International Institute for Applied Systems Analysis • A-2361 Laxenburg • Austria
Tel: +43 2236 807 • Fax: +43 2236 71313 • E-mail: info@iiasa.ac.at • Web: www.iiasa.ac.at

INTERIM REPORT IR-97-041 / July

Limit Theorems for Stationary Distributions of Birth-and-Death Processes

Yu. Kaniovski (kaniov@iiasa.ac.at)
G. Pflug (pflug@iiasa.ac.at)

Approved by
Govanni Dosi (dosi@iiasa.ac.at)
TED Project, *IIASA*

Interim Reports on work of the International Institute for Applied Systems Analysis receive only limited review. Views or opinions expressed herein do not necessarily represent those of the Institute, its National Member Organizations, or other organizations supporting the work.

Abstract

Birth-and-death processes or, equivalently, finite Markov chains with three-diagonal transition matrices proved to be adequate models for processes in physics [12], biology [4,5], sociology [13] and economics [1,3,10]. The analysis in this case quite often relies on the stationary distribution of the chain. Representing it as a Gibbs distribution, we study its limit behavior as the number of states increases.

We show that the limit nests on the set of global minima of the limit Gibbs potential. If the set consists of a finite number k of singletons a_i where the second derivatives α_i of the potential are positive, the limit distribution assigns probability

$$\frac{1/\sqrt{\alpha_i}}{\sum_{j=1}^{k} 1/\sqrt{\alpha_j}}$$

to a_i. When at some points the second derivative is zero, the limit distribution nests only on them, we describe it explicitly. If the set of minima consists of a finite number of singletons and intervals, the limit distribution concentrates only on intervals. We obtain a formula for it.

Key Words: birth-and-death process, stationary distribution, Gibbs distribution, global minimum.

Contents

Limit Theorems for Stationary Distributions of Birth-and-Death Processes

Yu. Kaniovski (kaniov@iiasa.ac.at)
G. Pflug (pflug@iiasa.ac.at)

1 Motivation and Formulation of the Problem

Imagine a population whose evolution is governed by a Markov chain. We shall be dealing with a time-homogeneous Markov chain $\xi_N^t,\ t \geq 0$, assuming a finite number of values $0, 1, \ldots, N-1$, the states of the population. Let

$$p_i^{(N)} = P\{\xi_N^{t+1} = i+1 | \xi_N^t = i\},\ \ q_i^{(N)} = P\{\xi_N^{t+1} = i-1 | \xi_N^t = i\},$$

$$r_i^{(N)} = P\{\xi_N^{t+1} = i | \xi_N^t = i\},$$

where $p_i^{(N)} + q_i^{(N)} + r_i^{(N)} = 1$ for every i. Thus, the probability transition matrix is three-diagonal. Such random processes are called birth-and-death processes [8, p. 50]. Indeed, the transition from i to $i+1$ can be interpreted as birth (emergence) of one more object (say, an economic agent) of a certain type. While the transition from i to $i-1$ means death (disappearance) of such an object. Set $\zeta_N^t = \xi_N^t / N,\ t \geq 0$. This chain nests on $[0, 1)$. If ξ_N^t describes the evolution through time of the absolute value, for example, the number of agents who have adopted a certain technology, then ζ_N^t captures the dynamic of the relative quantity corresponding to this value, say, the proportion (share) of agents who have adopted this technology. In applications people look at what happens to the population in the long run, that is, as time goes to infinity. Thus we have to turn to the stationary distribution D_N of the chains. It exists and is uniquely defined by the following relations (see [8, p. 51])

$$D_N = \{d_i^{(N)},\ i = 0, 1, \ldots, N-1\},\ \ d_i^{(N)} = \lim_{t\to\infty} P\{\xi_N^t = i\} =$$

$$\lim_{t\to\infty} P\{\zeta_N^t = i/N\},\ \ d_i^{(N)} = d_0^{(N)} \prod_{j=1}^{i} \frac{p_{j-1}^{(N)}}{q_j^{(N)}},\ i = 1, 2, \ldots, N-1,$$

$$d_0^{(N)} = \left[1 + \sum_{i=1}^{N-1} \prod_{j=1}^{i} \frac{p_{j-1}^{(N)}}{q_j^{(N)}}\right]^{-1}$$

if $p_{i-1}^{(N)} > 0$ and $q_i^{(N)} > 0$ for $1 \leq i \leq N-1$. Quite often it is important to know the behavior of the stationary distribution as N increases. At this point one can set

$$\Xi_N(x) = \begin{cases} d_i^{(N)} & \text{for } \frac{i}{N} \leq x < \frac{i+1}{N},\ 0 \leq i \leq N-2, \\ d_{N-1}^{(N)} & \text{for } 1 - 1/N \leq x \leq 1, \end{cases}$$

and to look at the limit behavior of $\Xi_N(\cdot)$ as $N \to \infty$ (see, for example, [9]). More conventionally, one represents $\Xi_N(\cdot)$ as a Gibbs distribution, that is,

$$\Xi_N(i/n) = d_0^{(N)} \exp[-N\Phi_N(i/N)] \tag{1.1}$$

with $\Phi_N(\cdot)$ called the Gibbs potential (see [1], p. 57),

$$\Phi_N(x) = \begin{cases} 0 & \text{for } 0 \le x < 1/N, \\ -\frac{1}{N}\sum_{j=1}^{i} \ln \frac{p_{j-1}^{(N)}}{q_j^{(N)}} & \text{for } \frac{i}{N} \le x < \frac{i+1}{N},\ 1 \le i \le N-2, \\ -\frac{1}{N}\sum_{j=1}^{N-1} \ln \frac{p_{j-1}^{(N)}}{q_j^{(N)}} & \text{for } 1 - 1/N \le x \le 1. \end{cases}$$

Throughout the paper we assume that for every N there is a unique stationary distribution D_N of the chain ζ_N^t, $t \ge 0$, and we are looking at its limit as $N \to \infty$. Conceptually we are interested in the limit behavior (in the sense of distributions) when first time goes to infinity and then the size of the system N also increases without bound.

