\cdot), y(\cdot))$ is a single solution of the Cauchy problem (2.13), (2.16) with $t_0 = 0$.

In what follows, N denotes the set of all equilibria in the initial bimatrix game, that is, pairs $(\bar{x}, \bar{y})$ satisfying (2.5), (2.6). We shall be concerned with the selection of equilibria through central paths. The equilibrium selection property will be understood as an attribute of sets Z^0 starting the central paths. Namely, a set $Z^0 \subset \text{int } S_{n-1} \times \text{int } S_{m-1}$ will be said *to select an equilibrium* $(\bar{x}, \bar{y}) \in N$ *at initial time* t_0 if Z^0 is a set of uniqueness for t_0, and for every central path $(x(\cdot), y(\cdot))$ starting in Z^0 at t_0 it holds that $\lim_{t \to \infty}(x(t), y(t)) = (\bar{x}, \bar{y})$.

In section 5 we show that, under some natural nondegeneracy conditions, a connected set of uniqueness selects an equilibrium (Theorem 5.3); we also give a regularity condition ensuring that the whole state space $\text{int } S_{n-1} \times \text{int } S_{m-1}$ selects an equilibrium for any initial t_0 (the global equilibrium selection property). In section 6 we specify the first result for the case where A and B are square and nondegenerate and the interior of $S_{n-1} \times S_{m-1}$ contains a single equilibrium $(\bar{x}, \bar{y})$. Namely, we show that a set selecting $(\bar{x}, \bar{y})$ lies arbitrarily close to the interior of $S_{n-1} \times S_{m-1}$ if the initial time t_0 is sufficiently large. Thus, the central paths converge to $(\bar{x}, \bar{y})$ from "almost all" points provided a starting time is sufficiently large; the latter condition, in terms of the learning model outlined in section 3, implies that at the start of the evolution the populations have learned well enough.

5 General selection properties

Our preliminary goal is to show that, as t goes to infinity, $Z(t \mid \psi_1, \psi_2)$ converges to N. We shall exploit the next lemma known in the theory of analytic centers for linear programming.

Lemma 5.1 (Sonnevend, 1986) *Let $b \in R^k$, $c \in R^l$, F be a $k \times l$ matrix,*

$$\mu = \max\{c^T x \ : \ Fx \leq b\} > -\infty,$$

and

$$x(r) = \text{argmax}\left\{c^T x + r\sum_{j=1}^{k} \log(b_j - (Fx)_{(j)})\right\} \quad (r > 0).$$

Then

$$\mu - c^T x(r) \leq (k+1)r.$$

Below $\mid \cdot \mid$ stands for the Euclidean norm.

Corollary 5.1 *For every $(x_*, y_*) \in Z(t \mid \psi_1, \psi_2)$ we have*

$$p_A(x_*, y_*) \geq \max_{x \in S_{n-1}} p_A(x, y_*) - \frac{n + \mid \psi_1 \mid (n-1)}{t}, \tag{5.1}$$

$$p_B(x_*, y_*) \geq \max_{y \in S_{m-1}} p_A(x_*, y) - \frac{m + \mid \psi_2 \mid (m-1)}{t}. \tag{5.2}$$

Proof. Let $c^T = y_*^T A + g_1^T$ and

$$\bar{x}_* = \text{argmax } \{c^T x \ : \ x \in S_{n-1}\}.$$

By Lemma 5.1 we have

$$c^T \bar{x}_* \geq \max_{x \in S_{n-1}} c^T x - \frac{n}{t}.$$

Due to the form of $p_A(t, x, y)$ and $p_A(x, y)$ (see (2.7), (2.3)), it holds that

$$\mid c^T \bar{x}_* - c^T x_* \mid \leq \max_{x \in S_{n-1}} \frac{\mid \psi_1 \mid\mid x \mid}{t} \leq \frac{\mid \psi_1 \mid (n-1)}{t}.$$

Hence, referring to the form of $p_A(x, y)$, (2.3), we obtain (5.1). Similarly, using (2.4), (2.8), we arrive at (5.2). □

Let $d(w, Y)$ and $d(X, Y)$ stand, respectively, for the distance of an element w to a set Y, and the semidistance from a set X to Y in R^k,

$$d(w, Y) = \inf_{y \in Y} \mid w - y \mid, \quad d(X, Y) = \sup_{x \in X} d(x, Y).$$

Theorem 5.1 *Let Ψ be a bounded set in $R^{n-1} \times R^{m-1}$. Then*

$$\lim_{t \to \infty} \sup_{(\psi_1, \psi_2) \in \Psi} d(Z(t \mid \psi_1, \psi_2), N) = 0.$$

Proof. Assume that, to the contrary,

$$d(Z(t_i \mid \psi_{1,i}, \psi_{2,i}), N) \geq \epsilon > 0$$

for some $t_i \to \infty$, $(\psi_{1,i}, \psi_{2,i}) \in \Psi$. Then there are $(x_{*i}, y_{*i}) \in Z(t_i \mid \psi_{1,i}, \psi_{2,i})$ such that

$$d((x_{*i}, y_{*i}), N) \geq \epsilon/2.$$

Therefore, as one can easily show by contradiction, there exist $x_i \in S_{n-1}$, $y_i \in S_{m-1}$ and $\delta > 0$ such that either

$$p_A(x_{*i}, y_{*i}) < p_A(x_i, y_{*i}) - \delta,$$

or

$$p_B(x_{*i}, y_{*i}) < p_A(x_{*i}, y_i) - \delta.$$

For large i this is not possible due to Corollary 5.1. □.

