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OPTIMAL COMPENSATION PROGRAMMES IN WATER CONTROL DISTRIBUTION

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I. Introduction

Let a region R be given, having water control system WR. The remaining part of the region R will be called environmental system Ξ.

The system WR is a controlled set of water sources (i.e. reservoirs) and users. All processes in both systems WR and Ξ are considered in a period of integer time t, $t = 1,\ldots,s,\ldots,k$.

Suppose, in the region R at the s^{th} stage, a production programme $\Gamma^s = \left(\mathring{P}^s, \mathring{Y}^s\right) = \left(\mathring{P}_o^s, \mathring{P}_1^s, \ldots, \mathring{P}_i^s, \ldots, \mathring{P}_N^s; \mathring{Y}_1^s, \ldots, \mathring{Y}_j^s, \ldots, \mathring{Y}_M^s\right)$ and a programme $E^s = (B^s)$ are given, where $\mathring{P}^s$ and $\mathring{Y}^s$ are subprogrammes, while B^s is a value equivalent to the programme Γ^s (the total income obtained if Γ^s is realized).

The element $\mathring{P}_i^s$, $i = 1,N$ of the programme $\mathring{P}^s$ is the amount of the i^{th} output which is to be produced in the region R by means of available water resources. $\mathring{P}_o^s$ is the amount of water distributed as mandatory releases, i.e. municipal water supply, low-flow augmentation and other needs unconnected with the programme Γ^s.

Realization of any element $\mathring{P}_i^s$, $i = 1,N$ of the subprogramme P^s is carried out by producing the output $\mathring{P}_{iu}^s$ using u production units $u = \overline{1,\ell}$ belonging to the system WR. To produce output $\mathring{P}_{iu}^s$ every unit needs an amount of water $\mathring{X}_{iu}^s$. If $\mathring{X}_{iu}^s$ is changed in the interval $(\underline{X}_{iu}^s, \overline{X}_{iu}^s)$, then the output of the unit is changed in the interval $(\underline{P}^s_{iu}, \bar{P}^s_{iu})$. Obviously, $\overset{o}{X}{}^s_{iu} \in (\underline{X}^s_{iu}, \bar{X}^s_{iu})$ and $\overset{o}{P}{}^s_{iu} \in (\underline{P}^s_{iu}, \bar{P}^s_{iu})$. Apparently, to produce output $\overset{o}{P}{}^s_i$ in the system WR, the following conditions have to be satisfied:

$$(1) \qquad \overset{o}{P}{}^s_i = \sum_{u=1}^{\ell} \overset{o}{P}{}^s_{iu} , \qquad i = \overline{1,N} \quad .$$

It is assumed that the programme P^s is realized if the equality (2) is fulfilled:

$$(2) \qquad P^s_i = \overset{o}{P}{}^s_i \quad , \qquad i = \overline{1,N} \quad ,$$

where P^s_i is the real output and $\overset{o}{P}{}^s_i$ is the planning output.

The element $\overset{o}{Y}{}^s_j$, $j = 1,M$ of the subprogramme Y^s is the quantity of j^{th} output which is to be produced in the system Ξ from the j^{th} unit without using water resources. The output of the j^{th} unit varies in the interval $(\underline{Y}^s_j, \bar{Y}^s_j)$, and hence $\overset{o}{Y}{}^s_j \in (\underline{Y}^s_j, \bar{Y}^s_j)$.

In the cases when the condition (2) cannot be fulfilled because of the shortage or flood in the system WR, the region is required to realize the programme E^s instead of Γ^s. (As was already mentioned above, both programmes are equivalent in terms of total income.) If

$$\sum_{i=1}^{N} \Delta P^s_i = \sum_{i=1}^{N} \left(\overset{o}{P}{}^s_i - P^s_i \right) \geq 0 \quad ,$$

it is necessary to calculate

$$\sum_{j=1}^{M} y^s_j = \sum_{j=1}^{M} \left(Y^s_j - \overset{o}{Y}{}^s_j \right) \geq 0$$

in such a way that the programmes Γ^s and E^s are equivalent.

This condition will be written in the following abbreviated form:

$$(3) \qquad \left[\left(\sum_{i=1}^{N} \Delta P_i^s \geq 0\right) \rightarrow \left(\sum_{j=1}^{M} y_j^s \geq 0\right)\right] \rightarrow (\Gamma^s \sim E^s) \quad .$$

The programme $y^s = (y_i^s, \ldots, y_j^s, \ldots, y_m^s)$ is called compensation programme and (3) the compensation condition.

The elements of the subprogramme P^s can be divided into two groups: elements producing output that cannot be compensated (i.e. replaced by its equivalent value), and elements producing output that can be compensated. The water demands of the first kind of elements are included in $\overset{o}{P}{}_{o}^{s}$.

In some cases, condition (3) can be modified in the following way. Suppose that the decision maker (DM) in the region has at his disposal funds y_f^s, $0 \leq y_f^s \leq \bar{y}_f^s$ for buying the unproduced output from other regions. This means that in the region there is the possibility for producing output $\sum_{j=1}^{M} y_j^s$ and/or for buying it with y_f^s. This condition will be written as follows:

$$(4) \qquad \left[\left(\sum_{i=1}^{N} \Delta P_i^s \geq 0\right) \rightarrow \left(\sum_{j=1}^{M} y_j^s \geq 0\right) \vee (y_f^s \geq 0)\right] \rightarrow (\Gamma^s \sim E^s) \quad .$$

The conditions (3) or (4) are required by the national planning authorities of many countries. Their quantitative description and introduction into water optimization models is of great interest. One of the possible ways to do this is discussed below.

