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Optimal Control Synthesis in Grid Approximation Schemes

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Abstract

Grid approximation schemes for constructing value functions and optimal feedbacks in problems of guaranteed control are proposed via theory of generalized (minimax, viscosity) solutions of Hamilton-Jacobi equations. Value functions in optimal control problems are usually nondifferentiable and corresponding feedbacks have the discontinuous switching character. Therefore, constructions of generalized gradients for local hulls of different types are used in finite difference operators which approximate value functions. Optimal feedbacks are synthesized by extremal shift in the direction of generalized gradients. Both problems of constructing the value function and control synthesis are solved simultaneously in the unique grid scheme. The interpolation problem is analyzed for grid values of optimal feedbacks. Questions of correlating spatial and temporal meshes are examined. Significance of quasiconvexity properties is clarified for the linear dependence of space-time grids.

The proposed grid schemes for solving optimal guaranteed control problems can be applied for models arising in mechanics, mathematical economics, differential and evolutionary games.

Contents

1	Dynamics and Hamilton-Jacobi Equations	2
2	Subdifferentials of Local Convex Hulls and Finite Difference Operators	6
3	Properties of Operators with Generalized Gradients and Convergence of Approximation Schemes	14
4	Algorithms for Computing Values of Operator F	22
5	Constructions of Nonsmooth Analysis in Finite Difference Operators	26
6	Mean Square Generalized Gradients	30
7	Optimal Control Synthesis and Generalized Gradients in Grid Schemes	35

Optimal Control Synthesis in Grid Approximation Schemes

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Introduction

In this paper we propose grid schemes for constructing value functions and optimal feedbacks in problems of guaranteed control. It is known in the theory of optimal control and differential games that value functions are usually nondifferentiable and the corresponding optimal synthesis has discontinuous properties on switching surfaces. The theory of generalized (minimax, viscosity) solutions of Hamilton-Jacobi equations ([Crandall, Lions, 1983, 1984], [Subbotin, 1980, 1995]) provides the instrument for operating with nondifferentiable value functions. Different constructions of nonsmooth analysis such as directional derivatives, Dini subdifferentials are used for describing stability (viability) properties of value functions at points of nondifferentiability. For constructing generalized solutions approximation schemes of different types were proposed ([Lax, 1954], [Godunov, 1959], [Oleinik, 1959], [Fleming, 1961], [Hopf, 1965], [Kruzhkov, 1965], [Crandall, Lions, 1984], [Souganidis, 1985], [Bardi, Falcone, 1990] [Bardi, Osher, 1991], [Osher, Shu, 1991], [Tarasyev, 1994], [Tarasyev, Ushakov, Uspenskii, 1994]), and their convergence was proved. In the present paper we use constructions of local (convex, concave, linear) hulls for approximation of value functions. The corresponding finite difference operators are based on notions of generalized gradients - subdifferentials of convex hulls, superdifferentials of concave hulls, gradients of linear hulls.

There exists the adjoint problem to synthesize optimal feedbacks using approximations of the value function. If the value function is differentiable then it coincides with the classical solution of the Hamilton-Jacobi equation and the optimal synthesis can be constructed by extremal aiming in the direction of gradients. For exactly known (or known with the high accuracy) nonsmooth value functions optimal feedbacks can be designed by the method of extremal shift of a trajectory to accompanying points of local extremum ([Krasovskii, 1985], [Krasovskii, Subbotin, 1974, 1988], [A.N. Krasovskii, N.N. Krasovskii, 1995]). The principle of extremal aiming in the direction of quasigradients defined with the help of Yosida-Moreau transformations (see [Garnysheva, Subbotin, 1994]) also can be used for finding optimal synthesis. Let us note that the mentioned methods require the exact calculation of the value function or the high accuracy of its approximations.

In the present paper it is proposed to combine in the unique algorithm the approximation scheme for constructing the value function and the principle of extremal shift in the direction of generalized gradients of local (convex, concave, linear) hulls.

In space-time grid realizations of approximation schemes the value function, generalized gradients and corresponding optimal feedbacks are calculated only at nodes of the

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fixed grid. But for constructing optimal trajectories which can slide between these nodes it is necessary to interpolate extremal values of control parameters to internal points. Different types of interpolations: piecewise constant, piecewise minimum, piecewise linear, are examined and their properties are indicated. In this connection the question on correlation of spatial and temporal grids is studied. In the general case it is necessary to have the density of the spatial mesh of the higher order accuracy than the density of the temporal mesh. For the linear dependence of space-time grids the impact of quasiconvexity properties is analyzed.

The elaborated grid approximation schemes for constructing value functions and optimal feedbacks can be used for analysis of applied problems of mechanics, mathematical economics and evolutionary biology.

1 Dynamics and Hamilton-Jacobi Equations

Let us consider a control system described on a time interval $T = [t_0, \vartheta]$ by a vector differential equation

$$\begin{aligned} \dot{x} &= f(t, x, u, v) = h(t, x) + B(t, x)u + C(t, x)v \\ x &\in R^n, \quad u \in P \subset R^p, \quad v \in Q \subset R^q \end{aligned} \tag{1.1}$$

Here x is the n -dimensional phase vector of the system, u, v are nonfixed parameters in compact convex sets P, Q and can be generated on different principles - open-loop (programming control), closed-loop (feedback principle), stochastic principle (random variables).

We consider the minimax statement of the control problem (control with the guarantee) when control parameter u is synthesized on the feedback principle in order to minimize the functional

$$\gamma(x(\cdot)) = \sigma(x(\vartheta)) \tag{1.2}$$

on trajectories $x(\cdot)$ of system (1.1) while v can be induced in different ways and realized in the most unfavorable form. So the problem is to find a positional control (feedback) $U^0 = U^0(t, x)$ that provides an external minimum in the minimax relation

$$w(t_*, x_*) = \min_U \max_{x(\cdot) \in X(t_*, x_*, U)} \sigma(x(\vartheta)) \tag{1.3}$$

and to determine the value $w(t_*, x_*)$, called the optimal guaranteed result or the value of the game. Here by the symbol $X(t_*, x_*, U)$ we denote the set of trajectories of system (1.1) generated in the sense of [Krasovskii, Subbotin, 1974, 1988] by a positional control $U = U(t, x)$ and various realizations of parameter $v = v(t)$ from the initial position (t_*, x_*) .

The function $(t_*, x_*) \rightarrow w(t_*, x_*)$ linking initial positions (t_*, x_*) and optimal guaranteed results $w(t_*, x_*)$ is called the value function. The value function w plays the key role in solving the control problem (1.1), (1.2) and constructing the guaranteeing optimal feedback $(t, x) \rightarrow U^0(t, x)$.

Let us note that such statement provides guaranteeing optimal feedbacks which often are very flexible and can give solution in other senses. Many applied control problems arising in mechanics, economics, evolutionary biology can be interpreted in terms of optimal control with guarantee: control in mechanical systems, pursuit-evasion games, games against the nature, bimatrix games.

We assume that function $f(t, x, u, v)$ on the right-hand side of system (1.1) is satisfied the following conditions.

(F1) Uniform continuity in all variables.

(F2) The Lipschitz condition by variable x

$$\|f(t, x_1, u, v) - f(t, x_2, u, v)\| \leq L_1(D)\|x_1 - x_2\|$$

for all $(t, x_i) \in D$, $i = 1, 2$, $u \in P$, $v \in Q$.

(F3) Extendability of solutions: there exists a constant κ such that

$$\|f(t, x, u, v)\| \leq \kappa(1 + \|x\|)$$

for all $(t, x, u, v) \in T \times R^n \times P \times Q$.

(F4) The Lipschitz continuity with respect to variable t

$$\|f(t_1, x, u, v) - f(t_2, x, u, v)\| \leq L_2(D)|t_1 - t_2|$$

for all $(t_j, x) \in D$, $j = 1, 2$, $u \in P$, $v \in Q$.

Here D is a compact set, $D \subset T \times R^n$.

The function $\sigma(x)$ in the payoff functional (1.2) is assumed to be

(Σ) Lipschitz continuous

$$|\sigma(x_1) - \sigma(x_2)| \leq L_3(D_\vartheta)\|x_1 - x_2\|$$

for all $x_k \in D_\vartheta$, $k = 1, 2$, where D_ϑ is a compact set, $D_\vartheta \subset R^n$.

Let us turn our attention to the value function $(t, x) \rightarrow w(t, x)$. Since there exists the saddle point in the “small game”

$$\begin{aligned} \max_{v \in Q} \min_{u \in P} \langle s, f(t, x, u, v) \rangle &= \min_{u \in P} \max_{v \in Q} \langle s, f(t, x, u, v) \rangle = \\ &= \langle s, h(t, x) \rangle + \min_{u \in P} \langle s, B(t, x)u \rangle + \max_{v \in Q} \langle s, C(t, x)v \rangle = H(t, x, s) \end{aligned} \quad (1.4)$$

then the theorem on alternative [Krasovskii, Subbotin, 1974, 1988] implies the existence of the saddle point in the original game posed in the classes of “pure” feedbacks $(t, x) \rightarrow U(t, x)$, $(t, x) \rightarrow V(t, x)$

$$w(t_*, x_*) = \min_U \max_{x(\cdot) \in X(t_*, x_*, U)} \sigma(x(\vartheta)) = \max_V \min_{y(\cdot) \in Y(t_*, x_*, V)} \sigma(y(\vartheta))$$

Here the symbol $Y(t_*, x_*, V)$ denotes the set of trajectories $y(\cdot)$ generated by a positional strategy $V = V(t, x)$ from the initial position (t_*, x_*) .

The function $(t, x, s) \rightarrow H(t, x, s)$ defined by the saddle point (1.4) is called the Hamiltonian of the dynamical system (1.1).

It is known that the dynamic programming principle [Bellman, 1957] is valid for the value function w

$$\begin{aligned} w(t, x) &= \min_U \max_{x(\cdot) \in X(t, x, U)} w(s, x(s)) = \\ &= \max_V \min_{y(\cdot) \in Y(t, x, V)} w(s, y(s)) \end{aligned} \quad (1.5)$$

for all $(t, x) \in T \times R^n$, $t \leq s \leq \vartheta$.

One can verify that the value function $w(t, x)$ is Lipschitz continuous and is consequently differentiable almost everywhere. At points (t, x) of differentiability of the value function w the dynamic programming principle (1.5) turns into the so-called Bellman-Isaacs equation - first order partial differential equation of the Hamilton-Jacobi type

$$\frac{\partial w}{\partial t}(t, x) + H(t, x, \frac{\partial w}{\partial x}(t, x)) = 0 \quad (1.6)$$

From (1.5) it also follows that the value function w satisfies the boundary condition

$$w(\vartheta, x) = \sigma(x) \quad (1.7)$$

for all $x \in R^n$.

The core characteristic of the value function w is the so-called property of u and v -stability [Krasovskii, Subbotin, 1974, 1988] which provides the weak invariance [Aubin, 1990] of epigraph (hypograph, Lebesgue sets) of the value function with respect to differential inclusions relating to dynamical system (1.1). Accurately the property of u -stability is formulated in the following way.

(PS) A function w is called u -stable at a point (t_*, x_*) if for all control parameters $v \in Q$ and numbers $\varepsilon > 0$ there exist a number $\delta > 0$ and a trajectory $x(\cdot) = (x(t), t_*) \leq t \leq t_* + \delta, x(t_*) = x_*$ of the differential inclusion

$$\dot{x}(t) \in F(t, x(t), v) \quad (1.8)$$

$$F(\tau, y, v) = \{f \in R^n : f = f(\tau, y, u, v), u \in P\}$$

such that the inequality

$$w(t, x(t)) \leq w(t_*, x_*) + \varepsilon(t - t_*) \quad (1.9)$$

takes place for all $t \in [t_*, t_* + \delta]$.

A function w which satisfies the property of u -stability (1.9) at all points is called u -stable.

Inequality (1.9) means that any trajectory $x(\cdot)$ of differential inclusion (1.8) survives in epigraph of function w .

The property of v -stability is formulated in the dual form.

One can prove (see [Krasovskii, Subbotin, 1974, 1988]) that properties u and v stability together with boundary condition (1.7) uniquely determine value function w . So they form a block of necessary and sufficient conditions.

Properties of u and v -stability can be formulated in different equivalent ways. The most preferable is the infinitesimal form in which constructions of nonsmooth analysis appear. In terms of directional derivatives these properties were formulated in the work [Subbotin, 1980] and the notion of generalized (minimax) solution of Hamilton-Jacobi equation coinciding with the value function was introduced. The notion of viscosity solution is presented in the works [Crandall, Lions, 1983, 1984] where stability properties are expressed in terms of Dini subdifferentials and superdifferentials. Viscosity solutions in application to differential games were studied in [Barron, Evans, Jensen, 1984].

In this paper for describing stability properties and defining generalized solutions of Hamilton-Jacobi equations - value functions, we use notions of conjugate derivatives (see [Subbotin, Tarashev, 1985]).

Definition 1.1 *A Lipschitz continuous function $w(t, x)$ is called a generalized (minimax) solution of the boundary value problem (1.6), (1.7) - the value function of control problem (1.1), (1.2), if the differential inequalities*

$$\inf_{s \in R^n} \sup_{h \in R^n} (\langle s, h \rangle - \partial_- w(t, x) | (1, h) - H(t, x, s)) \geq 0 \quad (1.10)$$

$$\sup_{s \in R^n} \inf_{h \in R^n} (\langle s, h \rangle - \partial_+ w(t, x) | (1, h) - H(t, x, s)) \leq 0 \quad (1.11)$$

are fulfilled for all $(t, x) \in [t_0, \vartheta)$ and the boundary condition (1.7) holds.

Here lower and upper directional derivatives of function w at point (t, x) in direction $(1, h)$ are defined by relations

$$\partial_- w(t, x)|(1, h) = \liminf_{\delta \downarrow 0} \frac{w(t + \delta, x + \delta h) - w(t, x)}{\delta}$$

$$\partial_+ w(t, x)|(1, h) = \limsup_{\delta \downarrow 0} \frac{w(t + \delta, x + \delta h) - w(t, x)}{\delta}$$

At points where function w is differentiable, inequalities (1.10), (1.11) turn into Hamilton-Jacobi equation (1.6) and so can be considered as its generalization.

Below we propose approximation schemes for constructing the value function $(t, x) \rightarrow w(t, x)$ and designing the optimal feedback $(t, x) \rightarrow U^0(t, x)$. Finite-difference operators used in these schemes are essentially based on constructions of upper and lower conjugate derivatives D^* , D_* from differential inequalities (1.10), (1.11)

$$D^* w(t, x)|(s) = \sup_{h \in R^n} (\langle s, h \rangle - \partial_- w(t, x)|(1, h)) \quad (1.12)$$

$$D_* w(t, x)|(s) = \inf_{h \in R^n} (\langle s, h \rangle - \partial_+ w(t, x)|(1, h)) \quad (1.13)$$

In order to realize approximation schemes it is necessary to restrict constructions on a compact domain $G_r \subset T \times R^n$, $r > 0$ which we define in the following way.

Denote by the symbol $X(t_*, x_*)$ the set of solutions $x(\cdot)$ of the differential inclusion

$$\dot{x} \in F(t, x(t)), \quad t \in [t_*, \vartheta], \quad x(t_*) = x_* \quad (1.14)$$

Here

$$F(\tau, y) = \{f \in R^n : f = f(\tau, y, u, v), \quad u \in P, \quad v \in Q\} \quad (1.15)$$

is the set of velocities of system (1.1).

Consider a set G which is strongly invariant with respect to differential inclusion (1.14)

(G1) If $(t_*, x_*) \in G$, then $(t, x(t)) \in G$ for all $x(\cdot) \in X(t_*, x_*)$, $t \in [t_*, \vartheta]$.

According to condition (F3) there exist compact domains G satisfying the principle of strong invariance (G1).

Let

$$K = \max_{(t, x, u, v) \in G \times P \times Q} \|f(t, x, u, v)\| \quad (1.16)$$

be the maximum velocity of system (1.1) in domain G .

By condition (F3) velocity K is restricted as follows

$$K \leq \max_{(t, x) \in G} \kappa(1 + \|x\|)$$

Let us introduce now the domain G_r by the following invariance conditions

(G2) $G_r \subset G$.

(G3) If $(t_*, x_*) \in G_r$, then $(t, x_* + (t - t_*)B_r) \subset G_r$, for all $t \in [t_*, \vartheta]$.