Consider an intuition which is (with different degrees of rigour) behind the analysis in many of applied papers on this issue. Set $\Delta\zeta_N^t = \zeta_N^{t+1} - \zeta_N^t$. Then

$$E(\Delta\zeta_N^t | \zeta_N^t = i/N) = \frac{1}{N}(p_i^{(N)} - q_i^{(N)}),$$

$$E\Big[(\Delta\zeta_N^t)^2 | \zeta_N^t = i/N\Big] = \frac{1}{N^2}(p_i^{(N)} + q_i^{(N)}).$$

Let $p_i^{(N)} = f_N(i/N)$ and $q_i^{(N)} = g_N(i/N)$. For $u \ge 0$ define a step-function

$$x_N^M(u) = \zeta_N^{M+i} \quad \text{if} \quad \frac{i}{N} \le u < \frac{i+1}{N},$$

where M is a positive integer, so $x_N^M(0) = \zeta_N^M$. Let there exist Lipschitz functions $f(\cdot)$ and $g(\cdot)$ such that

$$\lim_{N\to\infty} \sup_{x\in[0,1]} \Big[|f_N(x) - f(x)| + |g_N(x) - g(x)|\Big] = 0. \tag{1.2}$$

If ζ_N^M weakly converges as $N \to \infty$, $M \to \infty$ (that is, first M goes to infinity, then N goes to infinity) to a random variable ζ^*, we can show that for every finite $T > 0$ the random processes $x_N^M(\cdot)$ weakly converge on $C[0,T]$ as $N \to \infty, M \to \infty$ to the curve $x_{\zeta^*}(\cdot)$ belonging to $[0,1]$. (The argument is similar to the one given in §3 of Chapter II of [5].) The limit satisfies the relations

$$\frac{dx_{\zeta^*}}{dt} = f(x_{\zeta^*}) - g(x_{\zeta^*}), \quad x_{\zeta^*}(0) = \zeta^*. \tag{1.3}$$

By $C[0,T]$ we designate the space of continuous on $[0,T]$ functions endowed with the topology of uniform convergence.

Thus, *if ζ^* is a weak limit point for ζ_N^t as $N \to \infty$, $t \to \infty$, then the weak limit of $x_N^M(\cdot)$ as $N \to \infty$, $M \to \infty$ satisfies (1.3) provided that (1.2) holds true.*

Since the chain ζ_N^t belongs to $[0,1)$, we have that $g(0) = 0$ and $f(1) = 0$. By continuity this implies: $f(\cdot) - g(\cdot) \ge 0$ in a neighborhood of 0 and $f(\cdot) - g(\cdot) \le 0$

in a neighborhood of 1. Thus, $[0,1]$ is an invariant set for the differential equation involved in (1.5).

If M is sufficiently large, then the distributions of ζ_N^{M+i}, $i \geq 0$, are arbitrarily close to the stationary one for a fixed N. Hence, the distributions of $x_N^M(s)$ and $x_N^M(t)$ almost coincide for all $0 \leq s < t < \infty$. This implies that $x_{\zeta^*}(s)$ and $x_{\zeta^*}(t)$ are equally distributed for all $s \neq t$ and this distribution is ζ^*. Also, for a fixed x_0, $x_{x_0}(t)$ converges to a singular point of (1.3) as $t \to \infty$. The singular points are solutions of the equation

$$f(x) = g(x). \tag{1.4}$$

Hence, $x_{\zeta^*}(t)$ deterministically converges (and, consequently, weakly) to a singular point as $t \to \infty$. Since T can be arbitrarily large and the distribution of $x_{\zeta^*}(t)$ is ζ^* for all $t \geq 0$, we conclude that *the weak limits of ζ_N^t (as $N \to \infty$, $t \to \infty$) can nest only on the set of singular points,* that is, with probability one

$$f(\zeta^*) = g(\zeta^*) \tag{1.5}$$

provided that (1.2) holds true.

If there is a single solution of (1.4) on $[0,1]$, then (1.5) characterizes completely the limit distribution, which is deterministic. But if there is more than one solution of (1.4) on $[0,1]$, the characterization is unsufficiently precise. For example, some of the solutions are stable in terms of the dynamic system (1.3), others are not. The criterion (1.5) does not distinguish between such points, although our intuition suggests that unstable singular points should not be attained by the limit. Furthermore, if (1.4) holds for an interval, (1.5) does not allow to characterize the distribution generated by ζ^* on this interval. Thus, we need a more delicate instrument than (1.5) to analyze the limit behavior of stationary distributions as $N \to \infty$.

2 Convergence to the Global Minimum of the Limit Potential

At this section we first look at the limit behavior of

$$P_N^\epsilon = P\{\Phi_N(\mu_N) - \Phi_N^* \geq \epsilon\}.$$

Here ϵ designates a positive number; μ_N stands for a random variable such that

$$P\{\mu_N = i/N\} = d_i^{(N)},\ i = 0, 1, \ldots, N-1;$$

$\Phi_N^* = \min_{x \in [0,1]} \Phi_N(\cdot)$. Since $p_{i-1}^{(N)} > 0$ and $q_i^{(N)} > 0$ for $i = 1, 2, \ldots, N-1$, Φ_N^* is a finite value.

By (1.1) at every state i/N where $\Phi_N(\cdot)$ exceeds its minimal value, the stationary probability wipes out as $N \to \infty$ faster than at a state where the minimum is attained. This intuition is confirmed by the following statement which can be thought of as a large deviation result for the random variables $\Phi_N(\mu_N) - \Phi_N^*$.

Theorem 2.1. $P_N^\epsilon \leq N \exp(-N\epsilon)$ for every $\epsilon > 0$.

Proof. We have that

$$P_N^\epsilon = \sum_{i:\, \Phi_N(i/N) - \Phi_N^* \geq \epsilon} d_i^{(N)} =$$

$$d_0^{(N)} \sum_{i:\, \Phi_N(i/N)-\Phi_N^* \geq \epsilon} \exp[-N\Phi_N(i/N)] =$$

$$d_0^{(N)} \exp(-N\Phi_N^*) \sum_{i:\, \Phi_N(i/N)-\Phi_N^* \geq \epsilon} \exp\{-N[\Phi_N(i/N) - \Phi_N^*]\} \leq$$

$$d_0^{(N)} \exp(-N\Phi_N^*) \sum_{i:\, \Phi_N(i/N)-\Phi_N^* \geq \epsilon} \exp(-N\epsilon) \leq$$

$$d_0^{(N)} \exp(-N\Phi_N^*) N \exp(-N\epsilon).$$

To accomplish the proof it is enough to notice that $P\{\mu_N = x_N^*\} = d_0^{(N)} \exp(-N\Phi_N^*) \leq 1$ for every x_N^* such that $\Phi_N(x_N^*) = \Phi_N^*$.

Corollary 2.1. $P_N^\epsilon \to 0$ as $N \to \infty$ for every $\epsilon > 0$.

Set

$$F_N(x) = \begin{cases} p_{i-1}^{(N)}/q_i^{(N)} & \text{for } [Nx] = i-1, \ x \in [0,1), \\ p_{N-1}^{(N)}/q_N^{(N)} & \text{for } x = 1. \end{cases}$$

Let there exist a function $F(\cdot)$ such that

$$\sup_{x\in[0,1]} |F_N(x) - F(x)| = \sigma_N \to 0 \tag{2.1}$$

as $N \to \infty$. From now on we shall assume that $\ln F(\cdot)$ is Riemann integrable on $[0,1]$. Then

$$\Phi(x) = -\int_0^x \ln F(u)du$$

is a continuous function for $0 \leq x \leq 1$. It is differentiable on $(0,1)$. We call this function the limit Gibbs potential.

