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Psychological Stability of Solutions in the Multiple Criteria Decision Problems

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Abstract

In interactive programming, a choice behaviour of the decision maker may differ depending on a proximity of current solution to satisfactory values of the objectives. An interactive approach proposed in this paper allows the decision maker to use different search principles depending on his/her perception of the achieved values of the objectives and trade-offs. While an analysis of values of the objectives may guide the initial search for a final solution, it can be replaced by trade-off evaluations at some later stages of interactive decision making. Such an approach allows the decision maker to change search principles, and to identify a psychologically stable solution of the multiple criteria decision problem.

Vector Optimization; Interactive Decision Making; Psychological Stability; Trade-offs; Efficiency

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1 Introduction

A decision making problem considered in the literature (Bell,Raiffa, Tversky (1988); French (1986)) in its most general formulation may take the form of the following vague statement:

$$\text{"given a set of decisions, choose the best one according to decision making circumstances"} \tag{1}$$

Seldom is problem (1) further restrained by requiring that a choice be made from the efficient decisions. This condition, however, appears in the normative formulation of decision making problems which usually involves specification of a vector optimization model:

$$\text{"max"} f(x) \text{ s.t. } x \in X_0 \subseteq X\,, \tag{2}$$

where $f : X \to R^k$, $f = (f_1, f_2, ..., f_k)$, is a vector of the objective functions $f_i : X \to R$, X_0 is the set of feasible solutions (admissible decisions), and "max" stands for the operator of determining all efficient solutions of X_0. With the notation $f(x) = y$, $f(X_0) = Z$, we call y an *outcome* and Z an *outcome set*. Thus, problem (1) is translated into a problem of the generation of efficient solutions according to a predetermined set of objectives. Such a formulation allows one to identify "the best" decision only if decision making circumstances, in terms of relationships between the outcomes of the decisions, are fully specified. Information about these relationships can be acquired and processed in an interactive fashion (Wierzbicki (1980); Chankong,Haimes (1983); Yu (1985); Steuer (1986)). However, in the framework of interactive decision making, detailed information (such as provided by the trade-offs) and deep insight into mutual relationships between solutions of (2), if not correctly structured, may cause the unstable choice behaviour of a decision maker (DM). He/she may reverse previously expressed preferences, may become inclined towards moving in the direction of high relative gains (trade-offs) in certain criteria at the expense of moderate losses in other criteria neglecting the previously accepted levels of the objective function values, or may change his/her risk profile. It is also

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well known that certain solutions to (2) may have infinite trade-offs. This, as well as finite but very high trade-offs either indicate improperness of a normative model or reveal what we call *lack of psychological stability* of its solutions.

We propose that in order to control for the aberrations of choice behaviour both *qualitative* and *quantitative* aspects of a psychological stability of solutions should be taken into account when solving normative model (2). We propose also that psychological stability, augmented with some other forms of assessment of the DM's preferences, is used to guide a process of selecting "the best" efficient solution to (2), which might be interpreted as a normative manifestation of a solution to problem (1). Formal definitions of efficiency and trade-offs are given in Section 2 whereas a definition of psychological stability is presented in Section 4.

The purpose of this paper is to show how the notion of psychological stability effectively complements solving decision making problems in an interactive manner. We present an interactive technique which implements the principle of psychological stability combining evaluation of the trade-offs with evaluation of the objectives in a search for a single solution of a normative incarnation of (1).

The paper is organised as follows. In the next section we give preliminary definitions. In Section 3 we review the literature pertinent to trade-offs and interactive decision making. In Section 4 we discuss the notion of psychological stability in more detail and identify its relationship to the trade-offs. In Section 5 we operationalize our approach. Section 6 contains theoretical foundations of the methodology and its operation is illustrated with an example in Section 7. The paper concludes with a discussion in Section 8.

2 Preliminaries

Throughout the paper, for the sake of simplicity, we present all the results in terms of outcomes y, elements of an outcome set $Z = f(X_0)$. However, since Z is rarely given explicitly, usually computations are to be made in terms of the solutions x, as defined in (2).

We start with the necessary notation and formal definitions.

Let $\bar{y} \in Z$, $Z \subseteq \mathcal{R}^k$, where Z is an outcome set. For $i = 1, ..., k$, we denote:

$$Z_i^<(\bar{y}) = \{y \in Z \mid y_i < \bar{y}_i\,,\ y_l \geq \bar{y}_l\,,\ l = 1, ..., k\,,\ l \neq i\}\,,$$

Definition 2.1 *Let* $\bar{y} \in Z$. **Global trade-off** $T_{ij}^G(\bar{y})$ *involving objectives* i *and* j, $i, j = 1, ..., k$, $i \neq j$, *is defined as*

$$\sup_{y \in Z_j^<(\bar{y})} \frac{y_i - \bar{y}_i}{\bar{y}_j - y_j}\,.$$

By the usual convention, we assume that if $Z_j^<(\bar{y}) = \emptyset$, then $T_{ij}^G(\bar{y}) = -\infty$, $j, i = 1, ..., k$, $j \neq i$.

