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A DECISION ANALYTIC APPROACH TO RIVER BASIN POLLUTION CONTROL

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Summary

Often several cities discharge their sewage into a single river. Such cities are concerned with the water quality of the river as it flows past them and with the costs associated with treatment. A utility function for each city describes the trade-offs. Preferences often conflict: the cities downstream wish that those upstream would treat their waste more intensively. Ambient conditions in the river are uncertain but can be expressed in terms of probability densities.

The paper describes approaches to finding admissible solutions to the multi-interest-group water resource problem using Paretian analysis and optimal control theory. An example is solved, and extensions are discussed.

Introduction

Water flowing through a river often serves several purposes, and several decision makers affect its quality. For instance, several cities may discharge their processed sewage into the same reach of the river. Each city acts in its own interest, often to the detriment of those downstream; when waste handling capacity decreases, or when water quality deteriorates, conflicts result. The formation of regional water resource authorities is one attempt to coordinate water quality management, but such agencies often become embroiled in questions of basin-wide efficiency (Clough and Bayer [2]) and ignore the conflicting preferences of their constituency. When regulatory agencies make decisions, they are faced with:

1) a number of interest groups,

2) the multiple objectives of these interest groups,

3) limited information on how different decisions affect the groups,

4) uncertainties (e.g. stream flow, treatment plant efficiencies, effluent discharges).

A host of administrative, regulatory, judicial, and advisory bodies are involved in the control of aqueous pollutants. They--along with federal agencies, cities, and various private and public intervenors--participate in a complex decision-making process with a highly controversial set of alternatives. Yet regional and local agencies often have a severely limited capacity to conduct their own analyses of many of the issues they are called upon to decide.

This is not to say that no analytic techniques for political decision making are available. There has been much recent interest in models of political conflict, as expressed in the use of such tools as metagame theory (Hipel, Ragade, and Unny [8]) and vote trading (Haefele [7]); yet there are remarkably few economic techniques for modeling the political decision-making process. Consequently, many researchers use Paretian analysis to gain insight into the outcomes of primary interest.

A decision is said to be Pareto-admissible if it is feasible from a technological viewpoint and if there is no other decision that is preferred by some interest group(s) and is not less preferred by any other. In other words, a change from a Pareto-admissible decision that makes one interest group better off would (by definition) make another interest group worse off. The Paretian model is used to generate the set of Pareto-admissible outcomes; political, economic, and technological constraints limit the set of feasible alternatives.

Paretian analysis resembles what is called "multi-objective planning" in the literature on applied micro-economics and cost-benefit analysis. The two types of analysis may be similar mathematically, but differ in that conventional multi-objective planning usually considers trade-offs among abstractly defined social objectives (Cohon and Marks [3], Monarchi, Kisiel and Duckstein [11], and Vemuri [16]; whereas Paretian analysis considers the interest groups actually involved in trying to influence a public decision, or affected by it, and estimates the preference orderings of these interest groups.

Dorfman and Jacoby [4] constructed a Paretian model of water pollution control of a hypothetical river basin where municipal, industrial, and governmental interest groups influence the final decision. In their model all the important ambient conditions, such as stream flow, were deterministic. Schaumberg [14] has applied the Paretian model to the decision-making process that determines the ambient and effluent air quality standards for Syracuse, New York; in this study, wind speed and direction was determined by a discrete probability density. In both studies, functions were linearized so that linear programming optimization routines could be used. Further, in order to convert all costs and benefits into monetary terms, "alternative cost" and "willingness to pay" techniques were used. Gros [6] used a multi-attribute utility function to estimate the preferences of each interest group involved in a nuclear power plant siting problem. In addition, he studied the effect of uncertainty in the regulatory process on the final power plant siting decision.

The present analysis extends the previous efforts. We consider once again the problem of several cities discharging sewage into a single river. The cities have multiple objectives; multi-attribute utility functions express their trade-offs among them. Stream flows are uncertain and are given by continuous probability density functions. There is no need to linearize the technological relations, because optimal control techniques are used to determine the admissible decisions. The resulting aqueous waste control strategy for each city depends upon current flow rates and pollution conditions, on the probability of these states occurring, and on people's preferences.

This is not the first study to apply optimal control techniques to water resource problems (see, for example, Young and Beck [17]). Typically, such research has been concerned with finding abatement control trajectories that deviate as little as possible from a quality standard chosen a priori, and has tended to ignore the preferences of each of the groups involved. This analysis attempts to include these preferences explicitly.

Problem Formulation

Consider n cities in a region that discharge their treated sewage into a river (see Figure 1). City i discharges wastes y_i into the river and monitors local water quality in terms of dissolved oxygen w_i and biological oxygen demand x_i. Water quality (w_{n+1}, x_{n+1}) is measured by the regional water quality authority as the river flows out of its jurisdiction. The ambient river conditions can be described by a quantity q, which has daily and seasonal periodicity, whose probability density function p(q) is known.