II. Formulation of the Problem

Suppose a multi-reservoir system WR is given having u production units. By means of these units N outputs are produced.

With every unit belonging to WR (e.g. enterprises, crops, etc.) the following function, $f_{iu}(X_{iu})$,[1] can be associated. This function represents the dependence between the loss of the u^{th} unit producing the output i at stage s and the amount of water distributed to this unit.

Two types of the function, $f'_{iu}(X_{iu})$ and $f''_{iu}(X_{iu})$, shown in Figure 1 and Figure 2 respectively, are of substantial interest.

The function $f''_{iu}(X_{iu})$ differs from the function $f'_{iu}(X_{iu})$ in the way in which the water is distributed to the u^{th} user. The variable X_{iu} for the function $f''_{iu}(X_{iu})$ is determined in the following way:

$$(5)\qquad X_{iu} = \begin{cases} > 0 \,, & \text{if } X_{iu} > \underline{X}_{iu} \\ 0 \,, & \text{if } 0 \leq X_{iu} \leq \underline{X}_{iu} \end{cases} .$$

The condition (5) follows from the natural characteristics of some users. For example in agriculture, if the amount of water distributed to a given crop is less than a certain boundary, then the harvest approaches zero. Hence, if the amount of water distributed to the u^{th} unit is $\underline{X}_{iu}$, then it is better to give this crop the amount of water equaling zero and to distribute the total amount of water to the other units.

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[1]All indexes s are omitted because they pertain to every variable considered in this paper.

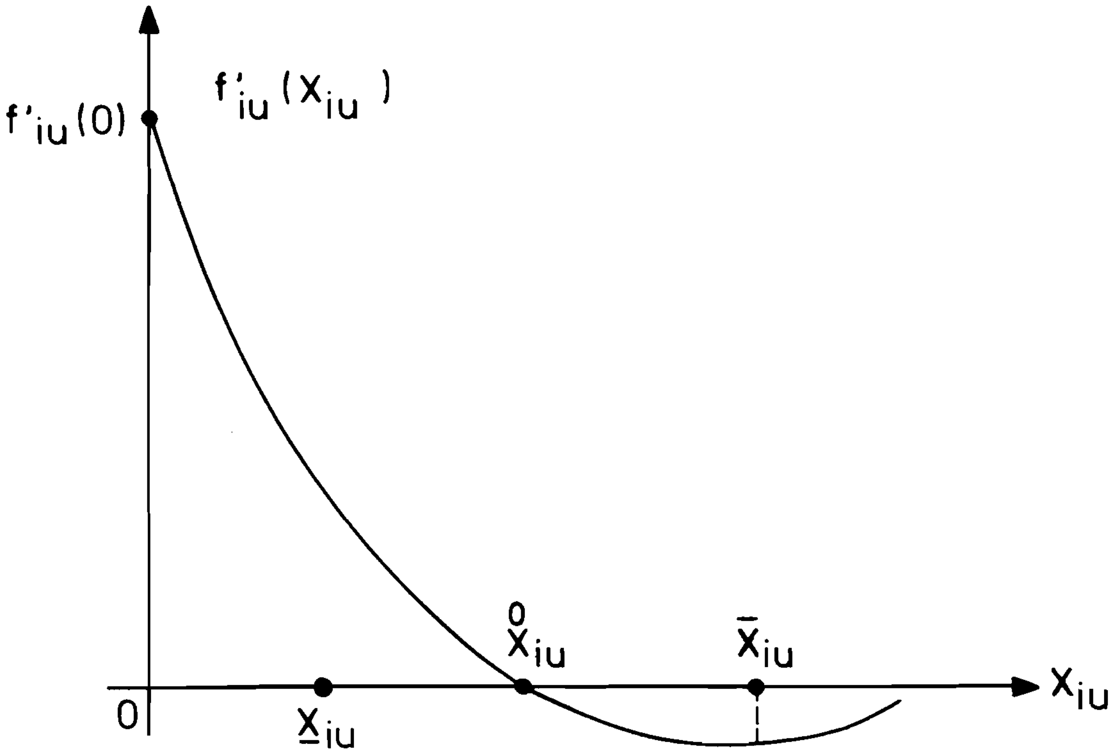

FIGURE 1. THE FIRST TYPE OF THE LOSS FUNCTION.