Here parameter r and ball B_r are connected with dynamics (1.1) and its characteristics $F(t, x)$ (1.15), K (1.16) by relations

$$r > K, \quad B_r = \{b \in R^n : \|b\| \leq r\}$$

$$F(t, x) \subset B_r, \quad (t, x) \in G_r$$

Let us indicate properties of the Hamiltonian $(t, x, s) \rightarrow H(t, x, s) : G_r \times R^n \times R^n \rightarrow R$ in the domain G_r which follow from conditions (F1)-(F4) and relation (1.4):

(H1) Uniform continuity in all variables.

(H2) The Lipschitz condition by variable x

$$|H(t, x_1, s) - H(t, x_2, s)| \leq L_1(G_r) \|s\| \|x_1 - x_2\|$$

for all $(t, x_i) \in G_r, i = 1, 2, s \in R^n$.

(H3) The Lipschitz condition by variable s

$$|H(t, x, s_1) - H(t, x, s_2)| \leq r \|s_1 - s_2\|$$

for all $(t, x) \in G_r, s_j \in R^n, j = 1, 2$.

(H4) The Lipschitz condition by variable t

$$|H(t_1, x, s) - H(t_2, x, s)| \leq L_2(G_r) \|s\| |t_1 - t_2|$$

for all $(t_k, x) \in G_r, k = 1, 2, s \in R^n$.

(H5) Positive homogeneity by variable s

$$H(t, x, \lambda s) = \lambda H(t, x, s)$$

for all $(t, x, s) \in G_r \times R^n, \lambda \geq 0$.

2 Subdifferentials of Local Convex Hulls and Finite Difference Operators

We introduce now finite difference operators with constructions of nonsmooth analysis: subdifferentials of local convex hulls and superdifferentials of local concave hulls for approximating generalized solutions of Hamilton-Jacobi equations - value functions.

Let $t \in T, t + \Delta \in T, t < \vartheta, \Delta > 0, (t, x) \in G_r$. Assume that at time $t + \Delta$ a Lipschitz continuous function $u(\cdot)$ is given in the domain $D_{t+\Delta} = \{x \in R^n : (t + \Delta, x) \in G_r, t + \Delta \in T\}$ and $L = L(D_{t+\Delta})$ is its Lipschitz constant. This function is considered in the subsequent constructions as an approximation of the solution $x \rightarrow w(t + \Delta, x)$ at time $t + \Delta$. We define operator $u \rightarrow F(t, \Delta, u)$ approximating the Hamilton-Jacobi equation in the neighborhood of a point $(t, x) \in G_r$ by the formula that can be interpreted as a generalization of Hopf's formula [Hopf, 1965], [Bardi, Osher, 1991] or of the programming maximin formula [Krasovskii, Subbotin, 1974], [Ushakov, 1981] connected with inequalities for conjugate derivatives (1.10), (1.11)

$$\begin{aligned} v(x) &= F(t, \Delta, u)(x) = \\ &= f(x) + \sup_{y \in O(x, r\Delta)} \max_{s \in Df(y)} \{ \Delta H(t, x, s) + f(y) - f(x) - \langle s, y - x \rangle \} \end{aligned} \quad (2.1)$$

Here the function $x \rightarrow v(x) : D_t \rightarrow R$ is treated as an approximation of the solution $x \rightarrow w(t, x)$ in the domain $D_t = \{s \in R^n : (t, x) \in G_r, t \in T\}$.

The set $O(x, r\Delta)$ is a neighborhood of point x of radius $r\Delta, r > K, \Delta > 0, (t, x) \in G_r$

$$O(x, r\Delta) = \{y \in R^n : \|y - x\| < r\Delta\}$$

The function $y \rightarrow f(y) : \overline{O}(x, r\Delta) \rightarrow R$ is the local convex hull of the function $y \rightarrow u(y)$ in the closed neighborhood $\overline{O}(x, r\Delta)$

$$\begin{aligned} f(y) &= \inf \left\{ \sum_{k=1}^{n+1} \alpha_k u(y_k) : y_k \in \overline{O}(x, r\Delta), \alpha_k \geq 0, k = 1, \dots, n+1, \right. \\ &\quad \left. \sum_{k=1}^{n+1} \alpha_k y_k = y, \sum_{k=1}^{n+1} \alpha_k = 1 \right\}, \quad y \in \overline{O}(x, r\Delta) \end{aligned} \quad (2.2)$$

$$\overline{O}(x, r\Delta) = \{y \in R^n : \|y - x\| \leq r\Delta\}$$

The set $Df(y)$ is the subdifferential [Clarke, 1983], [Rockafellar, 1970] of the convex function f at a point y , $y \in O(x, r\Delta)$

$$Df(y) = \{s \in R^n : f(z) - f(y) \geq \langle s, z - y \rangle, \quad z \in \overline{O}(x, r\Delta)\} \quad (2.3)$$

Let us note that the inequality

$$f(y) - f(x) - \langle s, y - x \rangle \leq 0, \quad y \in O(x, r\Delta), \quad s \in Df(y)$$

takes place in definition (2.1).

Let us consider properties of local convex hulls and subdifferentials.

Lemma 2.1 1. *The estimate*

$$|f(z) - f(y)| \leq L \left(1 + \frac{r\Delta + \|y - x\|}{r\Delta - \|y - x\|} \right) \|z - y\| \quad (2.4)$$

takes place for the convex hull $f : \overline{O}(x, r\Delta) \rightarrow R$, $z \in \overline{O}(x, r\Delta)$, $y \in O(x, r\Delta)$.

For $y = x$ it implies

$$|f(z) - f(x)| \leq 2L\|z - x\|$$

2. *The function $f : \overline{O}(x, K\Delta) \rightarrow R$ satisfies the Lipschitz condition with constant $L(1 + (r + K)/(r - K))$.*

3. *Subgradients $s \in Df(y)$, $y \in O(x, r\Delta)$ satisfy the inequality*

$$\|s\| \leq L \left(1 + \frac{r\Delta + \|y - x\|}{r\Delta - \|y - x\|} \right)$$

In particular, the following relations hold

$$\|s\| \leq 2L, \quad s \in Df(x)$$

$$\|s\| \leq L \left(1 + \frac{r + K}{r - K} \right), \quad s \in Df(y), \quad y \in \overline{O}(x, K\Delta)$$

Proof.

Let us estimate the difference $f(z) - f(y)$, $z \in \overline{O}(x, r\Delta)$, $y \in O(x, r\Delta)$. According to definition of the convex hull (2.2) for a point $y \in O(x, r\Delta)$ and arbitrary number $\varepsilon > 0$ there exist points $y_k \in \overline{O}(x, r\Delta)$ and coefficients $\alpha_k \geq 0$, $\sum_{k=1}^{n+1} \alpha_k y_k = y$, $\sum_{k=1}^{n+1} \alpha_k = 1$ such that

$$f(z) - f(y) < f(z) - \sum_{k=1}^{n+1} \alpha_k u(y_k) + \varepsilon$$

Let

$$z_k = y_k + (z - y) - \frac{(y_k - y)\|z - y\|}{h(y, \beta)}, \quad k = 1, \dots, n + 1$$

$$h(y, \beta) = (r^2\Delta^2 - \|y - x\|^2 \sin^2 \beta)^{1/2} + \|y - x\| \cos \beta$$

$$\cos \beta = \frac{\langle z - y, x - y \rangle}{\|z - y\| \|x - y\|}$$

One can verify that

$$z = \sum_{k=1}^{n+1} \alpha_k z_k, \quad z_k \in \overline{O}(x, r\Delta), \quad k = 1, \dots, n + 1$$

Then

$$\begin{aligned} f(z) - f(y) &< \sum_{k=1}^{n+1} \alpha_k u(z_k) - \sum_{k=1}^{n+1} \alpha_k u(y_k) + \varepsilon \leq \\ L\|z - y\| + L\|z - y\| \sum_{k=1}^{n+1} \frac{\alpha_k \|y_k - y\|}{(r\Delta - \|y - x\|)} + \varepsilon &\leq \\ L\|z - y\| \left(1 + \frac{r\Delta + \|y - x\|}{r\Delta - \|y - x\|} \right) + \varepsilon \end{aligned}$$

since

$$\|y_k - y\| \leq \|y_k - x\| + \|x - y\| \leq r\Delta + \|y - x\|$$

Changing places of y and x and eliminating ε we obtain the estimate (2.4).

The rest inequalities of Lemma 2.1 follow from estimate (2.4). \square

Lemma 2.2 *Assume that function $y \rightarrow \xi(y) : \overline{O}(x, r\Delta) \rightarrow R$ is convex, Lipschitz continuous and the following relation holds*

$$\xi(y) > \xi(y_0), \quad y \in \overline{O}(x, K\Delta), \quad y \neq y_0, \quad y_0 \in \partial\overline{O}(x, K\Delta)$$

$$\partial\overline{O}(x, K\Delta) = \{y \in \overline{O}(x, K\Delta) : \|y - x\| = K\Delta\}, \quad r > K$$

Then there exist a sequence $\{y_m\}$, $y_m \in O(x, K\Delta)$, $\lim_{m \rightarrow \infty} y_m = y_0$, a sequence l_m , $l_m \in D\xi(y_m)$ and a vector $l_0 \in D\xi(y_0) \subset R^n$, $\lim_{m \rightarrow \infty} l_m = l_0$ such that inequalities

$$\xi(y) - \xi(y_0) \geq \langle l_0, y - y_0 \rangle \geq 0 \quad (2.5)$$

are valid for all $y \in \overline{O}(x, K\Delta)$.

Proof

Since function $\xi(y)$ satisfies the Lipschitz condition then for any $\varepsilon > 0$ there exists a point z_ε such that the following relations hold

$$\|y_0 - z_\varepsilon\| = \frac{\varepsilon}{\lambda}, \quad z_\varepsilon \in O(x, K\Delta), \quad \xi(z_\varepsilon) \leq \xi(y_0) + \varepsilon$$

Here λ is a Lipschitz constant of function $\xi(y)$ on the set $\overline{O}(x, K\Delta)$.

Define function $\psi(y, \varepsilon)$ by the relation

$$\begin{aligned} \psi(y, \varepsilon) &= \frac{(y - z_\varepsilon)^2}{\chi(\alpha)} - 1, \quad \alpha = \alpha(y, \varepsilon) \\ \chi(\alpha) &= ((K^2\Delta^2 - (x - z_\varepsilon)^2 \sin^2 \alpha)^{1/2} + \|x - z_\varepsilon\| \cos \alpha)^2 \\ \cos \alpha &= \frac{\langle y - z_\varepsilon, x - z_\varepsilon \rangle}{\|y - z_\varepsilon\| \|x - z_\varepsilon\|} \end{aligned}$$

Function $y \rightarrow \psi(y, \varepsilon)$ is strictly convex, differentiable and nonpositive

$$\psi(y, \varepsilon) \leq 0, \quad y \in \overline{O}(x, K\Delta)$$

It has the strict minimum at point z_ε

$$\psi(y, \varepsilon) > \psi(z_\varepsilon, \varepsilon) = -1, \quad y \in \overline{O}(x, K\Delta), \quad y \neq z_\varepsilon$$

Furthermore, function $\psi(y, \varepsilon)$ receives zero values at the boundary

$$\psi(y, \varepsilon) = 0, \quad y \in \partial\overline{O}(x, K\Delta)$$

Gradients $\nabla\psi(y, \varepsilon)$ of function $\psi(y, \varepsilon)$ by variable y satisfy the inequality

$$\|\nabla\psi(y, \varepsilon)\| \leq 2\frac{\lambda}{\varepsilon}, \quad y \in O(x, K\Delta)$$

Compose the function

$$\gamma(y, \varepsilon) = \xi(y) + 2\varepsilon\psi(y, \varepsilon)$$

One can verify relations

$$\gamma(z_\varepsilon, \varepsilon) = \xi(z_\varepsilon) - 2\varepsilon \leq \xi(y_0) + \varepsilon - 2\varepsilon < \xi(y_0) = \gamma(y_0, \varepsilon)$$

Besides

$$\gamma(y_0, \varepsilon) = \xi(y_0) \leq \xi(y) = \gamma(y, \varepsilon), \quad y \in \partial\overline{O}(x, K\Delta)$$

Combining these inequalities we obtain the relation

$$\gamma(z_\varepsilon, \varepsilon) < \gamma(y, \varepsilon), \quad y \in \partial\overline{O}(x, K\Delta), \quad z_\varepsilon \in O(x, K\Delta)$$

Therefore, there exists an internal minimum point $y_\varepsilon \in O(x, K\Delta)$

$$\gamma(y_\varepsilon, \varepsilon) \leq \gamma(y, \varepsilon), \quad y \in \overline{O}(x, K\Delta)$$

Necessary conditions of minimum for convex function $y \rightarrow \gamma(y, \varepsilon)$ at point $y_\varepsilon \in O(x, K\Delta)$ implies

$$0 \in D\gamma(y_\varepsilon, \varepsilon)$$

Here $D\gamma(y_\varepsilon, \varepsilon)$ is the subdifferential of function $y \rightarrow \gamma(y, \varepsilon)$ at point y_ε . The last inclusion means that there exists a subgradient $l_\varepsilon \in D\xi(y_\varepsilon)$ satisfying the condition

$$l_\varepsilon = -2\varepsilon\nabla\psi(y_\varepsilon, \varepsilon)$$

According to definition of subdifferentials we have inequalities

$$\begin{aligned} \xi(y) - \xi(y_\varepsilon) &\geq \langle l_\varepsilon, y - y_\varepsilon \rangle = \langle -2\varepsilon\nabla\psi(y_\varepsilon, \varepsilon), y - y_\varepsilon \rangle \geq \\ &2\varepsilon(\psi(y_\varepsilon, \varepsilon) - \psi(y, \varepsilon)) \geq 2\varepsilon\psi(y_\varepsilon, \varepsilon) \geq \\ &2\varepsilon\psi(z_\varepsilon, \varepsilon) = -2\varepsilon, \quad y \in \overline{O}(x, K\Delta) \end{aligned} \quad (2.6)$$

Let us consider a sequence $\varepsilon_m \downarrow 0$, $m \rightarrow \infty$ and introduce notations $y_m = y_{\varepsilon_m}$, $l_m = l_{\varepsilon_m} \in D\xi(y_m)$.

Since $\|y_m - x\| < K\Delta$ then

$$\|l_m\| = \|-2\varepsilon_m\nabla\psi(y_m, \varepsilon_m)\| \leq 2\varepsilon_m 2\frac{\lambda}{\varepsilon_m} = 4\lambda$$

Thus, sequences $\{y_m\}$, $\{l_m\}$ are bounded. Without loss of generality assume that they converge. So there exist a limit point $y_* \in \overline{O}(x, K\Delta)$ and a limit vector $l_0 \in R^n$

$$y_* = \lim_{m \rightarrow \infty} y_m, \quad l_0 = \lim_{m \rightarrow \infty} l_m$$

Let us show that $y_0 = y_*$. Assuming the contrary $y_0 \neq y_*$ and passing to the limit in inequality (2.6) by $m \rightarrow \infty$ we obtain the relation

$$\xi(y) - \xi(y_*) \geq 0, \quad y \in \overline{O}(x, K\Delta)$$

In particular, $\xi(y) - \xi(y_0) \geq 0$. It contradicts to the condition of the strict minimum $\xi(y_0) < \xi(y_*)$. Hence, $y_0 = y_*$.

Let us prove that $l_0 \in D\xi(y_0)$. By definition of subdifferentials we have the inequality

$$\xi(y) - \xi(y_m) \geq \langle l_m, y - y_m \rangle, \quad y \in \overline{O}(x, r\Delta)$$

Passing to the limit by $m \rightarrow \infty$ we obtain the relation

$$\begin{aligned} \xi(y) - \xi(y_0) &\geq \langle l_0, y - y_0 \rangle, \quad y \in \overline{O}(x, K\Delta) \\ y_0 &\in \partial\overline{O}(x, K\Delta) \subset O(x, r\Delta) \end{aligned}$$

which means $l_0 \in D\xi(y_0)$.

Furthermore, passing to the limit in relation (2.6) while $\varepsilon_m \downarrow 0$ we get the second necessary inequality in (2.5). \square

Using Lemma (2.1) and Lemma (2.2) we indicate now properties of operator F (2.1).