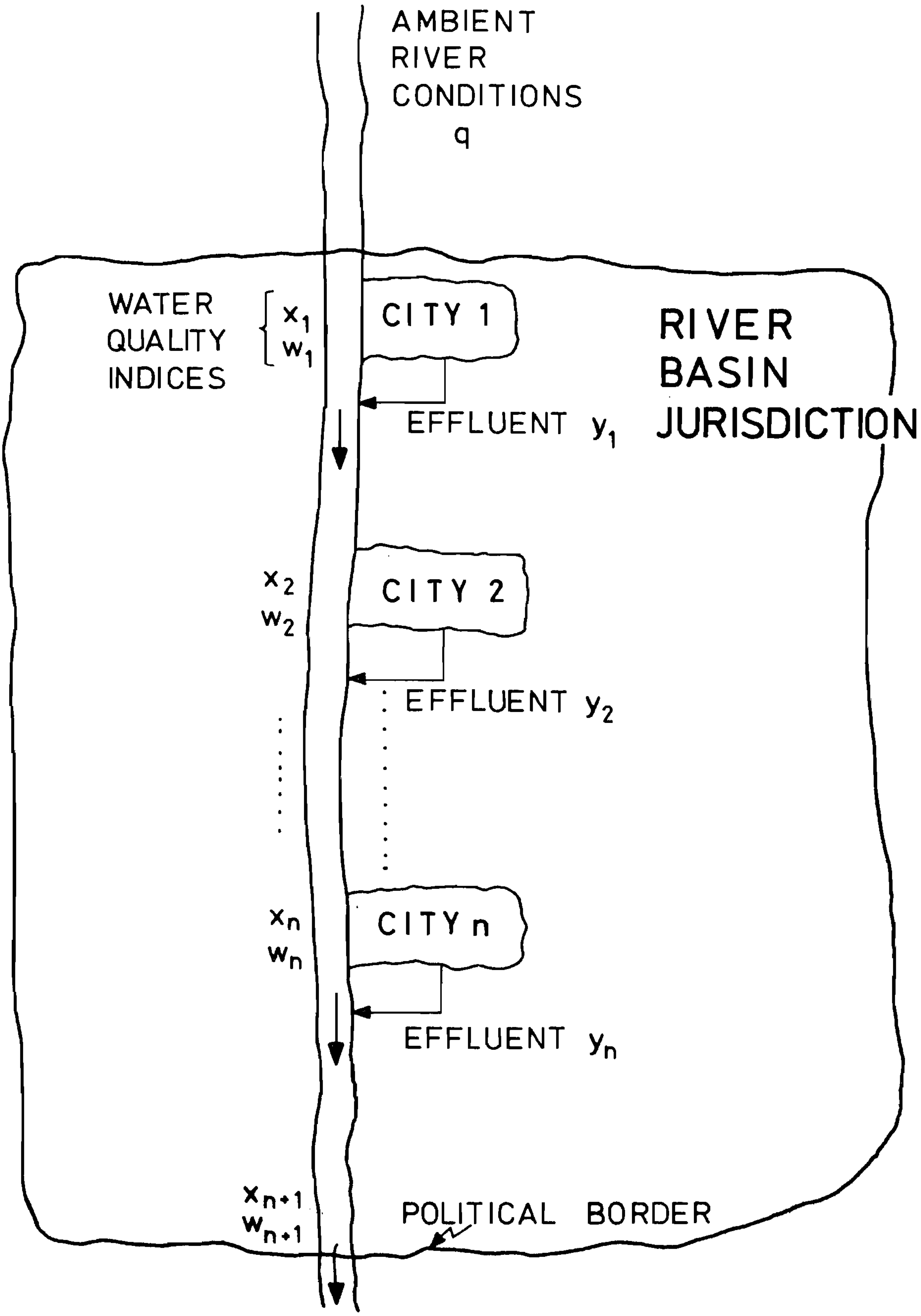

FIGURE 1. RIVER BASIN WITH n POLLUTERS

The regional authority is responsible for advising the cities on waste treatment policies to maintain reasonable water quality at each city and at the border. These cities, however, have different preferences for the quality of the water as it flows past them and for money they spend for treatment, and the authority wishes to take these into account when recommending treatment policies. Let $u_i[w_i, x_i, z_i(y_i)]$ be the utility function value for city i for water quality values of w_i and x_i at the city, given that the city pays $z_i(y_i)$ to limit its discharge to y_i tons of BOD.[1] This utility function describes the relative preferences of having different values of w_i, x_i, and $z_i(y_i)$. In addition, for situations involving uncertainty, the expected value of the utility function is the appropriate guide for decision making (von Neumann and Morgenstern [12], and Maass [10]). Therefore, under stochastic river conditions, and where the treatment level depends on river conditions, city i wants $U_i = \int u_i[w_i, x_i, z_i(y_i)]\ p(q)\ dq$ to be maximized. (The greater the expected value, the more preferred the situation.) Since there is no treatment at the border (and hence no cost) the utility function at the border is $u_{n+1}(w_{n+1}, x_{n+1})$.

Let us consider the relations that describe BOD and DO levels along the river. The values at city 1 depend only on ambient conditions and can be expressed as:

$$x_1 = c_1(q)$$

$$w_1 = d_1(q) \quad .$$

We are assuming that q describes whatever needs to be known about the ambient conditions. In some problems q can be a value describing flow rate, DO level and BOD level above city 1. If these levels are functions of flow rate, then q represents just the flow rate, the other levels being given by functions of the flow rate.

The relations that describe BOD and DO levels elsewhere along the river depend on the treatment upstream. The value of BOD at city i + 1, x_{i+1}, and of DO at city i + 1, w_{i+1}, can be expressed as follows:

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[1] In the formulation we ignore water treatment before use as an alternative to waste water treatment and side payments between cities.

$$x_{i+1} = c_{i+1}(x_i, y_i, q)$$

$$w_{i+1} = d_{i+1}(x_i, w_i, y_i, q) \quad .$$

These last two equations have the form of the classical Streeter-Phelps equations [15] when the simple transformation from flow rate q to flow time between cities is made. It is understood that these equations can represent more accurate relationships than the Streeter-Phelps by simply adding to their arguments. For later analytic convenience let us write these relationships such that the functions equal zero:

$$C_{i+1}(x_i, x_{i+1}, y_i, q) = c_{i+1}(x_i, y_i, q) - x_{i+1} = 0$$

$$D_{i+1}(x_i, w_i, w_{i+1}, y_i, q) = d_{i+1}(x_i, w_i, y_i, q) - w_{i+1} = 0 \quad .$$

In summary, each city wishes to maximize the expected value of its utility, $U_i = \int u_i[w_i, x_i, z_i(y_i)]\ p(q)\ dq$. Each city can change only its treatment level y_i and try to influence upstream cities to treat their sewage more intensively. This is subject to constraints $C_i = 0$ and $D_i = 0$ which describe the conditions at downstream points. Pareto-admissible decisions can be found as follows.