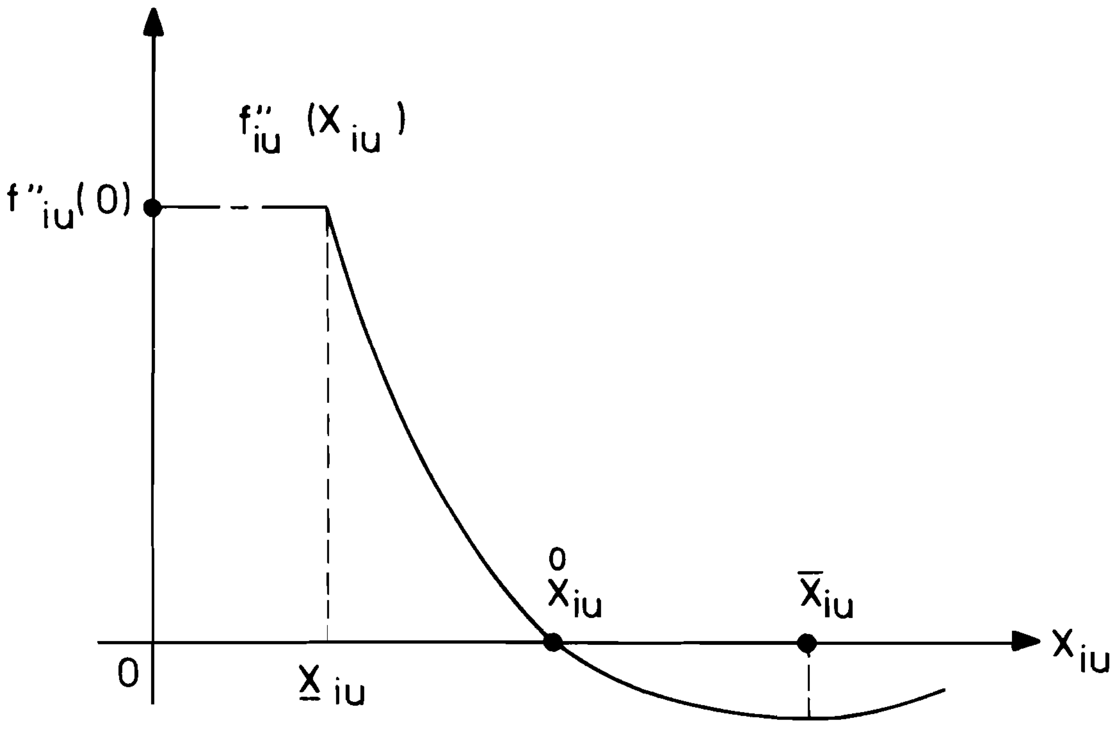

FIGURE 2. THE SECOND TYPE OF THE LOSS FUNCTION. (IN THE INTERVAL $(0, \underline{X}_{iu})$ ONLY THE VALUES 0 AND $\underline{X}_{iu}$ ARE ADMISSIBLE FOR THE ARGUMENT X_{iu}.

It is assumed that all u indexes of the units in the system WR belong to a set $\mu = \{1,\ldots,\ell\}$, i.e. $u \in \mu$. This set is divided into two subsets: μ_1 and μ_2, where $\mu_1 \cup \mu_2 = \mu$ and $\mu_1 \cap \mu_2 \neq 0$. For all units having the first type of function $f'_{iu}(X_{iu})$ the index $u \in \mu_1$ and for the rest of the units $u \in \mu_2$.

With each production unit belonging to the system Ξ (using no water from WR) can be associated the function $g_j(y_j)$, $j = 1,\ldots,M$, for every stage s. This function reflects the connection between additional output from the j^{th} unit at the s^{th} stage and increasing y_j. The variable y_j could be interpreted as the amount of raw materials, labour, energy, or it could be a synthesis of all the factors together. Apparently, this function is similar to the one shown in Figure 3. Because of the restricted production capacity of the units, the maximum additional output from any unit cannot be in excess of $\bar{g}_j$. This output is obtained when $y_j = \bar{y}_j$.

The expenditure connected with producing an output $g_j(y_j)$ by the j^{th} unit belonging to the system Ξ is denoted by $\psi_j(y_j)$.

When one uses the compensation condition (4) a function $\psi_f(y_f)$ is introduced. This function quantifies the preferences of the DM as to the extent of using these financial resources. One possible function of this type is shown in Figure 4.

After these preliminary notations, let us consider the problems connected with putting into practice the compensation idea given in (4).

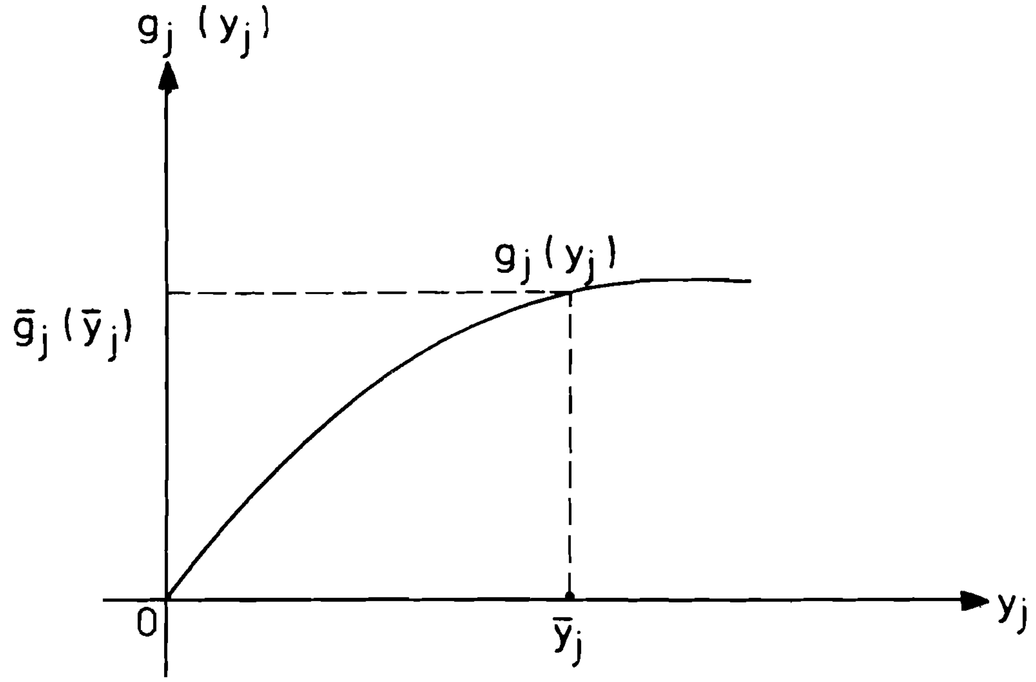

FIGURE 3. ADDITIONAL OUTPUT $g_j(y_j)$ OF THE COMPENSATION UNITS IN THE SYSTEM Ξ.