Let $\bar{y} \in Z$. The following are commonly accepted definitions of various types of efficiency.

The outcome $\bar{y} \in Z$ is
weakly efficient
if there is no y, $y \in Z$, such that $y_i > \bar{y}_i$, $i = 1, ..., k$,
efficient
if $y_i \geq \bar{y}_i$, $i = 1, ..., k$, $y \in Z$, implies $y = \bar{y}$,
and (following Geoffrion (1968)) **properly efficient**
if it is efficient and there exists a finite number $M > 0$ such that for each i we have

$$\frac{y_i - \bar{y}_i}{\bar{y}_j - y_j} \leq M$$

for some j such that $y_j < \bar{y}_j$, whenever $y \in Z$ and $y_i > \bar{y}_i$.

A trivial observation, but not out of place in view of the results of Section 6 is that an efficient outcome is also weakly efficient, but the opposite is not true. Contrary to other definitions of trade-offs, we do not require outcome $\bar{y}$ for which a trade-off is being calculated, to be efficient. It is easy to show that if Z is convex and $\bar{y}$ is not weakly efficient, then a trade-off does not exist. For nonconvex Z, trade-offs can exist for the outcomes which are not weakly efficient, as demonstrated by the case where Z is a finite set.

3 An Overview of Literature

A trade-off is defined for a particular solution to (2), and for a selected pair of the objectives (components of vector function $f(x)$). A trade-off specifies an amount by which one criterion value increases (*gain*) while the other decreases (*loss*) when moving from one solution to another. Usually two types of trade-offs are considered: *point-to-point* trade-offs and *global* trade-offs.

A point-to-point trade-off is normally represented as a ratio of relative value increase in one criterion per one unit of value decrease in a reference criterion when a particular solution is replaced by another *given* solution.

A global trade-off for given $\bar{y}$ is calculated as a supremum of all point-to-point trade-offs defined for pairs of solutions $\bar{y}, y$ such that all but one objective in y have values greater or equal to those in $\bar{y}$. Hence, a global trade-off specifies the least upper bound on an increase of one criterion value relative to a unit decrease of another criterion value occurring while moving from a particular solution of (2) in a direction where all the remaining criteria do not decrease. A formal definition of global trade-off was given in Section 2.

Simply calculating supremum over all point-to-point trade-offs (which is the definition of a *gain-to-loss ratio*, (Kaliszewski (1994)) is obviously not equivalent to determining a global trade-off. In many instances a finite gain-to-loss ratio does not exist whereas a global trade-off does (cf ibidem). Thus, it is a global trade-off that is analyzed in many publications.

Point-to-point trade-offs are used when the potential candidates for the final solution are simply detectable, as in linear models where often only vertices of a feasible set are considered. Given two different solutions, point-to-point trade-off is immediately calculable.

In contrast to this, determining a global trade-off involves computations. So far, most of the research on the evaluation of global trade-off was focused on the derivation of trade-off information for a given efficient solution to (2). This problem was addressed in Khun-Tucker (1951); Haimes,Chankong (1979); Wierzbicki (1990); Sakawa,Yano (1990); Halme (1992), Henig,Buchanan (1992); Kaliszewski (1993,1994). On the other hand, characterization of the efficient solutions to (2) by corresponding classes of global trade-offs, or the generation of efficient solutions with preimposed bounds on global trade-offs did not receive prominent treatment in the literature. This is despite the fact that there is an obvious link between weighting coefficients in the linear scalarization of a problem (2) and bounds imposed on global trade-offs in this problem (cf Section 6, Theorem 6.2). The other link involves use of the weighted Tchebycheff method (Steuer (1986)) in solving problem (2) and it was observed first in Wierzbicki (1990) (cf Section 6, Theorem 6.4). Wierzbicki's preliminary results were later generalised in Kaliszewski, Michalowski (1995, 1997). In Section 6 we present those results.

Interactive decision making shares a common feature of a dialog scenario of alternative stages of converting problem (2) into a single criterion optimization problem and solving it (Gardiner, Steuer (1994)). Changes in problem parameters are reflected by generating different forms of information which evaluation by the DM prompts to inflict further changes in parameters of (2). A rough taxonomy of interactive decision making methods involves: *distinctions among decision outcomes*, *treatment of trade-off*, and *manipulation of a reference point*. An overview and evaluation of interactive procedures may be found in Evans (1984); Goicoechea et al. (1982); Michalowski (1987); Szidarovszky et al. (1986); and Steuer (1986), among others.

Distinctions among outcomes were considered earlier in lexicographic programming (Ignizio (1982)), and this concept has been introduced into interactive decision making through the partition of outcomes. Partitioning of outcomes is one of the features of the interactive approach inspired by the fact that a process of choice and judgment can be improved by enlarging the context of evaluation. In the method given in Chankong,Haimes (1983) information about the partition of outcomes is utilised to construct a proxy function which is applied to the selection of the most preferred decision. In the method of Benayoun et al. (1971), an optimisation problem is solved in order to identify a candidate for the most preferred solution. The function being optimised is built using weights derived from the information about partition of outcomes. In the methods proposed in Michalowski (1988) and Michalowski,Szapiro (1989,1992) the same information helps to displace points of reference and to guide the direction of the search for the most preferred solution. To this group of methods there also belongs the classical method by Geoffrion et al. (1972), where one attempts to elicit information about differential properties of DM implicit utility function and uses it to guide the search for the most preferred decision.

