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On One Aspect of Science Policy Based on An Uncertain Model

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Contents

Abstract

We discuss one aspect of the allocation of new scientists to teaching and research careers. In the past, this allocation problem was treated on the basis of a model with kown constant parameters and as a classical open-loop optimal control problem with the allocation ratio as the sole control variable. The utility function in that treatment took both short term and long term goals into account. Here we allow for uncertainty in the possibly time-varying system parameters, and we account for the possibility of new scientists going into careers other than teaching and research. We treat the allocation problem not as an optimal control one but rather as one of robust control, insensitive to uncertainties, in order to assure desired numbers of teachers and scientists within a computable horizon.

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1 Introduction

The problem of allocation of resources to carry out scientific activities for attaining certain economic and social objectives has been the concern not only of economists, but also of policy makers, and sociopolitical scientists. As both teaching and research scientists contribute directly or indirectly to the scientific progress and hence the economic growth of a society, it is important for policy makers and administrators to allocate public funds in the "best" possible manner. However, in their allocation of resources, including human resources, to scientific activities, they are confronted with a multitude of choices such as the choice among fields of scientific activities and projects to support, as well as the choice among educational and research institutions.

Among others, Intriligator [1], Intriligator and Smith [2], Stoikov [3], Bolt, Koltun, and Levine [4] have investigated one aspect of the problem of choice, namely, the allocation of new doctoral scientists between teaching and research careers. This aspect of the problem has attracted much interest because an inadequate allocation of new scientists teaching in higher education may weaken the educational process. Equally serious is the situation when private and public research establishments do not have a sufficient number of new scientists to carry out important research projects which directly affect the economic growth of a society.

In [1, 2], the authors studied the problem of allocation of new scientists by formulating it as a constrained optimal control problem with the numbers of research scientists and teaching scientists as state variables and the allocation proportion as control variable. The objective function to be maximized is a welfare function which consists of a future welfare component and an intermediate welfare component; the allocation proportion is constrained to lie within two specified limits. In this paper, we attempt to address the allocation problem of new scientists between teaching and research in a manner different from that used in [1, 2]. With uncertainty present in the system parameters and the control constraints, we wish to determine the fractions of new scientists which must be induced to go into teaching and research careers in order to achieve, within a given or at least computable time interval, desired numbers of each or a prescribed number of one or the other. These fractions are obtained as functions of the current numbers of educators and researchers, utilizing the robust control theory developed in [5-8] and summarized in the Appendix.

2 Problem Formulation

In this section, we investigate three variations of the problem of allocating new scientists between teaching and research careers based on the model discussed in [1-4]. However, instead of formulating them as optimal control problems, we investigate them as stabilization problems with the desired number(s) of one or both of state variables specified. The two state variables are $E(t)$, the number of teaching scientists, and $R(t)$, the number of research scientists, at time t. The two control variables $v(t)$ and $w(t)$ are the fractions of new scientists at time t that can be influenced by a policy maker to select teaching and research careers, respectively.

In order to arrive at a simple model for analysis, we assume that all teaching scientists are engaged in full-time teaching and all research scientists are engaged in full-time research. Actually, as pointed out by Intriligator and Smith [2], teaching and research are complementary, they may also compete for the scientists' time and effort. Furthermore, we let $g(t)$ denote the average number of new scientists produced by a teaching scientist, and let $\delta_1(t)$ and $\delta_2(t)$ represent, respectively, the fractional rates of decrease of teaching scientists and research scientists due to death, retirement, or change of career.

2.1 A Teaching Scientist Inducement Model

In this subsection, we model the situation in which only the desired number of teaching scientists is specified. Based on the current number of teaching scientists, $E(t)$, we wish to obtain the fraction of new scientists at time t that should be induced by a policy maker to become teaching scientists so that the prescribed desired number of teaching scientists can be achieved within a given interval of time. Consider the following growth model of teaching scientists

$$\dot{E}(t) = v(t)g(t)E(t) - \delta_1(t)E(t), \tag{1}$$

where $\dot{E}(t) \overset{\triangle}{\rightarrow} = \frac{dE(t)}{dt}$. In (1), some elements of uncertainty have been introduced in the system parameters. The positive functions $g(\cdot)$ and $\delta_1(\cdot)$ are assumed to be unknown but bounded with known bounds. They are assumed to be of the form

$$\begin{aligned} g(\cdot) &= g^* + \triangle g(\cdot), \\ \delta_1(\cdot) &= \delta_1^* + \triangle\delta_1(\cdot), \end{aligned} \tag{2}$$

with the uncertain functions $\triangle g(\cdot)$ and $\triangle\delta_1(\cdot)$ satisfying

$$\begin{aligned} |\triangle g(t)| &\leq \overline{\triangle g} < g^*, \\ |\triangle\delta_1(t)| &\leq \overline{\triangle\delta}_1 < \delta_1^*, \end{aligned} \tag{3}$$

where g^*, $\overline{\triangle g}$, δ_1^*, $\overline{\triangle\delta}_1$, are known positive constants. The control $v(t)$ represents the fraction of new scientists that a policy maker can influence to become teaching scientists through the use of grants, scholarships, salaries, promotion opportunities, and other forms of inducements. Since a policy cannot 'force' a new scientist to enter a career path regardless of the inducements used, $v(t)$ is constrained by

$$0 \leq v_0(t) \leq v(t) \leq v_1(t) \leq 1. \tag{4}$$

We allow for uncertainty in the bounds on $v(t)$ by assuming $v_0(\cdot)$ and $v_1(\cdot)$ to be of the form

$$v_0(\cdot) \stackrel{\triangle}{\rightarrow} = v_0^* + \triangle v_0(\cdot), \qquad v_1(\cdot) \stackrel{\triangle}{\rightarrow} = v_1^* + \triangle v_1(\cdot),$$

with

$$|\triangle v_0(t)| \leq \overline{\triangle v}_0 \leq v_0^* \quad \text{and} \quad |\triangle v_1(t)| \leq \overline{\triangle v}_1,$$

where v_0^*, v_1^*, $\overline{\triangle v}_0$, and $\overline{\triangle v}_1$ are known positive constants.

In view of the fact that $v_0(t) \geq 0$, and $v_1(t) \leq 1$, and since we must assure the desired outcome for all possible realizations of $v(t)$, we must constrain it to

$$0 \leq v_0^* - \overline{\triangle v}_0 \leq v_0^* + \overline{\triangle v}_0 \leq v(t) \leq v_1^* - \overline{\triangle v}_1 \leq v_1^* + \overline{\triangle v}_1 \leq 1. \tag{5}$$

It is worth pointing out that since we do not take $v_1(t) = 1$, we allow for some new scientists entering fields other than teaching.

Consider the transformations

$$\begin{aligned} x_1(t) &\stackrel{\triangle}{\rightarrow} = \ln \frac{E(t)}{E^*}, \\ \eta_1(t) &\stackrel{\triangle}{\rightarrow} = v(t) - \frac{1}{2}(v_0^* + \overline{\triangle v}_0 + v_1^* - \overline{\triangle v}_1), \end{aligned} \tag{6}$$

where E^* is the desired number of teaching scientists. Using (6), equation (1) becomes[1]

$$\dot{x}_1 = (\eta_1 + \eta_1^*)g - \delta_1, \tag{7}$$

$$\eta_1^* \stackrel{\triangle}{\rightarrow} = \frac{1}{2}(v_0^* + \overline{\triangle v}_0 + v_1^* - \overline{\triangle v}_1) \tag{8}$$

and

$$|\eta_1(t)| \leq \overline{\rho}_1,$$

with

$$\overline{\rho}_1 \stackrel{\triangle}{\rightarrow} = \frac{1}{2}(v_1^* - v_0^* - \overline{\triangle v}_0 - \overline{\triangle v}_1). \tag{9}$$

For any given real number $a > 0$, (7) may be written as

$$\dot{x}_1 = -ax_1 + g^*\eta_1 + g^*e_1, \tag{10}$$

where

$$e_1 \stackrel{\triangle}{\rightarrow} = \frac{1}{g^*}[(\eta_1 + \eta_1^*)\triangle g - \triangle\delta_1 + (\eta_1^* g^* - \delta_1^*) + ax_1]. \tag{11}$$

Thus,

$$|e_1| \leq h_{01} + h_{11}|x_1| + h_{21}|\eta_1|,$$

where

$$h_{01} \stackrel{\triangle}{\rightarrow} = \frac{1}{g^*}(|\eta_1^* g^* - \delta_1^*| + \overline{\triangle\delta}_1 + \eta_1^* \overline{\triangle g}),$$

$$h_{11} \stackrel{\triangle}{\rightarrow} = \frac{a}{g^*}, \quad \text{and} \quad h_{21} \stackrel{\triangle}{\rightarrow} = \frac{\overline{\triangle g}}{g^*}.$$

[1]Henceforth, for the sake of brevity, we omit the argument t

Utilizing the results of [5-8], the fraction v of new scientists to be induced to select a teaching career so that x_1 converges at a given exponential rate α, to a given neighborhood of zero (corresponding to $E = E^*$), regardless of the realizations of the uncertain elements $\triangle g,\ \triangle\delta_1,\ \triangle v_0,$ and $\triangle v_1,$ is $v = \eta_1 + \eta_1^*$ with

$$\eta_1 = p(x_1) = -\rho_1 sat\left(\varepsilon^{-1} g^* P x_1\right) - \widetilde{\rho}_1 sat\left(\widetilde{\rho}_1^{-1}\gamma\left(|x_1|\right) g^* P x_1\right) \tag{12}$$

where

$$\rho_1 \stackrel{\triangle}{\to} = \frac{h_{01}}{1-h_{21}}, \qquad \widetilde{\rho}_1 \stackrel{\triangle}{\to} = \overline{\rho}_1 - \rho_1, \qquad \gamma \stackrel{\triangle}{\to} = \frac{\sigma + \mu^{-1} h_{11}^2}{2(1-h_{21})},$$

and, σ and μ are two arbitrarily chosen positive numbers. The saturation function $sat(\cdot)$ is given by

$$sat(z) = \begin{cases} z, & \text{if} \quad |z| \le 1 \\ \frac{z}{|z|}, & \text{if} \quad |z| > 1, \end{cases}$$