Notice that (1.2) implies that $F(x) = f(x)/g(x)$ if $g(x) > 0$. Furthermore, since $\Phi'(x) = -\ln F(x)$, we obtain that $\Phi'(x) = -\ln[f(x)/g(x)]$. Consequently, all singular points of $\Phi(\cdot)$ satisfy (1.4) if (1.2) holds true. Hence (1.5) relates to the necessary condition of extremum for $\Phi(\cdot)$. The result to be given in this section sharpens the characterization provided by (1.5) showing that the limit distributions nest on the set of global minima of $\Phi(\cdot)$. Notice, that under (1.2) each point of minimum of $\Phi(\cdot)$ turns out to be a stable attractor of the differential equation involved in (1.3). Thus, the description of the limits in terms of global minima proves to be sharper than any one based on the analysis of stability of the limit differential equation. Because the set of global minima may contain a point where $F(\cdot)$ is discontinuous, the characterization given here generalizes the one based on the necessary condition. Since $\Phi_N(x)$ is almost an integral sum for $\Phi(x)$, intuitively the result we are going to obtain follows from Corollary 2.1.

let $[a]$ be the integer part of a real number a. By $o_N(1)$ we shall designate nonnegative sequences, not necessarily equal, converging to 0 as $N \to \infty$. We say that a function $F(\cdot)$ is Hölder on $[a,b]$ if there is $\gamma \in (0,1]$ such that

$$|F(x) - F(y)| \leq L|x-y|^\gamma$$

for every $x, y \in [a,b]$. Here L is called the Hölder constant, γ is called the Hölder exponent. If $\gamma = 1$ the function is Lipschitz and L is its Lipschitz constant.

Since for $x < y$

$$\Phi(y) - \Phi(x) = -\int_x^y \ln F(u) du,$$

we obtain the following result.

Lemma 2.1. Assume that for some $0 \le a < b \le 1$ the function $F(\cdot)$ is continuous and positive on $[a,b]$. Then for every $x, y \in [a,b]$, $x < y$

$$|\Phi(y) - \Phi(x)| \le \max_{u \in [x,y]} |\ln F(u)||y - x|.$$

Lemma 2.2. Let (2.1) holds true. Assume that for some $0 \le a < b \le 1$ the function $F(\cdot)$ is Hölder and positive on $[a,b]$. Then for every $x, y \in [a,b]$

$$\Phi_N(x) - \Phi_N(y) = \Phi(x) - \Phi(y) + \Delta(N, x, y),$$

where

$$|\Delta(N,x,y)| \le (b-a)\Big\{N^{-\gamma}\frac{L}{c_{[a,b]}}[1 + o_N(1)] + \frac{\sigma_N}{c_{[a,b]}}\Big\} +$$

$$2N^{-1}C_{[a,b]},$$

$$c_{[a,b]} = \min_{x \in [a,b]} F(x), \qquad C_{[a,b]} = \max_{x \in [a,b]} |\ln F(x)|,$$

L stands for the Hölder constant of $F(\cdot)$ on $[a,b]$.

Proof. By hypothesis, the function $F(\cdot)$ is continuos and positive on $[a,b]$. Hence $0 < c_{[a,b]} \le \bar{c}_{[a,b]} < \infty$ and $\ln(\cdot)$ is a Lipschitz function on $[c_{[a,b]}, \bar{c}_{[a,b]}]$ whose Lipschitz constant does not exceed $1/c_{[a,b]}$. Here $\bar{c}_{[a,b]} = \max_{x \in [a,b]} F(x)$. Thus, the constants involved in the estimate for $\Delta(N, x, y)$ exist.

Let $x < y$, then

$$\Phi_N(y) - \Phi_N(x) - \Phi(y) + \Phi(x) =$$

$$-\frac{1}{N}\sum_{i=[Nx]+1}^{[Ny]} \ln F_N(i/N) + \int_x^y \ln F(u) du =$$

$$-\sum_{i=[Nx]+1}^{[Ny]} \int_{i/N}^{i/N+1/N} \ln F_N(i/N) dv + \int_x^y \ln F(u) du =$$

$$-\sum_{i=[Nx]+1}^{[Ny]-1} \int_{i/N}^{i/N+1/N} [\ln F_N(i/N) - \ln F(u)] du -$$

$$\frac{1}{N} \ln F_N([Ny]/N) + \int_{[Ny]/N}^{y} \ln F(u) du. \tag{2.2}$$

Notice that

$$\ln F_N(i/N) = \ln F(i/N) + \ln\Big[1 + \frac{F_N(i/N) - F(i/N)}{F(i/N)}\Big] \tag{2.3}$$

and for $[Nx] + 1 \leq i \leq [Ny] - 1$

$$|\ln F(i/N) - \ln F(u)| \leq \frac{1}{c_{[a,b]}}|F(i/N) - F(u)| \leq$$

$$\frac{L}{c_{[a,b]}}N^{-\gamma} \text{ if } u \in [i/N, i/N + 1/N]. \tag{2.4}$$

Since $\ln(1 + x) \leq x$ and by (2.1)

$$\sup_{[Nx]\leq i\leq[Ny]} \Big|\frac{F_N(i/N) - F(i/N)}{F(i/N)}\Big| \leq \sigma_N / c_{[a,b]},$$

we obtain that

$$\Big|\ln\Big[1 + \frac{F_N(i/N) - F(i/N)}{F(i/N)}\Big]\Big| \leq \frac{\sigma_N}{c_{[a,b]}}. \tag{2.5}$$

The statement of the lemma follows from (2.2) – (2.5).

Remark 2.1. In the proof we actually used the rate of uniform convergence of $F_N(\cdot)$ to $F(\cdot)$ only on $[a, b]$.

Let for an $\epsilon > 0$

$$X_N^\epsilon = \{x \in [0,1] : \ \Phi_N(x) - \Phi_N^* < \epsilon\},$$

$$X^\epsilon = \{x \in [0,1] : \ \Phi(x) - \Phi^* < \epsilon\},$$

where $\Phi^* = \min_{x\in[0,1]} \Phi(x)$.

The following statement follows from Corollary 2.1.

Theorem 2.2. If for every $\epsilon > 0$ there is a real $\epsilon' > 0$ and a positive integer N' depending on ϵ and such that $X^\epsilon \supseteq X_N^{\epsilon'}$ for $N \geq N'$, then

$$P\{\Phi(\mu_N) - \Phi^* < \epsilon\} \to 1 \quad \text{as} \quad N \to \infty.$$

By Lemma 2.1 the hypothesis of Theorem 2.2 holds true if on $[0, 1]$ the function $F(\cdot)$ is Hölder and positive. But there are less restrictive conditions ensuring this hypothesis.

Theorem 2.2 establishes weak convergence of $\Phi(\mu_N)$ to Φ^* as $N \to \infty$. To obtain weak convergence of μ_N to the set $X^* = \{x \in [0,1] : \ \Phi(x) = \Phi^*\}$ (that is, when the Euclidean distance between them goes weakly to zero), we need additionally some regularity condition.