Property 2.1 *Operator F (2.1) is finitely defined for all Lipschitz continuous functions u and the following estimates are valid*

$$\min_{y \in \overline{O}(x, r\Delta)} u(y) - 2LK\Delta \leq F(t, \Delta, u)(x) \leq \max_{y \in \overline{O}(x, K\Delta)} u(y) \quad (2.7)$$

Proof.

Consider a Lipschitz continuous function $u : \overline{O}(x, r\Delta) \rightarrow R$. Let $y \rightarrow f(y)$ be the convex hull of function u on the set $\overline{O}(x, r\Delta)$. According to Lemma (2.1) subdifferential $Df(y)$, $y \in O(x, r\Delta)$ is a bounded set and, hence, is a convex compactum.

Function

$$s \rightarrow (\Delta H(t, x, s) + f(y) - f(x) - \langle s, y - x \rangle) : Df(y) \rightarrow R$$

is a continuous one on the compactum $Df(y)$. Hence, maximum in (2.1) is well-defined.

Let $y \in O(x, r\Delta)$, $s \in Df(y)$. We estimate now the expression

$$R = R(t, \Delta, x, y, s) = \Delta H(t, x, s) + f(y) - f(x) - \langle s, y - x \rangle$$

We have

$$|\Delta H(t, x, s)| \leq K\|s\| \leq K \langle s, \frac{s}{\|s\|} \rangle$$

Then

$$\begin{aligned} R &\leq f(y) - f(x + \Delta K \frac{s}{\|s\|}) + \langle s, x + \Delta K \frac{s}{\|s\|} - y \rangle + \\ &f(x + \Delta K \frac{s}{\|s\|}) - f(x) \leq f(x + \Delta K \frac{s}{\|s\|}) - f(x) \end{aligned}$$

since

$$(x + \Delta K \frac{s}{\|s\|}) \in \overline{O}(x, K\Delta) \subset \overline{O}(x, r\Delta)$$

Hence,

$$\begin{aligned} F(t, \Delta, u)(x) &= f(x) + \sup_{y \in O(x, r\Delta)} \max_{s \in Df(y)} R(t, \Delta, x, y, s) \leq \\ \max_{\|y\| \leq 1} f(x + \Delta K y) &\leq \max_{y \in \overline{O}(x, K\Delta)} f(y) \leq \max_{y \in \overline{O}(x, K\Delta)} u(y) \end{aligned}$$

Thus, value $F(t, \Delta, u)(x)$ is bounded above and finitely defined.

Let us estimate value $F(t, \Delta, u)(x)$ from below. According to Lemma (2.1) we have the necessary relations for $y = x$, $s \in Df(x)$

$$\begin{aligned} F(t, \Delta, u)(x) &\geq f(x) + \Delta H(t, x, s) \geq \\ \min_{y \in \overline{O}(x, r\Delta)} u(y) - K\Delta\|s\| &\geq \min_{y \in \overline{O}(x, r\Delta)} u(y) - 2LK\Delta \quad \square \end{aligned}$$

Property 2.2 *The following equalities are valid for operator F*

$$\begin{aligned}
 F(t, \Delta, u)(x) &= f(x) + \sup_{y \in O(x, r\Delta)} \max_{s \in Df(y)} R(t, \Delta, x, y, s) = \\
 &= f(x) + \sup_{y \in O(x, K\Delta)} \max_{s \in Df(y)} R(t, \Delta, x, y, s) = \\
 &= f(x) + \max_{y \in \overline{O}(x, K\Delta)} \max_{s \in Df(y)} R(t, \Delta, x, y, s) \quad (2.8)
 \end{aligned}$$

Thus, supremum in definition of operator F (2.1) on the set $O(x, r\Delta)$ coincides with supremum on the set $O(x, K\Delta)$, $r > K$ and is realized on the set $\overline{O}(x, K\Delta)$.

Proof.

Evidently supremum on the set $O(x, r\Delta)$ is not less than supremum on the set $O(x, K\Delta)$, $r > K$. Let us prove the inverse inequality.

For this purpose we estimate the difference

$$dif = \sup_{y \in O(x, K\Delta)} \max_{s \in Df(y)} R(t, \Delta, x, y, s) - \sup_{y \in O(x, r\Delta)} \max_{s \in Df(y)} R(t, \Delta, x, y, s)$$

For $\varepsilon > 0$ let us choose $y_\varepsilon \in O(x, r\Delta)$, $l_\varepsilon \in Df(y_\varepsilon)$ such that

$$\begin{aligned}
 dif &\geq \sup_{y \in O(x, K\Delta)} \max_{s \in Df(y)} R(t, \Delta, x, y, s) - \\
 &\Delta H(t, x, l_\varepsilon) - f(y_\varepsilon) + f(x) + \langle l_\varepsilon, y_\varepsilon - x \rangle - \varepsilon
 \end{aligned}$$

Consider the function

$$y \rightarrow \xi(y) = f(y) - \langle l_\varepsilon, y \rangle$$

on the set $\overline{O}(x, K\Delta)$. Let

$$y_0 = \arg \min_{y \in \overline{O}(x, K\Delta)} \xi(y)$$

Two cases are possible.

Case 1. Let $y_0 \in O(x, K\Delta)$. Then $l_\varepsilon \in Df(y_0)$ since the relation

$$f(y) - f(y_0) \geq \langle l_\varepsilon, y - y_0 \rangle, \quad y \in \overline{O}(x, K\Delta), \quad y_0 \in O(x, K\Delta)$$

is valid.

In this case we continue the estimate

$$\begin{aligned}
 dif &\geq \Delta H(t, x, l_\varepsilon) + f(y_0) - f(x) - \langle l_\varepsilon, y_0 - x \rangle - \\
 &\Delta H(t, x, l_\varepsilon) - f(y_0) + f(x) + \langle l_\varepsilon, y_\varepsilon - x \rangle - \varepsilon \geq \\
 &f(y_0) - f(y_\varepsilon) + \langle l_\varepsilon, y_0 - y_\varepsilon \rangle - \varepsilon \geq -\varepsilon
 \end{aligned}$$

Since $\varepsilon > 0$ is an arbitrary number then we obtain the necessary inequality $dif \geq 0$.

Case 2. Let $y_0 \in \partial \overline{O}(x, K\Delta)$ and there are no other minimum points y_{min} of function $\xi(y)$ such that $y_{min} \in O(x, K\Delta)$. Let us prove that point y_0 is the unique minimum point of function $\xi(y)$ in this case.

Assuming the contrary

$$\xi(y_1) = \xi(y_0), \quad y_1 \neq y_0, \quad y_1 \in \partial \overline{O}(x, K\Delta)$$

and using the convexity property of function $\xi(y)$ we obtain the inequality

$$\lambda \xi(y_1) + (1 - \lambda) \xi(y_0) \geq \xi(\lambda y_1 + (1 - \lambda) y_0), \quad 0 < \lambda < 1$$

and, hence,

$$\xi(y_1) = \xi(y_0) \geq \xi(\lambda y_1 + (1 - \lambda)y_0)$$

So we obtain that point

$$y(\lambda) = \lambda y_1 + (1 - \lambda)y_0, \quad y(\lambda) \in O(x, K\Delta)$$

is also a minimum point of function $\xi(y)$ and come to the contradiction.

Thus, point y_0 is the unique minimum point and by Lemma 2.2 there exist a sequence $y_m, y_m \in O(x, K\Delta)$, $\lim_{m \rightarrow \infty} y_m = y_0$, a sequence $l_m, l_m \in D\xi(y_m)$ and a vector $l_0 \in D\xi(y_0)$, $\lim_{m \rightarrow \infty} l_m = l_0$ such that for all $y \in \overline{O}(x, K\Delta)$ the inequality

$$\xi(y) - \xi(y_0) \geq \langle l_0, y - y_0 \rangle \geq 0$$

takes place.

In other words, there exist a vector $s_0 \in Df(y_0)$ and a sequence $s_m, s_m \in Df(y_m)$, $\lim_{m \rightarrow \infty} s_m = s_0$ such that for all $y \in \overline{O}(x, K\Delta)$ the following relations

$$f(y) - f(y_0) - \langle l_\varepsilon, y - y_0 \rangle \geq \langle s_0 - l_\varepsilon, y - y_0 \rangle \geq 0$$

are valid.

Taking into account the Lipschitz continuity (H3) of the Hamiltonian H we obtain the estimate

$$\begin{aligned} dif &\geq \Delta H(t, x, s_0) + f(y_0) - f(x) - \langle s_0, y_0 - x \rangle - \\ &\Delta H(t, x, l_\varepsilon) - f(y_\varepsilon) + f(x) + \langle l_\varepsilon, y_\varepsilon - x \rangle - 2\varepsilon \geq \\ &-\Delta K \|s_0 - l_\varepsilon\| + (f(y_0) - f(y_\varepsilon) - \langle l_\varepsilon, y_0 - y_\varepsilon \rangle) + \langle s_0 - l_\varepsilon, x - y_0 \rangle - 2\varepsilon \geq \\ &\langle s_0 - l_\varepsilon, (x - K\Delta \frac{(s_0 - l_\varepsilon)}{\|s_0 - l_\varepsilon\|}) - y_0 \rangle - 2\varepsilon \geq -2\varepsilon \end{aligned}$$

since

$$(x - K\Delta \frac{(s_0 - l_\varepsilon)}{\|s_0 - l_\varepsilon\|}) \in \overline{O}(x, K\Delta)$$

Arbitrariness of number $\varepsilon > 0$ implies the necessary inequality $dif \geq 0$.

Let us prove that external supremum in definition (2.1) of operator F is realized on the set $\overline{O}(x, K\Delta)$.

For a sequence $\{\varepsilon_m\}$, $\varepsilon_m \downarrow 0$, $m \rightarrow \infty$ let us choose sequences $y_m, y_m \in O(x, K\Delta)$, $s_m, s_m \in Df(y)$ such that

$$F(t, \Delta, u)(x) \leq f(x) + (\Delta H(t, x, s_m) + f(y_m) - f(x) - \langle s_m, y_m - x \rangle) + \varepsilon_m \quad (2.9)$$

Sequence $\{y_m\}$ is bounded. According to Lemma 2.1 sequence $\{s_m\}$ is also bounded

$$\|s_m\| \leq L \left(1 + \frac{r + K}{r - K}\right)$$

Without loss of generality let us assume that sequences $\{y_m\}$, $\{s_m\}$ converge. So there exist a point $y_0 \in \overline{O}(x, K\Delta)$ and a vector $s_0 \in R^n$ such that

$$y_0 = \lim_{m \rightarrow \infty} y_m, \quad s_0 = \lim_{m \rightarrow \infty} s_m$$

Let us show that $s_0 \in Df(y_0)$. Really, since $s_m \in Df(y_m)$, $y_m \in O(x, K\Delta) \subset O(x, r\Delta)$ then for all $y \in \overline{O}(x, r\Delta)$ the following inequality is valid

$$f(y) - f(y_m) \geq \langle s_m, y - y_m \rangle$$

Passing to the limit by $m \rightarrow \infty$ in this inequality we obtain relations

$$f(y) - f(y_0) \geq \langle s_0, y - y_0 \rangle, \quad y \in \overline{O}(x, r\Delta), \quad y_0 \in \overline{O}(x, K\Delta) \subset O(x, r\Delta)$$

The last inequality means that

$$s_0 \in Df(y_0), \quad y_0 \in \overline{O}(x, K\Delta)$$

Passing also to the limit by $m \rightarrow \infty$ in relation (2.9) we derive inequalities

$$\begin{aligned} F(t, \Delta, u)(x) &\leq f(x) + (\Delta H(t, x, s_0) + f(y_0) - f(x) - \langle s_0, y_0 - x \rangle) \leq \\ &f(x) + \sup_{y \in \overline{O}(x, K\Delta)} \max_{s \in Df(y)} \{\Delta H(t, x, s_0) + f(y_0) - f(x) - \langle s_0, y_0 - x \rangle\} \end{aligned}$$

Besides, the inverse inequality is evidently fulfilled. Hence, all inequalities turn into equalities. Therefore, the external supremum is realized on the set $\overline{O}(x, K\Delta)$. \square

Property 2.3 *Relation (2.1) in definition of operator F is the programming maximin formula on local convex hulls and can be regarded as generalization of Hopf's formula*

$$\begin{aligned} F(t, \Delta, u)(x) &= \sup_{s \in R^n} \{\langle s, x \rangle + \Delta H(t, x, s) - f^*(s)\} = \\ &\max_{q(\cdot)} \min_{p(\cdot)} f(x + \Delta h(t, x) + \int_t^{t+\Delta} B(t, x)p(\tau)d\tau + \int_t^{t+\Delta} C(t, x)q(\tau)d\tau) = \\ &\max_{q \in Q} \min_{p \in P} f(x + \Delta(h(t, x) + B(t, x)p + C(t, x)q)) \end{aligned} \quad (2.10)$$

Here

$$f^*(s) = \sup_{y \in \overline{O}(x, r\Delta)} \{\langle s, y \rangle - f(y)\}$$

is the conjugate function

$$\tau \rightarrow p(\tau) : [t, t + \Delta) \rightarrow P, \quad \tau \rightarrow q(\tau) : [t, t + \Delta) \rightarrow Q$$

are Lebesgue measurable programming controls.

Proof.

Taking into account Property 2.1 we have the following formula for operator F

$$F(t, \Delta, u)(x) = \max_{y \in \overline{O}(x, K\Delta)} \max_{s \in Df(y)} \{\Delta H(t, x, s) + f(y) - \langle s, y - x \rangle\}$$

Properties of subdifferentials of convex functions imply relations

$$s \in Df(y) \iff \langle s, y \rangle - f(y) = f^*(s)$$

Hence,

$$F(t, \Delta, u)(x) = \max_{y \in \overline{O}(x, K\Delta)} \max_{s \in Df(y)} \{\langle s, x \rangle + \Delta H(t, x, s) - f^*(s)\}$$

Using the scheme of proof of Property 2.2 one can find out that maximum on the set

$$\{s \in R^n : s \in Df(y), \quad y \in \overline{O}(x, K\Delta)\}$$

coincides with supremum on the space R^n

$$\begin{aligned} F(t, \Delta, u)(x) &= \sup_{s \in R^n} \{ \langle s, x \rangle + \Delta H(t, x, s) - f^*(s) \} = \\ &= \sup_{R^n} \max_{q \in Q} \min_{p \in P} \{ \langle s, x \rangle + \Delta(h(t, x) + B(t, x)p + C(t, x)q) - f^*(s) \} \end{aligned}$$

The last relation generalizes the Hopf's formula [Hopf, 1965], [Bardi, Osher, 1991] and according to permutability of operators

$$\sup_{R^n}, \max_{q \in Q}, \min_{p \in P}$$

coincides with the programming maximin formula [Krasovskii, Subbotin, 1974], [Ushakov, 1981] for the convex hull $y \rightarrow f(y)$

$$F(t, \Delta, u)(x) = \max_{q \in Q} \min_{p \in P} f(x + \Delta(h(t, x) + B(t, x)p + C(t, x)q))$$

3 Properties of Operators with Generalized Gradients and Convergence of Approximation Schemes

Approximation schemes for Hamilton-Jacobi equations were considered in the framework of the theory of viscosity solutions [Crandall, Lions, 1984], [Souganidis, 1985]. Sufficient conditions providing convergence of approximation schemes were formulated for finite difference operators. Explicit approximation schemes with operators of Lax-Friedrichs type were analyzed in the work [Crandall, Lions, 1984]. Sufficient conditions of convergence of approximation schemes were given in the work [Souganidis, 1985] and implicit approximation schemes with Lax-Friedrichs operators were developed.

We formulate now these sufficient conditions for convergence of approximation schemes and check them for operator F (2.1) based on constructions of subdifferentials of local convex hulls.

Theorem 3.1 *Finite difference operator $u \rightarrow F(t, \Delta, u)$ (2.1) based on subdifferentials of local convex hulls satisfies the following conditions.*

(F1) *Compatibility : for $\Delta = 0$ the map F is the identity operator*

$$F(t, 0, u)(x) = u(x), \quad x \in D_t \tag{3.1}$$

(F2) *Continuity : Mapping $(t, \Delta) \rightarrow F(t, \Delta, u)$ is continuous.*

(F3) *Additivity with constants : for all points $x \in D_t$ and constants $a \in R$ the equality*

$$F(t, \Delta, u + a)(x) = F(t, \Delta, u)(x) + a \tag{3.2}$$

takes place.