For $\alpha > 0$,

$$P = \frac{(\alpha - a) + \sqrt{(\alpha - a)^2 + \sigma\mu g^{*2}}}{\sigma g^{*2}}$$

is the positive root of the Riccati equation

$$\sigma g^{*2} P^2 - 2(\alpha - a)P - \mu = 0.$$

The positive scalar ε is chosen sufficiently small to satisfy

$$\varepsilon < \frac{\alpha c^2}{h_{01}}$$

where c is a positive number such that

$$\frac{\sqrt{P}\left[\sigma\mu + h_{11}^2\right] c}{2\mu(1-h_{21})} \le \widetilde{\rho}_1.$$

For simulation purposes, we select

$$a = 0.5, \quad \alpha = 1, \quad \mu = 1, \quad \sigma = 0.5, \quad E^* = 100, \quad g^* = 1.5, \quad \delta_1^* = 0.35, \quad \overline{\triangle g} = 0.15,$$

$$\overline{\triangle\delta}_1 = 0.035, \quad v_0^* = 0.1, \quad v_1^* = 0.55, \quad \overline{\triangle v}_0 = 0.01, \quad \text{and} \quad \overline{\triangle v}_1 = 0.01$$

It follows that

$$\eta_1^* = 0.325, \quad h_{01} = 0.1475, \quad h_{11} = \frac{1}{3}, \quad h_{21} = 0.1,$$

$$\overline{\rho}_1 = 0.215, \quad \rho_1 = 0.1639, \quad \tilde{\rho}_1 = 0.0511, \quad P = 1.4868,$$

so that we may use $\varepsilon = 0.05$.

Two initial values of E, namely, $E = 30$ and $E = 150$ are used in the simulations, and we consider two realizations of the parameter uncertainties:

$$\begin{cases} \triangle g & = -\overline{\triangle g}\cos 2\pi t \\ \triangle\delta_1 & = \overline{\triangle\delta}_1 \cos 2\pi t \end{cases} \tag{13}$$

$$\begin{cases} \triangle g & : \textit{random variable} \in [-\overline{\triangle g},\ \overline{\triangle g}] \\ \triangle\delta_1 & : \textit{random variable} \in [-\overline{\triangle\delta_1},\ \overline{\triangle\delta_1}] \end{cases} \tag{14}$$

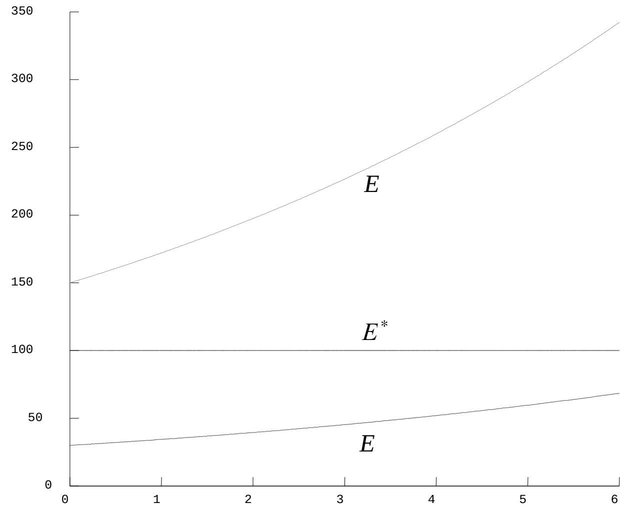

Figure 1: (a) Number of teaching scientists. Time, t (years).

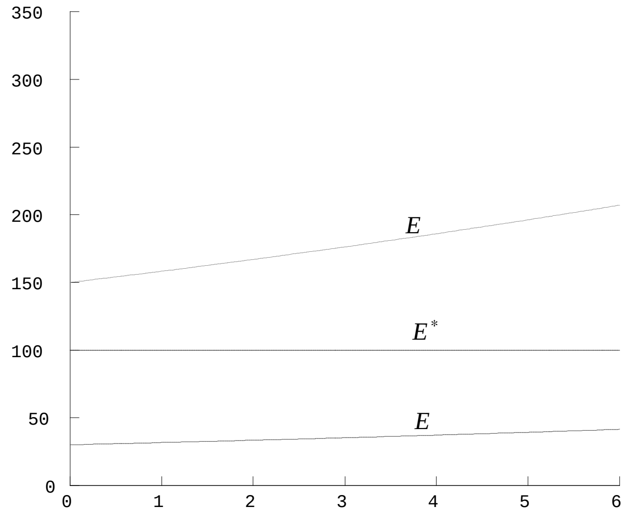

Figure 1: (b) Number of teaching scientists. Time, t (years).

Figure 1(a) shows the time histories of the numbers of teaching scientists, $E(t)$, as a result of inducing the constant fraction $v(t) \equiv \eta_1^*$ of new scientists to select a teaching career in the *absence* of uncertainties. Figure 1(b) displays the time histories of the numbers of teaching scientists as a result of the policy maker utilizing the above mentioned strategy, but subject to the realization of the uncertain disturbances (13).

Figures 2(a) and 2(b) correspond to the realization (13) of uncertain elements while Figures 3(a) and 3(b) correspond to the realization (14).

In Figure 2(a), we display the time histories of the numbers of teaching scientists as a result of utilizing the proposed control strategy (12) as shown in Figure 2(b). Similarly, Figure 3(a) shows the time histories of the numbers of teaching scientists as a result of employing the proposed control strategy (12) as shown in Figure 3(b). We observe that the proposed fraction of new scientists induced by the policy maker to select a teaching career results in the number of teaching scientists reaching a level 'very near' the desired level E^* within fewer than three years.

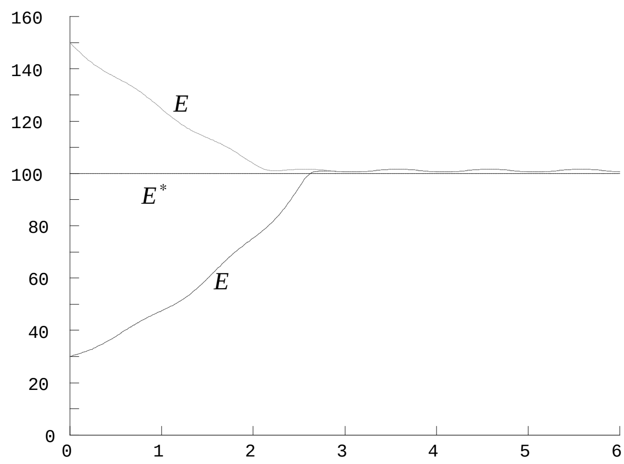

Figure 2: (a)Number of teaching scientists. Time, t (years).

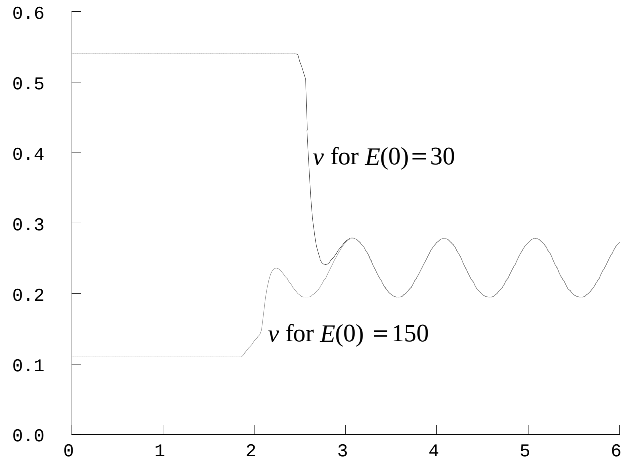

Figure 2: (b)Fraction of new scientists. Time, t (years).

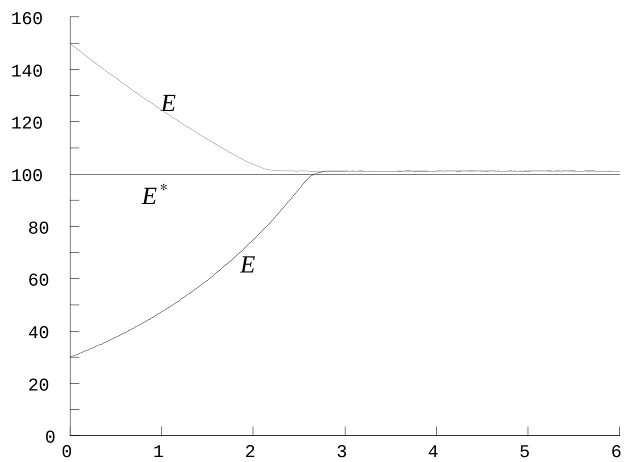

Figure 3: (a)Number of teaching scientists. Time, t (years).

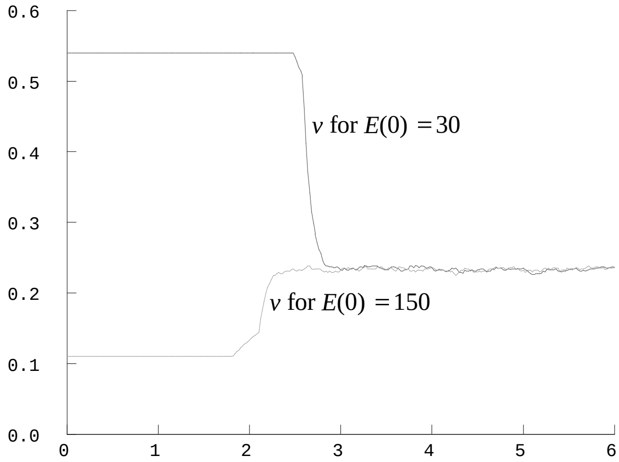

Figure 3: (b)Fraction of new scientists. Time, t (years).