Since $\Phi(\cdot)$ is a continuous function, the set X^* is closed. From now on we shall be assuming that it consists of a finite number of connected components: singletons a_i, $i = 1, 2, \ldots, k$, and intervals $[b_j, c_j]$, $j = 1, 2, \ldots, l$. Also, let there be continuous functions $\psi_i(\cdot)$ and $\Psi_j(\cdot)$ such that:

$\Phi(x) = \Phi^* + \psi_i(x - a_i)$ in a neighborhood of a_i

and

$\Phi(x) = \Phi^* + \Psi_j(\min[x - b_j, \max(0, x - c_j)])$ in a neighborhood of $[b_j, c_j]$.
We call them growth functions, if they are decreasing for negative values of the argument and increasing for positive values of the argument. Also, $\psi_i(0) = \Psi_j(0) = 0$.

Theorem 2.3. Let hypothesis of Theorem 2.2 hold true, the set X^* consist of a finite number of connected components possessing growth functions. Then μ_N weakly converges to X^* as $N \to \infty$.

Proof. For a given $\epsilon > 0$ there are positive numbers $\delta_i^-(\epsilon)$, $\delta_i^+(\epsilon)$, $\Delta_j^-(\epsilon)$ and $\Delta_j^+(\epsilon)$ such that

$$\Phi^* + \epsilon = \psi_i(a_i - \delta_i^-(\epsilon)) = \psi_i(a_i + \delta_i^+(\epsilon)) =$$

$$\Psi_j(b_j - \Delta_j^-(\epsilon)) = \Psi_j(c_j + \Delta_j^+(\epsilon))$$

for all possible i and j. Also, by continuity and monotonicity of the growth functions

$$\delta_i^-(\epsilon) \to 0, \quad \delta_i^+(\epsilon) \to 0, \quad \Delta_j^-(\epsilon) \to 0, \quad \Delta_j^+(\epsilon) \to 0 \tag{2.6}$$

as $\epsilon \to 0$.

Then

$$P\{\Phi(\mu_N) - \Phi^* < \epsilon\} = \sum_{i=1}^{k} p_\epsilon^{i,N} + \sum_{j=1}^{l} P_\epsilon^{j,N}, \tag{2.7}$$

provided that ϵ is so small that the intervals $\Big(a_i - \delta_i^-(\epsilon), a_i + \delta_i^+(\epsilon)\Big)$, $i = 1, 2, \ldots, k$, and $\Big(b_j - \Delta_j^-(\epsilon), c_j + \Delta_j^+(\epsilon)\Big)$, $j = 1, 2, \ldots, l$, do not overlap. Here

$$p_\epsilon^{i,N} = P\{\mu_N \in \Big(a_i - \delta_i^-(\epsilon), a_i + \delta_i^+(\epsilon)\Big)\}$$

and

$$P_\epsilon^{j,N} = P\{\mu_N \in \Big(b_j - \Delta_j^-(\epsilon), c_j + \Delta_j^+(\epsilon)\Big)\}.$$

Since in (2.7) the value ϵ can be arbitrarily small, the statement of the theorem follows from Theorem 2.2 and (2.6).

Theorem 2.3 states that all weak limits of D_N as $N \to \infty$ are concentrated with probability one in X^*. It might happen that some of the limits put zero weights on certain connected components of X^*. Now we shall calculate the probabilities that the limits assign to different connected components of X^* and identify conditions of uniqueness of the limit of D_N.

3 Local Limit Theorems

By $o_\epsilon(1)$ we shall designate nonnegative values, not necessarily equal, converging to 0 as $\epsilon \to 0$. Also, $\Delta(\epsilon, N)$ stands for nonnegative values, not necessarily equal, such that $\lim_{\epsilon\to 0}\lim_{N\to\infty} \Delta(\epsilon, N) = 0$.

Lemma 3.1 Let

1) for some $\delta > 0$ the function $F(\cdot)$ be Lipschitz on $[a_i - \delta, a_i + \delta]$ and $|F_N(x) - F(x)| \le c/N$ for every x from this interval;

2) for some $\gamma_i \geq 2$ and $\alpha_i > 0$

$$\lim_{\sigma\to 0} \sup_{|u|\leq\sigma} |\frac{\psi_i(u)}{|u|^{\gamma_i}} - \alpha_i| = 0.$$

If there is a sequence $\{x_N^i\}$ such that $\Phi_N(x_N^i) = \Phi_N^*$ and $x_N^i \to a_i$ as $N \to \infty$, then

$$\lim_{\epsilon\to 0} \lim_{N\to\infty} \frac{p_\epsilon^{i,N} \alpha_i^{1/\gamma_i} \gamma_i}{d_0^{(N)} \exp(-N\Phi^*) N^{1-1/\gamma_i} 2\Gamma(1/\gamma_i)} = 1,$$

where $\Gamma(\cdot)$ designates the complete gamma-function,

$$\Gamma(z) = \int_0^\infty \exp(-u) u^{z-1} du.$$

Proof. We have that

$$p_\epsilon^{i,N} = d_0^{(N)} \sum_{j/N\in I(\epsilon)} \exp[-N\Phi_N(j/N)] =$$

$$d_0^{(N)} \exp(-N\Phi^*) N A(\epsilon, N), \tag{3.1}$$

where

$$A(\epsilon, N) = \frac{1}{N} \sum_{j/N\in I(\epsilon)} \exp\{-N[\Phi_N(j/N) - \Phi^*]\},$$

$$I(\epsilon) = \Big(a_i - \delta_i^-(\epsilon), a_i + \delta_i^+(\epsilon)\Big).$$

Applying Lemma 2.2, we obtain

$$A(\epsilon, N) = \frac{1}{N} \sum_{j/N\in I(\epsilon)} \exp\{-N[\Phi(j/N) - \Phi^*]\} r_1(j, N), \tag{3.2}$$

where

$$r_1(j, N) = \exp\{-N[\Phi_N(a_i) - \Phi(a_i) + \Delta(N, j/n, a_i)]\}, \tag{3.3}$$

$$|\Delta(N, j/N, a_i)| \leq o_\epsilon(1) N^{-1} [\frac{L+c}{c_{I(\epsilon)}} + o_\epsilon(1)]. \tag{3.4}$$

In the latter estimate we took into account that the set $I(\epsilon)$ is an interval shrinking to zero as $\epsilon \to 0$, which implies

$$c_{I(\epsilon)} \to 1 \text{ and } C_{I(\epsilon)} \to 0.$$

By hypothesis $x_N^i \in I(\epsilon)$ for all sufficiently large N, hence applying Lemma 2.2

$$\Phi_N(x_N^i) - \Phi_N(a_i) = \Phi(x_N^i) - \Phi(a_i) + \Delta(N, x_N^i, a_i).$$