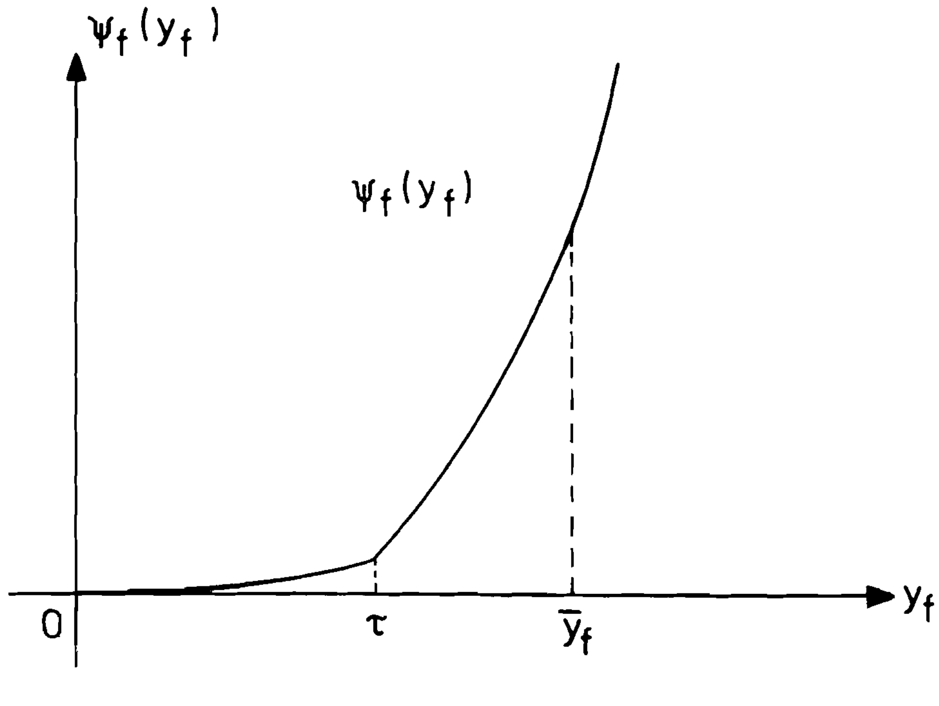

FIGURE 4. FINANCIAL COMPENSATION OUTPUT. THE INTERVALS $(0, \tau)$ AND $(\tau, \bar{y}_f)$ REFLECT THE DIFFERENT PREFERENCES OF THE DM ABOUT USING FINANCIAL RESOURCES AS COMPENSATION OUTPUT.

In [1] it was pointed out that optimum control of a multi-reservoir system can be carried out on two hierarchical levels. On the first level the optimum amount of water released from the reservoir at all stages of time is determined. On the second level this amount of water is distributed amongst the units.

Suppose the optimum amount of water W^*, allocated to the units at the s^{th} stage, is distributed to the set of canals $V = (1,2,\ldots,p,\ldots,r)$. Every canal has a restricted capacity C_p, $p \in V$. In addition, there are N units producing output in the system WR.

The goal of a system control on the second level is to distribute the amount of water W^* amongst the units in such a manner that:

a) the total loss in the system will be minimized,

b) the compensation condition (4) will be fulfilled,

c) all the physical constraints described below are satisfied.

This goal could be formularized in terms of non-linear programming by the following two-step optimization problem. In the first step, the amount of water W^*, allocated to all the units $u \in \mu$ in the system WR, is distributed to every unit, taking into account all the constraints except for compensation condition (4). In terms of non-linear programming, this problem (called problem A) is as follows:

Minimize

(6) $$F(X) = \sum_{u \in \mu_1} f'_{iu}(X_{iu}) + \sum_{u \in \mu_2} f''_{iu}(X_{iu})$$

subject to

(7) $$\sum_{u \in U_p} X_{iu} \leq C_p \quad , \quad p \in V$$

(8) $$\sum_{u \in \mu} X_{iu} = W^*$$

(9) $$\underline{X}_{iu} \leq X_{iu} \leq \bar{X}_{iu} \quad , \quad u \in \mu_1$$

(10) $$X_{iu} \leq \bar{X}_{iu} \quad , \quad u \in \mu_2$$

(11) $$X_{iu} = \begin{cases} > 0 & , \text{ if } X_{iu} > \underline{X}_{iu} \\ 0 & , \text{ otherwise} \end{cases} \quad , \quad \forall u \in \mu_2 \quad .$$

By means of (7) the amount of water distributed amongst the units is restricted in accordance with the capacity of the p^{th} canal, $p \in V$.

The next constraint (8) requires the whole amount of water released from the reservoir at the s^{th} stage to be distributed amongst the units $u \in \mu$ in the system WR. This constraint allows the entire planning problem to be decomposed into problems A and B.

The lower boundary of all the variables X_{iu} reflects a mandatory demand of unit output at the stage s.

Solving problem A, particularly compensation of the losses in the system WR, is carried out because the absolute minimum of the functions $f'_{iu}(X_{iu})$ and $f'_{iu}(X_{iu})$ is reached at the point having "negative loss" (see Figures 1 and 2).