2.2 A Research Scientist Inducement Model

In this subsection, we model the situation in which only the desired number of research scientists is specified. The number of teaching scientists E is regarded as a *known* function of time. This assumption is quite reasonable since information on the number of teaching scientists at any instance of time can be obtained rather easily. Thus we may write

$$E(t) = E^* + \triangle E(t)$$

where $\triangle E(\cdot)$ is a known function satisfying $|\triangle E(t)| \leq \overline{\triangle E}$, and E^*, $\overline{\triangle E}$ are known constants. Consider the following growth model of research scientists

$$\dot{R}(t) = w(t)g(t)E(t) - \delta_2(t)R(t). \tag{15}$$

As in the previous model, (15) also allows for uncertainty in the system parameters. The function $g(\cdot)$ is defined in (2), while the function $\delta_2(\cdot)$ is assumed to be of the form

$$\delta_2(\cdot) = \delta_2^* + \triangle\delta_2(\cdot),$$

with

$$|\triangle\delta_2(t)| \leq \overline{\triangle\delta_2} < \delta_2^*,$$

where δ_2^* and $\overline{\triangle\delta_2}$ are known constants. The control $w(t)$ represents the fraction of new scientists a policy maker should induce to choose the career of research scientists through the implementation of various forms of inducement policy at his disposal. We allow again for the imperfect effect of inducements by considering the constraint

$$0 \leq w_0(t) \leq w(t) \leq w_1(t) \leq 1. \tag{16}$$

In line with the previous model, we also allow for uncertainty in the bounds on $w(t)$ by assuming $w_0(\cdot)$ and $w_1(\cdot)$ to be of the form

$$w_0(\cdot) \overset{\triangle}{\rightarrow} = w_0^* + \triangle w_0(\cdot), \qquad w_1(\cdot) \overset{\triangle}{\rightarrow} = w_1^* + \triangle w_1(\cdot),$$

with

$$|\triangle w_0(t)| \leq \overline{\triangle w_0}, \qquad |\triangle w_1(t)| \leq \overline{\triangle w_1},$$

where w_0^*, w_1^*, $\overline{\triangle w_0}$, and $\overline{\triangle w_1}$ are known positive constants. In view of the constraint (16), as well as the need to assure the desired outcome for all possible realizations of $w(t)$, we must constrain $w(t)$ by

$$0 \le w_0^* - \overline{\triangle w_0} \le w_0^* + \overline{\triangle w_0} \le w(t) \le w_1^* - \overline{\triangle w_1} \le w_1^* + \overline{\triangle w_1} \le 1. \tag{17}$$

Consider the transformations

$$\begin{aligned} x_2(t) &\stackrel{\triangle}{\to} = \frac{R(t) - R^*}{R^*}, \\ \eta_2(t) &\stackrel{\triangle}{\to} = w(t) - \frac{1}{2}(w_0^* + \overline{\triangle w_0} + w_1^* - \overline{\triangle w_1}), \end{aligned} \tag{18}$$

where R^* is the desired number of research scientists. In view of (17) and (18), we have

$$|\eta_2(t)| \le \overline{\rho}_2,$$

where

$$\overline{\rho}_2 \stackrel{\triangle}{\to} = \frac{1}{2}(w_1^* - w_0^* - \overline{\triangle w_0} - \overline{\triangle w_1}). \tag{19}$$

Using (18), equation (15) is transformed to

$$\dot{x}_2 = -\delta_2^* x_2 + \frac{E^* g^*}{R^*}(\eta_2 + e_2)$$

where

$$e_2 \stackrel{\triangle}{\to} = \eta_2^* - \frac{R^*\delta_2^*}{g^*E^*} - \frac{\triangle\delta_2 R^*}{g^*E^*}(1 + x_2) + \eta_2\frac{\triangle E}{E^*} + \frac{\triangle g}{g^*}(\eta_2 + \eta_2^*)(1 + \frac{\triangle E}{E^*}) + \eta_2^*\frac{\triangle E}{E^*},$$

and

$$\eta_2^* \stackrel{\triangle}{\to} = \frac{1}{2}(w_0^* + \overline{\triangle w_0} + w_1^* - \overline{\triangle w_1}). \tag{20}$$

Thus,

$$|e_2| \le h_{02} + h_{12}|x_2| + h_{22}|\eta_2|,$$

where

$$h_{02} \stackrel{\triangle}{\to} = |\eta_2^* - \frac{R^*\delta_2^*}{g^*E^*}| + \eta_2^*(\frac{\overline{\triangle g}}{g^*} + \frac{\overline{\triangle E}}{E^*} + \frac{\overline{\triangle g}}{g^*}\frac{\overline{\triangle E}}{E^*} + \frac{R^*\overline{\triangle\delta_2}}{g^*E^*},$$

$$h_{12} \stackrel{\triangle}{\to} = \frac{\overline{\triangle\delta_2}R^*}{g^*E^*} \quad \text{and} \quad h_{22} \stackrel{\triangle}{\to} = (\frac{\overline{\triangle g}}{g^*} + \frac{\overline{\triangle E}}{E^*} + \frac{\overline{\triangle g}}{g^*}\frac{\overline{\triangle E}}{E^*}).$$

Utilizing the results of [5-8], the fraction w of new scientists to be induced to become research scientists so that x_2 converges at a given exponential rate α, to a given neighborhood of zero (corresponding to $R = R^*$), regardless of the realizations of $\triangle g$, $\triangle\delta_2$, $\triangle w_0$, and $\triangle w_1$, is $w = \eta_2 + \eta_2^*$ with

$$\eta_2 = p(x_2) = -\rho_2 sat\left(\frac{E^*g^*}{R^*\varepsilon}Px_2\right) - \widetilde{\rho}_2 sat\left(\frac{E^*g^*}{R^*\widetilde{\rho}_2}\gamma\left(|x_2|\right)Px_2\right), \tag{21}$$

where

$$\rho_2 \stackrel{\triangle}{\to} = \frac{h_{02}}{1 - h_{22}}, \qquad \widetilde{\rho}_2 \stackrel{\triangle}{\to} = \overline{\rho}_2 - \rho_2, \qquad \gamma \stackrel{\triangle}{\to} = \frac{\sigma + \mu^{-1}h_{12}^2}{2(1 - h_{22})},$$

and, σ and μ are two arbitrarily chosen positive numbers, and $sat(\cdot)$ is the saturation function. For $\alpha > 0$,

$$P = \frac{(\alpha - \delta_2^*) + \sqrt{(\alpha - \delta_2^*)^2 + \sigma\mu \frac{E^{*2} g^{*2}}{R^{*2}}}}{\sigma \frac{E^{*2} g^{*2}}{R^{*2}}}$$

is the positive root of the Riccati equation

$$\sigma\left(\frac{E^{*2} g^{*2}}{R^{*2}}\right) P^2 - 2(\alpha - \delta_2^*)P - \mu = 0.$$

The positive scalar ε is chosen sufficiently small to satisfy

$$\varepsilon < \frac{\alpha c^2}{h_{02}}$$

where c is a positive number such that

$$\frac{E^* g^* \sqrt{P}\left[\sigma\mu + h_{12}^2\right] c}{2R^*\mu(1 - h_{22})} \leq \widetilde{\rho}_2.$$

For simulation purposes, we select

$$\alpha = 1, \quad \mu = 1, \quad \sigma = 0.5, \quad E^* = 100, \quad \overline{\triangle E} = 10, \quad R^* = 120, \quad g^* = 1.5, \quad \delta_2^* = 0.3,$$

$$\overline{\triangle g} = 0.15, \quad \overline{\triangle \delta_2} = 0.03, \quad w_0^* = 0.1, \quad w_1^* = 0.4, \quad \overline{\triangle w_0} = 0.01, \quad \text{and} \quad \overline{\triangle w_1} = 0.01$$

It follows that

$$\eta_2^* = 0.25, \quad h_{02} = 0.0865, \quad h_{12} = 0.024, \quad h_{22} = 0.21,$$

$$\overline{\rho}_2 = 0.14, \quad \rho_2 = 0.1095, \quad \tilde{\rho}_2 = 0.0305, \quad P = 2.339,$$

so that we may use $\varepsilon = 0.025$.

Two initial values of R, $R(0) = 30$ and $R(0) = 150$, are used in the simulations, and we consider the realization of the uncertain elements

$$\begin{cases} \triangle g & = -\overline{\triangle g} \cos 2\pi t \\ \triangle \delta_2 & = -\overline{\triangle \delta_2} \sin 2\pi t, \end{cases} \tag{22}$$

and three prescribed functions $E(t)$, namely,

$$E(t) = 100, \qquad E(t) = E^* + \frac{(3-t)}{3}\overline{\triangle E}, \qquad \text{and} \quad E(t) = E^* - \frac{(3-t)}{3}\overline{\triangle E}.$$

Figure 4(a) shows the time histories of the number of research scientists $R(t)$ corresponding to $E(t) \equiv 100$, as a result of utilizing the strategy $w(t) \equiv \eta_2^*$ in the *absence* of uncertainties. Figure 4(b) depicts the time histories of the number of research scientists corresponding to $E(t) \equiv 100$ and realization (22) of uncertain disturbances, as a result of employing $w(t) \equiv \eta_2^*$.

Figure 5(a) displays the time histories of the number of research scientists corresponding to $E(t) \equiv 100$ and realization (22), as a result of the employment of strategy (21) as shown in Figure 5(b).

In Figures 6–7, we consider $E(t)$ as the linear functions given above. Figures 6(a) and 6(b) correspond to the first linear function $E(t)$ and uncertainty realization (22).