Since $\Phi_N(x_N^i) - \Phi_N(a_i) \leq 0$ and $\Phi(x_N^i) - \Phi(a_i) \geq 0$, this relation implies

$$|\Phi_N(a_i) - \Phi(a_i)| \leq \frac{3}{2} |\Delta(N, x_N^i, a_i)|. \tag{3.5}$$

Taking into account that

$$\lim_{x\to 0}[\exp(x)-1]/x=1, \tag{3.6}$$

by (3.2) – (3.5) we obtain

$$[1-\Delta(\epsilon,N)]B(\epsilon,N)\le A(\epsilon,N)\le[1+\Delta(\epsilon,N)]B(\epsilon,N), \tag{3.7}$$

where

$$B(\epsilon,N)=\frac{1}{N}\sum_{j/N\in I(\epsilon)}\exp\{-N[\Phi(j/N)-\Phi^*]\}.$$

Notice that

$$B(\epsilon,N)=\sum_{j/N\in I(\epsilon)}\int_{j/N}^{j/N+1/N}\exp\{-N[\Phi(j/N)-\Phi(a_i)]\}du=$$

$$\sum_{j/N\in I(\epsilon)}\int_{j/N}^{j/N+1/N}\exp\{-N[\Phi(u)-\Phi(a_i)]\}r_2(j,N)du,$$

where

$$r_2(j,N)=\exp\{-N[\Phi(j/N)-\Phi(u)]\}.$$

Since $C_{I(\epsilon)}\to 0$ as $\epsilon\to 0$, by (3.6) and Lemma 2.1 we conclude that

$$|r_2(j,N)-1|\le\Delta(\epsilon,N).$$

Thus,

$$[1-\Delta(\epsilon,N)]C(\epsilon,N)\le B(\epsilon,N)\le[1+\Delta(\epsilon,N)]C(\epsilon,N), \tag{3.8}$$

where

$$C(\epsilon,N)=\sum_{j/N\in I(\epsilon)}\int_{j/N}^{j/N+1/N}\exp\{-N[\Phi(u)-\Phi(a_i)]\}du.$$

Notice that

$$|C(\epsilon,N)-D(\epsilon,N)|=\Delta(\epsilon,N), \tag{3.9}$$

where

$$D(\epsilon,N)=\int_{I(\epsilon)}\exp\{-N[\Phi(u)-\Phi(a_i)]\}du,$$

$$\Delta(\epsilon,N)=|\int_{\bar{I}(\epsilon)}^{[\bar{I}(\epsilon)N]/N+1/N}\exp\{-N[\Phi(u)-\Phi(a_i)]\}du-$$

$$\int_{\underline{I}(\epsilon)}^{[\underline{I}(\epsilon)N]/N+1/N}\exp\{-N[\Phi(u)-\Phi(a_i)]\}du|,$$

$$\underline{I}(\epsilon)=a_i-\delta_i^-(\epsilon)\ \text{ and }\ \bar{I}(\epsilon)=a_i+\delta_i^+(\epsilon),$$

$$\Delta(\epsilon, N) \leq 2/N. \tag{3.10}$$

By hypothesis 2) for all sufficiently small ϵ

$$D(\epsilon, N) = \int_{I(\epsilon)} \exp[-N\psi_i(u - a_i)]du,$$

$$\int_{I(\epsilon)} \exp\{-N[\alpha_i + o_\epsilon(1)]|u - a_i|^{\gamma_i}\}du \leq D(\epsilon, N) \leq$$

$$\int_{I(\epsilon)} \exp\{-N[\alpha_i - o_\epsilon(1)]|u - a_i|^{\gamma_i}\}du, \tag{3.11}$$

where

$$o_\epsilon(1) = \sup_{u \in I(\epsilon)} |\frac{\psi_i(u - a_i)}{\alpha_i |u - a_i|^{\gamma_i}} - 1|.$$

Furthermore,

$$\left(a_i - \left\{\frac{\epsilon}{\alpha_i}[1 - o_\epsilon(1)]\right\}^{1/\gamma_i}, a_i + \left\{\frac{\epsilon}{\alpha_i}[1 - o_\epsilon(1)]\right\}^{1/\gamma_i}\right) \subseteq I(\epsilon) \subseteq$$

$$\left(a_i - \left\{\frac{\epsilon}{\alpha_i}[1 + o_\epsilon(1)]\right\}^{1/\gamma_i}, a_i + \left\{\frac{\epsilon}{\alpha_i}[1 + o_\epsilon(1)]\right\}^{1/\gamma_i}\right). \tag{3.12}$$

Notice that for $\sigma > 0$

$$\int_0^\sigma \exp(-N\alpha u^\gamma)du = \frac{1}{N^{1/\gamma}\alpha^{1/\gamma}\gamma} \int_0^{N\alpha\sigma^\gamma} \exp(-v)v^{1/\gamma - 1}dv =$$

$$\frac{\Gamma(1/\gamma)[1 - \Delta(\sigma, N)]}{N^{1/\gamma}\alpha^{1/\gamma}\gamma},$$

provided that $N > 0,\ \alpha > 0$ and $\gamma > 0$. Increasing the right-hand side of (3.11) by integrating over the larger set involved in (3.12) and decreasing the left-hand side of (3.11) by integrating over the smaller set involved in (3.12), we obtain that

$$\frac{2\Gamma(1/\gamma_i)}{N^{1/\gamma_i}\alpha_i^{1/\gamma_i}\gamma_i}[1 - \Delta(\epsilon, N)] \leq D(\epsilon, N) \leq$$

$$\frac{2\Gamma(1/\gamma_i)}{N^{1/\gamma_i}\alpha_i^{1/\gamma_i}\gamma_i}[1 + \Delta(\epsilon, N)]. \tag{3.13}$$

Since $\gamma_i \geq 2$,

$$\lim_{N \to \infty} \frac{1/N}{1/N^{1/\gamma_i}} = 0.$$

Taking this into account, by (3.1), (3.7) – (3.10) and (3.13), we obtain the statement of the lemma.