As a result of solving problem A, the optimal value $F(X^*)$ is obtained, i.e. the total loss in the system WR at the s^{th} stage. This loss is compensated through over-planned production by the units in system Ξ. The whole over-planned production has to take place with minimum expenditure. In terms of mathematical programming, the second step, obtaining the optimal compensation programme (Problem B), is as follows:

Minimize

$$(12) \qquad \phi(y) = \left[\sum_{j=1}^{M} \psi_j(y_j) + \psi_f(y_f) \right]$$

subject to

$$(13) \qquad \sum_{j=1}^{M} g_j(y_j) + y_f = \begin{cases} F(X^*) \quad , & \text{if } F(X^*) > 0 \\ 0 \quad , & \text{if } F(X^*) \leq 0 \end{cases}$$

$$(14) \qquad 0 \leq y_j \leq \bar{y}_j \quad , \quad j=1,M$$

$$(15) \qquad 0 \leq y_f \leq \bar{y}_f \quad .$$

The optimal values $g_j(y_j^*)$, $j = \overline{1,M}$, and y_f^*, obtained at the s^{th} stage by solving (12), (13), (14) and (15), are components of the optimal compensation programme which is to be fulfilled in the region R.

III. Computational Procedure

Although problems A and B described in the previous section are relatively simple ones, in many cases substantial computational difficulties may be encountered. These difficulties are due to the non-linear constraint (11). In the cases when the functions $f_{iu}(X_{iu})$, $\psi_j(y_j)$ and $\psi_f(y_f)$ are concave and $g_j(y_j)$ is linear, problems A and B could easily be solved by the method proposed in [2]. Moreover, this method allows finding a parametrical solution for both problems A and B when W^* and $F(X^*)$ are changed.

IV. Illustrative Example

The methodology described here is illustrated by the following numerical example.[2]

In the system WR, 5 outputs $i = \overline{1,5}$ are produced using 10 units $u = \overline{1,10}$ (Figure 5). For compensation of the unproduced output in the system WR, four compensation units are available for the system Ξ (N=4). At the stage s = 7 the amount of water from the reservoir allocated to the units is 21.778.

Using the data available, the following functions $f'_{iu}(X_{iu})$, $f''_{iu}(X_{iu})$, $\psi_j(y_j)$, $\psi_f(X_f)$ and $g_j(y_j)$ are obtained.

$$(16)\quad f'_{iu}(X_{iu}) = \begin{cases} d^{(1)}_{iu} + p^{(1)}_{iu}X_{iu} + a^{(1)}_{iu}X^2_{iu}, & \text{if } X_{iu} \leq \overset{o}{X}_{iu} \\ d^{(2)}_{iu} + p^{(2)}_{iu}X_{iu} + a^{(2)}_{iu}X^2_{iu}, & \text{if } X_{iu} \geq \overset{o}{X}_{iu} \end{cases}$$

u = 2,3,4,8,10;

$$(17)\quad f''_{iu}(X_{iu}) = \begin{cases} \alpha^{(1)}_{iu} + \beta^{(1)}_{iu}X_{iu} + \gamma^{(1)}_{iu}X^2_{iu}, & \text{if } X_{iu} \leq \overset{o}{X}_{iu} \\ \alpha^{(2)}_{iu} + \beta^{(2)}_{iu}X_{iu} + \gamma^{(2)}_{iu}X^2_{iu}, & \text{if } X_{iu} \geq \overset{o}{X}_{iu} \end{cases}$$

u = 1,5,6,7,9;

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[2] The data used here are taken from the investigation of the Iskar River basin.

$$(18) \qquad \psi_j(y_j) = a_j y_j^2 , \qquad j = 1,4$$

$$(19) \qquad \psi_j(y_j) = a_j(e^{b_j y_j} - 1) , \qquad j = 2,3$$

$$(20) \qquad \psi_f(X_f) = \begin{cases} a_f(e^{b_f y_f} - 1) , & \text{if } 0 \leq X_f \leq \tau \\ d_f + p_f X_f + q_f X_f^2 , & \text{if } \tau \leq X_f \leq \overline{X}_f \end{cases}$$

$$(21) \qquad g_j(y_j) = h_j y_j .$$

All the coefficients of functions (16), (17), (18), (19), (20) and (21) are given in Table 1, 2, 3 and 4.

The optimum results obtained by the method described in [2] are shown in Tables 5 and 6.