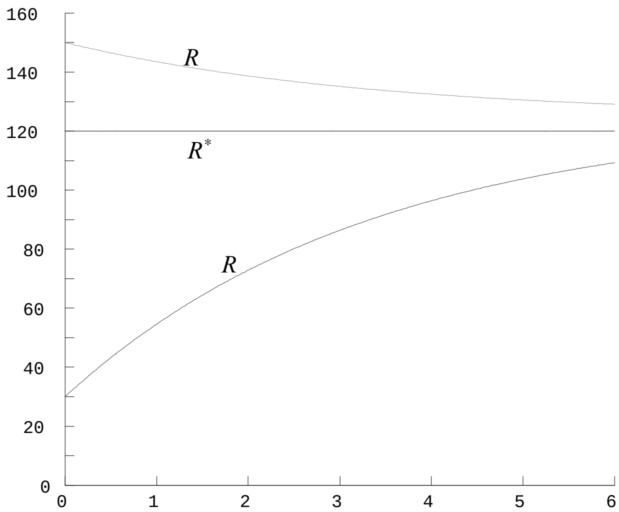

Figure 4: (a)Number of research scientists. Time, t (years).

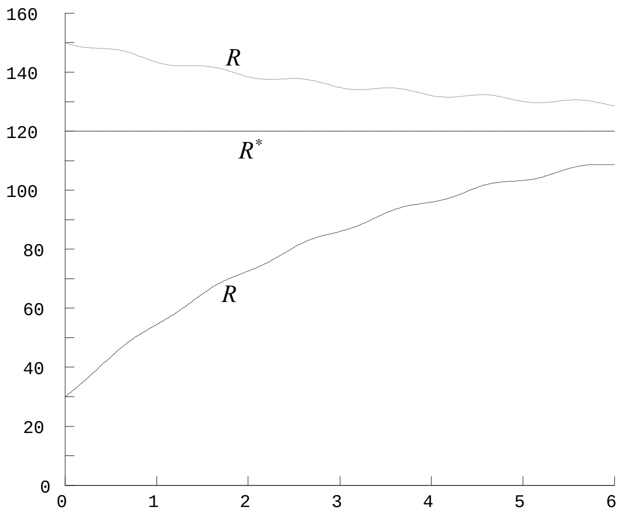

Figure 4: (b)Number of research scientists. Time, t (years).

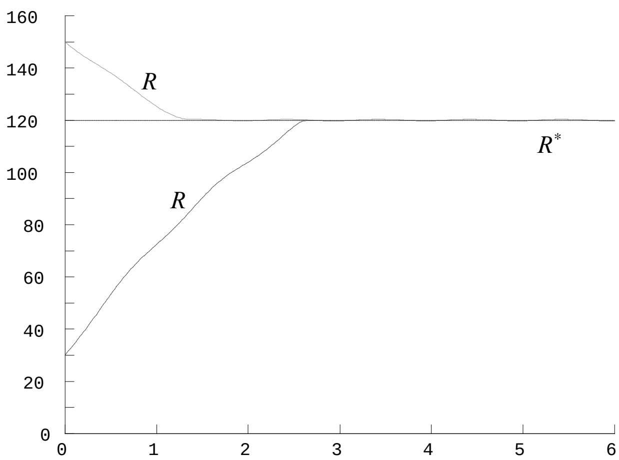

Figure 5: (a) Number of research scientists. ($E(t) \equiv 100$).

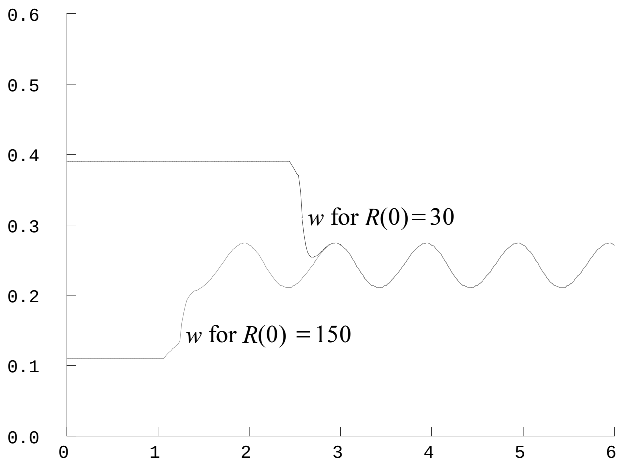

Figure 5: (b)Fraction of new scientists. Time, t (years).

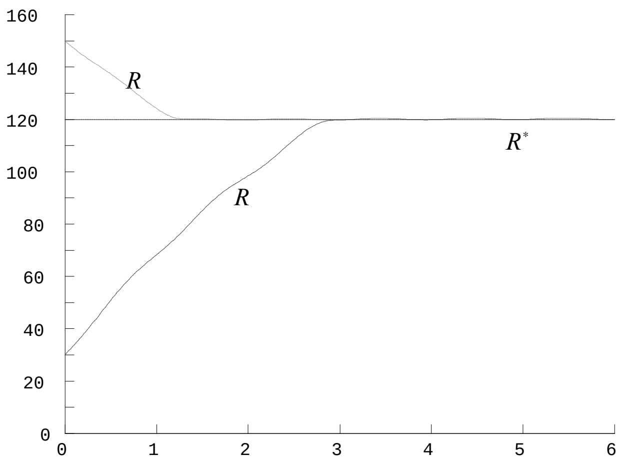

Figure 6: (a) Number of research scientists. ($E(t)$ increases linearly from 90 to 110 over 6 years). Time, t (years)

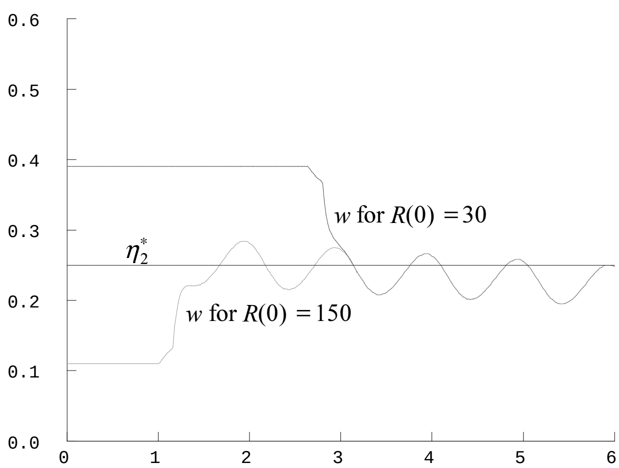

Figure 6: (b)Fraction of new scientists. Time, t (years).

Figure 6(a) shows $R(t)$ as a result of utilizing the proposed control (21) shown in Figure 6(b). Figures 7(a) and 7(b) correspond to the second linear function $E(t)$ and uncertainty realization (22). Figure 7(a) depicts $R(t)$ as a result of using the proposed control (21) displayed in Figure 7(b).

We observe that $E(t)$, the population size of teaching scientists, influences the rate of change of research scientists. As expected, if the initial number of research scientists is far below the desired number, that is $R(0) \ll R^*$, then the larger the number of teaching scientists, the faster is the rate of increase of the research scientists. Furthermore, if the population size of teaching scientists decreases linearly with time, then the policy maker has to increase his inducement efforts (i.e. the period of maximum inducement).

2.3 A Teaching and Research Scientist Inducement Model

In this subsection, we model the situation in which the desired numbers of both teaching and research scientists, E^* and R^*, are prescribed. With uncertainty present in the system parameters and control constraints, we inquire after fractions of new scientists which must be induced to enter academic and research careers, respectively, in order to achieve, within a computable time interval, the prescribed numbers of teaching and research scientists.

Consider the following model of teaching and research scientists

$$\begin{aligned} \dot{E}(t) &= v(t)g(t)E(t) - \delta_1(t)E(t) \\ \dot{R}(t) &= w(t)g(t)E(t) - \delta_2(t)R(t) \end{aligned} \tag{23}$$

Here, the notations used are same as those employed in the previous two subsections. The state variables $E(t)$ and $R(t)$ represent, respectively, the numbers of teaching scientists and research scientists at time t. The control variables $v(t)$ and $w(t)$ are the fractions of new scientists that should be induced to take up teaching and research careers, respectively. The functions $g(\cdot)$, $\delta_1(\cdot)$ and $\delta_2(\cdot)$ are assumed to be unknown positive functions with known bounds. In particular, they are of the form

$$\begin{aligned} g(t) &= g^* + \triangle g(t), \\ \delta_1(t) &= \delta_1^* + \triangle \delta_1(t), \\ \delta_2(t) &= \delta_2^* + \triangle \delta_2(t), \end{aligned}$$

with

$$\begin{aligned} |\triangle g(t)| &\leq \overline{\triangle g} < g^* \\ |\triangle \delta_1(t)| &\leq \overline{\triangle \delta}_1 < \delta_1^* \\ |\triangle \delta_2(t)| &\leq \overline{\triangle \delta}_2 < \delta_2^*, \end{aligned}$$

where g^*, δ_1^*, δ_2^*, $\overline{\triangle g}$, $\overline{\triangle \delta}_1$, and $\overline{\triangle \delta}_2$ are known positive constants. Since a new scientist cannot be forced to enter a particular career path regardless of the inducements employed by policy makers, the controls are constrained by

$$\begin{aligned} 0 \leq v_0(t) \leq v(t) \leq v_1(t), \\ 0 \leq w_0(t) \leq w(t) \leq w_1(t), \end{aligned} \tag{24}$$

where we allow for uncertainty in the bounds by assuming

$$\begin{aligned} v_0(t) &= v_0^* + \triangle v_0(t), \qquad v_1(t) = v_1^* + \triangle v_1(t), \\ w_0(t) &= w_0^* + \triangle w_0(t), \qquad w_1(t) = w_1^* + \triangle w_1(t), \end{aligned} \tag{25}$$

with

$$|\triangle v_0(t)| \leq \overline{\triangle v}_0, \qquad |\triangle v_1(t)| \leq \overline{\triangle v}_1,$$
$$|\triangle w_0(t)| \leq \overline{\triangle w}_0, \qquad |\triangle w_1(t)| \leq \overline{\triangle w}_1, \tag{26}$$

where v_0^*, v_1^*, w_0^*, w_1^*, $\overline{\triangle v}_0$, $\overline{\triangle v}_1$, $\overline{\triangle w}_0$, and $\overline{\triangle w}_1$ are known positive constants. We also model the situation in which new scientists may enter careers other than teaching and research by imposing

$$v_1(t) + w_1(t) \leq 1. \tag{27}$$

In view of (25), (27) leads to

$$v_1^* + \overline{\triangle v}_1 + w_1^* + \overline{\triangle w}_1 \leq 1. \tag{28}$$

Furthermore, on considering the control constraints with uncertain bounds (24)–(26), and the need for the controls to assure the desired behavior for all possible realizations of (24), we must constrain them to

$$v_0^* + \overline{\triangle v}_0 \leq v(t) \leq v_1^* - \overline{\triangle v}_1,$$
$$w_0^* + \overline{\triangle w}_0 \leq w(t) \leq w_1^* - \overline{\triangle w}_1 \tag{29}$$

resulting in constraints (5) and (17).