Remark 3.1. The argument given above allows for a generalization of Lemma 3.1. If, instead of hypothesis 2), we require that there are pairs $\gamma_i^+ \geq 2,\ \alpha_i^+ > 0$ and $\gamma_i^- \geq 2,\ \alpha_i^- > 0$ such that

$$\lim_{\sigma\to 0}\sup_{0<u\leq\sigma}|\frac{\psi_i(u)}{u^{\gamma_i^+}} - \alpha_i^+| = 0 \text{ and } \lim_{\sigma\to 0}\sup_{-\sigma\leq u<0}|\frac{\psi_i(u)}{(-u)^{\gamma_i^-}} - \alpha_i^-| = 0,$$

the statement modifies as follows:
if $\gamma_i^+ = \gamma_i^- = \gamma_i$,

$$\lim_{\epsilon\to 0}\lim_{N\to\infty}\frac{p_\epsilon^{i,N}(\alpha_i^+)^{1/\gamma_i}(\alpha_i^-)^{1/\gamma_i}\gamma_i}{d_0^{(N)}\exp(-N\Phi^*)N^{1-1/\gamma_i}[(\alpha_i^+)^{1/\gamma_i}+(\alpha_i^-)^{1/\gamma_i}]\Gamma(1/\gamma_i)} = 1;$$

if $\gamma_i^+ \neq \gamma_i^-$,

$$\lim_{\epsilon\to 0}\lim_{N\to\infty}\frac{p_\epsilon^{i,N}\alpha(i)^{1/\gamma(i)}\gamma(i)}{d_0^{(N)}\exp(-N\Phi^*)N^{1-1/\gamma(i)}\Gamma(1/\gamma(i))} = 1,$$

where $\gamma(i) = \max(\gamma_i^+, \gamma_i^-)$,

$$\alpha(i) = \begin{cases} \alpha_i^+ & \text{if } \gamma_i^+ > \gamma_i^-, \\ \alpha_i^- & \text{if } \gamma_i^+ < \gamma_i^-. \end{cases}$$

Lemma 3.1 and Remark 3.1 allow to describe the limit of D_N which is unique in the situation when the set of minima consists of a finite number of singletons. *The distribution nests on the subset of points of global minima with the highest γ_i^+ or γ_i^-.* To avoid bulky formulations we shall give this result only for the case when $\gamma_i^+ = \gamma_i^- = 2$ for all i.

Theorem 3.1. Let

1) for every $\epsilon > 0$ there be a real $\epsilon' > 0$ and a positive integer N' depending on ϵ and such that $X^\epsilon \supseteq X_N^\epsilon$ for $N \geq N'$;

2) $X^* = \{a_i\},\ i = 1, 2, \ldots, k$;

3) in a neighborhood of each a_i the function $F(\cdot)$ be Lipschitz and $F_N(\cdot)$ deviates from $F(\cdot)$ at most by c/N for some $c > 0$;

4) for each a_i there be positive numbers α_i^+ and α_i^- such that

$$\lim_{\sigma\to 0}\sup_{0<u\leq\sigma}|\frac{\psi_i(u)}{u^2} - \alpha_i^+| = 0 \text{ and } \lim_{\sigma\to 0}\sup_{-\sigma\leq u<0}|\frac{\psi_i(u)}{u^2} - \alpha_i^-| = 0.$$

Then D_N weakly converges to a limit that assigns to a_i the probability

$$\frac{1/\sqrt{\alpha_i^+} + 1/\sqrt{\alpha_i^-}}{\sum_{j=1}^k(1/\sqrt{\alpha_j^+} + 1/\sqrt{\alpha_j^-})}.$$

Proof. By Theorem 2.3 and (2.7) we obtain

$$|\sum_{i=1}^k p_\epsilon^{i,N} - 1| = o_N(1).$$

By Remark 3.1 this implies that

$$\Big[\frac{1}{\sum_{i=1}^k (1/\sqrt{\alpha_i^+} + 1/\sqrt{\alpha_i^-})} - \Delta(\epsilon, N)\Big] \leq$$

$$d_0^{(N)} \exp(-N\Phi^*) N^{1/2} \frac{\Gamma(1/2)}{2} \leq$$

$$\Big[\frac{1}{\sum_{i=1}^k (1/\sqrt{\alpha_i^+} + 1/\sqrt{\alpha_i^-})} + \Delta(\epsilon, N)\Big].$$

Hence, applying Remark 3.1 again,

$$\lim_{\epsilon \to 0} \lim_{N \to \infty} p_\epsilon^{i,N} = \frac{1/\sqrt{\alpha_i^+} + 1/\sqrt{\alpha_i^-}}{\sum_{j=1}^k (1/\sqrt{\alpha_j^+} + 1/\sqrt{\alpha_j^-})}.$$

The theorem is proved.

Remark 3.2. A similar result was obtained by Hwang [9] even for the multivariate case. However, he considers Gibbs distributions with potentials that do not depend on N.

Now let us proceed to the analysis of the limit distributions on the intervals. The following result is a counterpart of Lemma 3.1 in this case.

Lemma 3.2. Let

1) for some $\delta > 0$ the function $F(\cdot)$ be Lipschitz on $[b_j - \delta, b_j] \cup [c_j, c_j + \delta]$, and $|F_N(x) - F(x)| \leq c/N$ for every x from this set;

2) there be $\kappa_j > 1$ and $\beta_j > 0$ such that $\Psi_j(u) \geq \beta_j |u|^{\kappa_j}$ for all sufficiently small u;

3) there be a Hölder on $[b_j, c_j]$ function $\phi_j(\cdot)$ such that

$$\lim_{N \to \infty} N \sup_{i/N \in [b_j, c_j]} \Big|\frac{p_{i-1}^{(N)}}{q_i^{(N)}} - 1 - \frac{1}{N}\phi_j(i/N)\Big| = 0.$$

If there is a sequence $\{x_N^j\}$ such that $\Phi_N(x_N^j) = \Phi_N^*$ and $x_N^j \to [b_j, c_j]$ as $N \to \infty$, then

$$\lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{P_\epsilon^{j,N}}{d_0^{(N)} \exp(-N\Phi^*) N d_j(c_j)} = 1$$

and for every $x \in [b_j, c_j]$

$$\lim_{N \to \infty} \frac{P\{\mu_N \in [b_j, x]\}}{d_0^{(N)} \exp(-N\Phi^*) N d_j(x)} = 1,$$

where

$$d_j(x) = \int_{b_j}^x \exp[\bar\phi_j(u)] du, \quad \bar\phi_j(u) = \int_{b_j}^u \phi_j(v) dv.$$

Proof. We have that

$$P_\epsilon^{j,N} = T_-(\epsilon, N) + T(N) + T_+(\epsilon, N), \tag{3.14}$$

where

$$T_-(\epsilon, N) = d_0^{(N)} \sum_{i/N \in (b_j - \Delta_j^-(\epsilon), b_j)} \exp[-N\Phi_N(i/N)],$$

$$T(N) = d_0^{(N)} \sum_{i/N \in [b_j, c_j]} \exp[-N\Phi_N(i/N)],$$

$$T_+(\epsilon, N) = d_0^{(N)} \sum_{i/N \in (c_j, c_j + \Delta_j^+(\epsilon))} \exp[-N\Phi_N(i/N)].$$