In other approaches, the implicit partition of the outcomes is replaced by an analysis of trade-offs between them. Point-to-point trade-off evaluation was also used in Zionts,Wallenius (1983) to control a review of the adjacent extreme points of the (polyhedral) set of admissible solutions. In principle, all methods which make use of comparison of two or more solutions (via comparing corresponding outcomes) can be viewed as an indirect application of point-to-point trade-off information.

Manipulation of points of reference is another form of preference information retrieval used in interactive decision making. A change of reference may be accomplished through the displacement of an ideal point (Zeleny (1982), or some other points of reference (Lewandowski, Wierzbicki (1989); Michalowski (1988); Michalowski, Szapiro (1992); Zeleny (1982). A point of reference may be specified by a model (Nakayama, Sawaragi (1984), or it may be elicited from a decision maker (Korhonen, Laakso (1986); Lewandowski, Wierzbicki (1989); Michalowski, Szapiro (1989,1992). The framework of reference point displacements, its operational possibilities, measurements of achievement, and research directions to follow were discussed in Lewandowski, Wierzbicki (1989) and Zeleny (1982).

A dialog scenario of an interactive decision making usually is structured around either evaluation of the outcomes or trade-offs. So far there were not attempts to explore a complementary character of these two evaluations and to incorporate them into a single dialog scenario framework.

4 Psychological Stability and Trade-offs

The multiple criteria decision making (MCDM) literature usually assumes that a search for a final solution is guided by a single underlying principle (Lewandowski, Wierzbicki (1989), Steuer (1986), Zeleny (1982)). This principle is expressed as a desire to obtain some satisfactory (i.e. satisficing or optimal) values of the objectives. At the same time, behavioral decision making research demonstrates that people often simplify their choices and emphasize changes as value carriers (Kahneman, Tversky (1990)). Within the MCDM paradigm this should be interpreted as evidence of the existence of more than one underlying principle of a final solution search. Preferred values of the objective functions and a desire to reach them guide this search until generated solutions significantly improve the DM's satisfaction. When a certain level of satisfaction is reached, values of the objective functions are no longer good discriminants between efficient solutions, and they may be considered as a component which is being "shared" by the solutions. In order to account for what Kahneman, Tversky (1990) call an *isolation effect*, the DM should be given an opportunity to focus on the attributes which distinguish among the solutions - namely global trade-offs. Hence, the DM, being satisfied with the achieved values of the objectives, continues a search for a final solution which, when subjected to some unforseen perturbations, will not (in a subjective sense) significantly disturb the *status quo* of the outcomes. Such a solution has a property of being *psychologically stable.*

A concept of psychological stability needs to be specified, and this specification can be conveniently accomplished using the notion of global trade-offs.

Definition 4.1 *We say that a solution x is* **psychologically stable** *if it has* **satisfactory** *values of the objective functions and has* **acceptable** *global trade-offs.*

This is a purely *qualitative* definition and one should not propose any specific values for acceptable global trade-offs since they depend on a scaling of the criteria.

Moreover, the notion of the DM's acceptability involves subjective judgements and the objective reality captured according to the assumptions of the normative model (2). However, as shown in Section 5, using methodological developments of trade-offs bounding, we are able to incorporate the notion of psychological stability into interactive decision making and combine it with a search for satisfactory values of the objectives. It is worth to note here that setting bounds on trade-offs may be considered as an equivalent to implicit and partial specification of the DM's utility function. Such an equivalence is clear when the DM's preferences are elicited with respect to the values of objective functions. Observe that a bound specified by the DM and imposed on a trade-off results in elimination of all solutions which trade-offs violate this bound. This amounts to a statement that solutions eliminated in such a way have lower utility than at least one solution which trade-off satisfies a bound. As stated in the previous section, trade-off information is used in a dialog scenario of an interactive decision making. There are two possible modes of utilising trade-off information. In the first *passive* mode, for a given efficient solution of (2) global trade-offs are calculated and then used by the DM in his/her judgments. In the second *active* mode, a search is made for solutions which satisfy a certain preimposed pattern of global trade-offs created following the DM's preference profile. Such a search can be combined with a method of generating (weakly) efficient solutions to (2) and can allow for changes in the DM's search principle depending on the vicinity of the currently generated outcomes from the desired values of the objectives.