Consider the transformation of state and control variables

$$x(t) \stackrel{\triangle}{\to} = \frac{E(t) - E^*}{E^*}, \quad y(t) \stackrel{\triangle}{\to} = \frac{R(t) - R^*}{R^*},$$
$$\eta_1(t) \stackrel{\triangle}{\to} = v(t) - \eta_1^*, \quad \eta_2(t) \stackrel{\triangle}{\to} = w(t) - \eta_2^*,$$

where η_1^* and η_2^* are defined in (8) and (20), respectively.

The system (23) becomes

$$\dot{x} = (\eta_1 + \eta_1^*) g(1 + x) - \delta_1(1 + x)$$
$$\dot{y} = \frac{E^*}{R^*}(\eta_2 + \eta_2^*) g(1 + x) - \delta_2(1 + y), \tag{30}$$

and (29) yields

$$|\eta_1(t)| \leq \overline{\rho}_1, \quad |\eta_2(t)| \leq \overline{\rho}_2,$$

where $\overline{\rho}_1$ and $\overline{\rho}_2$ are defined in (9) and (19) respectively.

(30) may be written in the form

$$\dot{z} = Az + B_1\eta_1 + B_2\eta_2 + B_1 e_1 + B_2 e_2 \tag{31}$$

where

$$z \stackrel{\triangle}{\to} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A \stackrel{\triangle}{\to} = \begin{pmatrix} \eta_1^* g^* - \delta_1^* & 0 \\ 0 & -\delta_2^* \end{pmatrix}, \quad B_1 \stackrel{\triangle}{\to} = \begin{pmatrix} g^* \\ 0 \end{pmatrix}, \quad B_2 \stackrel{\triangle}{\to} = \begin{pmatrix} 0 \\ \frac{g^* E^*}{R^*} \end{pmatrix},$$

$$e_1 \stackrel{\triangle}{\to} = \eta_1 x + \eta_1^* + \frac{\eta_1 \triangle g}{g^*}(1 + x) + \frac{\eta_1^* \triangle g}{g^*}(1 + x) - \frac{\delta_1^*}{g^*} - \frac{\triangle \delta_1}{g^*}(1 + x)$$

$$e_2 \stackrel{\triangle}{\to} = \eta_2^* - \frac{R^* \delta_2^*}{E^* g^*} - \frac{R^* \triangle \delta_2}{E^* g^*}(1 + y) + (\eta_2 + \eta_2^*)x + \frac{\triangle g}{g^*}(1 + x)(\eta_2 + \eta_2^*)$$

Thus,

$$|e_1| \leq k_{01} + k_{11}\|z\| + k_{21}|\eta_1| \quad \text{and} \quad |e_2| \leq k_{02} + k_{12}\|z\| + k_{22}|\eta_2|,$$

where

$$k_{01} \overset{\triangle}{\to} = |\eta_1^* - \frac{\delta_1^*}{g^*}| + \eta_1^* \frac{\overline{\triangle g}}{g^*} + \frac{\overline{\triangle \delta_1}}{g^*},$$

$$k_{11} \overset{\triangle}{\to} = \overline{\rho}_1 (1 + \frac{\overline{\triangle g}}{g^*}) + \eta_1^* \frac{\overline{\triangle g}}{g^*} + \frac{\overline{\triangle \delta_1}}{g^*}, \qquad k_{21} \overset{\triangle}{\to} = \frac{\overline{\triangle g}}{g^*},$$

and

$$k_{02} \overset{\triangle}{\to} = |\eta_2^* - \frac{R^* \delta_2^*}{E^* g^*}| + \eta_2^* \frac{\overline{\triangle g}}{g^*} + \frac{\overline{\triangle \delta_2} R^*}{E^* g^*},$$

$$k_{12} \overset{\triangle}{\to} = (\overline{\rho}_2 + \eta_2^*)(1 + \frac{\overline{\triangle g}}{g^*}) + \frac{\overline{\triangle \delta_2} R^*}{E^* g^*}, \qquad k_{22} \overset{\triangle}{\to} = \frac{\overline{\triangle g}}{g^*}.$$

Utilizing the results of [5–8], the proposed fractions v and w of new scientists that should be induced to select teaching and research careers, respectively, so that system (31) is uniformly exponentially convergent with given rate α to a given neighborhood of zero (corresponding to $E = E^*$ and $R = R^*$), for any realization of the uncertain disturbances $\triangle g,\ \triangle \delta_1,\ \triangle \delta_2,\ \triangle v_0,\ \triangle v_1,\ \triangle w_0,$ and $\triangle w_1,$ are given by

$$v = \eta_1 + \eta_1^*, \qquad w = \eta_2 + \eta_2^*$$

with

$$\eta_i = p_i(z) = -\rho_i sat\left(\varepsilon^{-1} B_i^T P z\right) - \widetilde{\rho}_i sat\left(\widetilde{\rho}_i^{-1} \gamma_i(\|z\|) B_i^T P z\right), \quad i = 1,\ 2 \tag{32}$$

where

$$\rho_i \overset{\triangle}{\to} = \frac{k_{0i}}{1 - k_{2i}}, \qquad \widetilde{\rho}_i \overset{\triangle}{\to} = \overline{\rho}_i - \rho_i, \qquad \gamma_i\left(\|z\|\right) \overset{\triangle}{\to} = \frac{\sigma + 2\mu^{-1} k_{1i}^2\left(\|z\|\right)}{2(1 - k_{2i})}, \quad i = 1,\ 2.$$

The positive numbers σ and μ may be chosen arbitrarily, and P is a positive-definite symmetric matrix which satisfies the Riccati equation

$$P(A + \alpha I) + (A + \alpha I)^T P - \sigma P B B^T P + \mu I = 0 \tag{33}$$

where $B = (B_1\ B_2)$. Since the controllability matrix of $(A,\ B)$ has maximum rank, the existence of a positive-definite solution to (33) is assured. The positive scalar ε is chosen sufficiently small to satisfy

$$\varepsilon < \varepsilon^* = \frac{\alpha c^{*2}}{k_0}$$

with

$$c^* \overset{\triangle}{\to} = \min\left[c_1,\ c_2\right], \quad k_0 \overset{\triangle}{\to} = \sum_{i=1}^{2} k_{0i}$$

and $c_i > 0$ is such that

$$\lambda_i \left[\sigma\mu + 2k_{1i}^2(\lambda c_i)\right] c_i \leq 2\mu(1 - k_{2i})\widetilde{\rho}_i.$$

where

$$\lambda_i \overset{\triangle}{\to} = \lambda_{\max}(B_i^T P B_i)^{\frac{1}{2}}, \quad \lambda \overset{\triangle}{\to} = \lambda_{\min}(P)^{-\frac{1}{2}}, \quad i = 1,\ 2.$$

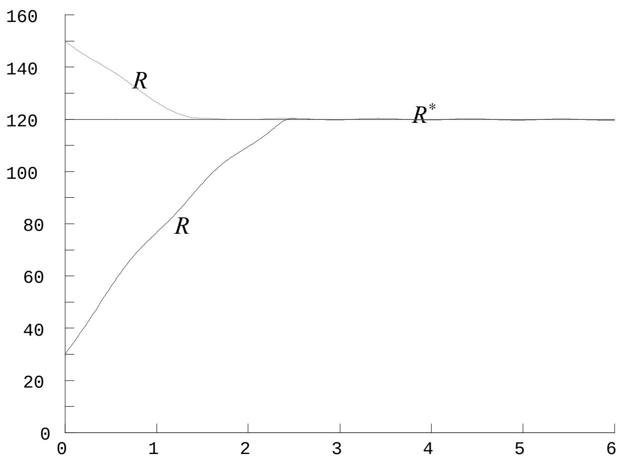

Figure 7: (a) Number of research scientists. ($E(t)$ increases linearly from 90 to 110 over 6 years). Time, t (years)

For simulation purposes, we choose

$$\alpha = 1, \quad \mu = 1, \quad \sigma = 0.5, \quad E^* = 100, \quad R^* = 120, \quad g^* = 1.5, \quad \delta_1^* = 0.35, \quad \delta_2^* = 0.3,$$

$$v_0^* = 0.1, \quad v_1^* = 0.55, \quad w_0^* = 0.1, \quad w_1^* = 0.4, \quad \overline{\triangle g} = 0.15, \quad \overline{\triangle\delta}_1 = 0.035, \quad \overline{\triangle\delta}_2 = 0.03,$$

$$\overline{\triangle v}_0 = 0.01, \quad \overline{\triangle v}_1 = 0.01, \quad \overline{\triangle w}_0 = 0.01, \quad \overline{\triangle w}_1 = 0.01 \ .$$

It follows that

$$\eta_1^* = 0.325, \quad \eta_2^* = 0.25, \quad \overline{\rho}_1 = 0.215, \quad \overline{\rho}_2 = 0.14, \quad \rho_1 = 0.1639, \quad \rho_2 = 0.0656,$$

$$\tilde{\rho}_1 = 0.0511, \quad \tilde{\rho}_2 = 0.0744, \quad p_{11} = 2.3936, \quad p_{12} = 0, \quad p_{22} = 2.3392$$

so that we may use $\varepsilon = 0.01$.