(F4) *Boundedness : there exists a constant $C_1 \geq 0$ such that for all $x \in D_t$ the following inequality holds*

$$|F(t, \Delta, u)(x) - u(x)| \leq C_1, \quad C_1 = (r + 2K)L\Delta \tag{3.3}$$

(F5) *Monotonicity :*

$$\begin{aligned} &\text{if } u(x) \geq v(x) \text{ for all } x \in D_{t+\Delta} \\ &\text{then } F(t, \Delta, u)(x) \geq F(t, \Delta, v)(x) \text{ for all } x \in D_t \end{aligned} \tag{3.4}$$

(F6) Exponential growth : there exists a constant $C_2 \geq 0$ such that the inequality

$$\|F(t, \Delta, u)\|_{D_t} \leq \exp(C_2\Delta)(\|u\|_{D_{t+\Delta}} + C_2\Delta) \quad (3.5)$$

is valid.

Here

$$\begin{aligned} \|F(t, \Delta, u)\|_{D_t} &= \max_{x \in D_t} |F(t, \Delta, u)(x)| \\ \|u\|_{D_{t+\Delta}} &= \max_{x \in D_{t+\Delta}} |u(x)| \end{aligned}$$

By virtue of positive homogeneity (H5) of the Hamiltonian H one can assume $C_2 = 0$.

(F7) Lipschitz continuity by variable x : there exists a constant C_3 such that for all $x_i \in D_t$, $i = 1, 2$ the Lipschitz condition holds

$$|F(t, \Delta, u)(x_1) - F(t, \Delta, u)(x_2)| \leq \exp(C_3\Delta)L\|x_1 - x_2\| \quad (3.6)$$

Here

$$C_3 = L_1(G_r) \left(1 + \frac{r + K}{r - K}\right)$$

$L_1(G_r)$ is a Lipschitz constant (H2) of the Hamiltonian H , L is a Lipschitz constant of the function u on the set $D_{t+\Delta}$.

(F8) Generator type condition : there exists a constant C_4 such that for all twice differentiable functions $\varphi : D_{t+\Delta} \rightarrow R$ and points $x \in D_t \subset D_{t+\Delta}$ the following estimate holds

$$\left| \frac{F(t, \Delta, \varphi)(x) - \varphi(x)}{\Delta} - H(t, x, \nabla\varphi(x)) \right| \leq C_4\|\partial^2\varphi\|\Delta \quad (3.7)$$

$$C_4 = r^2 + 2Kr \left(2 + \frac{r + K}{r - K}\right)$$

Here

$$\|\partial^2\varphi\| = \sum_{i,j}^n \left\| \frac{\partial^2\varphi}{\partial x_i \partial x_j} \right\|, \quad \left\| \frac{\partial^2\varphi}{\partial x_i \partial x_j} \right\| = \max_{y \in D_{t+\Delta}} \left| \frac{\partial^2\varphi(y)}{\partial x_i \partial x_j} \right|$$

Proof.

Arguments in proves of conditions (F1)-(F7) are similar. Therefore, omiting proves of conditions (F1)-(F4), (F6), (F7) we give the proof of the monotonicity condition (F5). We also present the proof of the generator type condition (F8) for Hamilton-Jacobi equation (1.6).

Proof of condition (F5).

Let us introduce the following notations. By symbols $f(y)$ and $h(y)$ we denote convex hulls of functions $y \rightarrow u(y)$ and $y \rightarrow v(y)$ respectively on the set $\overline{O}(x, r\Delta)$. Remind that $u(y) \geq v(y)$ and, hence, $f(y) \geq h(y)$, $y \in \overline{O}(x, r\Delta)$.

Let us estimate the difference

$$dif = F(t, \Delta, u)(x) - F(t, \Delta, v)(x)$$

Let $\varepsilon > 0$, $y_\varepsilon \in O(x, K\Delta)$, $l_\varepsilon \in Dh(y_\varepsilon)$ be such that

$$dif \geq F(t, \Delta, u)(x) - \Delta H(t, x, l_\varepsilon) - h(y_\varepsilon) + \langle l_\varepsilon, y_\varepsilon - x \rangle - \varepsilon$$

Consider the function $\xi(y) = f(y) - \langle l_\varepsilon, y \rangle$ on the set $\overline{O}(x, K\Delta)$. Let

$$y_0 = \arg \min_{y \in \overline{O}(x, K\Delta)} (f(y) - \langle l_\varepsilon, y \rangle)$$

Two cases are possible.

Case 1. Let $y_0 \in O(x, K\Delta)$. Then $l_\varepsilon \in Df(y_0)$, since

$$f(y) - f(y_0) \geq \langle l_\varepsilon, y - y_0 \rangle, \quad y \in \overline{O}(x, K\Delta), \quad y_0 \in O(x, K\Delta)$$

Hence, we have relations

$$\begin{aligned} dif &\geq \Delta H(t, x, l_\varepsilon) + f(y_0) - \langle l_\varepsilon, y_0 - x \rangle + \\ &\Delta H(t, x, l_\varepsilon) - h(y_\varepsilon) + \langle l_\varepsilon, y_\varepsilon - x \rangle - \varepsilon \geq \\ &f(y_0) - h(y_\varepsilon) - \langle l_\varepsilon, y_0 - y_\varepsilon \rangle - \varepsilon \geq \\ &h(y_0) - h(y_\varepsilon) - \langle l_\varepsilon, y_0 - y_\varepsilon \rangle - \varepsilon \geq -\varepsilon \end{aligned}$$

According to the arbitrariness of $\varepsilon > 0$ we obtain the necessary inequality

$$F(t, \Delta, u)(x) - F(t, \Delta, v)(x) \geq 0$$

Case 2. Let $y_0 \in \partial\overline{O}(x, K\Delta)$. In this case we can repeat arguments of the proof of Property 2.2 and obtain the monotonicity condition.

Proof of condition (F8).

By the symbol $f(y)$ we denote the convex hull of the function $y \rightarrow \varphi(y)$ on the set $\overline{O}(x, K\Delta)$. Let $y \in O(x, K\Delta)$, $l \in Df(y)$.

We estimate at first the expression

$$\|l - \nabla\varphi(x)\| = \langle l - \nabla\varphi(x), \frac{l - \nabla\varphi(x)}{\|l - \nabla\varphi(x)\|} \rangle$$

Let point z be given by the relation

$$z = y + \alpha \frac{(l - \nabla\varphi(x))}{\|l - \nabla\varphi(x)\|}, \quad 0 < \alpha \leq K\Delta - \|y - x\|$$

One can check that $z \in \overline{O}(x, K\Delta)$, $\alpha = \|z - y\|$.

Then according to definition of a subgradient $l \in Df(y)$ we get relations

$$\begin{aligned} \|l - \nabla\varphi(x)\| &= \langle l - \nabla\varphi(x), \frac{(z - y)}{\|z - y\|} \rangle = \\ &\frac{1}{\|z - y\|} (\langle l, z - y \rangle - \langle \nabla\varphi(x), z - y \rangle) \leq \\ &\frac{1}{\|z - y\|} (f(z) - f(y) - \langle \nabla\varphi(x), z - y \rangle) \end{aligned}$$

Let

$$\varepsilon > 0, \quad y_k \in \overline{O}(x, r\Delta), \quad \alpha_k \geq 0, \quad \sum_{k=1}^{n+1} \alpha_k y_k = y, \quad \sum_{k=1}^{n+1} \alpha_k = 1$$

be such that

$$f(y) \geq \sum_{k=1}^{n+1} \alpha_k \varphi(y_k) - \varepsilon \|z - y\|$$

Then

$$\|l - \nabla\varphi\| \leq \frac{1}{\|z - y\|} (f(z) - \sum_{k=1}^{n+1} \alpha_k \varphi(y_k) - \langle \nabla\varphi(x), z - y \rangle) + \varepsilon$$

Let

$$z_k = y_k + (z - y) - \frac{(y_k - y)\|z - y\|}{h(y, \alpha)}, \quad k = 1, \dots, n + 1$$

$$h(y, \alpha) = (r^2 \Delta^2 - \|y - x\|^2 \sin^2 \alpha)^{1/2} + \|y - x\| \cos \alpha$$

$$\cos \alpha = \frac{\langle z - y, y - x \rangle}{\|z - y\| \|y - x\|}$$

One can verify that

$$z = \sum_{k=1}^{n+1} \alpha_k z_k, \quad z_k \in \overline{O}(x, r\Delta), \quad k = 1, \dots, n + 1$$

According to the Lagrange mean value theorem applied to the function $y \rightarrow \varphi(y)$ we have relations

$$\|l - \nabla \varphi(x)\| \leq \frac{1}{\|z - y\|} \left(\sum_{k=1}^{n+1} \alpha_k \varphi(z_k) - \sum_{k=1}^{n+1} \alpha_k \varphi(y_k) - \langle \nabla \varphi(x), z - y \rangle \right) + \varepsilon =$$

$$\frac{1}{\|z - y\|} \left(\sum_{k=1}^{n+1} \alpha_k \langle \nabla \varphi(w_k), (z - y) - \frac{(y_k - y)\|z - y\|}{h(y, \alpha)} \rangle - \sum_{k=1}^{n+1} \alpha_k \langle \nabla \varphi(x), z - y \rangle \right) + \varepsilon$$

$$w_k = y_k + \vartheta_k(z - y), \quad 0 < \vartheta_k < 1, \quad w_k \in \overline{O}(x, r\Delta)$$

Taking into account the equality

$$\sum_{k=1}^{n+1} \alpha_k \langle \nabla \varphi(x), y_k - y \rangle = \langle \nabla \varphi(x), \sum_{k=1}^{n+1} \alpha_k y_k - y \rangle = 0$$

we have the estimate

$$\|l - \nabla \varphi(x)\| \leq \frac{1}{\|z - y\|} \left(\sum_{k=1}^{n+1} \alpha_k \langle \nabla \varphi(w_k) - \nabla \varphi(x), (z - y) - \frac{(y_k - y)\|z - y\|}{h(y, \alpha)} \rangle \right) + \varepsilon$$

We continue estimation by the Lagrange mean value theorem applied to the function $y \rightarrow \nabla \varphi(y)$

$$\|l - \nabla \varphi(x)\| \leq \|\partial^2 \varphi\| \sum_{k=1}^{n+1} \alpha_k \|w_k - x\| \left(1 + \frac{\|y_k - y\|}{h(y, \alpha)} \right) + \varepsilon \leq$$

$$\|\partial^2 \varphi\| r \left(1 + \frac{r + K}{r - K} \right) \Delta + \varepsilon$$

Due to arbitrariness of $\varepsilon > 0$ we obtain the inequality

$$\|l - \nabla \varphi(x)\| \leq \|\partial^2 \varphi\| r \left(1 + \frac{r + K}{r - K} \right) \Delta \quad (3.8)$$

We estimate now the following expression

$$\begin{aligned} & \left| \frac{F(t, \Delta, \varphi)(x) - \varphi(x)}{\Delta} - H(t, x, \nabla\varphi(x)) \right| \leq \\ & \frac{1}{\Delta} |f(x) - \varphi(x)| + \left| \frac{1}{\Delta} \sup_{y \in O(x, K\Delta)} \max_{s \in Df(y)} \{ \Delta H(t, x, s) + \right. \\ & \left. f(y) - f(x) - \langle s, y - x \rangle \} - H(t, x, \nabla\varphi(x)) \right| \end{aligned}$$

Let us evaluate the first term. Let

$$\varepsilon > 0, \quad x_k \in \overline{O}(x, r\Delta), \quad \alpha_k \geq 0, \quad \sum_{k=1}^{n+1} \alpha_k x_k = x, \quad \sum_{k=1}^{n+1} \alpha_k = 1$$

be such that

$$\frac{1}{\Delta} |f(x) - \varphi(x)| \leq \frac{1}{\Delta} \left| \sum_{k=1}^{n+1} \alpha_k \varphi(x_k) - \sum_{k=1}^{n+1} \alpha_k \varphi(x) \right| + \varepsilon$$

By the Lagrange mean value theorem we have

$$\frac{1}{\Delta} |f(x) - \varphi(x)| \leq \frac{1}{\Delta} \left| \sum_{k=1}^{n+1} \alpha_k \langle \nabla\varphi(t_k), x_k - x \rangle \right| + \varepsilon$$

$$t_k = x_k + \lambda_k(x_k - x), \quad 0 < \lambda_k < 1, \quad t_k \in \overline{O}(x, r\Delta), \quad k = 1, \dots, n+1$$

Since

$$\sum_{k=1}^{n+1} \alpha_k \langle \nabla\varphi(x), x_k - x \rangle = 0$$

then

$$\frac{1}{\Delta} |f(x) - \varphi(x)| \leq \frac{1}{\Delta} \left| \sum_{k=1}^{n+1} \alpha_k \langle \nabla\varphi(t_k) - \nabla\varphi(x), x_k - x \rangle \right| + \varepsilon$$

Again by the Lagrange mean value theorem we have

$$\begin{aligned} \frac{1}{\Delta} |f(x) - \varphi(x)| & \leq \frac{1}{\Delta} \|\partial^2\varphi\| \sum_{k=1}^{n+1} \alpha_k \|t_k - x\| \|x_k - x\| + \varepsilon \leq \\ & \|\partial^2\varphi\| r^2 \Delta + \varepsilon \end{aligned}$$

and, hence,

$$\frac{1}{\Delta} \|f(x) - \varphi(x)\| \leq \|\partial^2\varphi\| r^2 \Delta \tag{3.9}$$

Let us estimate the second term. For $\varepsilon > 0$ one can find a point $y_\varepsilon \in O(x, K\Delta)$ and a subgradient $l_\varepsilon \in Df(y_\varepsilon)$ such that

$$\begin{aligned} & \left| \frac{1}{\Delta} \sup_{y \in O(x, K\Delta)} \max_{s \in Df(y)} \{ \Delta H(t, x, s) + \right. \\ & \left. f(y) - f(x) - \langle s, y - x \rangle \} - H(t, x, \nabla\varphi(x)) \right| \leq \\ & |H(t, x, l_\varepsilon) - H(t, x, \nabla\varphi(x))| + \frac{1}{\Delta} |f(y_\varepsilon) - f(x) - \langle l_\varepsilon, y_\varepsilon - x \rangle| + \varepsilon = \\ & |H(t, x, l_\varepsilon) - H(t, x, \nabla\varphi(x))| + \frac{1}{\Delta} (f(x) - f(y_\varepsilon) - \langle l_\varepsilon, x - y_\varepsilon \rangle) + \varepsilon = \end{aligned}$$

In the last inequality we take into account that according to definition of subdifferentials the following relation holds

$$f(x) - f(y_\varepsilon) - \langle l_\varepsilon, x - y_\varepsilon \rangle \geq 0$$

Using (3.8) one can estimate the first term in this sum

$$|H(t, x, l_\varepsilon) - H(t, x, \nabla\varphi(x))| \leq \|\partial^2\varphi\|Kr \left(1 + \frac{r+K}{r-K}\right) \Delta \quad (3.10)$$

Let us evaluate the expression

$$\frac{1}{\Delta}(f(x) - f(y_\varepsilon) - \langle l_\varepsilon, x - y_\varepsilon \rangle)$$

Let

$$\varepsilon > 0, \quad y_k \in \overline{O}(x, r\Delta), \quad \alpha_k \geq 0, \quad \sum_{k=1}^{n+1} \alpha_k y_k = y_\varepsilon, \quad \sum_{k=1}^{n+1} \alpha_k = 1$$

be such that

$$\begin{aligned} & \frac{1}{\Delta}(f(x) - f(y_\varepsilon) - \langle l_\varepsilon, x - y_\varepsilon \rangle) \leq \\ & \frac{1}{\Delta}(f(x) - \sum_{k=1}^{n+1} \alpha_k \varphi(y_k) - \langle l_\varepsilon, x - y_\varepsilon \rangle) + \varepsilon \end{aligned}$$

Assume

$$x_k = y_k + (x - y_\varepsilon) + \frac{(y_k - y_\varepsilon)\|x - y_\varepsilon\|}{(r\Delta + \|x - y_\varepsilon\|)}$$