5 New Interactive Decision Making Approach

An interactive decision making approach proposed here allows the DM to use different search principles depending on his/her perception of the achieved values of the objectives and trade-offs [1]. One possibility is to search for satisfactory values of the objective functions paying no attention to the values of trade-offs. Another possibility is to search for solutions with acceptable values of trade-offs regardless of the values of the objective functions. It seems that none of these possibilities is consistently followed by the DM. While an analysis of values of the objectives may guide the initial search for a final solution to (2), it can be replaced by trade-off evaluations at some later stages of interactive decision making. Thus, the DM initially discriminates among efficient solutions of (2) with the help of outcomes, and terminates his/her search by identifying a psychologically stable efficient solution. Following Definition 4.1, a decision, represented by a solution to (2), is psychologically stable if it has satisfactory values of the objective functions and its change triggers acceptable *relative* changes in those values. Thus, searching for psychologically stable decisions can be conveniently operationalized through the constraints imposed on the values of the objective functions and bounds on trade-offs.

There is a large group of decision problems where a similar approach is widely used without reference to psychological stability. Namely, when dealing with a finite set of solutions (called alternatives) any pairwise comparison of outcomes involves

[1] Below we shall write "trade-off" meaning "global trade-off".

criteria level comparison but also it may, at DM discretion, involve evaluation of relative gains and losses in criteria levels. Given two outcomes y^1 and y^2, the criteria levels and point-to-point trade-off information is complete; i may be such that $y_i^2 - y_i^1 \geq 0$, and j may be such that $y_j^1 - y_j^2 > 0$ and the corresponding point-to-point trade-off is defined. Whether the ratio

$$\frac{y_i^2 - y_i^1}{y_j^1 - y_j^2}$$

is explicitly calculated, it is merely a technical question.

It is immediate to see that the above observation does not apply to cases where the set of solutions (outcomes) is not explicitly given. To account for this we propose to use the notion of trade-off which is applicable (and calculable) for sets of any nature.

The operation of a procedure with changing search principles which generates a psychologically stable solution is given below.

The VTB (*Value and Trade-offs Bounding*) **Technique**

1. Find an efficient solution $\bar{x}$ to (2).
2. Ask the DM to evaluate the candidate solution $\bar{x}$ in terms of objective function values and trade-offs by:

   a) specifying criteria which he/she would like to improve, to maintain at least at the current level, and where he/she is indifferent to changes;

   b) setting bounds on trade-offs.
3. The information gathered in Step **2** with respect to change of objective function values is represented by constraining a feasible set as in (Michalowski, Szapiro, 1992).
4. The information gathered in Step **2** with respect to values of upper bounds on trade-offs is represented by setting trade-off control parameters as in (Kaliszewski, Michalowski, 1997).
5. Determine an efficient solution x satisfying requirements specified in Step **3** and calculate trade-offs. Substitute this solution for $\bar{x}$.
6. If trade-offs of $\bar{x}$ satisfy the specified bounds, go to Step **2**, otherwise go to Step **7**.
7. Determine a subset of solutions with acceptable trade-offs which were elicited in Step **2**. Find an element of this subset which is "closest" (in the sense of, for example, Euclidean metric) to $\bar{x}$. Substitute this solution for $\bar{x}$
   Go to Step 2.

The VTB technique is flexible and does not impose normative priority on the importance of the analysis of values of the objective functions and the evaluation of trade-offs. However, it may be expected that in a search for a psychologically stable solution the DM first searches solutions with satisfactory outcomes and later discriminates among them by looking for acceptable trade-offs.

The seven steps of the VTB technique are quite natural in the context of decision making and do not require specific justification, with the possible exception of Step 7. An explanation for taking the "closest" solution to $\bar{x}$ from the determined subset is, that it cannot be guaranteed that there exists a solution which satisfies the DM's temporary preferences with respect to the values of objective functions *and* the trade-offs. The closeness is measured by some subjective measure of the DM.

Necessary specifications for the VTB technique are:

Stopping rule: The operation of VTB terminates in Step 2 whenever the DM is satisfied with the current solution $\bar{x}$.

In Step 2: Very large values for trade-offs bounds can be used as a default.

In Step 4: Trade-off control parameters are explained in the next section.

In Step 5: A general method of calculating trade-offs is given in Kaliszewski (1993,1994).

In Step 7: A method to generate a subset of solutions to (2) satisfying preimposed bounds on trade-offs is discussed in the next section.

6 Theoretical Foundations of the VTB Technique

This section presents the results of Kaliszewski, Michalowski (1995) which established a method to generate weakly efficient solutions with a *common* upper bound on trade-offs for a *selected* subset of all possible $n(n-1)/2$ trade-offs. Some refinements of those results (Kaliszewski, Michalowski (1997), see Theorem 6.6) allow to bound groups of trade-offs by *different* upper bounds, and we shall show how to generate properly efficient solutions with the required properties of their trade-offs (Theorem 6.8).

The section draws from other relevant research. Theorem 6.1 is a well known result by Geoffrion (Geoffrion (1968)) on generating properly efficient elements of convex outcome sets. Theorem 6.2 shows how weighting coefficients in a linear scalarizing method are related to bounds on values of trade-offs. Theorem 6.3 recalls an earlier result on scalarizing problem (2) by the modified Tchebycheff metric, and Theorem 6.4 identifies a relationship between a parameter ρ of this metric and bounds on trade-offs. Theorem 6.5 shows how a bound on a selected trade-off of weakly efficient elements of an outcome set Z can be preimposed. Finally, with Theorem 6.6 we easily arrive at a generalization (Theorem 6.7) of Theorem 6.4.