In our simulations, we use four pairs of initial values of E and R, namely, $(E(0), R(0)) = (30, 30), (150, 150), (30, 150)$, and $(150, 30)$, and we consider the following two realizations of the uncertain elements:

$$\begin{cases} \triangle g & = -\overline{\triangle g} \cos 2\pi t \\ \triangle\delta_1 & = \overline{\triangle\delta}_1 \cos 2\pi t \\ \triangle\delta_2 & = -\overline{\triangle\delta}_2 \sin 2\pi t \end{cases} \tag{34}$$

and

$$\begin{cases} \triangle g & : \textit{random variable} \in [-\overline{\triangle g}, \ \overline{\triangle g}] \\ \triangle\delta_1 & : \textit{random variable} \in [-\overline{\triangle\delta_1}, \ \overline{\triangle\delta_1}] \\ \triangle\delta_2 & : \textit{random variable} \in [-\overline{\triangle\delta_2}, \ \overline{\triangle\delta_2}] \end{cases} \tag{35}$$

Figures 8(a),(b) display, respectively, the time histories of the numbers of teaching and research scientists when constant fractions $v = \eta_1^*, \ w = \eta_2^*$ of new scientists are induced to become teaching and research scientists in the *absence* of uncertain disturbances.

In Figures 9-13, the realization (34) of the uncertain elements is used. Figures 9(a), 9(b) depict, respectively, the time histories of the numbers of teaching and research scientists when constant fractions $v = \eta_1^*, \ w = \eta_2^*$ of new scientists are induced to select teaching and research careers.

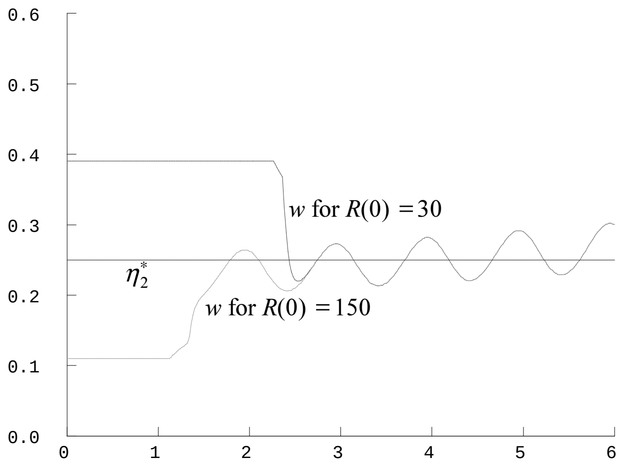

Figure 7: (b)Fraction of new scientists. Time, t (years).

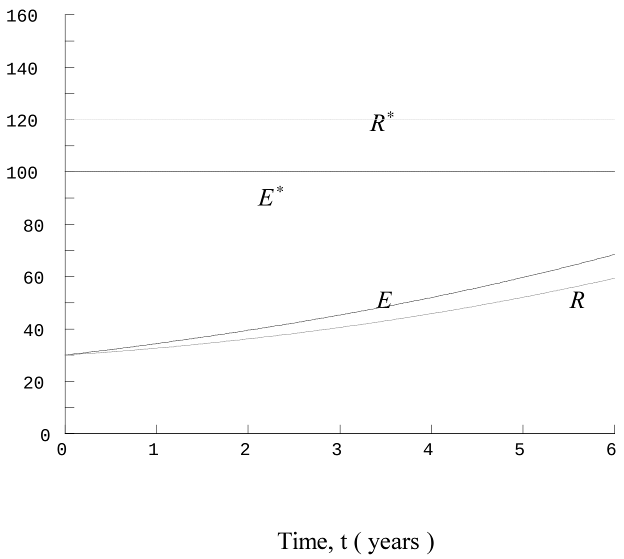

Figure 8: (a) Number of teaching and research scientists. Time, t (years)

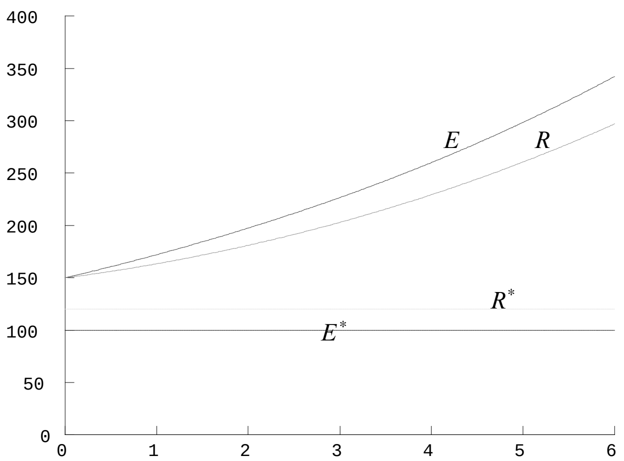

Figure 8: (b)Number of teaching and research scientists. Time, t (years).

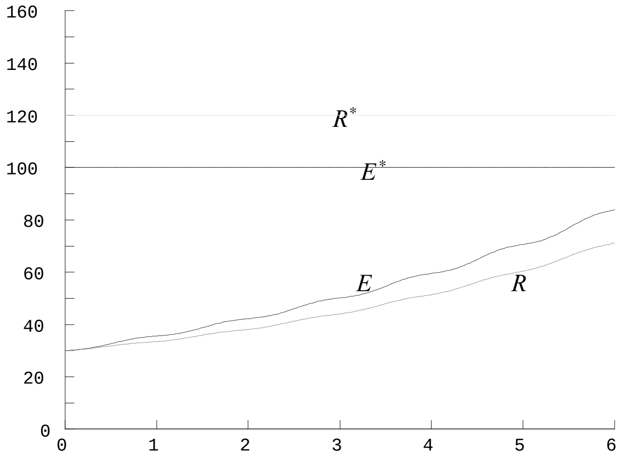

Figure 9: (a) Number of teaching and research scientists. Time, t (years)

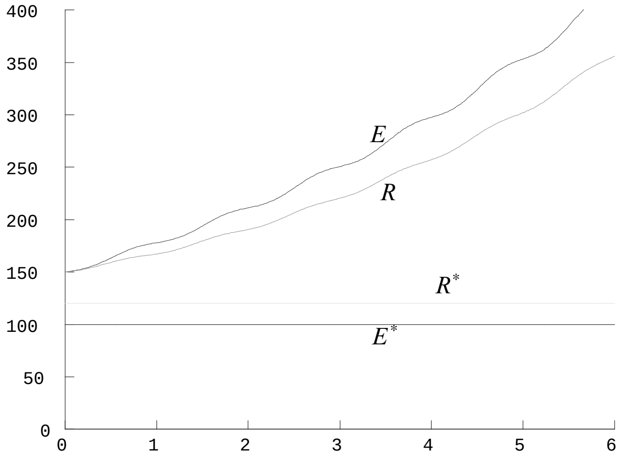

Figure 9: (b)Number of teaching and research scientists. Time, t (years).

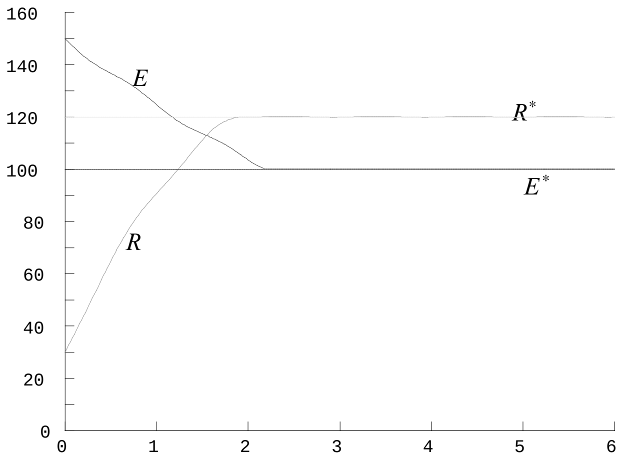

Figure 10: (a) Number of teaching and research scientists. Time, t (years)

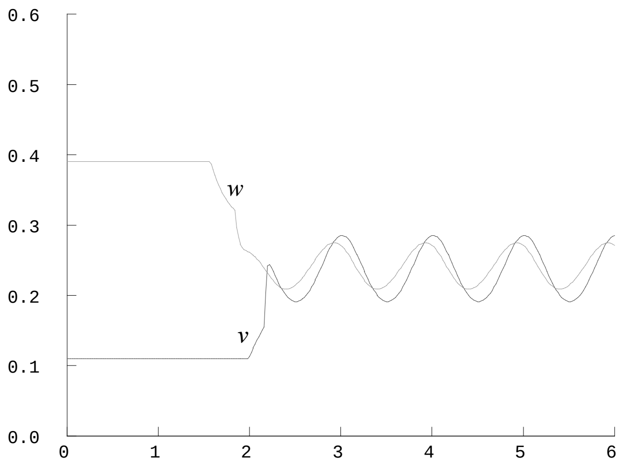

Figure 10: (b)Fraction of new scientists. Time, t (years).