One can verify that

$$x = \sum_{k=1}^{n+1} \alpha_k x_k, \quad x_k \in \overline{O}(x, r\Delta), \quad k = 1, \dots, n+1$$

Then by the Lagrange mean value theorem we obtain estimates

$$\begin{aligned} & \frac{1}{\Delta}(f(x) - f(y_\varepsilon) - \langle l_\varepsilon, x - y_\varepsilon \rangle) \leq \\ & \frac{1}{\Delta} \left(\sum_{k=1}^{n+1} \alpha_k \varphi(x_k) - \sum_{k=1}^{n+1} \alpha_k \varphi(y_k) - \langle l_\varepsilon, x - y_\varepsilon \rangle \right) + \varepsilon = \\ & \frac{1}{\Delta} \left(\sum_{k=1}^{n+1} \alpha_k \langle \nabla\varphi(p_k), (x - y_\varepsilon) - \frac{(y_k - y_\varepsilon)\|x - y_\varepsilon\|}{(r\Delta + \|x - y_\varepsilon\|)} \rangle - \right. \\ & \left. \langle l_\varepsilon, x - y_\varepsilon \rangle \right) + \varepsilon, \quad p_k = y_k + \mu(x_k - y_k) \\ & 0 < \mu_k < 1, \quad p_k \in \overline{O}(x, r\Delta), \quad k = 1, \dots, n+1 \end{aligned}$$

Taking into account the relation

$$\sum_{k=1}^{n+1} \alpha_k \langle \nabla\varphi(x), y_k - y_\varepsilon \rangle = 0$$

we have

$$\begin{aligned}
& \frac{1}{\Delta}(f(x) - f(y_\varepsilon) - \langle l_\varepsilon, x - y_\varepsilon \rangle) \leq \\
& \frac{1}{\Delta} \left(\sum_{k=1}^{n+1} \alpha_k \langle \nabla \varphi(p_k) - \nabla \varphi(x), x - y_\varepsilon \rangle + \right. \\
& \left. \sum_{k=1}^{n+1} \alpha_k \langle \nabla \varphi(x) - l_\varepsilon, x - y_\varepsilon \rangle - \right. \\
& \left. \sum_{k=1}^{n+1} \alpha_k \langle \nabla \varphi(p_k) - \nabla \varphi(x), y_k - y_\varepsilon \rangle \frac{\|x - y_\varepsilon\|}{(r\Delta + \|x - y_\varepsilon\|)} \right) + \varepsilon \leq \\
& \frac{1}{\Delta} (\|\partial^2 \varphi\| \|p_k - x\| \|x - y_\varepsilon\| + \|\partial^2 \varphi\| r \left(1 + \frac{r+K}{r-K}\right) \|x - y_\varepsilon\| \Delta + \\
& \|\partial^2 \varphi\| \|p_k - x\| \|x - y_\varepsilon\| \frac{\|x - y_\varepsilon\|}{(r\Delta + \|x - y_\varepsilon\|)}) + \varepsilon \leq \\
& \frac{1}{\Delta} \|\partial^2 \varphi\| \|x - y_\varepsilon\| (2\|p_k - x\| + r(1 + \frac{r+K}{r-K})\Delta) + \varepsilon \leq \\
& \Delta \|\partial^2 \varphi\| K(2r + r(1 + \frac{r+K}{r-K})) + \varepsilon
\end{aligned}$$

Due to arbitrariness of $\varepsilon > 0$ we obtain the inequality

$$\frac{1}{\Delta}(f(x) - f(y_\varepsilon) - \langle l_\varepsilon, x - y_\varepsilon \rangle) \leq \Delta \|\partial^2 \varphi\| K r \left(3 + \frac{r+K}{r-K}\right) \quad (3.11)$$

Combining inequalities (3.8), (3.11) we derive the necessary estimate

$$\begin{aligned}
& \left| \frac{F(t, \Delta, \varphi)(x) - \varphi(x)}{\Delta} - H(t, x, \nabla \varphi(x)) \right| \leq \\
& \|\partial^2 \varphi\| r^2 \Delta + \|\partial^2 \varphi\| K r \left(1 + \frac{r+K}{r-K}\right) \Delta + \|\partial^2 \varphi\| K r \left(3 + \frac{r+K}{r-K}\right) \Delta = \\
& (r^2 + 2Kr(2 + \frac{r+K}{r-K})) \|\partial^2 \varphi\| \Delta = C_4 \|\partial^2 \varphi\| \Delta \quad \square
\end{aligned}$$

Using results of [Crandall, Lions, 1984], [Souganidis, 1985] about sufficiency of conditions (F1)-(F8) for convergence of approximation schemes one can formulate the following proposition.

Theorem 3.2 *Let function w be the generalized solution of the boundary value problem (1.6), (1.7) in domain G_r and for partition $\Gamma = \{t_0 < t_1 < \dots < t_N = \vartheta\}$ of interval T with step $\Delta = t_{i+1} - t_i$, $i = 0, \dots, N-1$ the approximation scheme with operator F (2.1) be determined by formulas*

$$\begin{aligned}
& u_\Gamma(\vartheta, x) = \sigma(x), \quad x \in D_\vartheta \\
& u_\Gamma(t, x) = F(t, t_{i+1} - t, u_\Gamma(t_{i+1}, \cdot))(x) \\
& t \in [t_i, t_{i+1}), \quad x \in D_t, \quad i = 0, \dots, N-1
\end{aligned} \quad (3.12)$$

Then approximation scheme (3.12) converges to generalized solution w of problem (1.6), (1.7) with the estimate of convergence $\Delta^{1/2}$

$$\|u_\Gamma - w\|_{G_r} \leq C \Delta^{1/2} \quad (3.13)$$

Here

$$\|u_\Gamma - w\|_{G_r} = \max_{(t,x) \in G_r} |u_\Gamma(t, x) - w(t, x)|$$

In practice we use a grid realization of operator $F(t, \Delta, u)$. We define grid operator $F^*(t, \Delta, u)$ as a piecewise linear function whose graph vertices are situated at nodes of the fixed grid. For this purpose we need the following notations.

Let

$$(\tau, x_0) \in G_r, \quad \tau \in \Gamma, \quad h_i = \gamma_i \Delta > 0, \quad i = 1, \dots, n$$

We define the spatial grid $GR(\tau)$ by the formula

$$GR(\tau) = \left\{ y \in R^n : y = x_0 + \sum (m_1 h_1 e_1 + \dots + m_n h_n e_n), \quad (\tau, y) \in G_r, \right. \\ \left. m_i = 0, \pm 1, \pm 2, \dots, \quad i = 1, \dots, n \right\}$$

Here vectors e_i are basis vectors $e_i^j = 0, e_i^i = 1, i, j = 1, \dots, n, i \neq j$ in R^n .

Let D_τ^* be the convex hull of the grid $GR(\tau)$

$$D_\tau^* = \left\{ y \in R^n : y = \sum_{j=0}^n \alpha_j y_j, \quad y_j \in GR(\tau), \quad \alpha_j \geq 0, \quad \sum_{j=0}^n \alpha_j = 1 \right\}$$

Let us fix $t \in \Gamma, t + \Delta \in \Gamma$ and a simplex partition Ω of the n -dimensional cube.

Assume that at time $t + \Delta$ values of function u are given at nodes y_j of the spatial grid $GR(t + \Delta)$. Define function $u : D_{t+\Delta}^* \rightarrow R$ as piecewise linear interpolation of these values according to partition Ω . Assume that values of operator $F(t, \Delta, u)$ are calculated at nodes x_j of spatial grid $GR(t)$. We define operator F^* by piecewise linear interpolation of values of operator F

$$F^*(t, \Delta, u)(x) = \sum_{j=0}^n \alpha_j F(t, \Delta, u)(x_j) \tag{3.14}$$

$$x \in D_t^*, \quad x = \sum_{j=0}^n \alpha_j x_j, \quad x_j \in GR(t), \quad \alpha_j \geq 0, \quad j = 0, \dots, n, \quad \sum_{j=0}^n \alpha_j = 1$$

$$x_m = x_0 + \sum (k_1 h_1 e_1 + \dots + k_n h_n e_n), \quad m = 1, \dots, n, \quad k_i = 0, \pm 1, \quad i = 1, \dots, n$$

Coefficients $\alpha_j = \alpha_j(\Omega)$ and points $x_j = x_j(\Omega), j = 0, \dots, n$ are determined uniquely by partition Ω .

We can formulate for operator F^* the similar results as for operator F .

Theorem 3.3 *Operator F^* (3.14) satisfies conditions (F1)-(F8) with parameters*

$$C_1^* = (r + 2K + \sqrt{n} \max_i \gamma_i) L \Delta \tag{3.15}$$

$$C_2^* = C_2 = 0 \tag{3.16}$$

$$C_3^* = C_3 = L_1(G_r) \left(1 + \frac{r + K}{r - K} \right) \tag{3.17}$$

$$C_4^* = (C_4 + n \max_i \gamma_i^2 + \sqrt{n} K \max_i \gamma_i) \|\partial^2 \varphi\| + \\ \sqrt{n} L_1(G_r) \max_i \gamma_i \|\nabla \varphi\| \tag{3.18}$$

Theorem 3.4 *Approximation scheme with operator F^**

$$\begin{aligned}
 u_{\Gamma}^*(\vartheta, y) &= \sigma^*(y) = \sum_{j=0}^n \alpha_j \sigma(y_j), \quad y \in D_{\vartheta}^*, \quad y = \sum_{j=0}^n \alpha_j y_j \\
 \alpha_j &= \alpha_j(\Omega) \geq 0, \quad \sum_{j=0}^n \alpha_j = 1, \quad y_j = y_j(\Omega) \in GR(\vartheta) \\
 u_{\Gamma}^*(t, x) &= F^*(t, t_{i+1} - t, u_{\Gamma}^*(t_{i+1}, \cdot))(x) \\
 t &\in [t_i, t_{i+1}), \quad x \in D_t^*, \quad i = 0, \dots, N-1
 \end{aligned} \tag{3.19}$$

converges to the generalized solution w of the boundary value problem (1.6), (1.7) with the estimate of convergence

$$\|u_{\Gamma}^* - w\|_{G_r^*} \leq C^* \Delta^{1/2} \tag{3.20}$$

Here

$$\begin{aligned}
 \|u_{\Gamma}^* - w\|_{G_r^*} &= \max_{(t,x) \in G_r^*} |u_{\Gamma}^*(t, x) - w(t, x)| \\
 G_r^* &= \{(t, x) \in G_r : t \in T, x \in D_t^*\}
 \end{aligned}$$

Parameter C^* in (3.20) depends only on Lipschitz constants of the payoff function σ and the Hamiltonian H .

4 Algorithms for Computing Values of Operator F

We indicate now some properties of operator F which are necessary for computational algorithms. Let us introduce the following notations

$$\begin{aligned}
 F(t, \Delta, r_i, u)(x) &= f(r_i, x) + \max_{y \in \overline{O}(x, K\Delta)} \max_{s \in Df(y)} \{\Delta H(t, x, s) + \\
 &\quad f(r_i, y) - f(r_i, x) - \langle s, y - x \rangle\}, \quad i = 1, 2 \\
 F(t, \Delta, S, u)(x) &= f(S, x) + \max_{y \in \overline{O}(x, K\Delta)} \max_{s \in Df(y)} \{\Delta H(t, x, s) + \\
 &\quad f(S, y) - f(S, x) - \langle s, y - x \rangle\}
 \end{aligned}$$

Here $r_2 > r_1 > K$, the set $S = S(x, r_1, r_2, \Delta)$ is the convex polytope satisfying inclusions

$$\overline{O}(x, r_1\Delta) \subseteq S(x, r_1, r_2, \Delta) \subseteq \overline{O}(x, r_2\Delta)$$

Functions $f(S, \cdot)$, $f(r_i, \cdot)$ are convex hulls of the function $u(\cdot)$ on sets $S(x, r_1, r_2, \Delta)$, $\overline{O}(x, r_i\Delta)$, $i = 1, 2$ respectively.

Property 4.4 *Inequalities*

$$F(t, \Delta, r_1, u)(x) \geq F(t, \Delta, S, u)(x) \geq F(t, \Delta, r_2, u)(x) \tag{4.1}$$

are valid for operator F .

Proof.

It is clear that convex hulls are connected by relations

$$f(r_1, y) \geq f(S, y) \geq f(r_2, y), \quad y \in \overline{O}(x, r\Delta)$$

By definition of operator F the following equalities are valid

$$\begin{aligned} F(t, \Delta, r_i, u)(x) &= F(t, \Delta, r_i, f(r_i, \cdot))(x), \quad i = 1, 2 \\ F(t, \Delta, S, u)(x) &= F(t, \Delta, S, f(S, \cdot))(x) \end{aligned}$$

Combining these relations with monotonicity condition (F5) we obtain the necessary chain of inequalities (4.1). \square

A peculiarity of operator F consists in the mathematical programming problem (2.1). In the case when the convex hull f is a piecewise linear function and the Hamiltonian H is a piecewise linear, positively homogeneous function by the impulse variable s this mathematical programming problem can be reduced to solution of the series of linear programming problem.

Assume that function u is piecewise linear. Then convex hull $f(\cdot) = f(S, \cdot)$ of function u determined on convex polytope $S(x, r_1, r_2, \Delta)$ is also piecewise linear. In particular, in the neighborhood $O(x, K\Delta)$ we can represent function f in the following way

$$f(y) = \max_j \max_n (\langle l_n^j, y - y_j \rangle + f(y_j)), \quad j = 1, \dots, N_f, \quad n = 1, \dots, N_j$$

Here points y_j and vectors l_n^j satisfy the following condition: for any point $y \in O(x, K\Delta)$ there exists an index $i_0 \in J(y)$ such that for all $i \in J(y)$ the inclusion holds

$$coL(y, i) \subseteq coL(y, i_0)$$

$$J(y) = \{i : \max_n (\langle l_n^i, y - y_i \rangle + f(y_i)) = \max_j \max_n (\langle l_n^j, y - y_j \rangle + f(y_j))\}$$

$$L(y, i) = \{l = l_k^i : \langle l_k^i, y - y_i \rangle = \max_n \langle l_n^i, y - y_i \rangle\}$$

Subdifferential $Df(y)$ of function f at point $y \in O(x, K\Delta)$ is determined by the formula

$$Df(y) = coL(y, i_0(y))$$

Assume that at point $y_j, j = 1, \dots, N_f$ the following relation takes place

$$i_0 = i_0(y_j) = j$$

Then subdifferential $Df(y_j)$ of function f at point $y_j \in O(x, K\Delta)$ is a convex polytope

$$Df(y_j) = co\{l_n^j : n = 1, \dots, N_j\} = coL(y_j, j), \quad j = 1, \dots, N_f$$

It is clear that for any point $y \in O(x, K\Delta)$ there exists a point y_{j_0} such that

$$Df(y) \subseteq Df(y_{j_0}) \tag{4.2}$$

Let us suppose that the Hamiltonian H is a piecewise linear, positively homogeneous function with respect to variable s . For example, the Hamiltonian H satisfies these conditions if control constraints P, Q are convex polytopes.

Let p_k be vertices of polytope $B(t, x)P$ and L_p^k be cones of linearity

$$L_p^k = L_p^k(t, x) = \{s \in R^n : \langle s, p - p_k \rangle \geq 0, \quad p \in B(t, x)P\}, \quad k = 1, \dots, N_p$$

for the function

$$s \rightarrow \min_{p \in P} \langle s, B(t, x)p \rangle$$

Analogously, let q_m be vertices of polytope $C(t, x)Q$ and L_q^m be cones of linearity

$$L_q^m = L_q^m(t, x) = \{s \in R^n : \langle s, q - q_m \rangle \leq 0, \quad q \in C(t, x)Q\}, \quad m = 1, \dots, N_q$$

for the function

$$s \rightarrow \max_{q \in Q} \langle s, C(t, x)q \rangle$$

Property 4.5 *If the function u is piecewise linear, the Hamiltonian H is piecewise linear and positively homogeneous as a function of s then operator F can be calculated by the formula*

$$\begin{aligned} F = F(t, \Delta, S, u)(x) &= f(x) + \max_j \max_k \max_m \max_s \{ \Delta(\langle s, h(t, x) \rangle + \\ &\langle s, p_k \rangle + \langle s, q_m \rangle) + f(y_j) - f(x) - \langle s, y_j - x \rangle \} \\ s \in L_{j,k,m} &= Df(y_j) \cap L_p^k \cap L_q^m \end{aligned} \quad (4.3)$$

In this formula the set $L_{j,k,m} = L_{j,k,m}(t, x)$ is a convex polytope and the maximized function is linear by s . Thus, computation of operator F at point x is reduced to the series of linear programming problems.