Theorem 6.1 (Geoffrion (1968)) *Assume that Z is convex. An element $\bar{y} \in Z$ is properly efficient if and only if there exists a vector λ such that $\bar{y}$ solves the problem*

$$\max_i \sum_i \lambda_i y_i \tag{3}$$

for some $\lambda > 0$.

Theorem 6.2 (Kaliszewski (1994)) *Let* $\bar{y}$ *solve problem (3). Then*

$$T^G_{ij}(\bar{y}) \leq \max_{i \neq j} \frac{\lambda_j}{\lambda_i}$$

for all $i, j = 1, ..., k\,,\ i \neq j$.

Let y^* be an element in $\mathcal{R}^k$ such that $Z \subseteq y^* - int(R^k_+)$.

Theorem 6.3 (Choo, Atkins (1983); Wierzbicki (1986); Kaliszewski (1987,1994)) *An outcome* $\bar{y} \in Z$ *is properly efficient if and only if there exists a vector* $\lambda > 0$ *and a number* $\rho > 0$ *such that* $\bar{y}$ *solves*

$$\min_{y \in Z} \max_i \lambda_i ((y^*_i - y_i) + \rho e^k (y^* - y))\,, \tag{4}$$

where e^k *is a k-dimensional row vector whose all components are equal to 1.*

The following theorems give an operational base for a concept of psychological stability.

Theorem 6.4 (Wierzbicki (1990); Kaliszewski (1994)) *Suppose* $\bar{y}$ *solves problem (4) for some* $\lambda > 0$ *and* $\rho > 0$. *Then,*

$$T^G_{ij}(\bar{y}) \leq (1 + \rho)\rho^{-1}$$

for all $i, j = 1, ..., k\,,\ i \neq j$.

Theorem 6.5 (Kaliszewski,Michalowski (1995)) *An outcome* $\bar{y} \in Z$ *is weakly efficient and* $T^G_{ji}(\bar{y}) \leq (1 + \rho)\rho^{-1}$ *if and only if for some vector* $\lambda > 0$ *and number* $\rho > 0$ *it solves the following problem*

$$\min_{y \in Z} \max(\lambda_i ((1 + \rho)(y^*_i - y_i) + \rho (y^*_j - y_j))\,,\ \max_{l \neq i} \lambda_l (y^*_l - y_l))\,, \tag{5}$$

where $l = 1, ..., k$.

Let $I = \{1, ..., k\}$ and let $I_1 \subseteq I\,,\ I_2 = I \setminus I_1$.

Theorem 6.6 (Kaliszewski,Michalowski (1997)) *An outcome* $\bar{y} \in Z$ *is weakly efficient and* $T^G_{ti}(\bar{y}) \leq (1 + \rho_i)\rho_t^{-1}$ *for each* $i \in I_1$ *and each* $t \in I_1\,,\ t \neq i$, *if and only if there exist* $\lambda_i > 0$ *and* $\rho_i > 0\,,\ i \in I_1$, *such that* $\bar{y}$ *solves*

$$\min_{y \in Z} \max(\max_{i \in I_1} \lambda_i ((y^*_i - y_i) + \sum_{t \in I_1} \rho_t (y^*_t - y_t))\,,\ \max_{i \in I_2} \lambda_i (y^*_i - y_i))\,. \tag{6}$$

The following result generalizes Theorem 6.4.

Theorem 6.7 *Suppose $\bar{y}$ solves the following problem*

$$\min_{y\in Z}\max_{i}\lambda_i((y_i^*-y_i)+\sum_{t\in I}\rho_t(y_t^*-y_t))\,, \tag{7}$$

where $\lambda_i > 0$ and $\rho_i > 0$ for each i. Then,

$$T_{ti}^G(\bar{y}) \leq (1+\rho_i)\rho_t^{-1}$$

for each $i, t \in I$, $i \neq t$.

Proof The proof follows immediately from Theorem 6.6. Indeed, putting in this theorem $I_1 = I$, $I_2 = \emptyset$ we immediately get (7). Theorem 6.6 states that in this case $T_{ti}^G(\bar{y}) \leq (1+\rho_i)\rho_i^{-1}$ for all i and all $t \neq i$, which is the required result. □

The next theorem generalizes Theorem 6.3.