Let us constrain n of the n + 1 expected values of utility functions to some arbitrary value. Let the value for city j be the unconstrained one; the constrained values are written:

$$U_i = R_i \quad , \qquad i = 1,2,\ldots,j-1,j+1,\ldots,n+1 \quad .$$

The mathematical problem of finding Pareto-admissible decisions can be written:

maximize U_j

subject to

$$U_i = R \quad , \qquad i = 1,2,\ldots,j-1,j+1,\ldots,n+1$$

$$C_i = 0 \quad , \qquad i = 1,\ldots,n+1$$

$$D_i = 0 \quad , \qquad i = 1,\ldots,n+1 \quad .$$

To find a solution, optimal control theory can be used. A Hamiltonian can be defined as follows:

$$H = \sum_{i=1}^{n+1} \gamma_i u_i[w_i, x_i, z_i(y_i)]\; p(q)$$

$$+ \sum_{i=0}^{n} \lambda_{i+1} C_{i+1}(x_i, x_{i+1}, y_i, q)$$

$$+ \sum_{i=0}^{n} \mu_{i+1} D_{i+1}(x_i, w_i, w_{i+1}, y_i, q) \quad ,$$

where λ_{i+1} and μ_{i+1} are multipliers; the γ_i's are constants such that $\gamma_j = 1$ and $\gamma_i (i \neq j)$ are included, so that $U_i = R_i$. Of course, $z_{n+1} = 0$, $w_0 = 0$, $x_0 = 0$, and $y_0 = 0$, since these variables do not exist in the model. The necessary conditions[2] for the maximum (Bryson and Ho [1]) are:

$$\frac{\partial H}{\partial y_i} = 0 \quad , \qquad i = 1,\ldots,n$$

$$\frac{\partial H}{\partial x_i} = 0 \quad , \qquad i = 1,\ldots,n+1$$

$$\frac{\partial H}{\partial w_i} = 0 \quad , \qquad i = 1,\ldots,n+1$$

subject to

$$C_i = 0 \quad , \qquad i = 1,\ldots,n+1$$

$$D_i = 0 \quad , \qquad i = 1,\ldots,n+1$$

$$U_i = R_i \quad , \qquad i \neq j \ .$$

Of course, varying the values of R_i changes the solution.

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[2]The sufficient conditions, assuming H is a convex function of x, w, z and u, are: $\partial^2 H/\partial y_i^2 < 0$, $i = 1,\ldots,n$; $\partial^2 H/\partial x_i^2 < 0$, $\partial^2 H/\partial w_i^2 < 0$, $i = 1,\ldots,n+1$.

The solutions to these equations are a set of pollution control strategies, y_i's, which depend on ambient stream conditions q, the probability of a particular condition occurring p(q), the preferences u_i, and the pollution control of upstream cities. By varying the values of R_i, the complete set of Pareto-admissible solutions can be obtained. These solutions can be used in three ways: descriptively, predictively, and prescriptively. When used descriptively, Paretian analysis attempts to identify the benefits accruing to each interest group involved in the decision-making process, and to elucidate the trade-offs among them. The use of utility functions to quantify the benefits to each interest group provides a formal method of analyzing the crucial trade-off issues and considerations of uncertainty involved in any pollution control decision. Paretian analysis also provides a structure for comparing the benefits to the different interest groups and identifies the conflicts of interest among them.

As a predictive tool the Paretian model is used to generate a set of decisions that includes the ultimate decision. In the prescriptive mode, the model determines a set of Pareto-admissible alternatives that are then introduced into the decision-making process. The analysis offers a set of efficient plans from which the decision-making group can choose. The advantage of the model is that it forces the analyst to consider the benefit to each interest group. Alternatively, after a decision has been made the analyst can check whether it is admissible, and, if it is not, suggest an admissible decision that would make at least one interest group better off without making another worse off.

Example

Consider the river basin shown in Figure 2, with two cities that discharge wastes y_1 and y_2 into the river. Let us assume, for simplicity in this discussion, that BOD level is the only environmental indicator of interest; each city monitors the BOD level of the river as it flows past, and the regional water quality authority measures the BOD level at the border.

The river has a uniform distribution of stream flow in the month considered. The maximum flow is 8.1×10^6 m^3/day and the minimum is 4.05×10^6 m^3/day. There is an effluent discharge upstream which results in a constant ambient load of 2.7 tons BOD/day at city 1. City 2, with a population of 100,000, is 32.4 km downstream from city 1 (which also has a population of 100,000) and 40 km from the downstream border. The average cross-sectional area of the river is 125 m^2.