Proof.

One can verify that for $y \in O(x, K\Delta)$, $s \in L_{k,m}(x, y) = Df(S, y) \cap L_p^k \cap L_q^m$ the Hamiltonian H is determined by the formula

$$H(t, x, s) = \langle s, h(t, x) \rangle + \langle s, p_k \rangle + \langle s, q_m \rangle$$

Then

$$\begin{aligned} F(t, \Delta, S, u)(x) &= f(S, x) + \sup_{y \in O(x, K\Delta)} \max_{s \in Df(S, y)} \{ \Delta H(t, x, s) + \\ &f(S, y) - f(S, x) - \langle s, y - x \rangle \} = \\ &f(S, x) + \sup_{y \in O(x, K\Delta)} \max_k \max_m \max_s \{ \Delta(\langle s, h(t, x) \rangle + \langle s, p_k \rangle + \langle s, q_m \rangle) + \\ &f(S, y) - f(S, x) - \langle s, y - x \rangle \}, \quad s \in L_{k,m}(x, y) \end{aligned} \quad (4.4)$$

Let us assume by contradiction that in (4.4) there exists a point $y \in O(x, K\Delta)$ such that for all $j = 1, \dots, N_f$ the strict inequality holds

$$\begin{aligned} &\max_k \max_m \max_{s \in L_{k,m}(x, y)} \{ \Delta(\langle s, h(t, x) \rangle + \langle s, p_k \rangle + \langle s, q_m \rangle) + \\ &f(S, y) - \langle s, y - x \rangle \} > \\ &\max_k \max_m \max_{s \in L_{j,k,m}} \{ \Delta(\langle s, h(t, x) \rangle + \langle s, p_k \rangle + \langle s, q_m \rangle) + \\ &f(S, y_j) - \langle s, y_j - x \rangle \} \end{aligned} \quad (4.5)$$

Let parameters k_0, m_0, s_0 realize maxima in the left-hand side of (4.5). According to condition (4.2) for a point $y \in O(x, K\Delta)$ there exists a point y_{j_0} such that $Df(S, y) \subseteq Df(S, y_{j_0})$.

Hence,

$$\begin{aligned} s_0 &\in Df(S, y) \subseteq Df(S, y_{j_0}) \\ s_0 &\in L_{k_0, m_0}(x, y) \subseteq L_{j_0, k_0, m_0} \end{aligned}$$

According to definition of subdifferentials we have inequalities

$$\begin{aligned} f(S, y_{j_0}) - f(S, y) &\geq \langle s_0, y_{j_0} - y \rangle, \quad \text{since } s_0 \in Df(S, y) \\ f(S, y) - f(S, y_{j_0}) - f(S, y) &\geq \langle s_0, y - y_{j_0} \rangle, \quad \text{since } s_0 \in Df(S, y_{j_0}) \end{aligned}$$

Therefore,

$$f(S, y_{j_0}) - \langle s_0, y_{j_0} - x \rangle = f(S, y) - \langle s_0, y - x \rangle$$

and, hence,

$$\begin{aligned} & \Delta(\langle s_0, h(t, x) \rangle + \langle s_0, p_k \rangle + \langle s_0, q_m \rangle) + f(S, y) - \langle s_0, y - x \rangle = \\ & \Delta(\langle s_0, h(t, x) \rangle + \langle s_0, p_k \rangle + \langle s_0, q_m \rangle) + f(S, y_{j_0}) - \langle s_0, y_{j_0} - x \rangle \end{aligned}$$

The last equality contradicts to (4.5) and, hence, relation (4.3) is valid. \square

We give now simple computational formulas for operator F (2.1) on the elementary diamond of the phase space

$$\begin{aligned} S(x, r_1, r_2, \Delta) &= co\{x \pm \Delta\gamma_i e_i, \quad i = 1, \dots, n\} \\ r_1 &= \left(\sum_{i=1}^n \gamma_i^{-2} \right)^{-1/2}, \quad r_2 = \max_i \gamma_i, \quad K < r_1 \leq r_2 \\ \overline{O}(x, r_1 \Delta) &\subseteq S(x, r_1, r_2, \Delta) \subseteq \overline{O}(x, r_2 \Delta) \end{aligned}$$

In particular, for $\gamma_i = \gamma$, $i = 1, \dots, n$ these conditions mean

$$r_1 = \frac{\gamma}{\sqrt{n}}, \quad r_2 = \gamma$$

Function u is piecewise linear and can be determined on the set $S(x, r_1, r_2, \Delta)$ by relations

$$\begin{aligned} u(y) &= \sum_{j=0}^n \alpha_j u(y_j), \quad y \in S(x, r_1, r_2, \Delta) \quad (4.6) \\ y_0 &= x, \quad y_i = x \pm \Delta\gamma_i e_i, \quad i = 1, \dots, n \\ y &= \sum_{j=0}^n \alpha_j y_j, \quad \sum_{j=0}^n \alpha_j = 1, \quad \alpha_j \geq 0 \end{aligned}$$

The convex hull f is also piecewise linear and can be defined by formulas

$$\begin{aligned} f(x \pm \Delta\gamma_i e_i) &= u(x \pm \Delta\gamma_i e_i) \\ f(x) &= \min\{u(x), \min_i \left\{ \frac{1}{2}(u(x + \Delta\gamma_i e_i) - u(x - \Delta\gamma_i e_i)) \right\}\} \quad (4.7) \\ f(y) &= \sum_{j=0}^n \alpha_j f(y_j), \quad y \in S(x, r_1, r_2, \Delta) \\ y_0 &= x, \quad y_i = x \pm \Delta\gamma_i e_i, \quad i = 1, \dots, n \\ y &= \sum_{j=0}^n \alpha_j y_j, \quad \sum_{j=0}^n \alpha_j = 1, \quad \alpha_j \geq 0 \end{aligned}$$

Subdifferential $f(x)$ of function f at point x is a rectangular parallelepiped with sides parallel to coordinate axes

$$\begin{aligned} Df(x) &= co\{a_k : k = 1, \dots, 2^n\}, \quad a_k = (a_k^1, \dots, a_k^n) \\ a_k^i &= \pm \frac{f(x \pm \Delta\gamma_i e_i) - f(x)}{\Delta\gamma_i} = \pm \frac{u(x \pm \Delta\gamma_i e_i) - f(x)}{\Delta\gamma_i} \quad (4.8) \end{aligned}$$

Operator F is determined by the formula

$$\begin{aligned} F &= F(t, \Delta, S, u)(x) = f(x) + \Delta \max_{s \in Df(x)} H(t, x, s) = \\ &= f(x) + \Delta \max_k \max_m \max_s \{ \langle s, h(t, x) \rangle + \langle s, p_k \rangle + \langle s, q_m \rangle \} \quad (4.9) \end{aligned}$$

$$s \in L_{k,m}(t, x) = Df(x) \cap L_k^u(t, x) \cap L_m^v(t, x)$$

5 Constructions of Nonsmooth Analysis in Finite Difference Operators

We introduce finite difference operators similar to operator F (2.1) in which we use constructions of nonsmooth analysis. We indicate properties of these operators and show that they complement each other giving complete description of grid schemes for constructing value functions and optimal control synthesis.

At first we consider the finite difference operator G dual to operator F

$$\begin{aligned} G(t, \Delta, u)(x) &= g(x) + \inf_{y \in O(x, r\Delta)} \min_{s \in \overline{D}g(y)} \{ \Delta H(t, x, s) + g(y) - g(x) - \langle s, y - x \rangle \} = \\ &= g(x) + \min_{y \in \overline{O}(x, K\Delta)} \min_{s \in \overline{D}g(y)} \{ \Delta H(t, x, s) + g(y) - g(x) - \langle s, y - x \rangle \} \end{aligned} \quad (5.1)$$

$$t \in T, \quad t + \Delta \in T, \quad t < \vartheta, \quad \Delta > 0, \quad (t, x) \in G_r, \quad r > K$$

Here function $g(y) : \overline{O}(x, r\Delta) \rightarrow R$ is the local concave hull of function $u(y)$

$$\begin{aligned} g(y) &= \sup \left\{ \sum_{k=1}^{n+1} \alpha_k u(y_k) : y_k \in \overline{O}(x, r\Delta), \alpha_k \geq 0, k = 1, \dots, n \right. \\ &\quad \left. \sum_{k=1}^{n+1} \alpha_k = 1, y = \sum_{k=1}^{n+1} \alpha_k y_k \right\}, \quad y \in \overline{O}(x, r\Delta) \end{aligned} \quad (5.2)$$

The superdifferential $\overline{D}g(y)$ of function g at point $y \in O(x, r\Delta)$ is defined by the formula

$$\overline{D}g(y) = \{ s \in R^n : g(z) - g(y) \leq \langle s, z - y \rangle, \quad z \in \overline{O}(x, r\Delta) \} \quad (5.3)$$

Property 5.6 *Operators F and G are connected by the inequality*

$$G(t, \Delta, u)(x) \geq F(t, \Delta, u)(x), \quad x \in D_t \quad (5.4)$$

Proof.

It is obvious that

$$g(z) \geq u(z), \quad z \in \overline{O}(x, r\Delta)$$

Then

$$u(z) \leq g(z) \leq g(y) + \langle s, z - y \rangle, \quad s \in \overline{D}g(y), \quad z \in \overline{O}(x, r\Delta), \quad y \in O(x, K\Delta)$$

Let us introduce the linear function

$$z \rightarrow w(z) = g(y) + \langle s, z - y \rangle$$

By monotonicity condition (F5) we have necessary relations

$$\begin{aligned} F(t, \Delta, u)(x) &\leq F(t, \Delta, w)(x) = w(x) + \Delta H(t, x, s) + \\ &= w(y) - w(x) - \langle s, y - x \rangle + \Delta H(t, x, s) + g(y) - \langle s, y - x \rangle \leq G(t, \Delta, u)(x) \quad \square \end{aligned}$$

On the elementary diamond $S(x, r_1, r_2, \Delta)$ operator G is computed by formulas

$$\begin{aligned} G &= G(t, \Delta, S, u)(x) = g(x) + \Delta \min_{s \in \overline{D}g(x)} H(t, x, s) = \\ &= g(x) + \Delta \min_k \min_m \min_s \{ \langle s, h(t, x) \rangle + \langle s, p_k \rangle + \langle s, q_m \rangle \} \end{aligned} \quad (5.5)$$

$$\begin{aligned}
s \in L_{k,m}(t, x) &= \overline{D}g(x) \cap L_k^u(t, x) \cap L_m^v(t, x) \\
g(x \pm \Delta\gamma_i e_i) &= u(x \pm \Delta\gamma_i e_i), \quad i = 1, \dots, n \\
g(x) &= \max\{u(x), \max\{\frac{1}{2}(u(x + \Delta\gamma_i e_i) - (u(x - \Delta\gamma_i e_i)))\}\} \\
\overline{D}g(x) &= \text{co}\{b_k : k = 1, \dots, 2^n\}, \quad b_k = (b_k^1, \dots, b_k^n) \\
b_k^i &= \pm \frac{g(x \pm \Delta\gamma_i e_i) - g(x)}{\Delta\gamma_i} = \pm \frac{u(x \pm \Delta\gamma_i e_i) - g(x)}{\Delta\gamma_i}
\end{aligned}$$

Proposition 5.1 *The dual operator G has properties analogous to Properties 2.1 - 2.3, 4.4, 4.5 of operator F and satisfies conditions (F1) - (F8). Therefore, approximation schemes (3.12), (3.19) with operator G converge to the generalized solution w (value function) and the convergence estimate has the order $\Delta^{1/2}$.*

One can combine the lower operator F and the upper operator G by convolution with coefficients

$$\alpha_i(x) \geq 0, \quad i = 1, 2, \quad \alpha_1(x) + \alpha_2(x) = 1$$

into the universal operator E .

Assume that $f(x) < g(x)$ and coefficients $\alpha_i(x)$, $i = 1, 2$ determine the ratio of deviation of function u from its convex f and concave g hulls

$$\alpha_1(x) = \frac{g(x) - u(x)}{g(x) - f(x)}, \quad \alpha_2(x) = \frac{u(x) - f(x)}{g(x) - f(x)} \quad (5.6)$$

For coefficients (5.6) the universal operator F is defined by formulas

$$\begin{aligned}
E &= E(t, \Delta, S, u)(x) = \alpha_1(x)F(t, \Delta, S, u)(x) + \alpha_2(x)G(t, \Delta, S, u)(x) = \\
&u(x) + \Delta(\alpha_1(x) \max_{s \in Df(x)} H(t, x, s) + \alpha_2(x) \min_{s \in \overline{D}g(x)} H(t, x, s)) = \\
&u(x) + \Delta(\max_{s \in D_*u(x)} H(t, x, s) + \min_{s \in D^*u(x)} H(t, x, s)) \quad (5.7)
\end{aligned}$$

$$D_*u(x) = \alpha_1(x)Df(x), \quad D^*u(x) = \alpha_2(x)\overline{D}g(x) \quad (5.8)$$

Property 5.7 *The set $D_*u(x)$ coincides with the subdifferential and the set $D^*u(x)$ (5.8) - with the superdifferential of function u at point x in the sense of [Demyanov, 1974].*

Proof.

Remind that Demyanov's subdifferential $\partial_*u(x)$ and superdifferential $\partial^*u(x)$ of function u (4.6) at point x is defined by relations

$$\begin{aligned}
u(x+h) - u(x) &= \partial u(x)|(h) = \lim_{\delta \downarrow 0} \frac{u(x+\delta h) - u(x)}{\delta} = \\
\max_{s \in \partial_*u(x)} \langle s, h \rangle &+ \min_{s \in \partial^*u(x)} \langle s, h \rangle \quad (5.9)
\end{aligned}$$

We can obtain the necessary equalities $\partial_*u(x) = D_*u(x)$, $\partial^*u(x) = D^*u(x)$ by transforming the sum

$$\begin{aligned}
&\max_{s \in D_*u(x)} \langle s, h \rangle + \min_{s \in D^*u(x)} \langle s, h \rangle = \\
&\alpha_1(x) \max_{s \in Df(x)} \langle s, h \rangle + \alpha_2(x) \min_{s \in \overline{D}g(x)} \langle s, h \rangle = \\
&\frac{g(x) - u(x)}{g(x) - f(x)} \sum_{i=1}^n \frac{(u(x + \Delta\gamma_i e_i \text{sign}(h_i)) - f(x))h_i}{\gamma_i \Delta} -
\end{aligned}$$

$$\begin{aligned}
& \frac{u(x) - f(x)}{g(x) - f(x)} \sum_{i=1}^n \frac{(g(x) - u(x + \Delta\gamma_i e_i \text{sign}(h_i)))h_i}{\gamma_i \Delta} = \\
& \frac{1}{g(x) - f(x)} (-u(x)(g(x) - f(x)) \sum_{i=1}^n \frac{h_i}{\gamma_i \Delta} + \\
& (g(x) - f(x)) \sum_{i=1}^n \frac{u(x + \Delta\gamma_i e_i \text{sign}(h_i))h_i}{\gamma_i \Delta}) = \\
& \sum_{i=1}^n \frac{(u(x + \Delta\gamma_i e_i \text{sign}(h_i)) - u(x))h_i}{\gamma_i \Delta} = u(x + h) - u(x) \quad \square
\end{aligned}$$

Property 5.8 *The following relations for subdifferential $Df(x)$ and superdifferential $\overline{D}g(x)$ are valid*

$$c \in Df(x) \cap \overline{D}g(x) \neq \emptyset, \quad c = (c^1, \dots, c^n) \quad (5.10)$$

$$c^i = \frac{u(x + \Delta\gamma_i e_i) - u(x - \Delta\gamma_i e_i)}{2\gamma_i \Delta}, \quad i = 1, \dots, n \quad (5.11)$$

$$c = \frac{1}{M} \sum_{k=1}^M a_k = \frac{1}{M} \sum_{k=1}^M b_k, \quad M = 2^n \quad (5.12)$$

$$g(x) - f(x) \geq \Delta \max_{s \in Df(x)} H(t, x, s) - \Delta \min_{s \in \overline{D}g(x)} H(t, x, s) \geq 0 \quad (5.13)$$

Proof.