Now we shall prove that the payoffs currently gained on each central path started in a connected set of uniqueness converge to a single equilibrium value (see below Theorem 5.2). The proof is conditioned by the requirement that a domain, within which the central paths evolve, intersects a finite set of equilibria.

We shall use the following definition. Given a set of uniqueness, Z^0, for an initial time t_0, we shall say that a set $D \subset S_{n-1} \times S_{m-1}$ *contains* the central paths starting in Z^0 at t_0 if $(x(t), y(t) \in D$ holds for each central path $(x(\cdot), y(\cdot))$ starting in Z^0 at t_0. We denote by $p(x, y)$ the pair of the payoffs at $(x, y) \in S_{n-1} \times S_{m-1}$ in the initial bimatrix game, $p(x, y) = (p_A(x, y), p_B(x, y))$. Given a set $D \in S_{n-1} \times S_{m-1}$, we shall write $V(D)$ for the collection of the payoffs at the equilibra contained in D, $V(D) = \{p(x, y) \ : \ (x, y) \in N \cap D\}$. We put $V = V(S_{n-1} \times S_{m-1})$.

Lemma 5.2 *There exists a function* $\pi(\cdot) : (0, \infty) \mapsto (0, \infty)$ *such that for every* $\epsilon > 0$ *and every* $(\bar{x}, \bar{y}) \in S_{n-1} \times S_{m-1}$ *the inequality*

$$d(p(\bar{x}, \bar{y}), V) \geq \epsilon$$

implies either

$$p_A(\bar{x}, \bar{y}) < \max_{x \in S_{n-1}} p_A(x, \bar{y}) - \pi(\epsilon),$$

or

$$p_B(\bar{x}, \bar{y}) < \max_{y \in S_{m-1}} p_B(\bar{x}, y) - \pi(\epsilon).$$

Proof. If there is no such $\pi(\cdot)$, then there exist an $\epsilon > 0$ and a sequence $(\bar{x}_i, \bar{y}_i) \in S_{n-1} \times S_{m-1}$ for which the inequality

$$d(p(\bar{x}_i, \bar{y}_i), V) \geq \epsilon$$

implies either

$$p_A(\bar{x}_i, \bar{y}_i) - \max_{x \in S_{n-1}} p_A(x, \bar{y}_i) \to 0,$$

or

$$p_B(\bar{x}_i, \bar{y}_i) - \max_{y \in S_{m-1}} p_B(\bar{x}_i, y) \to 0.$$

For a condensation point $(\bar{x}, \bar{y})$ of the sequence $(\bar{x}_i, \bar{y}_i)$ we have

$$d(p(\bar{x}, \bar{y}), V) \geq \epsilon, \tag{5.3}$$

$$p_A(\bar{x},\bar{y}) = \max_{x\in S_{n-1}} p_A(x,\bar{y}),$$

$$p_B(\bar{x},\bar{y}) = \max_{y\in S_{m-1}} p_B(\bar{x},y).$$

The last two equalities show that $(\bar{x},\bar{y}) \in N$, and consequently, $p(\bar{x},\bar{y}) \in V$. The latter contradicts to (5.3). □.

Lemma 5.2 leads to the following intermediate convergence result.

Lemma 5.3 *Let Z^0 be a set of uniqueness for initial time t_0, $D \subset S_{n-1} \times S_{m-1}$ be closed, contain all central paths starting in Z^0 at t_0, and $V(D)$ be finite. Then for every central path $(x(\cdot), y(\cdot))$ starting in Z^0 at t_0 there exists a $v \in V(D)$ such that*

$$\lim_{t\to\infty} p(x(t), y(t)) = v.$$

Proof. Let W be the set of all limits $\lim_{i\to\infty} p(x(t_i), y(t_i))$ with $t_i \to \infty$. By Theorem 5.1 $W \subset V$. Due to the closedness of the set D, we have $W \subset D$. Therefore, $W \subset V(D)$. It remains to show that W is one-element. Assume, to the contrary, that W contains two different elements, w_1 and w_2. We have