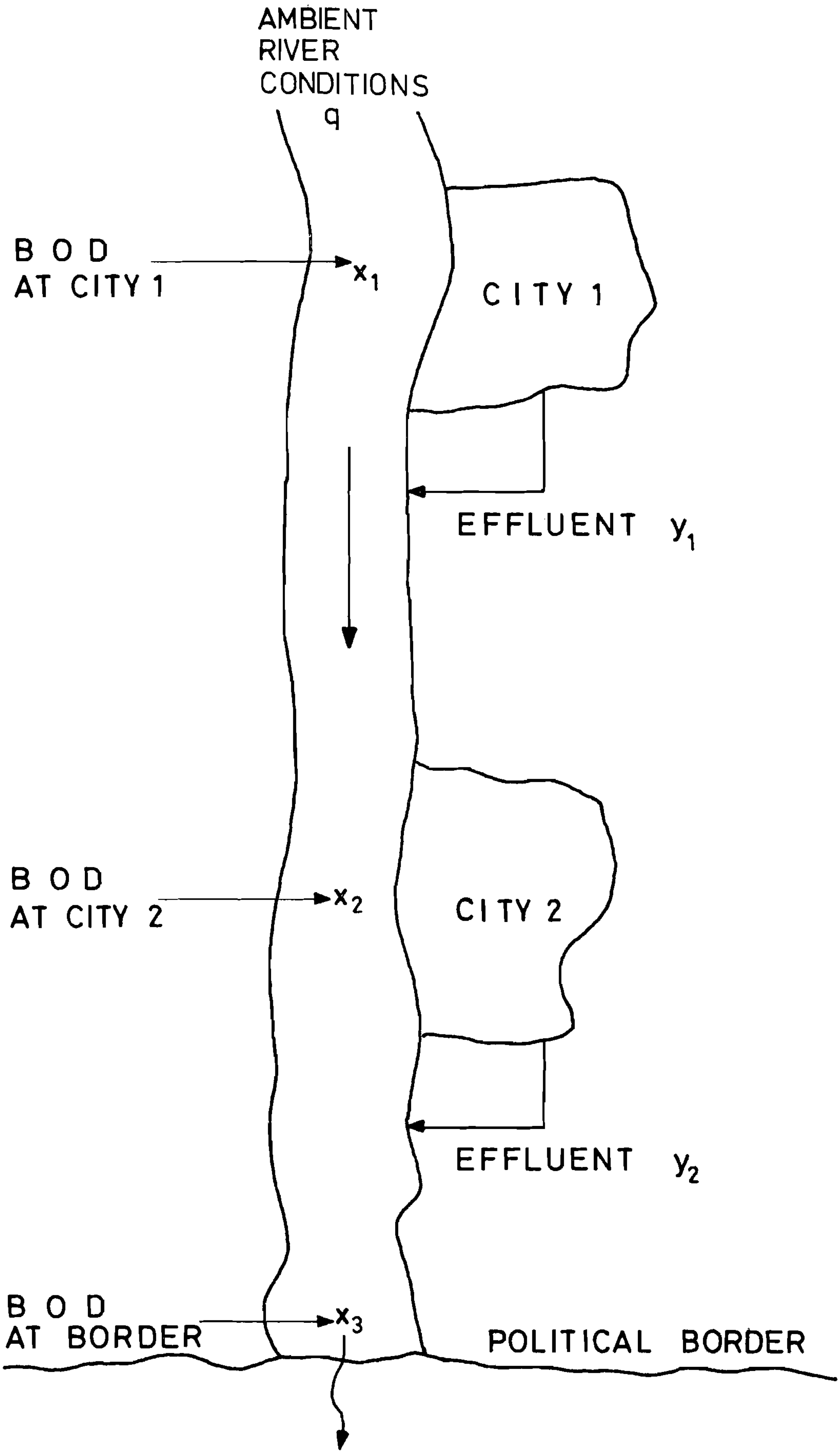

FIGURE 2. EXAMPLE PROBLEM WITH TWO CITIES AND POLITICAL BORDER

The classical Streeter-Phelps equations [15] can be solved to yield values of x_2 and x_3, the BOD concentrations at city 2 and the border, as functions of the ambient BOD levels, flow rate, and discharges y_1 and y_2. We can relate flow rate q to time taken for a unit volume to travel between the cities and the border in the following way:

$$q = \ell_2 A_2 / t_2 = \ell_3 A_3 / t_3 \quad ,$$

where

ℓ_2 is the distance, in metres, between city 1 and city 2,

t_2 is the time, in days, a unit volume takes to travel between city 1 and city 2,

A_2 is the average cross-sectional area, in m^2, of the river between city 1 and city 2,

with quantities subscripted 3 for the river between city 2 and the border.

The relationship of BOD at one point to BOD at a previous point can be written:

$$C_2 = (x_1 + y_1/q) \exp(-k\ell_2 A_2/q) - x_2 = 0$$

$$C_3 = (x_2 + y_2/q) \exp(-k\ell_3 A_3/q) - x_3 = 0 \quad ,$$

where

$$C_1 = c_1(q) - x_1 = 0$$

and k is the rate constant for BOD decay.

We assume that each city has a tertiary waste treatment plant. (In the next section we will show how the model can be used to decide on plant type and cost.) Figure 3 shows, for each city i, the relationship between operational treatment costs and discharge used in our example:

$$z_i(y_i) = \alpha y_i^{-\beta} \quad ,$$

with $\alpha = \$219 \times 10^6$, $\beta = 1.0$. The cost function is consistent with increasing marginal costs as treatment improves.