Notice that

$$T(N) = d_0^{(N)} \exp[-\Phi_N(b_j)] M(N), \tag{3.15}$$

where

$$M(N) = \sum_{i/N \in [b_j, c_j]} \exp\{-N[\Phi_N(i/N) - \Phi_N(b_j)]\} =$$

$$\sum_{i/N \in [b_j, c_j]} \prod_{s=[b_j N]+1}^{i} p_{s-1}^{(N)} / q_s^{(N)}.$$

By hypothesis 3) and (3.6), for every $i/N \in [b_j, c_j]$

$$[1 - o_N(1)] M(i, N) \leq \prod_{s=[b_j N]+1}^{i} p_{s-1}^{(N)} / q_s^{(N)} \leq$$

$$[1 + o_N(1)] M(i, N), \tag{3.16}$$

where

$$M(i, N) = \exp\Big[\sum_{s=[b_j N]+1}^{i} \frac{1}{N} \phi_j(s/N) \Big].$$

According to hypothesis 3), $\phi_j(\cdot)$ is a Hölder function on $[b_j, c_j]$, hence we obtain that

$$[1 - o_N(1)] \bar{\phi}_j(i/N) \leq M(i, N) \leq [1 + o_N(1)] \bar{\phi}_j(i/N). \tag{3.17}$$

Similarly, $\bar{\phi}_j(\cdot)$ is a Lipschitz function on $[b_j, c_j]$, consequently

$$[1 - o_N(1)] d_j(c_j) \leq \frac{1}{N} \sum_{i/N \in [b_j, c_j]} \exp[\bar{\phi}_j(i/N)] \leq$$

$$[1 + o_N(1)] d_j(c_j). \tag{3.18}$$

By (3.16) – (3.18) we obtain

$$\lim_{N \to \infty} \frac{M(N)}{N} = d_j(c_j),$$

which due to (3.15) implies that

$$\lim_{N\to\infty} \frac{T(N)}{d_0^{(N)} \exp[-\Phi_N(b_j)] N d_j(c_j)} = 1. \tag{3.19}$$

By (3.5) we conclude that

$$1 - o_\epsilon(1) \le \liminf_{N\to\infty} \exp\{-N[\Phi_N(b_j) - \Phi(b_j)]\} \le$$

$$\limsup_{N\to\infty} \exp\{-N[\Phi_N(b_j) - \Phi(b_j)]\} \le 1 + o_\epsilon(1).$$

Since $\exp\{-N[\Phi_N(b_j) - \Phi(b_j)]\}$ does not depend on ϵ, the latter inequalities imply that

$$\lim_{N\to\infty} \exp\{-N[\Phi_N(b_j) - \Phi(b_j)]\} = 1. \tag{3.20}$$

Slightly modifying the argument given in Lemma 3.1, we obtain that $|T_-(\epsilon, N)| + |T_+(\epsilon, N)|$ does not exceed

$$const \cdot d_0^{(N)} \exp(-N\Phi^*) N^{1-1/\kappa_j}$$

as $N \to \infty$. Taking into account (3.19) and (3.20), the terms $T_-(\epsilon, N)$ and $T_+(\epsilon, N)$ are asymptotically smaller than $T(N)$. This allows to derive the first statement of the lemma from (3.14), (3.19) and (3.20).

The second statement obtains by an argument similar to the one used in estimating $T(N)$.

The lemma is proved.

From Remark 3.1 and Lemma 3.2 we conclude that *if there are intervals among the connected components of the set of global minima, the limit distribution can nest only on them.* More formally we have the following statement.

Let us call a growth function like in Lemma 3.2 a power growth function.

Theorem 3.2. Let

1) for every $\epsilon > 0$ there be a real ϵ' and a positive integer N' depending on ϵ such that $X^\epsilon \supseteq X_N^{\epsilon'}$ for $N \ge N'$;

2) $X^* = \cup_{i=1}^k \{a_i\} \cup_{j=1}^l [b_j, c_j]$, where $l \ge 1$;

3) in a neighborhood of each a_i, b_j and c_j the function $F(\cdot)$ be Lipschitz and there be a constant $c > 0$ such that $F_N(\cdot)$ deviates from $F(\cdot)$ at most by c/N;

4) for each a_i and $[b_j, c_j]$ there be a power growth function;

5) for each $[b_j, c_j]$ there be a Hölder on this interval function $\phi_j(\cdot)$ such that

$$\lim_{N\to\infty} N \sup_{i/N \in [b_j, c_j]} \left|\frac{p_{i-1}^{(N)}}{q_i^{(N)}} - 1 - \frac{1}{N}\phi_j(i/N)\right| = 0.$$

Then D_N weakly converges as $N \to \infty$ to a limit such that for every $x \in [b_j, c_j]$

$$\lim_{N\to\infty} P\{\mu_N < b_j + x\} = \frac{\sum_{i:\, c_i < b_j} d_i(c_i) + d_j(x)}{\sum_{s=1}^l d_s(c_s)},$$

or, equivalently,

$$\lim_{N\to\infty} P\{\mu_N < b_j + x | \mu_N \in [b_j, c_j]\} = d_j(x)/d_j(c_j).$$

Proof. By Theorem 2.3 and (2.7) we obtain that

$$|\sum_{i=1}^{k} p_\epsilon^{i,N} + \sum_{j=1}^{l} P_\epsilon^{j,N} - 1| = o_N(1). \tag{3.21}$$

Since the growth functions are power ones, there are $\kappa > 1$ and $\beta > 0$ such that $\psi_i(u) \geq \beta|u|^\kappa$ for $i = 1, 2, \ldots, k$ provided that u is small. Arguing like in the proof of Lemma 3.1, we obtain that

$$\sum_{i=1}^{k} p_\epsilon^{i,N} \leq const \cdot d_0^{(N)} \exp(-N\Phi^*) N^{1-1/\kappa}.$$

This by the first statement of Lemma 3.2 shows that the impact of singletons is negligible in (3.21), that is,

$$|\sum_{j=1}^{l} P_\epsilon^{j,N} - 1| = \Delta(\epsilon, N). \tag{3.22}$$

Substituting here the expressions for $P_\epsilon^{j,N}$ from Lemma 3.2, we have that

$$1/\sum_{j=1}^{l} d_j(c_j) - \Delta(\epsilon, N) \leq d_0^{(N)} \exp(-N\Phi^*) N \leq$$

$$1/\sum_{j=1}^{l} d_j(c_j) + \Delta(\epsilon, N). \tag{3.23}$$

Which by the first statement of Lemma 3.2 implies that

$$|P_\epsilon^{j,N} - P^{(j)}| = \Delta(\epsilon, N), \tag{3.24}$$

where

$$P^{(j)} = \frac{d_j(c_j)}{\sum_{i=1}^{l} d_i(c_i)}.$$

By (3.22) – (3.24) and the second statement of Lemma 3.2 we obtain

$$\Big[\sum_{i:\, c_i < b_j} P^{(i)} + \frac{d_j(x)}{\sum_{s=1}^{l} d_s(c_s)} - \Delta(\epsilon, N)\Big] \leq$$

$$P\{\mu_N < b_j + x\} \leq \Big[\sum_{i:\, c_i < b_j} P^{(i)} + \frac{d_j(x)}{\sum_{s=1}^{l} d_s(c_s)} + \Delta(\epsilon, N)\Big],$$

which entails the statement of the theorem.