$$w_1 = \lim_{t_i\to\infty} p(x(t_i), y(t_i)) \quad t_i \to \infty,$$

$$w_2 = \lim_{\xi_i\to\infty} p(x(\xi_i), y(\xi_i)) \quad \xi_i \to \infty.$$

With no loss of generality, assume $t_i < \xi_i < t_{i+1}$. Set

$$\nu = \min\{| v_1 - v_2 | : \ v_1, v_2 \in V(D), \ v_1 \neq v_2\}. \tag{5.4}$$

Since $V(D)$ is finite, $\nu > 0$. For large i,

$$| w_1 - \lim_{t_i\to\infty} p(x(t_i), y(t_i)) | < \nu/4,$$

$$| w_2 - \lim_{\xi_i\to\infty} p(x(\xi_i), y(\xi_i)) | < \nu/4;$$

consequently, due to the continuity of $p(\cdot)$ and $(x(\cdot), y(\cdot))$ (see Lemma 4.1, (c)), we have

$$| p(x(\tau_i), y(\tau_i)) - w_1 | = \nu/2$$

for some $\tau_i \in (t_i, \xi_i)$. Then obviously

$$d(p(x(\tau_i), y(\tau_i)), V(D)) \geq \nu/2. \tag{5.5}$$

On the other hand, for a convergent subsequence $p(x(\tau_{i_j}), y(\tau_{i_j}))$ we have

$$\lim_{j\to\infty} p(x(\tau_{i_j}), y(\tau_{i_j})) \in W \subset V(D),$$

which contradicts to (5.5). □

The proof of our first principle result (Theorem 5.2) will use Lemma 5.3 and Lemma 5.5 given below. The latter is, in turn, preceded by technical Lemma 5.4 (in a sense complementary to Lemma 5.2).

Lemma 5.4 *There exists an increasing function* $\sigma(\cdot) : (0,\infty) \mapsto (0,\infty)$ *such that* $\lim_{\delta\to 0}\sigma(\delta) = 0$ *and for every* $\delta > 0$, $(\bar{x},\bar{y}) \in S_{n-1}\times S_{m-1}$ *the inequalities*

$$p_A(\bar{x},\bar{y}) \geq \max_{x\in S_{n-1}} p_A(x,\bar{y}) - \delta,$$

$$p_B(\bar{x},\bar{y}) \geq \max_{y\in S_{m-1}} p_B(\bar{x},y) - \delta$$

imply that

$$d(p(\bar{x},\bar{y}),V) \leq \sigma(\delta).$$

Lemma follows easily from Lemma 5.2.

Lemma 5.5 *Let the conditions of Lemma 5.3 be satisfied, and* $\sigma(\cdot)$ *be defined like in Lemma 5.4. Then*

(a) for every $t \geq t_0$ *there is a* $v(t) \in V$ *such that*

$$| p(x(t),y(t)) - v(t) | \leq$$
$$\sigma\left(\frac{n+ | \psi_1(t_0,x^0,y^0) | (n-1)}{t} + \frac{m+ | \psi_2(t_0,x^0,y^0) | (m-1)}{t}\right), \tag{5.6}$$

(b) it holds that

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} p(x(t),y(t)).$$

Proof. Statement (a) follows from Lemmas 5.1 and 5.4. The inequality (5.6) implies that

$$\lim_{t\to\infty} | p(x(t),y(t)) - v(t) | = 0.$$

The latter together with Lemma 5.3 prove (b). □

Theorem 5.2 *Let* Z^0 *be a connected set of uniqueness for initial time* t_0, $D \subset S_{n-1}\times S_{m-1}$ *be closed, contain all central paths starting in* Z^0 *at* t_0, *and* $V(D)$ *be finite. Then there exists a* $v \in V(D)$ *such that for all central paths* $(x(\cdot),y(\cdot))$ *starting in* Z^0 *at* t_0 *it holds that*

$$\lim_{t\to\infty} p(x(t),y(t)) = v. \tag{5.7}$$

Proof. Assume the contrary. In view of Lemma 5.3, we conclude that there are two central paths, $(x_1(\cdot),y_1(\cdot))$ and $(x_1(\cdot),y_1(\cdot))$, starting at t_0, respectively, from certain $(x_1^0,y_1^0) \in Z^0$ and $(x_2^0,y_2^0) \in Z^0$, which converge to different points,

$$w_1 = \lim_{t\to\infty} p(x_1(t),y_1(t)) \in V, \tag{5.8}$$

$$w_2 = \lim_{t\to\infty} p(x_2(t),y_2(t)) \in V, \tag{5.9}$$

$$w_1 \neq w_2. \tag{5.10}$$

Let $\lambda \mapsto (x^0(\lambda),y^0(\lambda)) : [0,1] \mapsto Z^0$ be a continuous function such that $(x^0(0),y^0(0)) = (x_1^0,y_1^0)$ and $(x^0(1),y^0(1)) = (x_2^0,y_2^0)$. Since Z^0 is connected, such a function exists. So far as $(x^0(\lambda), y^0(\lambda))$ lies in the interior of $S_{n-1} \times S_{m-1}$ for all $\lambda \in [0,1]$, there exists a $K > 0$ such that