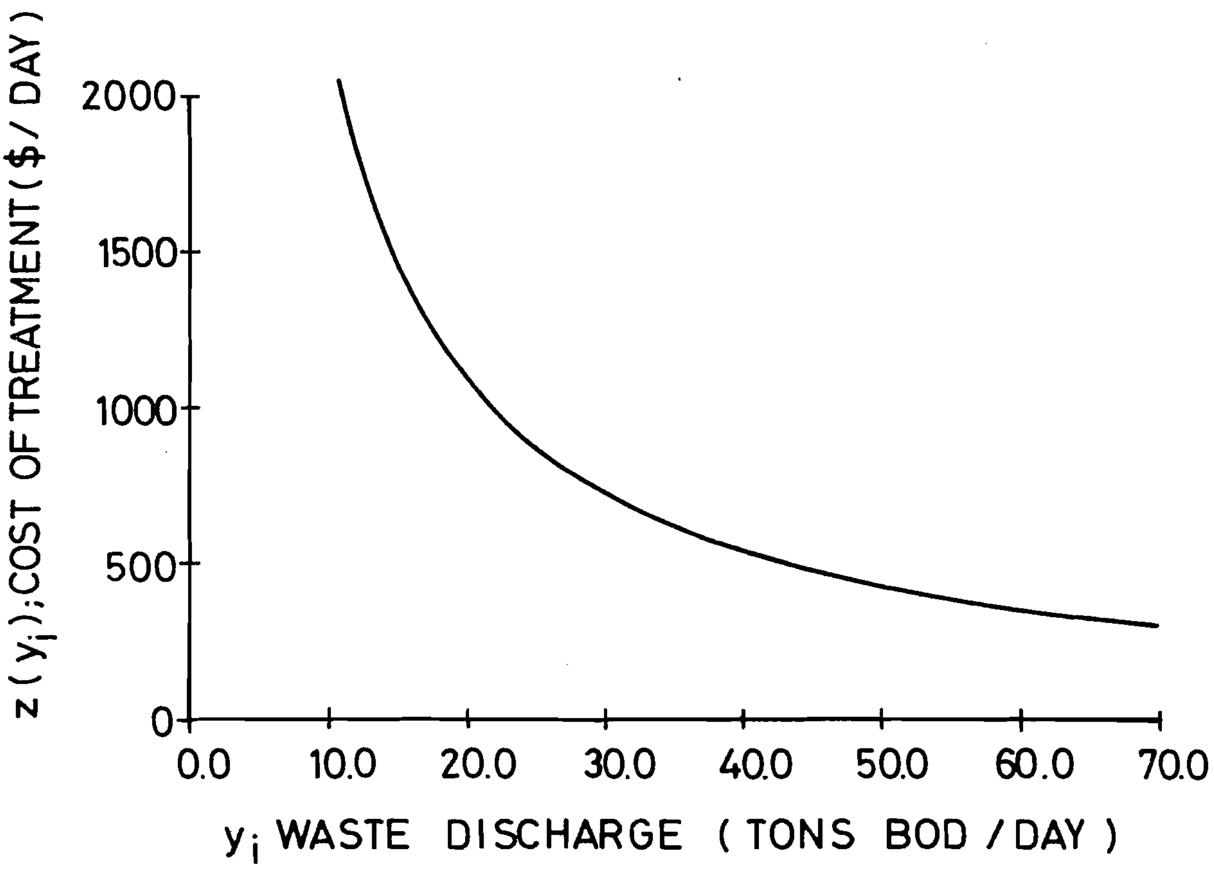

FIGURE 3. OPERATIONAL TREATMENT COSTS

If certain reasonable axioms are satisfied (Pratt, Raiffa, and Schlaifer [13]), it is possible to find a utility function for each interest group whose expected value is a guide for decision making. These utility functions express people's preferences for certain outcomes, along with their preferences in situations involving uncertainty. We assume that each city is concerned with the BOD of the river flowing past it, and with the costs it pays for treatment. We can represent the utility functions as follows:

$$u_1 = u_1[x_1, z_1(y_1)]$$

$$u_2 = u_2[x_2, z_2(y_2)]$$

$$u_3 = u_3(x_3) \quad .$$

Utility functions are generally scaled from 0 for the least preferred situation to 1 for the most preferred situation.

Much work has been done on developing techniques to assess utility functions, on finding properties of these functions, and on finding under what conditions utility functions can be written in simple forms (Keeney [9], and Gros [6]). Often, single-attribute utility functions exhibit risk aversion. This property can best be explained in terms of a simple example. Let us consider the lottery where, if a flipped (fair) coin turns up heads (probability .5), the most expensive treatment is used (in our example, this treatment costs $2000/day); and if the coin turns up tails (probability .5), the least expensive is used (costing $312). Alternatively, there is an offer of using a treatment whose cost is the numerical average of the most expensive and least expensive ($1156/day). Most people would choose the second alternative, the average treatment cost, instead of risking the lottery. This behavior is referred to as the assessor being risk averse over treatment costs, and the utility function can reflect this.

Let us assume that city 1 is risk averse, a reasonable assumption, and that its councilmen were interviewed so that their risk preferences could be quantified. A possible result of this questioning is that, when faced with a treatment cost of $1641, the councilmen were indifferent to the choice between this treatment for certain, and the lottery. Thus, the treatment costing $1641 is called the certainty equivalent of the lottery. Let us scale the single-attribute utility function for cost. The most preferred value, $312, should have a utility value of 1, so $u_z(312) = 1$; and the least preferred value, $2000, should have a utility value of 0, $u_z(2000) = 0$. Now \$1641, being the certainty equivalent of the lottery, has the same utility value as the expected utility value of the lottery: $u_z(1641) = .5u(2000) + .5u(312) = .5(1) + .5(0) = .5$. One form of single-attribute utility function that exhibits risk aversion is the exponential:

$$u_{z,i}(z_i) = a_i'\left\{1 - \exp\left[b_i(z_i - z_i^*)\right]\right\} \quad ,$$

where z_i^* is the most expensive treatment cost, and a_i' and b_i are constants. For city 1, with the above certainty equivalent, $a_1' = 1.0503$ and $b_1 = .0018$. (The exponential form exhibits certain risk preferences; in practice, the analyst confirms whether these hold before using this form.)