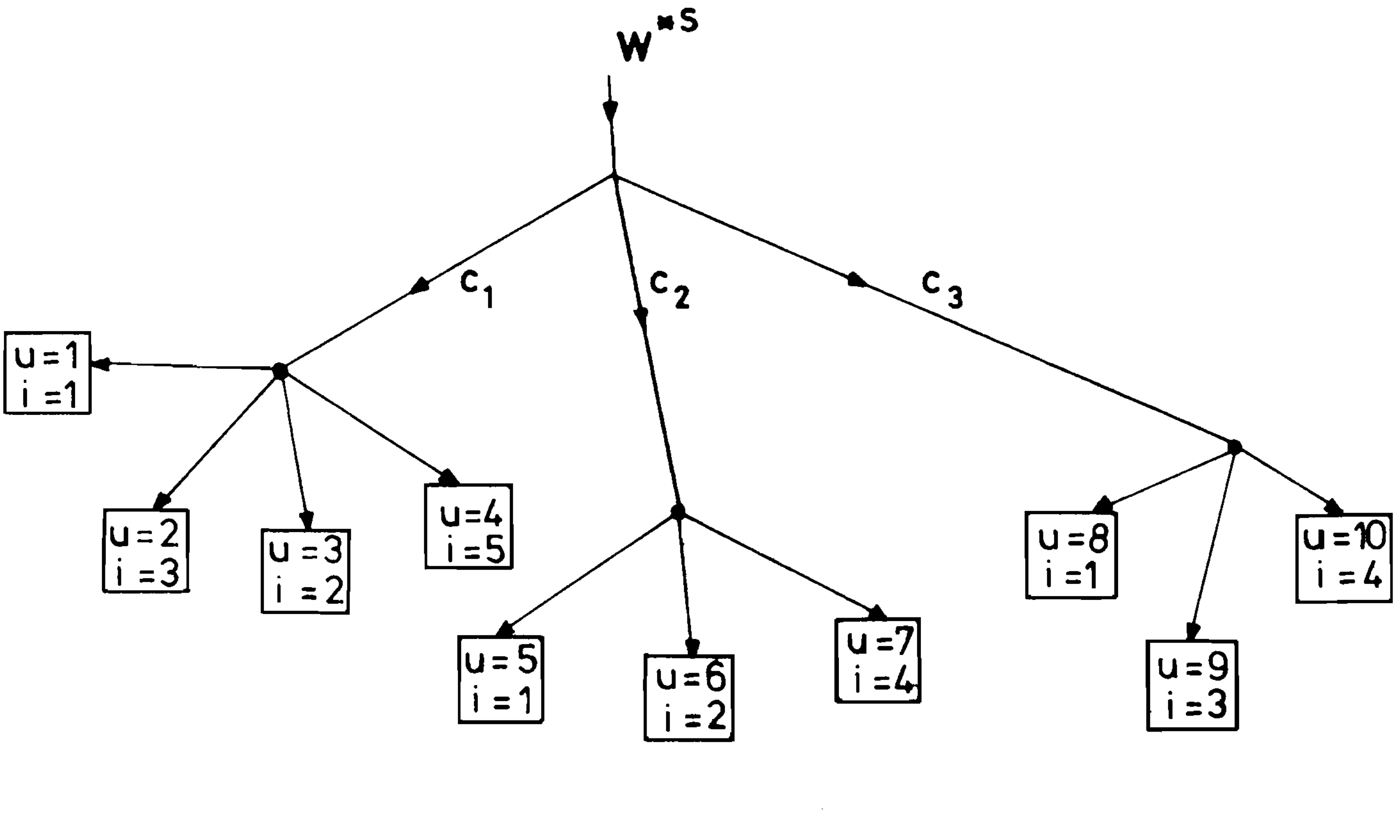


FIGURE 5.

Table 1. Coefficients of the functions $f'_{iu}(X_{iu})$.

i	u	$f_{iu}(X_{iu})$	d^1_{iu}	p^1_{iu}	a^1_{iu}	d^2_{iu}	p^2_{iu}	a^2_{iu}	$\underline{X}_{iu}$	$\overset{0}{X}_{iu}$	$\bar{X}_{iu}$
3	2	$f_{32}(X_{32})$	0.512	-0.348	0.0348	0.258	-0.240	0.050	0.500	1.800	2.400
2	3	$f_{23}(X_{23})$	0.784	-0.385	0.0385	0.338	-0.244	0.038	0.2500	1.900	3.200
5	4	$f_{54}(X_{54})$	0.350	-0.400	0.0500	0.064	-0.106	0.033	0.500	1.000	1.600
1	8	$f_{18}(X_{18})$	0.761	-0.440	0.0366	0.320	-0.218	0.033	0.505	2.100	3.300
4	10	$f_{4,10}(X_{4,10})$	1.820	-0.693	0.0288	1.120	-0.470	0.045	0.750	3.000	5.200

Table 2. Coefficients of the functions $f''_{iu}(X_{iu})$.

i	u	$f_{iu}(X_{iu})$	α^1_{iu}	β^1_{iu}	γ^1_{iu}	α^2_{iu}	β^2_{iu}	γ^2_{iu}	$\underline{X}_{iu}$	$\overset{0}{X}_{iu}$	$\bar{X}_{iu}$
1	1	$f_{11}(X_{11})$	1.330	-0.522	0.0325	0.475	-0.209	0.019	0	3.200	5.500
1	5	$f_{15}(X_{15})$	1.185	-0.474	0.0296	0.790	-0.410	0.050	0	3.100	4.100
2	6	$f_{26}(X_{26})$	0.458	-0.432	0.0432	0.154	-0.155	0.031	0	1.200	2.500
4	7	$f_{47}(X_{47})$	0.785	-0.422	0.0352	0.444	-0.266	0.036	0	2.300	3.700
3	9	$f_{39}(X_{39})$	0.761	-0.445	0.0145	1.410	-0.372	0.023	0	6.000	8.100

Table 3. Coefficients of the functions $\psi_j(y_j)$ and $g_j(y_j)$, $j = 1,2,3,4,f$.

j	a_j	b_j	d_j	p_j	q_j	τ	$\bar{y}_j$	h_j
1	2.600	-	-	-	-	-	0.7	2.26
2	1.525	1	-	-	-	-	1.0	5.00
3	2.100	1	-	-	-	-	0.8	4.20
4	4.10	-	-	-	-	-	0.9	6.30
f	0.4	1	0.1399	-1.8653	6.2178	0.3	0.4	1

Table 4. The capacity of the canals.