$$| \psi_1(t_0, (x^0(\lambda), y^0(\lambda)) |< K, \quad | \psi_2(t_0, (x^0(\lambda), y^0(\lambda)) |< K$$

for all $\lambda \in [0,1]$. Consequently, there is a $t_1 \geq t_0$ such that the right hand side of (5.6), with $(x^0, y^0) = (x^0(\lambda), y^0(\lambda))$ for an arbitrary $\lambda \in [0,1]$, is smaller than $\nu/4$; here ν is defined by (5.4). By Lemma 5.5 we conclude that for every $\lambda \in [0,1]$ and every $t \geq t_1$ there is a $v(t,\lambda) \in V$ such that

$$| p(x(t,\lambda), y(t,\lambda)) - v(t,\lambda) |< \nu/4, \tag{5.11}$$

and

$$\lim_{\tau\to\infty} p(x(\tau,\lambda), y(\tau,\lambda)) = \lim_{\tau\to\infty} v(\tau,\lambda); \tag{5.12}$$

here $(x(\cdot,\lambda), y(\cdot,\lambda))$ is the central path starting at t_0 from $x^0(\lambda), y^0(\lambda))$. Take a $t \geq t_1$. Due to the continuity of the function $\lambda \mapsto (x(t,\lambda), y(t,\lambda))$ (ensured by Lemma 4.1, (c)), there exists an $\epsilon > 0$ such that for every $\lambda \in [0,1]$ and all $\mu \in [0,1]$ from $B(\lambda,\epsilon)$, the open ϵ-neighborhood of λ, it holds that

$$| p(x(t,\mu), y(t,\mu) - p(x(t,\lambda), y(t,\lambda) |< \nu/4.$$

The latter implies that

$$| v(t,\mu) - v(t,\lambda) |< 3\nu/4,$$

which is equivalent to

$$v(t,\mu) = v(t,\lambda)$$

(see (5.4)). Building a finite family of neighborhoods $B(\lambda_j,\epsilon)$, $\lambda_j \in [0,1]$, $j = 1,...,k$, which covers $[0,1]$,

$$\cup_{j=1}^{k}(B(\lambda_j,\epsilon)) = [0,1],$$

we easily deduce that

$$v(t,\lambda) = v(t,0)$$

for all $\lambda \in [0,1]$. In particular, we have

$$v(t,1) = v(t,0).$$

Now we take into account the arbitrariness of t, and referring to (5.12) obtain that

$$\lim_{t\to\infty} p(x(t,1), y(t,1)) = \lim_{t\to\infty} p(x(t,0), y(t,0)).$$

The latter, in view of (5.8), (5.9), is equivalent to $w_1 = w_2$, which contradicts to (5.10). The contradiction completes the proof. □

Our general equilibrium selection theorem is justified by Theorem 5.2.

Theorem 5.3 *Let Z^0 be a connected set of uniqueness for initial time t_0, $D \subset S_{n-1} \times S_{m-1}$ be closed, contain the central paths starting in Z^0 at t_0, and for every different (x_1, y_1), $(x_2, y_2) \in N \cap D$ it holds that $p(x_1, y_1) \neq p(x_2, y_2)$. Then*

(a) Z^0 selects at time t_0 a certain $(\bar{x}, \bar{y}) \in N \cap D$, that is, for all central paths $(x(\cdot), y(\cdot))$ starting in Z^0 at t_0 it holds that

$$\lim_{t\to\infty}(x(t), y(t)) = (\bar{x}, \bar{y}); \tag{5.13}$$

(b) if $N \cap Z^0$ is nonempty, then $N \cap Z^0 = \{(\bar{x}, \bar{y})\}$.

Proof. By Theorem 5.2 there exists a $v \in V$ such that for every $(x^0, y^0) \in Z^0$ the central path $(x(\cdot), y(\cdot))$ starting at t_0 from (x^0, y^0) satisfies the equality (5.7). By assumption there is a single $(\bar{x}, \bar{y}) \in N$ such that $p(\bar{x}, \bar{y}) = v$. Assume that (5.13) is violated. Referring to Theorem 5.1, we conclude that for certain $(x^0, y^0) \in Z^0$ and $t_i \to \infty$ it holds that

$$\lim_{t \to \infty}(x(t), y(t)) = (\hat{x}, \hat{y}) \in N, \quad (\hat{x}, \hat{y}) \neq (\bar{x}, \bar{y}).$$