Theorem 6.8 (Kaliszewski,Michalowski (1997)) *An element $\bar{y} \in Z$ is properly efficient if and only if there exists a vector λ, $\lambda > 0$, and numbers ρ_i, $\rho_i > 0$, $i = 1, ..., k$, such that $\bar{y}$ solves*

$$\min_{y\in Z}\max_{i}\lambda_i((y_i^*-y_i)+\rho_i e^k(y^*-y))\,,$$

7 Illustrative Example

Let us consider the following problem:

$$\text{"max"} \quad \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{s.t. } X_0 = \left\{ x \;\middle|\; \begin{array}{rrcr} 16x_1 & +2x_2 & \leq & 444 \\ x_1 & +x_2 & \leq & 40 \\ 4x_1 & +16x_2 & \leq & 472 \\ & x_1 & \geq & 0 \\ & x_2 & \geq & 0 \end{array} \right\}.$$

The set Z is represented in Figure 1. We have:

$$\begin{array}{lll}
\text{for } y = \{a\}, & T_{12}^G(y) = 4\,, & T_{21}^G(y) = -\infty\,, \\
\text{for } y = (a,b), & T_{12}^G(y) = 4\,, & T_{21}^G(y) = 0.25\,, \\
\text{for } y = \{b\}, & T_{12}^G(y) = 1\,, & T_{21}^G(y) = 0.25\,, \\
\text{for } y = (b,c), & T_{12}^G(y) = 1\,, & T_{21}^G(y) = 1\,, \\
\text{for } y = \{c\}, & T_{12}^G(y) = 0.125\,, & T_{21}^G(y) = 1\,, \\
\text{for } y = (c,d), & T_{12}^G(y) = 0.125\,, & T_{21}^G(y) = 8\,, \\
\text{for } y = \{d\}, & T_{12}^G(y) = -\infty\,, & T_{21}^G(y) = 8\,,
\end{array}$$

where $(x, y) = \{t \mid t = \alpha x + (1-\alpha)y\,,\ 0 < \alpha < 1\}$.

Let us assume that the DM's choice behaviour, when driven only by a desire to maximize the values of the objective functions, is described by a proxy function $u(y) = \min\{6y_1\,, 27y_2\}$ which is being maximized. In order to start the VTB technique, it is necessary to generate an initial efficient solution. There are many

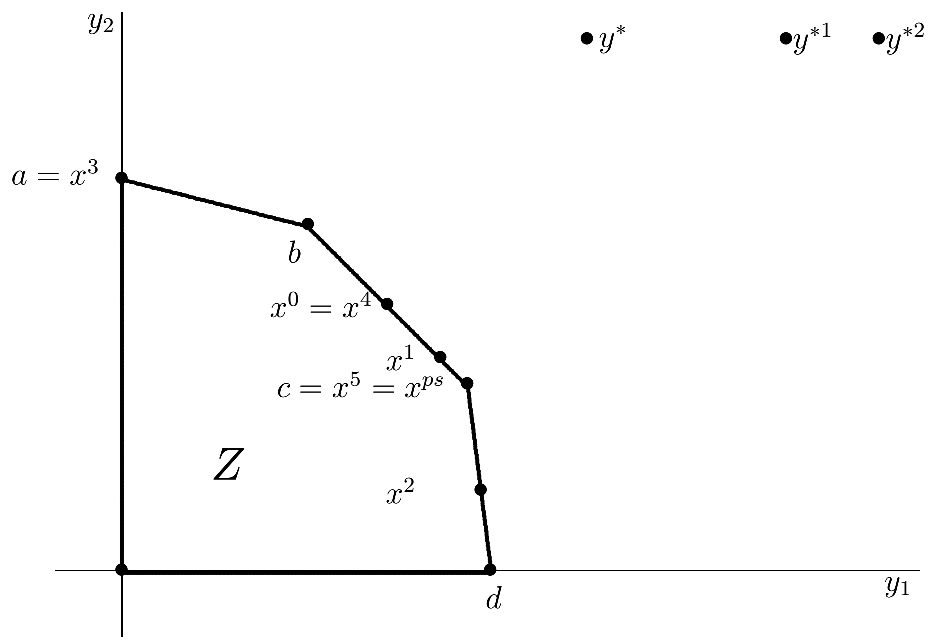

Figure 1 An example of operations of the VBT technique:
$a = (0, 29.5),\ b = (14, 26),\ c = (26, 14),\ d = (27.75, 0),$
$x^0 = (20, 20),\ x^1 = (24, 16),\ x^2 = (27, 6),$
$y^* = (40, 40),\ y^{*1} = (60, 40),\ y^{*2} = (67, 40).$

different ways to derive at solution $\bar{x}$ to (2). Here we assume that it is generated following Theorem 6.3, setting a small value for ρ, and putting $\lambda_1 = \lambda_2$ in order to generate a centrally located solution. Application of problem (5) also requires specification of y^* (since we shall vary y^*, its original value is denoted by $\tilde{y}$). Nevertheless, it should be noted here that the selection of a particular method of finding an efficient solution of (2) is irrelevant for the principles of the VTB technique.

Operation of the VTB technique is illustrated using the above sample problem[2].

1. Using problem (5), setting $\lambda_1 = \lambda_2 = 1$, $\rho = 0.001$, and specifying $y^* = \tilde{y} = (40, 40)$, we generate an efficient solution $x^0 = (20, 20)$ and we set $\bar{x} = x^0$.
According to Theorem 6.4, all trade-offs for the solution $\bar{x}$ are bounded by $(1+0.001)/0.001 = 1001$.