P	1	2	3
C_p	19.55	18.05	20.20

Table 5. Computational results for $W^* = 21.771$.
(optimal amount of water distributed to the units in the system WR)

W^*	X^*_{11}	X^*_{32}	X^*_{23}	X^*_{54}	X^*_{15}	X^*_{26}	X^*_{47}	X^*_{18}	X^*_{39}	$X^*_{4,10}$	$F(X^*)$
21.778	3.200	0.690	1.110	1.000	2.940	1.200	1.730	1.908	5.000	3.000	1.298

Table 6. Computational results for $W^* = 21.771$.
(optimal compensation output of the units in the system Ξ)

$F(X^*)$	y^*_1	y^*_2	y^*_3	y^*_4	y^*_f	$\psi(y^*) = \sum_{j=1}^{4} \psi_j(y^*_j) + \psi_f(y^*_f)$	$\Delta = \frac{F(X^*) - \psi(y^*)}{F(X^*)} 100$
1.298	0.0969	0	0	0.1713	0	0.1447	88.85

It is obvious that full compensation is impossible because some expenditure $\psi_j(y_j)$ is needed to produce compensation output $g_j(y_j)$. For example, Table 6 shows that the total loss $F(X^*)$ obtained, due to the shortage in the system WR, is 1.298. The system Ξ produces output $g(y) = \sum_{j=1}^{4} g_j(y_j) + y_f = 1.298$ but with total expenditure $\psi(y^*) = 0.1447$. This means that only

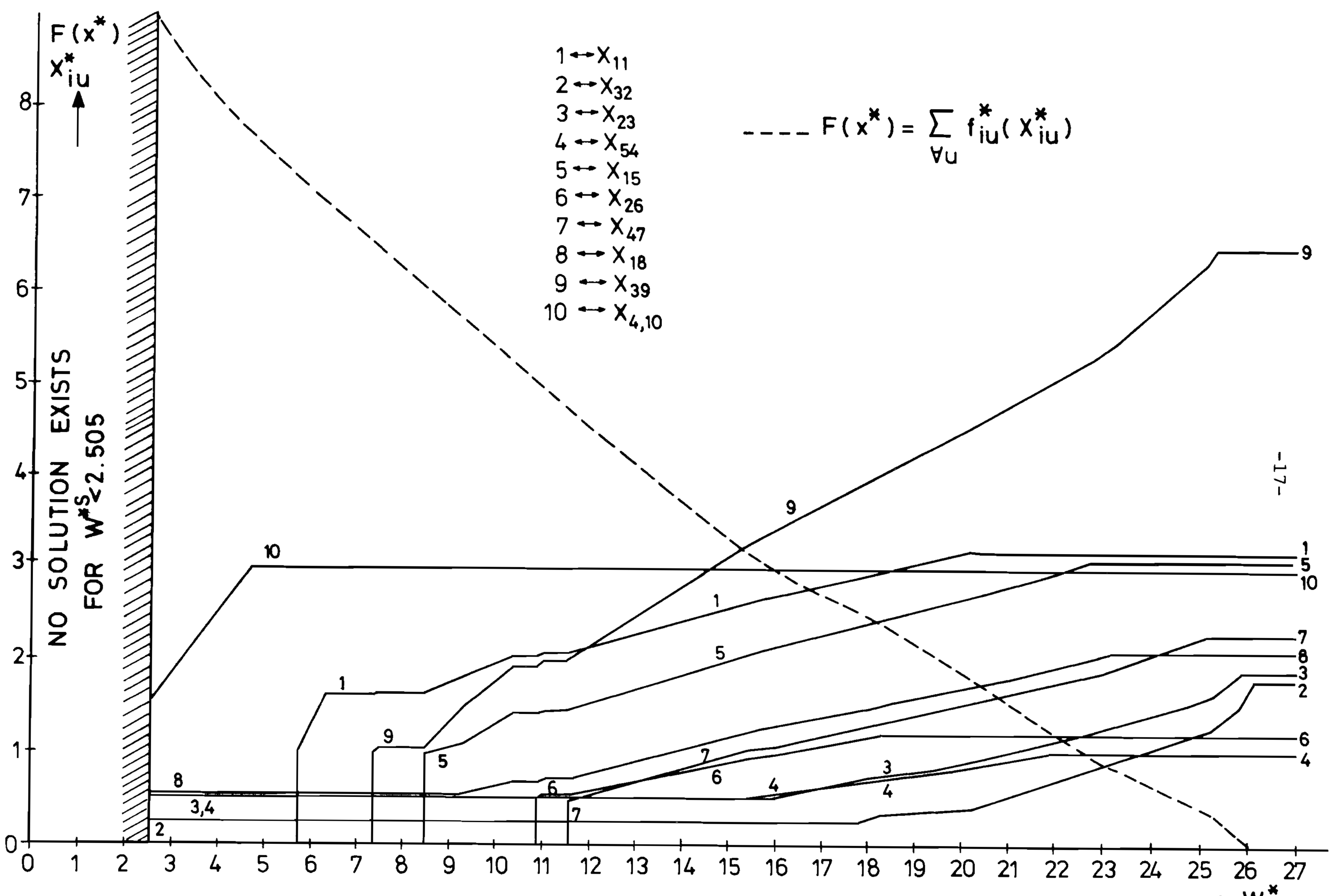
$$\frac{1.298 - 0.1447}{1.298} 100 = 88.85\%$$ of the loss in both systems WR and Ξ is compensated. The ratio $\frac{F(X^*) - \psi(y^*)}{F(X^*)}$ will be called compensation ability of the systems WR and Ξ.

The method used for solving problems A and B makes it possible to find the optimum solution when W^* and $F(X^*)$ are changed. The results obtained with this method are shown in Figure 6 and Figure 7 respectively.

In Figure 7, it can be seen that when a maximum loss $F(X^*) = 9.014$ in the system WR is obtained, the compensation ability of the systems WR and Ξ is only 56%.