By assumption $p(\hat{x}, \hat{y}) \neq v$. We obtained a contradiction with (5.7). Conjecture (a) is proved. Prove (b). Assume that (b) is untrue, that is, there is a $(\hat{x}, \hat{y}) \in N \cap Z^0$ different from $(\bar{x}, \bar{y})$. Let $(x(\cdot), y(\cdot))$ be the central path starting at t_0 from $(\hat{x}, \hat{y})$. Necessarily $(x(t), y(t)) = (\hat{x}, \hat{y})$. Hence

$$\lim_{t \to \infty} p(x(t), y(t)) = p(\hat{x}, \hat{y}) \neq v,$$

which contradicts to (5.7). □

In the remaining part of this section we consider a regular case characterized by the folowing

Condition 5.1 (*Regularity*). For all $t \geq 0$ and all $(x, y) \in$ int $S_{n-1} \times$ int S_{m-1} the matrix $H(t, y, z)$ is invertible; equivalently, $\mathcal{H} = [0, \infty) \times$ int $S_{n-1} \times$ int S_{m-1}.

Corollary 5.2 *Let Condition 5.1 be satisfied. Then* int $S_{n-1} \times$ int S_{m-1} *is a set of uniqueness for every initial time* $t_0 \geq 0$.

Proof. Take an arbitrary $(x^0, y^0) \in$ int $S_{n-1} \times$ int S_{m-1}. By Lemma 4.1 it is sufficient to state that there is a single solution of the Cauchy problem (2.13), (2.16). Since $H^{-1}(\cdot)$ is Lipschitz in a neighborhood of (x^0, y^0), there is an interval, on which a solution is defined, and at each interval of solution existence a solution is unique. Let $[t_0, t_*)$ $(t_* \leq \infty)$ be a maixium interval where a solution of (2.13), (2.16) is defined. Denote this solution by $(x(\cdot), y(\cdot))$. We must show that $t_* = \infty$. Assume, to the contrary, that $t_* < \infty$. So far as $(x(t), y(t))$ satisfies the central path equations (2.11), (2.12) for all $t \in [t_0, t_*)$, by Lemma 2.1 we conclude that there is a $\kappa > 0$ such that $x_i(t) \geq \kappa$, $1 - \sum_{i=1}^{n-1} x_i(t) \geq \kappa$, $y_j(t) \geq \kappa$, $1 - \sum_{j=1}^{m-1} y_j(t) \geq \kappa$ for all $t \in [0, t_*]$. Since $(x(\cdot), y(\cdot))$ solves the differential equation (2.13), $\dot{x}(\cdot)$ and $\dot{y}(\cdot)$ are bounded in a left neighborhood of t_*. Consequently, $(x(\cdot), y(\cdot))$ is Lipschitz in this neighborhood. Therefore there is the limit $(x_*, y_*) = \lim_{t \to 0+}(x(t), y(t))$ belonging obviously to the interior of $S_{n-1} \times S_{m-1}$. Then in a right neighborhood of t_* there exists a solution $(x_*(\cdot), y_*(\cdot))$ of the equation (2.13) satisfying the initial condition $(x_*(t_*), y_*(t_*) = (x_*, y_*)$. Thus the solution $(x(\cdot), y(\cdot))$ of the Cauchy problem (2.13), (2.16) can be extended to the right of t_*, which contradicts the definition of t_*. □

Let us give several simple conditions sufficient for Condition 5.1.

Lemma 5.6 *Let*

$$s^T \begin{pmatrix} 0 & A^T \\ B & 0 \end{pmatrix} s \leq 0 \tag{5.14}$$

for all $s \in R^{n+m-2}$. *Then Regularity Condition 5.1 is satisfied.*

Proof. For an arbitrary $(t, x, y) \in [0, \infty) \times \text{int } S_{n-1} \times \text{int } S_{m-1}$, and a nonzero $s \in R^{n+m-2}$ we have

$$\begin{aligned} s^T H(t, x, y) s &= s^T \begin{pmatrix} -D(x) & 0 \\ 0 & -D(y) \end{pmatrix} s + s^T \begin{pmatrix} 0 & tA^T \\ tB & 0 \end{pmatrix} s \\ &< s^T \begin{pmatrix} 0 & tA^T \\ tB & 0 \end{pmatrix} s < 0, \end{aligned}$$

implying $H(t,x,y)s \neq 0$. In view of the arbitrariness of s, we have that $H(t,x,y)$ is invertible. □

Corollary 5.3 *If $B = -A$, then Regularity Condition 5.1 is satisfied.*

Proof. For every $s = (s_1, s_2) \in R^{n-1} \times R^{m-1}$ we have

$$s^T \begin{pmatrix} 0 & A^T \\ B & 0 \end{pmatrix} s = s_1^T A^T s_2 + s_2^T B s_1 = (As_1)^T s_2 - (As_1)^T s_2 = 0,$$

that is, (5.14) holds. Now apply Lemma 5.6. □

Remark 5.1 If the initial bimatrix game is zero sum, that is, $B_0 = -A_0$, then we have $B = -A$; the inverse is, generally, untrue.