4 An Example

In Chapter 5 of [1] the following model is considered. There are M agents who have two choices, strategy 1 or strategy 2. They reevaluate their choices randomly. Namely, let

$$\eta(x) = P\{\pi_1(x) \geq \pi_2(1-x)\},$$

where $\pi_i(x)$ is a perceived random benefit of adopting alternative i when fraction x of agents are using it. Assume that each time instant only one agent is allowed to change his strategy. Let i agents are using strategy 1 at t. There are two possibilities: one of them switches to strategy 2, or one of $M-i$ agents who are using strategy 2 switches to strategy 1. Then either $i-1$ or $i+1$ agents will be using strategy 1 at $t+1$. Since the agent who switches is chosen by chance, we obtain the following transition probabilities:

$$P\{i \mapsto i-1\} = \frac{i}{M}[1-\eta(i/M)] \text{ and } P\{i \mapsto i+1\} = \frac{M-i}{M}\eta(i/M).$$

Thus we have arrived to the above Markov chain with $N = M+1$,

$$p_i^{(N)} = \Big(1 - \frac{i}{N-1}\Big)\eta(\frac{i}{N-1}) \text{ and } q_i^{(N)} = \frac{i}{N-1}\Big[1-\eta(\frac{i}{N-1})\Big].$$

It is ergodic if $\eta(x) \in (0,1)$ for every $x \in [0,1]$.

We have that

$$f_N(i/N) = \Big[1 - \frac{i}{N}(1+\frac{1}{N-1})\Big]\eta(\frac{i}{N}(1+\frac{1}{N-1}))$$

and

$$g_N(i/N) = \frac{i}{N}\Big(1+\frac{1}{N-1}\Big)\Big[1-\eta(\frac{i}{N}(1+\frac{i}{N-1}))\Big].$$

If $\eta(\cdot)$ is a Lipschitz function on $[0,1]$, then

$$\limsup_{N\to\infty} N \sup_{x\in[0,1]} [|f_N(x) - f(x)| + |g_N(x) - g(x)|] < \infty, \tag{4.1}$$

where

$$f(x) = (1-x)\eta(x) \quad \text{and} \quad g(x) = x[1-\eta(x)].$$

Since $f(\cdot) - g(\cdot)$ is positive in a neighborhood of 0 and it is negative in a neighborhood of 1, we conclude that there are solutions of the equation $f(x) = g(x), \quad x \in [0,1]$. They are *interior* points of $[0,1]$. The limits of D_N concentrate on, generally speaking, a subset of these solutions.

Notice that

$$F_N(i/N) = \frac{[1 - \frac{i}{N}(1+\frac{1}{N-1}) + \frac{1}{N-1}]\eta(\frac{i}{N}(1+\frac{1}{N-1}))}{\frac{i}{N}(1+\frac{1}{N-1})[1-\eta(\frac{i}{N}(1+\frac{i}{N-1}))]}$$

and

$$F(x) = \frac{(1-x)\eta(x)}{x[1-\eta(x)]}.$$

Because the denominators of these expressions are positive inside $[0, 1]$ and $\eta(\cdot)$ is a Lipschitz function, (4.1) implies that

$$\limsup_{N\to\infty} N \sup_{x\in[a,b]} |F_N(x) - F(x)| < \infty$$

for every $[a, b] \subset (0, 1)$. Also, $F(\cdot)$ is a Lipschitz function and $\ln F(\cdot)$ is a Riemann integrable function on $[0, 1]$.

Now, depending upon the structure of the set of global minima of the limit Gibbs potential, we can apply results given above.

5 Conclusions

The results obtained here show certain similarity of birth-and-death processes and generalized urn schemes [2]. Indeed, since the chain ζ_N^t and an urn process evolve in $[0, 1]$, they are stochastic replicator equations. In both cases the asymptotic analysis relies on the stability properties of some dynamic system associated with the process. In the case of singleton attractors this analogy works fully for urn processes. Their limits nest on the set of stable singular points. For birth-and-death processes it is not the case. Their limit distributions nest on a subset of stable attractors (namely, the points of global minima of the limit potential, or even a subset of this set) of the associated dynamic system. For urn processes each set of singular points having positive Lebesgue measure turns out to be an attractor [7], which is not the case for birth-and-death processes. There only intervals that consist of points of global minima support the limit. Finally, urn schemes generate time non-homogeneous Markov processes that are not ergodic, while the Markov chains corresponding to birth-and-death processes considered here are time homogeneous and ergodic.

These two mathematical objects have essentially the same area of application. In economics this includes learning processes. Conceptually the main difference between them is that the total size of a population involved in learning is growing in time in the case of an urn process, while it remains constant in the case of a birth-and-death process.

We believe that some of the results of Section 3 can be proved for a general annealing process [11].

REFERENCES

[1] Aoki, M. (1996). *New Approaches to Macroeconomic Modeling*, Cambridge University Press, Cambridge.

[3] Arthur, W. B., Y. M. Ermoliev, and Y. M. Kaniovski (1987). Adaptive Growth Processes Modeled by Urn Schemes. Cybernetics, **23**, pp. 779–789.

[3] Binmore, K. G., L. Samuelson, and R. Vaughan (1995). Musical Chairs: Modeling Noisy Evolution. Games and Economic Behavior, **11**, pp. 1–35.

[4] Daneubourg, J.-L., S. Aron, S. Goss, and J. M. Pasteels (1987). Error, Communication and Learning in Ant Societies. European Journal of Operational Research, **30**, pp. 168–172.

[5] Ewens, W. J. (1979). *Mathematical Population Genetics*, Springer-Verlag, Berlin.

[6] Gikhman, I. I., and A. V. Skorokhod (1975). *Theory of Random Processes*, Vol. III, Nauka, Moscow (in Russian).

[7] Hill, B. N., D. Lane, and W. Sudderth (1980). A Strong Law for Some Generalized Urn Processes. Ann. Prob., **8**, pp. 214–226.

[8] Hoel, P. G., S. C. Port, and Ch. J. Stone (1972). *Introduction to Stochastic Processes*, Houghton – Mifflin, Boston.

[9] Kirman, A. (1993). Ants, Rationality, and Recruitments. The Quaterly Journal of Economics, **108**, pp. 137–154.

[10] Hwang, Chii-Ruey (1980). Laplace's Method Revisited: Weak Convergence of Probability Measures. Ann. Prob., **6**, pp. 1177–1182.

[11] Kirkpatrick, S., C. D. Gelatt, Jr., and M. P. Vecchi (1983). Optimization by Simulated Annealing. Science, **220**, pp. 671–680.

[12] Risken, H. (1989). *The Fokker – Plank Equation*, Springer-Verlag, Berlin.

[13] Weidlich, W., and G. Haag (1983). *Concepts and Models of Quantitative Sociology*, Springer-Verlag, Berlin.