Let us compose the convolution of vectors a_k with coefficients

$$\gamma_k = \frac{1}{M}, \quad \gamma_k > 0, \quad \sum_{k=1}^M \gamma_k = 1, \quad \sum_{k=1}^M \gamma_k a_k \in Df(x)$$

According to definition of vectors a_k (4.8) we have relations

$$\sum_{k=1}^M a_k^i = 2^{n-1} \frac{u(x + \Delta\gamma_i e_i) - u(x - \Delta\gamma_i e_i)}{\gamma_i \Delta}, \quad i = 1, \dots, n$$

Hence,

$$\frac{1}{M} \sum_{k=1}^M a_k^i = \frac{u(x + \Delta\gamma_i e_i) - u(x - \Delta\gamma_i e_i)}{2\gamma_i \Delta}, \quad i = 1, \dots, n$$

Thus,

$$c = \frac{1}{M} \sum_{k=1}^M a_k \in Df(x)$$

Analogously one can prove that

$$c = \frac{1}{M} \sum_{k=1}^M b_k \in \overline{D}g(x)$$

Therefore, relations (5.10)-(5.12) are valid.

The first inequality in (5.13) follows from Property 5.6 $G \geq F$. The second inequality is fulfilled since the intersection of subdifferential and superdifferential is nonempty (5.10).

□

Property 5.9 *The universal operator E can be represented in the form of operators F and G*

$$E(t, \Delta, S, u)(x) = F(t, \Delta, S_\beta, u)(x) = G(t, \Delta, S_\beta, u)(x) \quad (5.14)$$

determined on diamond S_β

$$S_\beta = \text{co}\{x \pm \beta \Delta \gamma_i e_i, \quad i = 1, \dots, n\}$$

$$\beta = \frac{\Delta(D - d)}{g(x) - f(x)}, \quad 0 \leq \beta \leq 1$$

$$D = \max_{s \in Df(x)} H(t, x, s), \quad d = \min_{s \in Dg(x)} H(t, x, s)$$

Operator E provides in many cases the precise value of solution at point x for the guaranteed control problem with the simple motion and positively homogeneous payoff function

$$\frac{dy}{d\tau} = h(t, x) + B(t, x)p + C(t, x)q \quad (5.15)$$

$$\tau \in [t, t + \Delta], \quad y \in R^n, \quad p \in P, \quad q \in Q$$

$$J(y(\cdot)) = u(y(t + \Delta)) \quad (5.16)$$

The following boundary value problem for Hamilton-Jacobi equation (Riemann problem [Bardi, Osher, 1991]) corresponds to the guaranteed control problem (5.15), (5.16)

$$\begin{aligned} & \frac{\partial w(\tau, y)}{\partial \tau} + \left\langle \frac{\partial w(\tau, y)}{\partial y}, h(t, x) \right\rangle + \\ & \min_{p \in P} \left\langle \frac{\partial w(\tau, y)}{\partial y}, B(t, x)p \right\rangle + \max_{q \in Q} \left\langle \frac{\partial w(\tau, y)}{\partial y}, C(t, x)q \right\rangle = 0 \end{aligned} \quad (5.17)$$

$$w(t + \Delta, y) = u(y) \quad (5.18)$$

Let us note that operator F determines solution w of problem (5.17), (5.18) in the following cases.

1. Assume that the attainability set

$$AS(t, x, \Delta) = \{f \in R^n : f = x + \Delta(h(t, x) + B(t, x)p + C(t, x)q), \quad p \in P, \quad q \in Q\} \quad (5.19)$$

is contained in diamond S_β

$$AS(t, x, \Delta) \subseteq S_\beta \quad (5.20)$$

Then operator $F(t, \Delta, S_\beta, u)$ is the programming maximin and operator $G(t, \Delta, S_\beta, u)$ is the programming minimax. Hence, their coincidence provides relations

$$w(t, x) = F(t, \Delta, S_\beta, u) = G(t, \Delta, S_\beta, u) = E(t, \Delta, S, u) \quad (5.21)$$

2. If the boundary value function $y \rightarrow u(y)$ is convex then $\alpha_2 = 0$ and operator E turns into the programming maximin or Hopf's formula

$$\begin{aligned} w(t, x) &= E(t, \Delta, S, u) = F(t, \Delta, S_\beta, u) = \\ & u(x) + \Delta \max_{s \in Du(x)} H(t, x, s), \quad f(x) = u(x) \leq g(x) \end{aligned} \quad (5.22)$$

For the concave boundary value function $y \rightarrow u(y)$ coefficient α_1 is equal to zero and the dual relation takes place

$$\begin{aligned} w(t, x) &= E(t, \Delta, S, u) = G(t, \Delta, S_\beta, u) = \\ & u(x) + \Delta \min_{s \in Du(x)} H(t, x, s), \quad f(x) \leq u(x) = g(x) \end{aligned} \quad (5.23)$$

Proposition 5.2 *Values of the universal operator E are bounded by values of the lower operator F and the upper operator G*

$$F \leq E \leq G \quad (5.24)$$

Therefore, convergence of approximation schemes (3.12), (3.19) with operators F and G implies convergence of these schemes with operator E .

6 Mean Square Generalized Gradients

As it was mentioned above operators F , G , E provide in many cases precise values for the solution of the guaranteed control problem with the simple motion - Riemann problem for the corresponding Hamilton-Jacobi equation. In this sense they can be regarded as best approximations of solutions. But we should point out the algorithmic and computational complexity of these formulas containing the series of linear programming problems. Therefore, we propose now the finite difference operator with the simpler structure based on local linear hulls and their gradients (generalized gradients of the approximate solution) defined by the method of least squares. Let us note that similar constructions were considered in optimization theory [Batukhtin, Maiboroda, 1984], [Ermoliev, Norkin, Wets, 1995].

Let us fix a node $x \in GR(t)$ and its neighborhood of radius $r_1\Delta$, $r_1 = N\gamma$, $N \geq 1$ given in ρ_1 -metric

$$\bar{O}(x, r_1\Delta) = \{y \in D_{t+\Delta}^* : \rho_1(x, y) \leq r_1\Delta\} \quad (6.1)$$

$$\rho_1(x, y) = \max_i |x_i - y_i|, \quad i = 1, \dots, n \quad (6.2)$$

The number of nodes y_l of the grid $GR(t+\Delta)$ belonging to the neighborhood $\bar{O}_1(x, r_1\Delta)$ is determined as $M = (2N + 1)^n$. Assume that at nodes y_l values

$$U = \{u(y_l) : l = 1, \dots, M\}$$

of the function $y \rightarrow u(y)$ are given.

Let

$$y \rightarrow L(y) = \langle A, y \rangle + B : \bar{O}_1(x, r_1\Delta) \rightarrow R \quad (6.3)$$

be the hyperplane determined by the method of least squares with respect to the value set U

$$\min_{A, B} \sum_{l=1}^M (u(y_l) - (\langle A, y_l \rangle + B))^2 \quad (6.4)$$

Condition (6.3) leads to the system of linear equations with respect to parameters A , B of the hyperplane L

$$\sum_{l=1}^M \langle A, \xi_l \rangle \xi_l = \sum_{l=1}^M u(y_l) \xi_l, \quad \xi_l = y_l - x_0 \quad (6.5)$$

$$B = - \langle A, x_0 \rangle + u_0 \quad (6.6)$$

$$x_0 = x = \frac{1}{M} \sum_{l=1}^M y_l, \quad u_0 = \frac{1}{M} \sum_{l=1}^M u(y_l) \quad (6.7)$$

The system (6.5) is a symmetric one and can be written in the form

$$\Phi A = \psi, \quad \Phi = \{\varphi_{ij}\}, \quad \psi = (\psi_1, \dots, \psi_n) \quad (6.8)$$

$$\varphi_{ij} = \sum_{l=1}^M \xi_l^i \xi_l^j, \quad i, j = 1, \dots, n, \quad \psi = \sum_{l=1}^M u(y_l) \xi_l \quad (6.9)$$

The symmetry of the set $\overline{O}_1(x, r_1\Delta)$ with respect to coordinate planes provides the diagonal structure of matrix Φ

$$\varphi_{ii} = 2(1^2 + 2^2 + \dots + N^2)(2N + 1)^{n-1}h^2 = \frac{1}{3}N(N + 1)(2N + 1)^n h^2, \quad \varphi_{ij} = 0 \quad (6.10)$$

$$\psi_i = h \sum_{\eta} \sum_{k=1}^N k(u(x + khe_i + \eta) - u(x - khe_i + \eta)) \quad (6.11)$$

$$\eta = (\eta^1, \dots, \eta^n), \quad \eta^i = 0, \quad \eta^j = \pm l$$

$$h = \gamma\Delta, \quad j \neq i, \quad i, j = 1, \dots, n, \quad l = 0, \dots, N$$

Hence, vector A is determined by the formula

$$A = (A^1, \dots, A^n), \quad A^i = \frac{\psi_i}{\varphi_{ii}}, \quad i = 1, \dots, n \quad (6.12)$$

The principal point consists in the fact that the local linear hull is situated between convex and concave hulls in the neighborhood $\overline{O}_2(x, r_2\Delta)$ of a smaller radius $r_2 < r_1$ given in the metric ρ_2

$$\overline{O}(x, r_2\Delta) = \{y \in D_{t+\Delta}^* : \rho_2(x, y) \leq r_2\Delta\} \quad (6.13)$$

$$\rho_2(x, y) = \sum_i^n |x_i - y_i| \quad (6.14)$$

Lemma 6.1 *Let radiuses r_1, r_2 of neighborhoods $\overline{O}_1(x, r_1\Delta), \overline{O}_2(x, r_2\Delta)$ be connected by the inequality*

$$r_2 \leq \frac{r_1}{3} \quad (6.15)$$

Then the local linear $L(y)$, concave $g(y)$ and convex $f(y)$ hulls constructed in the neighborhood $\overline{O}_1(x, r_1\Delta)$ satisfy estimates

$$f(y) \leq L(y) \leq g(y), \quad y \in \overline{O}_2(x, r_2\Delta) \quad (6.16)$$

in the neighborhood $\overline{O}_2(x, r_2\Delta)$.

Proof.

Let us remind that the convex hull $y \rightarrow f(y)$ and the concave hull $y \rightarrow g(y)$ of the function $y \rightarrow u(y)$ in the neighborhood $\overline{O}_1(x, r_1\Delta)$ are determined by relations

$$\begin{aligned} f(y) &= \inf \left\{ \sum_{k=1}^{n+1} \alpha_k u(y_k) : y_k \in \overline{O}_1(x, r_1\Delta), \right. \\ &\quad \left. \sum_{k=1}^{n+1} \alpha_k y_k = y, \quad \alpha_k \geq 0, \quad \sum_{k=1}^{n+1} \alpha_k = 1 \right\} \end{aligned} \quad (6.17)$$

$$\begin{aligned} g(y) &= \sup \left\{ \sum_{k=1}^{n+1} \beta_k u(y_k) : y_k \in \overline{O}_1(x, r_1\Delta), \right. \\ &\quad \left. \sum_{k=1}^{n+1} \beta_k y_k = y, \quad \beta_k \geq 0, \quad \sum_{k=1}^{n+1} \beta_k = 1 \right\} \end{aligned} \quad (6.18)$$

We prove at first that the linear hull $y \rightarrow L(y)$ of the function $y \rightarrow u(y)$ can be represented in the form analogous to (6.17), (6.18)

$$L(y) = \sum_{l=1}^M \gamma_l(y) u(y_l), \quad y \in \overline{O}_1(x, r_1 \Delta)$$

According to (6.5), (6.7) we have

$$\begin{aligned} L(y) &= \langle A, y \rangle + B = \langle A, y - x_0 \rangle + u_0 = \\ &= \frac{1}{M} \sum_{l=1}^M \left(u(y_l) + \sum_{i=1}^n \frac{3(y^i - x_0^i)(y_l^i - x_0^i)}{N(N+1)h^2} u(y_l) \right) \end{aligned}$$

Then for coefficients $\gamma_l(y)$, $l = 1, \dots, M$ we obtain expressions

$$\gamma_l(y) = \frac{1}{M} \left(1 + \sum_{i=1}^n \frac{3(y^i - x_0^i)(y_l^i - x_0^i)}{N(N+1)h^2} \right) \quad (6.19)$$

Let us prove that coefficients $\gamma_l(y)$ satisfy conditions

$$\sum_{l=1}^M \gamma_l(y) y_l = y, \quad \sum_{l=1}^M \gamma_l(y) = 1 \quad (6.20)$$

We have

$$\sum_{l=1}^M \gamma_l(y) = 1 + \frac{1}{M} \sum_{i=1}^n \frac{3(y^i - x_0^i)}{N(N+1)h^2} \sum_{l=1}^M (y_l^i - x_0^i) = 1$$

since according to (6.7) the equality

$$\sum_{l=1}^M (y_l^i - x_0^i) = 0$$

is valid.

Let us calculate the sum

$$\begin{aligned} \sum_{l=1}^M \gamma_l(y) y_l^j &= \frac{1}{M} \sum_{l=1}^M \left(1 + \sum_{i=1}^n \frac{3(y^i - x_0^i)(y_l^i - x_0^i)}{N(N+1)h^2} \right) y_l^j = \\ &= x_0^j + \frac{1}{M} \sum_{i=1}^n \frac{3(y^i - x_0^i)}{N(N+1)h^2} \sum_{l=1}^M (y_l^i - x_0^i) y_l^j = \\ &= x_0^j + \frac{1}{M} \sum_{i=1}^n \frac{3(y^i - x_0^i)}{N(N+1)h^2} \sum_{l=1}^M (y_l^i - x_0^i)(y_l^j - x_0^j) \end{aligned} \quad (6.21)$$

The last equality holds because

$$\sum_{l=1}^M (y_l^i - x_0^i) x_0^j = x_0^j \sum_{l=1}^M (y_l^i - x_0^i) = 0$$

In relation (6.21) all terms in the sum are equal to zero when $i \neq j$

$$\sum_{l=1}^M (y_l^i - x_0^i)(y_l^j - x_0^j) = 0$$

When $i = j$ we obtain the nonzero term

$$\frac{1}{M} \frac{3(y^j - x_0^j)}{N(N+1)h^2} \sum_{l=1}^M (y_l^j - x_0^j)(y_l^j - x_0^j) = y^j - x_0^j$$

Hence,

$$\sum_{l=1}^M \gamma_l(y) y_l^j = y^j$$

Thus, relations (6.20) are proved.

Let us find the set of points y for which coefficients $\gamma_l(y)$ are nonnegative

$$\gamma_l(y) \geq 0, \quad l = 1, \dots, M \quad (6.22)$$

From (6.22) it follows

$$1 + \sum_{i=1}^n \frac{3(y^i - x_0^i)(y_l^i - x_0^i)}{N(N+1)h^2} \geq 0, \quad l = 1, \dots, M$$

Coefficients $\gamma_l(y)$ achieve least values when

$$y_l^i - x_0^i = -\text{sign}(y^i - x_0^i)Nh$$

Then we obtain relations

$$\sum_{i=1}^n |y^i - x_0^i| \leq \frac{(N+1)h}{3} = \frac{r_1\Delta}{3} \left(1 + \frac{1}{N}\right) \quad (6.23)$$

Let us consider the neighborhood $\overline{O}_2(x, r_2\Delta)$ of point $x = x_0$ of radius $r_2\Delta$ in ρ_2 -metric

$$r_2\Delta = \frac{r_1\Delta}{3} \leq \frac{r_1\Delta}{3} \left(1 + \frac{1}{N}\right)$$

In this neighborhood for coefficients $\{\gamma_l(y)\}$ conditions (6.20) and (6.22) are fulfilled. Remind that these conditions hold for coefficients α_k, β_k in definitions (6.17), (6.18) of convex $f(y)$ and concave $g(y)$ hulls in which lower and upper bounds are calculated. Hence, estimates (6.16) are valid. \square

Remark 6.1 For the radius

$$r_3 = \frac{r_2}{n} = \frac{r_1}{3n} \quad (6.24)$$

the estimates

$$f(y) \leq L(y) \leq g(y), \quad y \in \overline{O}_1(x, r_3\Delta) \quad (6.25)$$

are valid in the neighborhood $\overline{O}_1(x, r_3\Delta)$ given in ρ_1 -metric.