We shall call a matrix *column disjunct* if its every row has a single nonzero element, and *sign constant in columns* if all its nonzero elements in a same column have a same sign. The matrices A and B will be called *identically column disjunct* if they are column disjunct and their nonzero elements are placed identically. The matrices A and B will be said to be *sign different* if their corresponding elements either have different signs, or one of them vanishes.

Lemma 5.7 *Let A and B be identically column disjunct and sign different. Then Regularity Condition 5.1 is satisfied.*

Proof. Take arbitrary $t \geq 0$, $(x,y) \in \text{int } S_{n-1} \times \text{int } S_{m-1}$. Suppose that $H(t,x,y)$ is not invertible. Then the linear combination of its rows with some coefficients, not all of which vanish, is zero. Denote these coefficients corresponding to rows 1,..., $n-1$, n, ... , $n+m-2$, respectively, by β_1, ..., β_{n-1}, γ_1, ..., γ_{m-1}. Consider the submatrices

$$H_B(t,x,y) = \begin{pmatrix} -D(x) \\ tB \end{pmatrix}, \quad H_A(t,x,y) = \begin{pmatrix} tA^T \\ -D(y) \end{pmatrix}.$$

(see (2.14)). In $H_A(t,x,y)$ the lower $(m-1) \times (m-1)$ matrice, $-D(y)$ (see (2.15)), is diagonal with negative diagonal elements. Hence it cannot be that $\beta_j = 0$ for all j. Let J be the (nonempty) set of all j such that $\beta_j \neq 0$. Take an $j \in J$ and consider the jth column in the matrix $H_B(t,x,y)$. All nonzero elements in the jth column of B have a same sign. Let them be positive (the opposite case is treated similarly). The single nonzero element $d_j(x)$ in the jth column of $-D(x)$ lies in the jth row. The sum of all elements of the jth column of tB with the coefficients γ_1, ..., γ_{m-1} plus $\beta_j d_i(x)$ is zero. Since $d_j(x)$ is negative and all elements of the jth column of B are nonnegative, there is a i such that $\gamma_i \neq 0$, its sign coincides with that of β_j, and $b > 0$ where b is the ith element in the jth column of B. Now consider the ith column of $H_A(t,x,y)$. The ith column of tA^T coincides with the transposed ith row of tA. Since A is column disjunct, this row contains a single nonzero element a The latter, as long as A and B are identically column disjunct, is placed like the single nonzero element in the ith row of B, that is, b. Consequently, a is placed on the jth column in A. Coming back to A^T, we conclude that the single nonzero element, a, in the ith column of A^T lies on the jth row of this matrice. In the ith column of $-D(y)$ the single nonzero element $d_i(y)$ belongs to the ith row. Therefore $\beta_j a + \gamma_i d_i(y) = 0$. So far as A and B are sign different and $b > 0$, we have $a < 0$. Furtheremore, obviously $d_i(y) < 0$. Consequently β_j and γ_i have the different signs. However, above we have obtained that their signs coincide. The contradiction completes the proof. □

Under Regularity Condition Theorem 5.3 obviously implies the following corollary. We shall say that an equilibrium $(\bar{x}, \bar{y})$ is *globally selected* if int $S_{n-1} \times$ int S_{m-1} selects $(\bar{x}, \bar{y})$ at every time $t_0 \geq 0$, that is, (5.13) holds for every central path $(x(\cdot), y(\cdot))$ starting at t_0 in int $S_{n-1} \times$ int S_{m-1}.

Corollary 5.4 *Let Regularity Condition 5.1 be satisfied, N be finite, and $p(x_1, y_1) \neq p(x_2, y_2)$ for every different $(x_1, y_1), (x_2, y_2) \in N$. Then*

(a) there is a unique globally selected equilibrium $(\bar{x}, \bar{y})$,

(b) the intersection $N \cap$ (int $S_{n-1} \times$ int S_{m-1}) is either empty, or contains the single element $(\bar{x}, \bar{y})$.

For the 2×2 case Corollary 5.4 and Lemma 5.6 imply a simple characterization of global equilibrium selection.

Corollary 5.5 *Let $n = n = 2$ and $AB \leq 0$ (see (3.4), (3.5), (3.1)). Then there is a unique globally selected equilibrium.*

An equilibrium lying in int $S_{n-1} \times$ int S_{m-1} will be called *interior*; an equilibrium which is not interior will be called *boundary*. The next simple examples show that a globally selected equilibrium can be interior or boundary (we refer to the notations (3.4), (3.5), (3.1)).

Example 5.1 Let

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We have $A = 0$, $B = -2$, $AB = 0$. The single equilibrium $(1/2, 1/2)$ is interior and by Corollary 5.5 globally selected.

Example 5.2 Let

$$A_0 = \begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 3 & 0 \\ 4 & 1 \end{pmatrix}$$

(a variant of the Prisoner's Dilemma). We have $A = B = 0$. The single equilibrium $(0, 0)$ is boundary and by Corollary 5.5 globally selected.