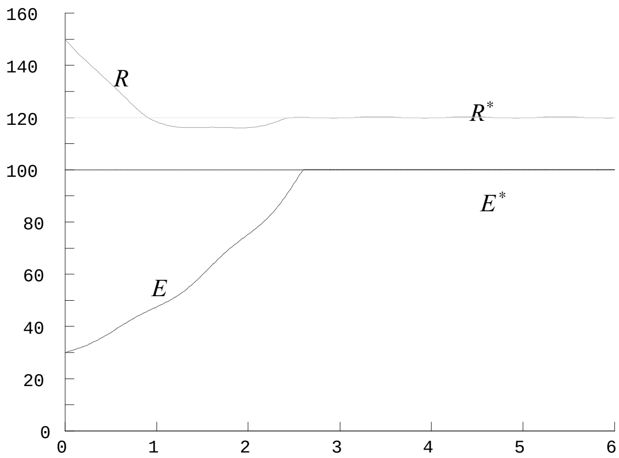

Figure 11: (a) Number of teaching and research scientists. Time, t (years)

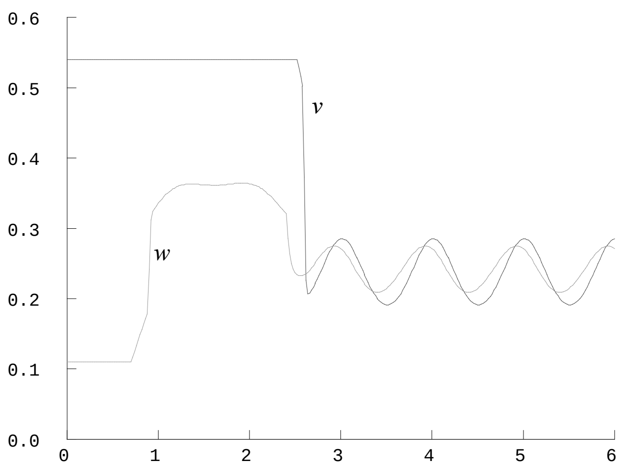

Figure 11: (b)Fraction of new scientists. Time, t (years).

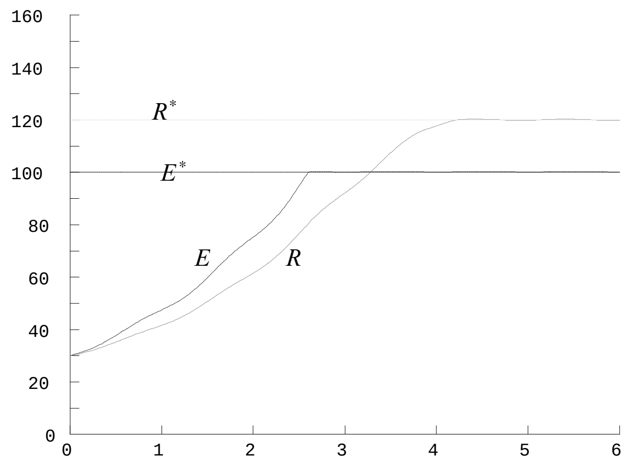

Figure 12: (a) Numbers of teaching and research scientists. Time, t (years)

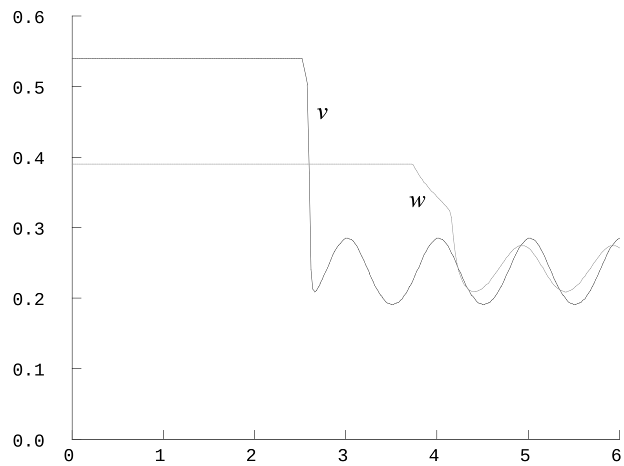

Figure 12: (b)Fractions of new scientists. Time, t (years).

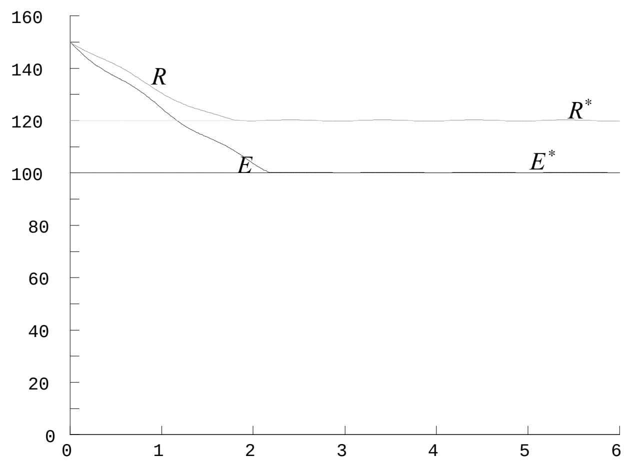

Figure 13: (a) Numbers of teaching and research scientists. Time, t (years)

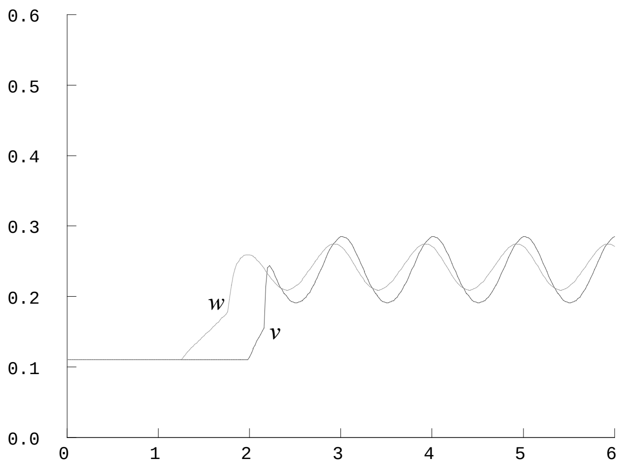

Figure 13: (b)Fractions of new scientists. Time, t (years).

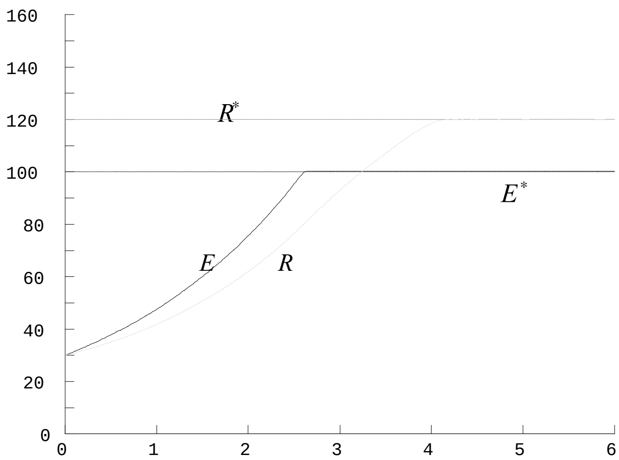

Figure 14: (a) Numbers of teaching and research scientists. Time, t (years)

Figures 10(a)-13(a) show the time histories of the numbers of teaching and research scientists as a result of utilizing the proposed control (32) displayed in Figures 10(b)-13(b).

In Figures 14–15, we consider realization (35) of the uncertain elements. Figures 14(a) and 15(a) display the time histories of the numbers of teaching and research scientists as a result of utilizing the control (32) shown in Figures 14(b) and 15(b).

We observe that if it is desired to attain prescribed numbers of teaching and research scientists, then the strategy of inducing constant fractions of new scientists to enter teaching and research careers is ineffective and hence leads to poor allocation of resources.

Among the Figures 10–13, Figures 11(a),(b) and 12(a),(b) are the most interesting. In Figure 11(a), we notice that a small number of teaching scientists can cause the number of research scientists to fall below the desired level R^* even though the latter begin with a large number. The policy maker takes about $1\frac{1}{2}$ years to rectify this problem by providing more inducement to new scientists so that they are influenced to select research careers. In Figure 12(a), it is observed that when both the numbers of teaching and research scientists are small, the population size of the research scientists lags behind that of the teaching scientists by about $1\frac{1}{2}$ years in approaching their respective desired levels.

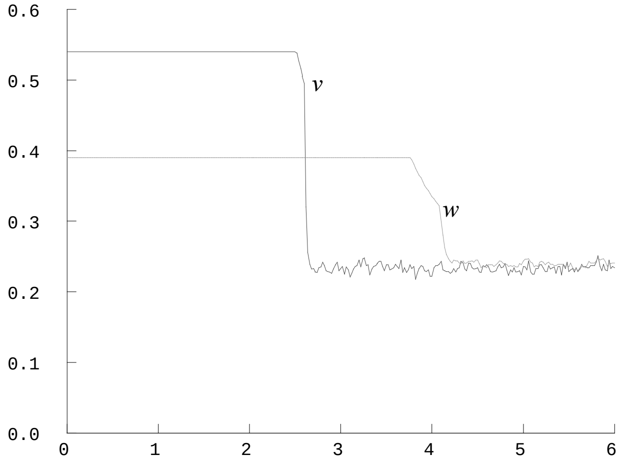

Figure 14: (b)Fractions of new scientists. Time, t (years).

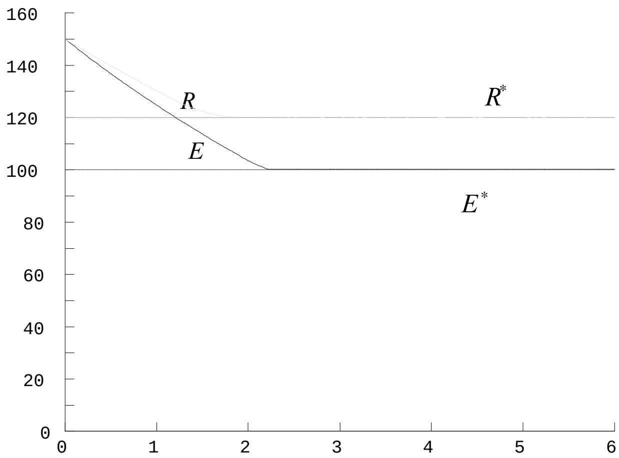

Figure 15: (a) Number of teaching and research scientists. Time, t (years)

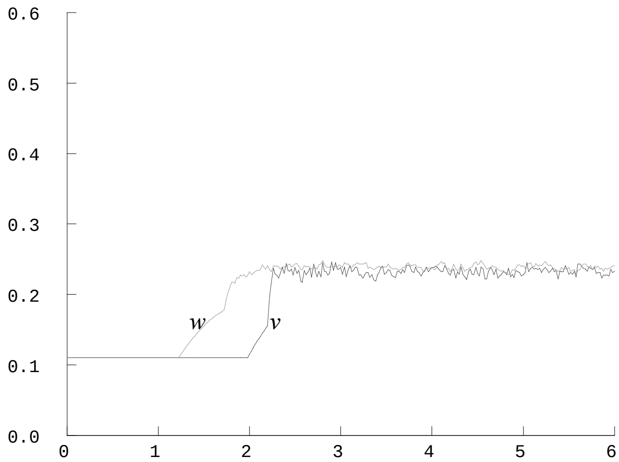

Figure 15: (b)Fractions of new scientists. Time, t (years).