These single-attribute utility functions are then combined for each city in a logical fashion, using responses from the councilmen to define the logic. If certain preference-independence properties are satisfied (Keeney [9]), the two-attribute utility function can be expressed in a sum form:

$$u_i = u_{x,i}(x_i) + u_{z,i}[z_i(y_i)] \quad .$$

In our example, it is assumed that the two-attribute utility function for each city can be expressed in this sum form, and the single-attribute utility functions in an exponential form; that is,

$$u_i = a_i\left\{1 - \exp\left[b_i(z_i - z_i^*)\right]\right\} + g_i\left\{1 - \exp\left[h_i(x_i - x_i^*)\right]\right\} \quad .$$

For this example, the following constants are used:

$a_1 = .79$	$a_2 = .79$	$a_3 = 0$
$b_1 = .0018$	$b_2 = .0018$	$b_3 = 0$
$z_1^* = \$2000/\text{day}$	$z_2^* = \$2000/\text{day}$	$z_3^* = 0$
$g_1 = .285$	$g_2 = .285$	$g_3 = 1.14$
$h_1 = .14$	$h_2 = .14$	$h_3 = .14$
$x_1^* = 15.0$ mg/l	$x_2^* = 15.0$ mg/l	$x_3^* = 15.0$ mg/l .

The single-attribute utility function curves for city 1 are shown in Figure 4.

As before, the Hamiltonian can be defined as:

$$H = \sum_{i=1}^{3} \gamma_i u_i p(q) + \sum_{i=1}^{3} \lambda_i C_i \quad .$$

The necessary conditions for a maximum are:

$$\frac{\partial H}{\partial y_i} = 0 \quad , \qquad i = 1,2$$

$$\frac{\partial H}{\partial x_i} = 0 \quad , \qquad i = 1,2,3$$

$$C_i = 0 \quad , \qquad i = 1,2$$

$$U_i = R_i \quad , \qquad i = 2,3$$

if $\gamma_1 = 1$ and γ_2, γ_3 are constants.

Results

The set of simultaneous nonlinear equations was solved for x_1, x_2, x_3, y_1, y_2, λ_1 and λ_2 as a function of q. Table 1 represents a typical control strategy A for our example. The expected utility function value for city 1 is .955; the values for city 2 and at the border are constrained to be $R_2 = .702$ and $R_3 = .722$, respectively. As the flow rate in the river decreases, the treatment at each city increases and the discharges, y_i, are reduced. The flow rate is related to the BOD level downstream through two mechanisms: (i) the lower the flow rate, the longer the flow time between the cities, and the greater the decay of BOD that can take place; (ii) the lower the flow rate, the lower the amount of dilution that takes place. From our results, the second effect dominates. Figure 5 illustrates the control trajectories and resulting water quality at the cities and the border. For other values of R_i, a different control strategy results; a second strategy B, with $R_2 = .953$ and $R_3 = .318$ was studied, and the results of the two

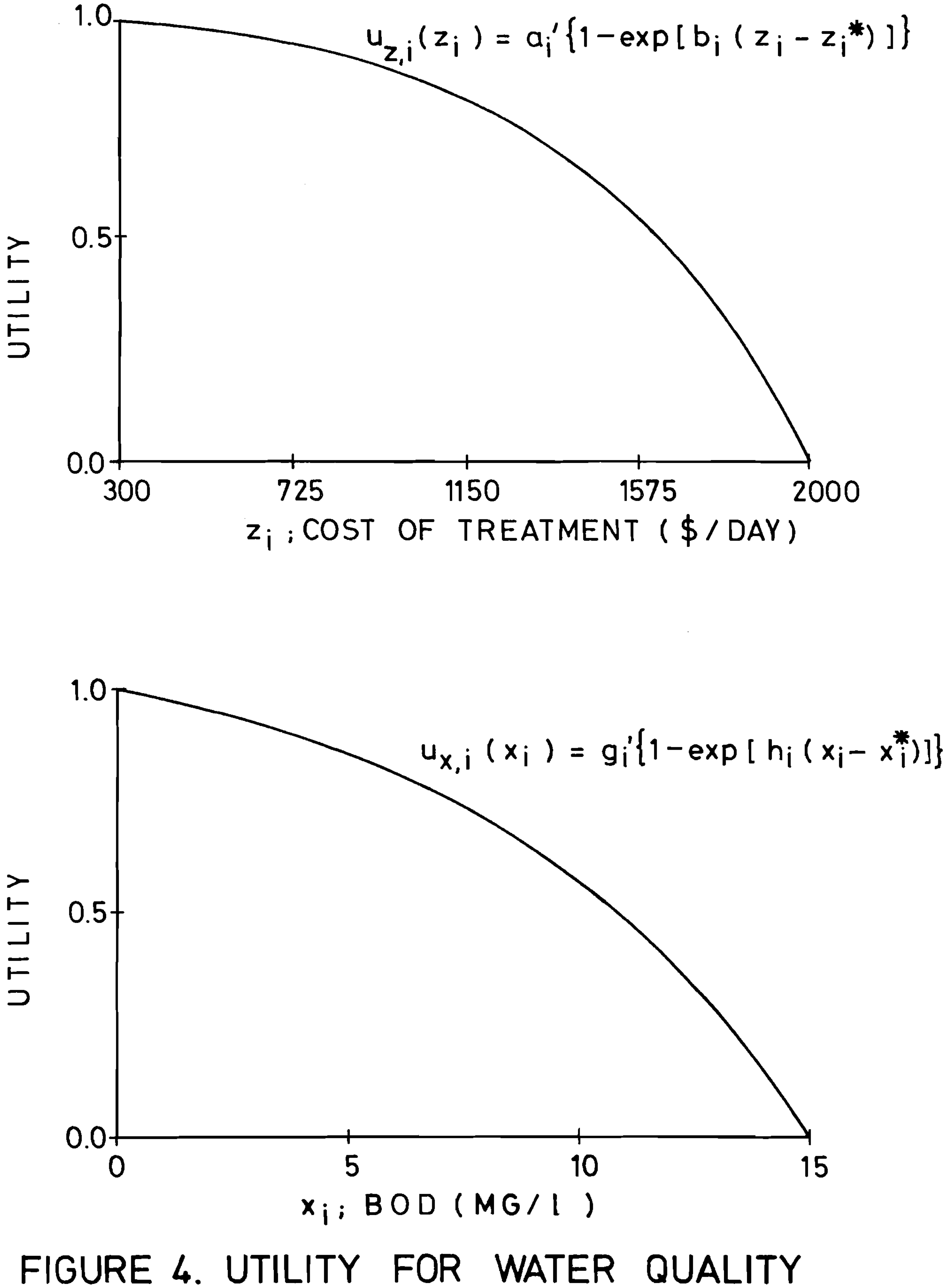

FIGURE 4. UTILITY FOR WATER QUALITY AND TREATMENT COSTS

Table 1. Control trajectories and water quality for Strategy A.