6 Selection of interior equilibrium

In this section we shall show that if matrices A and B are square and nondegenerate, and there is an interior equilibrium $(\bar{x}, \bar{y})$, then a set of uniqueness selecting $(\bar{x}, \bar{y})$ is arbitrarily close to the interior of $S_{n-1} \times S_{m-1}$ provided initial time t_0 is sufficiently large (Theorem 6.1) The proof is based on Theorem 5.3 and utilizes Lemma 4.1 and a criterion of viability.

Following Aubin, 1991, we call a set $F \in \mathcal{H}$ *viable* if for every $(t_0, x^0, y^0) \in F$ there exists a solution $(x(\cdot), y(\cdot))$ of the Cauchy problem (2.13), (2.16) defined on $[t_0, \infty)$ such that $(t, x(t), y(t)) \in F$ for all $t \in [t_0, \infty)$ (by Lemma 4.1 the above solution is unique). Lemma 4.1, (a), is obviously specified as follows.

Lemma 6.1 *Let $F = [t_0, \infty) \times Z^0 \in \mathcal{H}$ be viable. Then Z^0 is a set of uniqueness for initial time t_0 and contains the central paths starting in Z^0 at t_0.*

Let $G(t,x,y)$ stand for the right hand side of the equation (2.13). A standard viability criterion reads as follows (see Aubin, 1991, Theorems 1.2.1, 1.2.3).

Lemma 6.2 *A closed set $F \in \mathcal{H}$ is viable if and only if for every $(t,x,y) \in F$ it holds that*

$$(1, G(t,x,y)) \in T_F(t,x,y); \tag{6.1}$$

here $T_F(t,x,y)$ is the tangent cone to the set F at point (t,x,y),

$$T_F(t,x,y) =$$

$$\left\{(\tau,\xi) \in R^1 \times (R^{n-1} \times R^{m-1}) \;:\; \liminf_{\mu \to +0} \frac{d((t+\mu\tau,(x,y)+\mu\xi),F)}{\mu} = 0\right\}. \tag{6.2}$$

Lemma 6.3 *Let $n = m$, the matrices A and B be invertible, and there exists an interior equilibrium $(\bar{x},\bar{y})$. Then*

(a) it holds that

$$\bar{x} = -B^{-1}g_1, \quad \bar{y} = -(A^T)^{-1}h_2$$

(consequently, there are no other interior equilibria)

(b) for each $(t,x,y) \in \mathcal{H}$ the right hand side of the central path differential equation (2.13) is specified into

$$\begin{aligned} G(t,x,y) &= -H^{-1}(t,y,z)\begin{pmatrix} 0 & A^T \\ B & 0 \end{pmatrix}\begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix} \\ &= -t\begin{pmatrix} -D(x)/t & A^T \\ B & -D(y)/t \end{pmatrix}^{-1}\begin{pmatrix} 0 & A^T \\ B & 0 \end{pmatrix}\begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix}. \end{aligned} \tag{6.3}$$

Conjecture (a) follows from the definition of an equilibrium and the form of the payoffs $p_A(x,y)$, $p_B(x,y)$ (see (2.3), (2.4)); conjecture (b) is easily implied by (a).

Now we come back to Lemma 6.1 which provides a condition sufficient for Z^0 to be a set of uniqueness. Using Lemmas 6.2 and 6.3, we shall verify this condition for

$$Z^0 = \{(x,y) \in S_{n-1} \times S_{m-1} \;:\; \epsilon \le x^{(i)} \le 1-\epsilon,\ \epsilon \le y^{(j)} \le 1-\epsilon,$$

$$(1 \le i \le n-1,\ 1 \le j \le m-1)\}, \tag{6.4}$$

where $\epsilon > 0$. We see that as ϵ is sufficiently small, Z^0 covers "almost" the whole product $\mathrm{int}\, S_{n-1} \times \mathrm{int}\, S_{m-1}$.

Lemma 6.4 *Let the conditions of Lemma 6.3 be fulfilled, $(\bar{x},\bar{y})$ be the interior equilibrium, $\zeta > 0$ be such that*

$$\zeta \le \bar{x}^{(i)} \le 1-\zeta, \quad \zeta \le \bar{y}^{(j)} \le 1-\zeta \quad (1 \le i \le n-1, \quad 1 \le j \le m-1), \tag{6.5}$$

$$\epsilon \in (0, \min\{1/(n-1), \zeta/2\},$$

and Z^0 be defined by (6.4). Then there exists a $t_* \ge 0$ such that for every $t_0 \ge t_*$ the set $F = [t_0,\infty) \times Z^0$ lies in $\mathcal{H}$ and is viable.