3 Conclusion

We have examined one aspect of science policy based on a simple model with uncertain elements occurring in the system parameters and in the bounds of the control variables. Three allocation problems are formulated in this paper, all three are stabilization rather than optimization problems. In these problems, one or both of the desired levels of the state variables are specified, and stabilizing strategies in the form of fractions of new scientists to be induced by a policy maker to select teaching and research careers are obtained so that the desired numbers of teaching and research scientists can be achieved in given or at least computable time regardless of the realization of the uncertain elements. We observe that the use of constant strategies may lead to ineffective allocation of resources.

As mentioned earlier, teaching scientists often perform research tasks in addition to their teaching duties. This might be taken into account by adjusting the desired number of research scientists R^*. Let m denote the fraction of teaching scientists engaged in significant research. Then mE^* constitutes a portion of the desired number of research scientists. Hence, the desired number of scientists who enter a research rather than an academic career need be only $R^* - mE^*$.

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4 Appendix

A.1 Problem Statement

We consider uncertain systems described by

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{l} [B_i u_i(t) + \triangle F_i(t, x(t), u_i(t))] \tag{A.1}$$

where $t \in R$ is the time variable, $x(t) \in R^n$ is the state and $u_i(t) \in R^{m_i},\ i = 1, 2, \cdots l$ are control inputs. The continuous function f and the constant matrices $B_i,\ i = 1, 2, \cdots l$, are known; they define the *nominal system*

$$\dot{x}(t) = f(x(t)) + Bu(t), \tag{A.2}$$

where $u = (u_1, u_2, \cdots, u_l)^T, \quad B = (B_1, B_2, \cdots, B_l)$. All the uncertainty and time-dependence in the system is represented by the terms $\triangle F_i$ which are assumed to be continuous functions.

Each control input u_i is subject to a hard constraint of the form

$$\|u_i(t)\| \leq \overline{\rho}_i \tag{A.3}$$

where the bound $\overline{\rho}_i > 0$ is prescribed. We shall consider the control input u_i to be generated by a memoryless state feedback controller p_i, i.e.,

$$u_i(t) = p_i(x(t)) \tag{A.4}$$

The resulting *closed loop system* is described by

$$\dot{x}(t) = F(t, x(t)) \tag{A.5}$$

with

$$F(t, x) \stackrel{\triangle}{\to} = f(x) + \sum_{i=1}^{l} [B_i p_i(x) + \triangle F_i(t, x, p_i(x))]. \tag{A.6}$$

For any scalar $r \geq 0$, the ball of radius r is defined by $\mathcal{B}(r) \stackrel{\triangle}{\to} = \{\ x \in R^n\ :\ \|x\| \leq r\}$.

Consider any scalar $\alpha > 0$ and any set $\mathcal{A} \subset R^n$ containing a neighborhood of the origin.

Definition. System (A.5) is *uniformly exponentially convergent* to $\mathcal{B}(r)$ with rate α and *region of attraction* $\mathcal{A}$ if there exists a scalar $\beta \geq 0$ such that the following hold.

(i) *Existence of solutions.* For each $t_0 \in R$ and $x_0 \in \mathcal{A}$ there exists a solution $x(\cdot) :\ [\ t_0,\ t_1) \to R^n,\ t_0 < t_1$, of (A.5) with $x(t_0) = x_0$.

(ii) *Indefinite extension of solutions.* Every solution $x(\cdot) :\ [t_0,\ t_1) \to R^n$ of (A.5) with $x(t_0) \in \mathcal{A}$, has an extension $\overline{x}(\cdot) :\ [t_0,\ \infty) \to R^n$, i.e., $\overline{x}(t) = x(t)$ for all $t \in [t_0,\ t_1)$ and $\overline{x}(\cdot)$ is a solution of (A.5).

(iii) *Uniform exponential convergence of solutions.* If $x(\cdot) :\ [t_0,\ \infty) \to R^n$ is any solution of (A.5) with $x(t_0) \in \mathcal{A}$, then

$$\|x(t)\| \leq r + \beta \|x(t_0)\| \exp(-\alpha(t - t_0)) \quad \forall t \geq t_0.$$

The problem we wish to consider is as follows:

Problem Statement Consider a system described by (A.1) subject to control constraints (A.3) and let $\alpha > 0$ and $r \geq 0$ be specified scalars. Find memoryless state feedback controllers $p_i,\ i = 1, 2, \cdots, l$, which render the closed loop system (A.5) uniformly exponentially convergent to $\mathcal{B}(r)$ with rate α.

A.2 Assumptions on Uncertainty

The following assumption, which is sometimes referred to as a matching condition, is common in the literature on control of uncertain systems.

Assumption 1. For each $i = 1, 2, \cdots, l$, there is a function $e_i(\cdot)$ such that

$$\triangle F_i = B_i e_i. \tag{A.7}$$

Assumption 2. For each $i = 1, 2, \cdots, l$, there exist nonnegative scalars k_{0i}, k_{2i}, with

$$k_{2i} < 1 \tag{A.8}$$

and a continuous nondecreasing nonnegative function k_{1i} such that

$$\|e_i(t, x, u_i)\| \leq k_{0i} + k_{1i}(\|x\|)\|x\| + k_{2i}\|u_i\| \tag{A.9}$$

for all $t \in R,\ x \in R^n,\ u_i \in R^{m_i}$.

Assumption 3. For each $i = 1, 2, \cdots, l$,

$$\overline{\rho}_i > \frac{k_{0i}}{1 - k_{2i}}. \tag{A.10}$$

A.3 Constrained Control Assuring Exponential Convergence

The proposed control meeting the requirement of the Problem Statement in A.1 is

$$u_i = -\rho_i sat(\varepsilon^{-1} B_i^T Px) - \widetilde{\rho}_i sat(\widetilde{\rho_i}^{-1} \gamma_i(\|x\|) B_i^T Px), \quad i = 1, 2, \cdots, l, \tag{A.11}$$

where P satisfies

Assumption 4. There exist positive-definite symmetric matrices P and Q and a scalar $\sigma \geq 0$ which satisfy

$$2x^T Pf(x) \leq -2\alpha x^T Px - x^T Qx + \sigma x^T PBB^T Px \tag{A.12}$$

Remark. If the nominal system is linear and controllable, that is, the nominal system is

$$\dot{x} = Ax + Bu$$

with (A, B) controllable, then (A.12) is met for each positive-definite symmetric Q and each σ and $\alpha > 0$, since there exists a positive definite symmetric P which satisfies the Riccati equation

$$P(A + \alpha I) + (A + \alpha I)^T P - \sigma PBB^T P + Q = 0 \tag{A.13}$$

Furthermore,

$$\rho_i \overset{\triangle}{\rightarrow} = (1 - k_{2i})^{-1} k_{0i}, \tag{A.14}$$

γ_i is any continuous function which satisfies

$$\gamma_i(\|x\|) \geq \frac{\sigma}{2(1 - k_{2i})} + \frac{l k_{1i}(\|x\|)^2}{2\mu(1 - k_{2i})} \tag{A.15}$$

with

$$\mu \overset{\triangle}{\rightarrow} = \lambda_{\min}(Q), \tag{A.16}$$

and the saturation function sat is given by

$$sat(y) = \begin{cases} y, & \text{if} \quad |y| \leq 1 \\ \frac{y}{|y|}, & \text{if} \quad |y| > 1, \end{cases}$$

and

$$\widetilde{\rho}_i \stackrel{\triangle}{\rightarrow} = \overline{\rho}_i - \rho_i. \tag{A.17}$$

The positive real scalar ε is chosen sufficiently small to satisfy

$$\varepsilon < \varepsilon^* \stackrel{\triangle}{\rightarrow} = \frac{\alpha c^{*2}}{k_0} \tag{A.18}$$

with

$$c^* \stackrel{\triangle}{\rightarrow} = \min\{ \ c_i \ : \ i = 1, 2, \cdots, l\} \tag{A.19}$$

where $c_i > 0$ satisfies

$$\lambda_i \left[\sigma\mu + l k_{1i} (\lambda c_i)^2 \right] c_i \leq 2\mu(1 - k_{2i})\widetilde{\rho}_i \tag{A.20}$$

and

$$\lambda_i \stackrel{\triangle}{\rightarrow} = \lambda_{\max}(B_i^T P B_i)^{\frac{1}{2}}, \qquad \lambda \stackrel{\triangle}{\rightarrow} = \lambda_{\min}(P)^{-\frac{1}{2}}.$$

Before introducing the main result, consider any real scalar $c \geq 0$ and define the Lyapunov ellipsoid

$$\varepsilon(c) \stackrel{\triangle}{\rightarrow} = \{ \ x \in R^n \ : \ x^T P x \leq c^2 \}.$$

Theorem [8]. Consider an uncertain system described by (A.1), satisfying assumptions 1-4 and subject to bounded control given by (A.11). Then the resulting closed loop system (A.5) is uniformly exponentially convergent to $\mathcal{B}(r_\varepsilon)$ with rate α and region of attraction $\mathcal{A} = \varepsilon(c^*)$ where

$$r_\varepsilon = \left(\frac{\varepsilon k_0}{\alpha \lambda_{\min}(P)} \right)^{\frac{1}{2}}, \qquad k_0 = \sum_{i=1}^{l} k_{0i}$$

and

$$\beta = \left(\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \right)^{\frac{1}{2}}.$$