	q					
	8.1	6.75	5.79	5.06	4.50	4.05
y_1	31.30	28.81	26.79	25.16	23.79	22.62
y_2	17.10	16.02	15.16	14.45	13.85	13.33
x_1	3.33	4.00	4.66	5.34	6.00	6.67
x_2	4.21	4.67	5.10	5.50	5.89	6.25
x_3	6.32	7.04	7.72	8.36	8.96	9.55
λ_1	-.016	-.018	-.020	-.022	-.026	-.026
λ_2	-.016	-.017	-.019	-.021	-.023	-.025

$q = 10^8\ m^3$/day

y_i = tons BOD/day

x_i = mg/l BOD

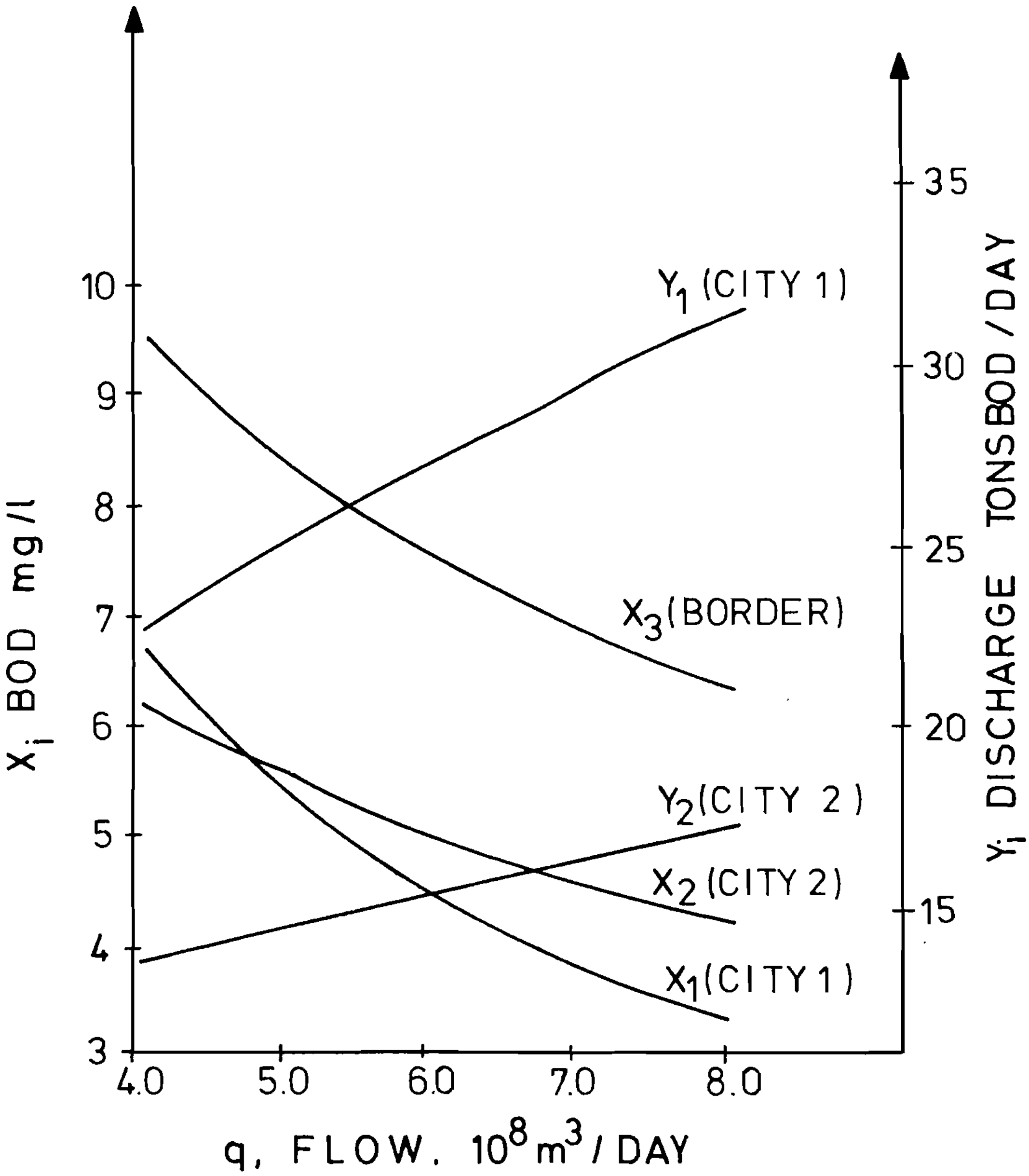

FIGURE 5. CONTROL STRATEGY A AND WATER QUALITY FOR TWO CITIES AND THE BORDER

strategies are compared in Table 2. Strategy B results in high BOD levels at the border with low treatment by city 2, whereas strategy A has lower BOD levels at the border at the expense of city 2. The pollution control authorities can regard the set of such control strategies, along with their associated expected utility functions, as aids in establishing their regulations.