Proof. The inequalities $1/(n-1) > \epsilon > 0$ imply that Z^0 lies in the interior of $S_{n-1} \times S_{m-1}$. Taking into account (2.15) we easily obtain that $D(x)$ and $D(y)$ are bounded on Z^0,

$$| D(x) | \leq K, \quad | D(y) | \leq K \quad ((x,y) \in Z^0)$$

for a certain constant K. Hence, for all t greater than a sufficiently large t_* we have

$$\left| \begin{pmatrix} -D(x)/t & A^T \\ B & -D(y)/t \end{pmatrix} - \begin{pmatrix} 0 & A^T \\ B & 0 \end{pmatrix} \right| < \delta \quad ((x,y) \in Z^0), \tag{6.6}$$

where δ is an arbitrarily chosen positive value. The second matrix on the left hand side is nondegenerate, since A and B are such by assumption. Therefore the first matrix on the left is nondegenerate too provided δ is sufficiently small. Assuming the latter, we conclude that $H(t,y,z)$ (see (2.14)) is nondegenerate for all $t \geq t_*$ and $(x,y) \in Z^0$. Consequently, $F \subset \mathcal{H}$ if $t_0 \geq t_*$. It remains to show that F is viable. Without loss of generality, we assume that for $t \geq t_*$ the matrices inverse to those from (6.6) are δ-close,

$$\left| \begin{pmatrix} -D(x)/t & A^T \\ B & -D(y)/t \end{pmatrix}^{-1} - \begin{pmatrix} 0 & (A^T)^{-1} \\ B^{-1} & 0 \end{pmatrix} \right| < \delta \quad ((x,y) \in Z^0).$$

The latter together with (6.3) imply that

$$G(t,x,y) = -t \left[\begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix} + \sigma(t,x,y) \right] \quad (t \geq t_*,\ (x,y) \in Z^0), \tag{6.7}$$

where with no loss of generality we assume

$$|\sigma(t,x,y)| \leq \delta \quad (t \geq t_*,\ (x,y) \in Z^0). \tag{6.8}$$

Now we shall verify the viability criterion (6.1). Take a $(t,x,y) \in F$ and assume $t_0 \geq t_*$. Let, first, $(x,y) \in \text{int } Z^0$. Then obviously $T_F(t,x,y) = \{1\} \times R^{n-1} \times R^{m-1}$ (see (6.2)), and the criterion (6.1) is satisfied. Let (x,y) belong to the boundary of Z^0. From (6.5) and (6.4) we easily deduce that the Euclidean $\zeta/(n-1)$-neighborhood of $(\bar{x},\bar{y})$ lies in Z^0. In other words,

$$(x + (\bar{x} - x)y + (\bar{y} - y)) + \sigma \in Z^0$$

for all $\sigma \in R^{n-1} \times R^{m-1}$ such that $| \sigma | < \zeta/(n-1)$. Assuming with no loss of generality that $\delta < \zeta/(n-1)$ and taking into account (6.7) and (6.8), we obtain that, in particular,

$$(x,y) + G(t,x,y)/t \in Z^0.$$

So far as Z^0 is, obviously, convex and $(x,y) \in Z^0$, we have

$$(x,y) + \mu G(t,x,y)/t \in Z^0 \quad (\mu \in [0,1]).$$

Consequently

$$(t,x,y) + (\mu, \mu G(t,x,y)/t) \in [t,\infty) \times Z^0 \subset F \quad (\mu \in [0,1]),$$

or

$$d((t,(x,y)) + \mu(1, G(t,x,y)), F) = 0 \quad (\mu \in [0,1])$$

yielding the desired inclusion (6.1). □

We are ready to formulate our final result on selection of the interior equilibrium.

Theorem 6.1 *Let $n = m$, the matrices A and B be invertible, there exist an interior equilibrium $(\bar{x}, \bar{y})$, and $\zeta > 0$ satisfy (6.5). Then for every $\epsilon \in (0, \min\{1/(n-1), \zeta/2\}$ there exists a $t_* \geq 0$ such that for every $t_0 \geq t_*$ the set Z^0 defined by (6.4) selects the interior equilibrium $(\bar{x}, \bar{y})$ at time t_0.*

Proof. By Lemma 6.4 there exists a $t_* \geq 0$ such that for every $t_0 \geq t_*$ the set $F = [t_0, \infty) \times Z^0$ lies in $\mathcal{H}$ and is viable. Therefore by Lemma 6.1 Z^0 is a set of uniqueness for above t_0. This set is convex and cosequently connected. Since F is viable, Z^0 contains the central paths starting in Z^0 at t_0. Hence by Theorem 5.3 Z^0 selects at time t_0 a certain Nash equilibrium belonging to $N \cap Z^0$. So far as Z^0 lies in the interior of $S_{n-1} \times S_{m-1}$, and by Lemma 6.3 $(\bar{x}, \bar{y})$ is a single interior equilibrium, $N \cap Z^0$ contains the single element $(\bar{x}, \bar{y})$. Consequently Z^0 selects $(\bar{x}, \bar{y})$ at time t_0. □

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