Conclusions

The results of this study indicate that there are some possibilities for investigating the mutual impact of the water control system WR and the system Ξ embracing WR. The term compensation ability is introduced to show the reaction of the systems WR and Ξ to the loss due to shortage or flood in the system WR.



F(x*)
X*iu
NO SOLUTION EXISTS
FOR W*s<2.505
1 ↔ X11
2 ↔ X32
3 ↔ X23
4 ↔ X54
5 ↔ X15
6 ↔ X26
7 ↔ X47
8 ↔ X18
9 ↔ X39
10 ↔ X4,10
F(x*) = ∑∀u f*iu(X*iu)
0 1 2 3 4 5 6 7 8
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27
W*

FIGURE 6.

$g_j(y_j^*) = h_j y_j^*$; $j = 1,2,3,4,f$ – OPTIMAL COMPENSATION OUTPUT OF THE SYSTEM Ξ

Δ – COMPENSATION ABILITY OF THE SYSTEM WR & Ξ

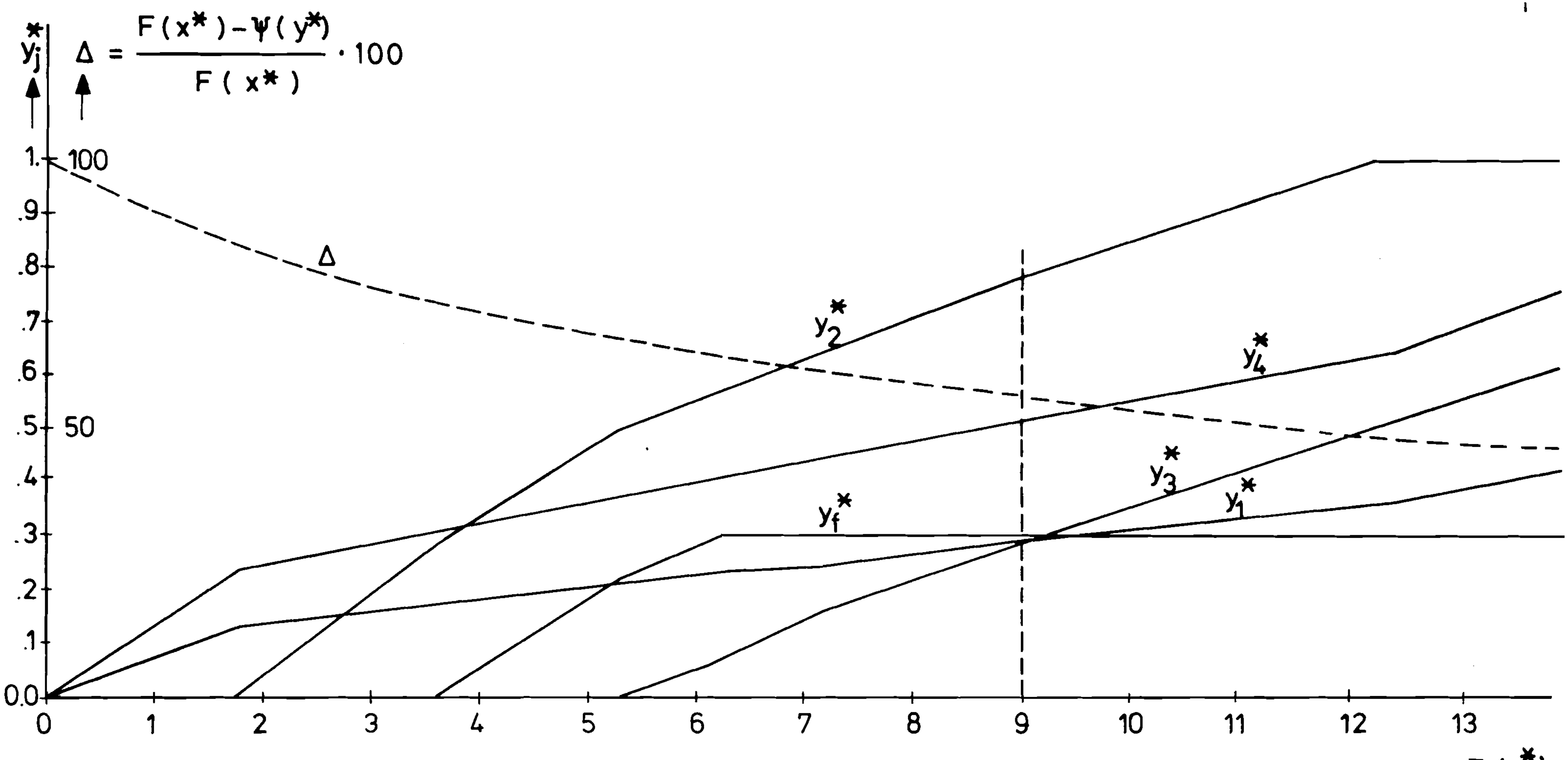


FIGURE 7.

The parametrical solution of the problem enables the DM to make more rational decisions by tracing all changes of the variables. Moreover, such a solution allows the DM to compare the results "at once" and to get clearer ideas for developing the systems WR and Ξ under different conditions.

References

[1] Gouevsky, I.V. "On Optimum Control of Multi-Reservoir Systems." Laxenburg, Austria, IIASA Research Report RR-74-25, December 1974.

[2] Gouevsky, I.V. "Three Algorithms for a Simple Nonlinear Programming Problem." Laxenburg, Austria, IIASA Working Paper, December 1974.