Extensions

There are many extensions of the model that result in simple additions to the mathematical format. Let us suppose that some higher pollution control authority requires the river quality at the border to meet some standard. Further, this standard is expressed in terms of a maximum BOD level at the border, $\overline{x}_{n+1}$. Mathematically, this can be expressed as

$$x_{n+1} - \overline{x}_{n+1} \leq 0$$

for all conditions. In order to incorporate this constraint in the analysis, the term $\phi(x_{n+1} - \overline{x}_{n+1})$ can be added to the Hamiltonian, where

$$\phi \begin{cases} \geq 0 \;, & \text{if } x_{n+1} - \overline{x}_{n+1} = 0 \\ = 0 \;, & \text{if } x_{n+1} - \overline{x}_{n+1} < 0 \end{cases} ,$$

and the necessary conditions could be found as before.[3]

The model, as presented in previous sections, is best used when each city already possesses waste treatment facilities. With minor additions, it can be extended to include capital investments. Let y_{i*} be the minimum amount of BOD discharged from city i; that is, $y_{i*} \leq y_i$ for all stream conditions. This minimum amount is the value the treatment plant must be designed to meet. Associated with this design are capital costs $z_{cap}(y_{i*})$, and operating costs $z_{op}(y_i, y_{i*})$ necessary to achieve an instantaneous discharge of y_i. Our

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[3]If the standard level is also to be found by the model, the utility functions must be augmented to include the standard level, and the additional necessary condition $\partial H/\partial \overline{x}_{n+1} = 0$ must be satisfied.

Table 2. Alternative control strategies.

	$\mathcal{E}x_i$ (mg/l)	$\mathcal{E}y_i$ (tons/day)	$\mathcal{E}z_i$ (\$/day)	$\mathcal{E}u_i$
Control Strategy A				
City 1	5.1	26.8	816.59	.955
City 2	5.38	15.2	1438.00	.702
Border	8.16	--	--	.722
Control Strategy B				
City 1	5.1	27.9	785.87	.960
City 2	5.57	39.3	557.02	.953
Border	12.78	--	--	.318

$\mathcal{E}$ = expected values

utility function can be expanded to $u_i(x_i, w_i, z_{cap}, z_{op})$ to include plant design variables. Similar to the previous extension, a term of the form $\psi(y_{i*} - y_i)$ would be added to the Hamiltonian, with

$$\psi \begin{cases} \geq 0 \quad , & \text{if } y_{i*} - y_i = 0 \\ = 0 \quad , & \text{if } y_{i*} - y_i < 0 \end{cases} .$$

Since y_{i*} is a variable, an additional necessary condition,

$$\frac{\partial H}{\partial y_{i*}} = 0 \quad ,$$

must be satisfied.

The problem could differ in other ways. Perhaps the preferences of city 1 depend upon what city 2 pays for its treatment. To include this factor, the cost to city 2 can be one of the arguments of the utility function of city 1, i.e. $u_1(w_1, x_1, z_1, z_2)$. Or, similarly, city 1 may be concerned with the BOD level at city 2, resulting in $u_1(w_1, x_1, x_2, z_1)$. It should be obvious that these and other changes can be incorporated in the model in a straightforward manner.

Conclusions

A method of finding Pareto-admissible solutions for the multi-interest-group water resource problem has been described. It was assumed that each group wants to maximize the expected value of its utility; that stream flows were stochastic; and that a pollution control authority exists. To find a solution, a set of simultaneous nonlinear equations had to be solved. The applicability of this type of analysis to large-scale problems depends upon obtaining realistic utility functions, and upon having computer techniques that can find solutions to problems with large numbers of simultaneous nonlinear equations.

Notation

A_i	cross-sectional area of river at city i, in m^2
a_i	weight for utility function $u_{z,i}$, in the multi-attribute utility function
α_i	weight for the single attribute utility function for BOD scaled from 0 to 1
b_i	parameter of $u_{z,i}$
BOD	biological oxygen demand
c_i	BOD control constraint for reach i of river
C_i	BOD control constraint for reach i of river expressed so constraint equals 0
DO	dissolved oxygen
d_i	DO control constraint for reach i of the river
D_i	DO control constraint for reach i expressed so constraint equals 0
g_i	weight for utility function $u_{x,i}$ in the multi-attribute utility function
g_i'	weight for the single attribute utility function for cost scaled from 0 to 1
H	the Hamiltonian
h_i	parameter of $u_{x,i}$
j	city for which the expected value of its utility function is maximized
k	rate constant for BOD decay, .23/day
ℓ_i	distance from city i - 1 to city i, in metres
p(q)	probability density function of q
q	ambient conditions in the river

R_i	constrained value of expected utility for city i
t_i	time taken for a unit volume of water to pass from city i - 1 to city i, in days
$u_{x,i}$	utility for water quality of city i
$u_{z,i}$	utility for treatment costs of city i
u_i	utility for quality and costs of city i
u_{n+1}	utility for quality at the border
w_i	DO concentration at city i, mg/l
x_i	BOD concentration at city i, mg/l
x_i^*	maximum possible BOD concentration at city i
$\overline{x}_{n+1}$	standard on BOD at border
y_i	effluent at city i, tons BOD/day
y_{i*}	minimum amount of y_i discharged from city i
z_i	cost of treatment to reduce effluent to y_i, \$/day
$z_{cap}(y_{i*})$	capital cost of a treatment plant at city i, in \$
$z_{op}(y_i, y_{i*})$	operating cost of a treatment plant at city i, in \$
z_i^*	maximum possible value of z_i
α	parameter of z_i cost function
β	parameter of z_i cost function
γ_i	multiplier for expected utility for city i
λ_i	multiplier for C_i
μ_i	multiplier for D_i
ϕ	multiplier for standard
ψ	multiplier for capital decisions
$\mathcal{E}$	expected value .

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