2. Following the proxy function, the DM is willing to increase the value of $f_1(x)$. The DM also sets the upper bounds on trade-offs as: $T^G_{12}(y) \leq +\infty$ and $T^G_{21}(y) \leq 4$.

3. The DM's preferences are expressed by adding to the set X_0 the constraint $f_1(x) \geq 20$.
Each time when we restrict the original problem, in order to avoid possible difficulties with the

[2]We use text written in small print to describe technical operations which are not explicit for the VTB technique.

interplay between the values of λ and ρ, we modify y^* in the following way

$$y_i^* = \tilde{y}_i + \begin{cases} \delta_i & \text{if } y_i \text{ is additionally restricted to be greater than } \delta_i\,, \\ 0 & \text{if there is no restriction on } y_i\,. \end{cases}$$

Hence, in this step we have, $y^* = y^{*1} = (60, 40)$.

4. Generation of outcomes with $T_{21}^G(y) > 4$ is (following Theorem 6.6) avoided by setting $\frac{1+\rho}{\rho} \le 4$ which gives $\rho \ge \frac{1}{3}$. If necessary, we shall use this value in calculations in step **7**.

5. We solve problem (5) for a modified X_0 and new y^*. We get $x = x^1 = (24, 16)$. According to Theorem 6.4, all trade-offs for the solution x^1 are bounded by $(1 + 0.001)/0.001 = 1001$.
The values of all trade-offs are equal to 1.
We set $\bar{x} = x^1$.

6. Trade-offs of $\bar{x}$ satisfy bounds specified by a DM.

In new iteration the steps of the VTB technique are repeated.
2. According to the proxy function, the DM is willing to increase a value of $f_1(x)$, and the new satisfactory level is 27. The bounds on trade-offs are not changed.

3. The constraint $f_1(x) \ge 27$ is added to X_0. Also, y^* is modified as $y^* = y^{*2} = (67, 40)$.

4. There is no change in trade-off bounds.

5. Problem (5) for a modified X_0 and modified y^* is solved. An efficient solution $x = x^2 = (27, 6)$ is reached.
According to Theorem 6.4, all trade-offs for the solution x^2 are bounded by $(1+0.001)/0.001 = 1001$.
The values of trade-offs are $T_{12}^G(f(x)) = 0.125$ and $T_{21}^G(f(x)) = 8$.
We set $\bar{x} = x^2$.

6. Bounds on trade-offs set by the DM, are not satisfied for $\bar{x}$.

7. In order to determine a subset of solutions with acceptable trade-offs, we solve the problem

$$\max_{y \in Z} \max(\lambda_1(1\frac{1}{3}(\tilde{y}_1 - y_1) + \frac{1}{3}(\tilde{y}_2 - y_2))\,, \lambda_2(\tilde{y}_2 - y_2))$$

for some $\lambda_1 > 0$, $\lambda_2 > 0$.
This time we search the whole set Z (by means of varying vector λ) and therefore we use the original value of y^*, namely $\tilde{y}$.
We set $\lambda^1 = (0.2, 0.8)$, $\lambda^2 = (0.5, 0.5)$, $\lambda^3 = (0.8, 0.2)$ and get $x^3 = (0, 29.5)$, $x^4 = (20, 20)$, $x^5 = (26, 14)$. The solution x^5 is closest to $\bar{x}$ (in the sense of the Euclidean meteric), thus $\bar{x} = x^5$. And finally in Step **2.** solution $\bar{x} = (26, 14)$ is being accepted as psychologically stable and a stopping rule is invoked.

8 Discussion

The above numerical example is rather simple but we believe that it has an illustrative power. Moreover, this example gives us a good reference point to discuss the relation of the VTB technique to other interactive decision making methods. The technique clearly belongs to the first group of the taxonomy discussed in Section 3, as it is based on a distinction among decision outcomes.

The VTB technique combines some aspects of classical approaches and novelty. Indeed, if we drop in it any references to trade-offs and in each iteration generate only one efficient solution, the operation of the VTB reduces to the STEM method (Benayoun R. et al (1971)). However, unlike the STEM method we do not require linearity of the problem (2).

Another extreme view at the VTB technique is to drop any reference to outcome levels. The resulting method (ie screening the entire population of efficient outcomes by a trade-off filter) remotely resembles the Zionts-Wallenius method (Zionts,Wallenius (1983)) which makes use of point-to-point trade-offs. Point-to-point trade-offs, similarly to the trade-offs we use, capture an information on relative behaviour of criteria. However, a point-to-point trade-off is calculated for two given efficient solutions, thus levels of criteria for each solution have to be explicitly known, which does not need to be a case in the VTB method.

Using only trade-off information to select the best decision, can be interpreted as probing of an underlying, but unknown even to the DM, utility function. Given a common bound on trade-offs, and discarding solutions with trade-offs having at least one value above this bound, is equivalent to the following implication:

Given two efficient solutions x^1 and x^2, and their respective outcomes y^1 and y^2, and the underlying utility function u (if exists) such that

$$\frac{y_i^1 - y_i^2}{y_j^2 - y_j^1} > \alpha$$

$(y_i^1 - y_i^2) \geq 0$, $(y_j^2 - y_j^1) > 0$, i,j *- any pair of indices,* $i \neq j$, α *- given trade-off bound. The above implies that*

$$u(y^1) > u(y^2) .$$

Possible indifference curves illustrating the above implication are shown in Figure 2.

Another possible interpretation of the 'trade-off only' aspect of the VTB technique positions it among the works attempting to discover DM's preferences. The first group of such works consists of all the formal and empirical studies on construction of DM's utility function - a complex and formal task which requires strong, hardly verifiable assumptions. The second group is formed of interactive decision making methods in which partial information on DM's preferences is elicited and successively processed until the selection of a final decision is possible. We discuss this topic at length in Section 3. The third group of works where the VTB technique actually belongs, is to derive an information about the outcome set as it is

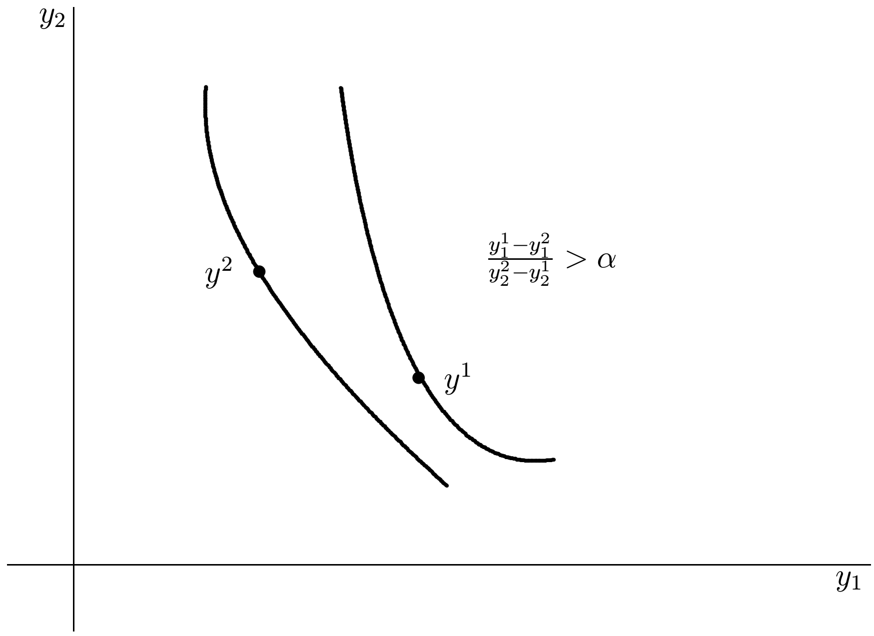

Figure 2 An example of the indifference curves

revealed by a given outcome, and to search for a possible improvement along the lines provided by the trade-off information.

It is hard to propose a laboratory numerical experiment for testing the VTB technique in the classical sense. At least two reasons can be given. First, with no formal model of DM (which is what we try to avoid in interactive decision making) no repetition of results is guaranteed. An approach often used is to test a method simulating DM's interactions by an *a priori* selected utility function (in the example of Section 7 we did it for illustrative purposes only). Results of such a testing are of limited use, since it is seldom (if at all) possible to identify the DM's behaviour in a rigid, scientific manner. The question of behavioural consistency as the time evolves is one of several crucial issues pertaining to numerical testing. Second, for the VTB technique we are not able to identify any benchmark method since the VTB operates in two 'dimensions' (criteria levels and trade-offs), whereas other comparable methods are 'uni-dimensional'.

9 Conclusions

In this paper we have presented a new approach to an interactive decision making. The novelty of the VTB technique lies in the premises of varying principles of searching for a final solution. Contrary to existing interactive methods, and accounting for a phenomenon of the *isolation effect*, we give the DM an opportunity to simplify a decision problem by neglecting the values of the objectives and instead focusing on their changes. As satisfactory (in the sense of Simon's aspiration levels (Simon (1956)) solutions are reached, a principle of improving them should no longer be applied. Then, our technique allows the DM to explore these satisfactory solutions using search principles which identify a solution with psychologically stable outcomes.

The advantage associated with the introduction of the notion of psychological stability to interactive decision making is that of having an approach which better suits a decision making environment. It is well known that the solutions of normative models are rarely directly implemented in practice. One of the reasons for such a situation is that a small perturbation of the solution may trigger a significant change in outcomes. The VTB technique addresses this issue by allowing the DM to search for a solution which guarantees that, if perturbed, changes of the values of the objective functions will be within acceptable bounds.

The methodology presented in this paper is very flexible because it is not contracted to a particular class of normative decision problems. Using the principles of the VTB technique, it is possible to analyze both continuous and discrete decision making situations, the only limitation being the possibility of solving problems (4) and (5). Moreover, the only formally required assumption is the existence of the element y^*. Such an assumption is not restrictive in most applications. We believe that the VTB technique opens new directions in interactive decision making research, where known behavioral biases of the DM can be addressed and accommodated regardless of the type of an underlying decision model. In that sense our approach is one of the few truly prescriptive decision making techniques.

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