Analogously, for the radius

$$r_4 = \frac{r_2}{\sqrt{n}} = \frac{r_1}{3\sqrt{n}} \quad (6.26)$$

inequalities

$$f(y) \leq L(y) \leq g(y), \quad y \in \overline{O}(x, r_4\Delta) \quad (6.27)$$

hold in the neighborhood $\overline{O}(x, r_4\Delta)$ given in Euclidean metric.

Assume

$$r_1 > 3\sqrt{n}K \quad (6.28)$$

and define the finite difference operator with the generalized gradient A of the local linear hull L

$$\begin{aligned} LA(t, \Delta, u)(x) &= F(t, \Delta, L)(x) = G(t, \Delta, L)(x) = \\ &u_0 + \Delta H(t, x, A) + \langle A, x - x_0 \rangle = u_0 + \Delta H(t, x, A) \end{aligned} \quad (6.29)$$

Relations (6.16), (6.28), (6.29) provide estimates for operators LA , F , G .

Lemma 6.2 *Let local hulls $L(y)$, $f(y)$, $g(y)$ be constructed in the neighborhood $\bar{O}_1(x, r_1\Delta)$ where parameter r_1 be determined by relation (6.28).*

Then values of operators LA , F and G at point x satisfy inequalities

$$F(t, \Delta, u)(x) \leq LA(t, \Delta, u)(x) \leq G(t, \Delta, u)(x) \quad (6.30)$$

Proof.

According to (6.16), (6.28) inequalities

$$f(y) \leq L(y) \leq g(y), \quad y \in \bar{O}(x, K\Delta) \quad (6.31)$$

are fulfilled in the neighborhood

$$\bar{O}(x, K\Delta) \subset \bar{O}_2(x, r_2\Delta), \quad r_2 = \frac{r_1}{3} > \sqrt{n}K$$

Remind that operators F and G satisfy the monotonicity condition. Therefore, relations (6.28), (6.31) imply the chain of necessary inequalities

$$\begin{aligned} F(t, \Delta, u)(x) &= F(t, \Delta, f)(x) \leq F(t, \Delta, L)(x) = \\ LA(t, \Delta, u)(x) &= G(t, \Delta, L)(x) \leq G(t, \Delta, g)(x) = G(t, \Delta, u)(x) \quad \square \end{aligned}$$

Lemma 6.2 implies the convergence result for approximation schemes with operator LA .

Proposition 6.1 *Values of the mean square operator LA are bounded by values of the lower operator F and the upper operator G (6.30). Therefore, convergence of approximation schemes (3.12), (3.19) with operators F and G implies convergence of these schemes with operator LA .*

Let us compare the introduced operators with the known constructions of control theory and theory of Hamilton-Jacobi equations.

Remark 6.2 *Operator LA coincides with Lax-Friedrichs operator [Lax, 1954], [Osher, Shu, 1991]*

$$\begin{aligned} LF(t, x, \Delta)(x) &= (1 - \sum_{i=1}^n \alpha_i)u(x) + \\ &\frac{1}{2} \sum_{i=1}^n \alpha_i (u(x + \gamma\Delta e_i) + u(x - \gamma\Delta e_i)) + \Delta H(t, x, c) \end{aligned} \quad (6.32)$$

on the elementary diamond

$$S(x, \gamma, \Delta) = \{y \in R^n : \sum_{i=1}^n |y_i - x_i| \leq \gamma \Delta\}$$

under conditions

$$\alpha_i = \frac{2}{2n+1}, \quad i = 1, \dots, n, \quad \gamma > 3\sqrt{n}K \quad (6.33)$$

Vector c in (6.32) is determined by relations (5.10), (5.11).

Remark 6.3 According to Property 2.3 operator F is the maximin formula on the local convex hull. Analogously, operator G is the minimax formula on the local concave hull.

Therefore, maximin PMM_* and minimax PMM^* operators (see, for example, [Krasovskii, Subbotin, 1974], [Pshenichnyi, 1969], [Ushakov, 1980], [Souganidis, 1985])

$$PMM_* = \max_{q \in Q} \min_{p \in P} u(x + \Delta(h(t, x) + B(t, x)p + C(t, x)q)) \quad (6.34)$$

$$PMM^* = \min_{p \in P} \max_{q \in Q} u(x + \Delta(h(t, x) + B(t, x)p + C(t, x)q)) \quad (6.35)$$

are connected with operators F, G by relations

$$F(t, \Delta, u)(x) \leq PMM_*(t, \Delta, u)(x) \leq PMM^*(t, \Delta, u)(x) \leq G(t, \Delta, u)(x) \quad (6.36)$$

Convergence of approximation schemes with operators F, G implies convergence of approximation schemes with operators PMM_*, PMM^* .

Remark 6.4 In operators F, G, E, LA dynamics of the attainability set $AS(t, x, \Delta)$ (5.19) is evaluated locally by the set $\overline{O}(x, r\Delta)$. If attainability sets are nonsymmetric with respect to points x then this evaluation is rather rough. In this case approximations in operators F, G, E, LA can be improved by finding the Chebyshev center \bar{x} and the Chebyshev radius \bar{r} of the attainability set $AS(t, x, \Delta)$ and introducing its neighborhoods $\overline{O}(\bar{x}, \bar{r}\Delta)$ in different metrics instead of the set $\overline{O}(x, r\Delta)$.

7 Optimal Control Synthesis and Generalized Gradients in Grid Schemes

We propose now modifications of the method of extremal shift [Krasovskii, 1985], [Krasovskii, Subbotin, 1974, 1988], [A.N. Krasovskii, N.N. Krasovskii, 1995] in the direction of generalized gradients of operators F, G, LA and prove optimality properties for designed trajectories. Let us note that rules of extremal aiming by quasigradients defined in the sense of Yosida-Moreau transformations were analyzed in the work [Garnysheva, Subbotin, 1994].

We give also the procedure realizing the extremal shift method in grid schemes. Values of generalized solutions of Hamilton-Jacobi equations, generalized gradients of local (linear, convex, concave) hulls and optimal feedbacks are calculated in parallel via the unique grid scheme.

In the grid scheme values of optimal feedbacks exist only at nodes. For constructing optimal trajectories which can slide between nodes of the grid it is necessary to solve the problem on interpolating values of optimal feedbacks to internal points. Different interpolants: piecewise constant, piecewise minimum, piecewise linear, are analyzed and their properties are studied. The question of correlation between spatial and temporal

grids is solved. In the general case the higher order density of the spatial mesh with respect to the temporal mesh provides optimality properties for designed trajectories. The quasiconvexity property of grid functions approximating generalized solutions (value functions) provides the linear dependence of space-time grids.

Let us consider the problem of synthesizing optimal guaranteed feedbacks $(t, x) \rightarrow U^0(t, x)$. For this purpose we use operator G or operator PMM^* which can be interpreted as minimax formulas defined on the local concave hull g or the local approximate function u respectively

$$G(t, \Delta, u)(x) = g(x) + \min_{y \in \overline{O}(x, K\Delta)} \min_{s \in \overline{Dg}(y)} \{ \Delta H(t, x, s) + g(y) - g(x) - \langle s, y - x \rangle \} = \min_{p \in P} \max_{q \in Q} g(y(t, x, \Delta, p, q)) \quad (7.1)$$

$$PMM^* = \min_{p \in P} \max_{q \in Q} u(y(t, x, \Delta, p, q)) \quad (7.2)$$

Here Euler spline y is determined by relations

$$y(t, x, \Delta, p, q) = x + \Delta(h(t, x) + B(t, x)p + C(t, x)q) \\ t \in T, \quad t + \Delta \in T, \quad t < \vartheta, \quad \Delta > 0, \quad (t, x) \in G_r, \quad r > K$$

Let us examine the approximation scheme with operator G (7.1) for partition Γ of interval T with step Δ

$$u_\Gamma(\vartheta, x) = \sigma(x), \quad x \in D_\vartheta \\ u_\Gamma(t, x) = G(t, t_{i+1} - t, u_\Gamma(t_{i+1}, \cdot))(x) \quad (7.3) \\ t \in [t_i, t_{i+1}), \quad x \in D_t, \quad i = 0, \dots, N - 1$$

Assume that the approximate function u_Γ is constructed at all points (t, x) , $t \in \Gamma$, $x \in D_t$. Let us define values of the optimal feedback $U^0 = U^0(t, x)$ at points (t, x) by the method of extremal shift using operator G and its generalized gradients - superdifferentials of local concave hulls

$$U^0 = U^0(t, x) = \arg \min_{p \in P} \langle s^0, B(t, x)p \rangle \quad (7.4)$$

$$s^0 = s^0(t, x, y^0) = \arg \min_{s \in \overline{Dg}(y^0)} \{ \Delta H(t, x, s) + g(y^0) - \langle s, y^0 - x \rangle \} \quad (7.5)$$

$$y^0 = y^0(t, x) = \arg \min_{y \in \overline{O}(x, K\Delta)} \min_{s \in \overline{Dg}(y)} \{ \Delta H(t, x, s) + g(y) - \langle s, y - x \rangle \} \quad (7.6)$$

Remark 7.1 According to the minimax formula (7.1) the optimal feedback U^0 (7.4) is determined also by relations

$$U^0 = U^0(t, x) = \arg \min_{p \in P} \max_{q \in Q} g(y(t, x, \Delta, p, q)) \quad (7.7)$$

Remark 7.2 The dual operators F , PMM_* can be used in approximation schemes for constructing optimal feedbacks $(t, x) \rightarrow V^0(t, x)$.

Lemma 7.1 For the optimal feedback U^0 (7.4) the basic inequality is valid on the Euler spline y

$$\max_{q \in Q} u_\Gamma(t + \Delta, y(t, x, \Delta, U^0, q)) \leq u_\Gamma(t, x) \quad (7.8)$$

Proof.

For any control parameter $q \in Q$ we have according to definitions of operator G and feedback U^0 the necessary chain of inequalities

$$\begin{aligned}
u_\Gamma(t + \Delta, y(t, x, \Delta, U^0, q)) &\leq g(y(t, x, \Delta, U^0, q)) \leq \max_{q \in Q} g(y(t, x, \Delta, U^0, q)) = \\
&\inf_{s \in R^n} (\langle s, x \rangle + \Delta(\langle s, h(t, x) \rangle + \langle s, B(t, x)U^0 \rangle + \\
\max_{q \in Q} \langle s, C(t, x)q \rangle - \inf_{y \in R^n} (\langle s, y \rangle - g(y))) &= \\
\min_{y \in \bar{O}(x, K\Delta)} \min_{s \in \bar{D}g(y)} (\langle s, x \rangle + \Delta(\langle s, h(t, x) \rangle + \langle s, B(t, x)U^0 \rangle + \\
\max_{q \in Q} \langle s, C(t, x)q \rangle - \langle s, y \rangle + g(y)) &= \\
\min_{s \in \bar{D}g(y^0)} (\langle s, x \rangle + \Delta(\langle s, h(t, x) \rangle + \langle s, B(t, x)U^0 \rangle + \\
\max_{q \in Q} \langle s, C(t, x)q \rangle - \langle s, y^0 \rangle + g(y^0)) &= \\
\Delta(\langle s^0, h(t, x) \rangle + \langle s^0, B(t, x)U^0 \rangle + \max_{q \in Q} \langle s^0, C(t, x)q \rangle - \\
\langle s^0, y^0 - x \rangle + g(y^0)) = \Delta H(t, x, s^0) - \langle s^0, y^0 - x \rangle + g(y^0) &= u_\Gamma(t, x) \quad \square
\end{aligned}$$

Inequality (7.8) can be transformed for a modification of the Euler spline y

$$y(t, x, \Delta, U^0, q(\cdot)) = x + \Delta(h(t, x) + B(t, x)U^0) + \int_t^{t+\Delta} C(\tau, x)q(\tau)d\tau$$

Here programming control $\tau \rightarrow q(\tau) : [t, t+\Delta] \rightarrow Q$ is a Lebesgue measurable function.

Lemma 7.2 *The optimal feedback U^0 (7.4) provides estimates*

$$\sup_{q(\cdot)} u_\Gamma(t + \Delta, y(t, x, \Delta, U^0, q(\cdot))) \leq u_\Gamma(t, x) + L_w L_2 K \Delta^2 \quad (7.9)$$

Proof.

Taking into account Lipschitz conditions (H2), (H4) for the Hamiltonian H , convexity of the control set Q and using the integral mean value theorem we obtain relations

$$\begin{aligned}
\int_t^{t+\Delta} C(\tau, x)q(\tau)d\tau &= \int_t^{t+\Delta} (C(\tau, x) - C(t, x))q(\tau)d\tau + \\
C(t, x) \int_t^{t+\Delta} q(\tau)d\tau &= \varepsilon(\Delta)\Delta + C(t, x)\Delta\bar{q}, \quad \bar{q} \in Q \\
\|\varepsilon(\Delta)\| &\leq L_2 K \Delta
\end{aligned}$$

Then the basic inequality (7.8) and the Lipschitz continuity of the approximate function u_Γ with the constant L_w imply the necessary inequality (7.9). \square

We apply now basic inequalities (7.8), (7.9) to derivation of optimality properties for Euler trajectories generated by feedback U^0

$$\begin{aligned}
x(\cdot) &= \{x(t, t_*, x_*, U^0, q(\cdot)) : t \in \Gamma \cap [t_*, \vartheta], t_* \in \Gamma, x_* \in D_{t_*}\} \quad (7.10) \\
x(t_{i+1}) &= x(t_i + \Delta) = x(t_i) + \Delta(h(t_i, x(t_i)) + B(t_i, x(t_i))U^0) + \\
&\int_{t_i}^{t_i+\Delta} C(\tau, x(t_i))q(\tau)d\tau, \quad x(t_*) = x_*, \quad t_i, t_{i+1} \in \Gamma
\end{aligned}$$

Theorem 7.1 For any partition Γ , initial position (t_*, x_*) and Lebesgue measurable control $\tau \rightarrow q(\tau) : [t_*, \vartheta] \rightarrow Q$ the trajectory (7.10) generated by feedback U^0 satisfies the estimate

$$\sigma(x(\vartheta)) \leq u_\Gamma(t_*, x_*) + L_w L_2 K(\vartheta - t_*)\Delta \quad (7.11)$$

and, hence, by Theorem 3.2

$$\sigma(x(\vartheta)) \leq w(t_*, x_*) + C\Delta^{1/2} + L_w L_2 K(\vartheta - t_*)\Delta \quad (7.12)$$

Fixing an arbitrary number $\varepsilon > 0$ one can indicate step Δ of partition Γ providing the estimate

$$\sigma(x(\vartheta)) < w(t_*, x_*) + \varepsilon$$

Proof.

By induction inequality (7.9) implies the estimate

$$u_\Gamma(t_i, x(t_i, t_*, x_*, U^*, q(\cdot))) \leq u_\Gamma(t_*, x_*) + (t_i - t_*)\Delta^{-1} L_w L_2 K \Delta^2$$

which provides relation (7.11) when $t_i = \vartheta$.

Inequality (7.12) follows from estimates (3.13), (7.11). \square

Remark 7.3 For trajectories $x(\cdot)$ generated by feedback U^P

$$U^P = U^P(t, x) = \arg \min_{p \in P} \max_{q \in Q} u_\Gamma(t + \Delta, y(t, x, \Delta, p, q)) \quad (7.13)$$

in the approximation scheme with minimax operator PMM^* (7.2) estimates (7.9), (7.12) are fulfilled.

In reality approximation scheme (7.3) can be realized only at nodes of grid $GR(t)$ but not in the whole domain D_t , $t \in \Gamma$. Let us assume that grid $GR(t)$, $t \in \Gamma$ is rectangular and uniform

$$\begin{aligned} GR(t) &= \{y \in D_t : y = \sum (m_1 e_1 + \dots + m_n e_n) \gamma \Delta\} \\ m_i &= 0, \pm 1, \pm 2, \dots, \quad i = 1, \dots, n \\ e_i &= (e_i^1, \dots, e_i^n), \quad e_i^i = 1, \quad e_i^j = 0, \quad i \neq j \end{aligned} \quad (7.14)$$

We define values of operator G only at nodes y_j of grid $GR(t)$ and interpolate them linearly into domain D_t^*

$$D_t^* = \{y \in R^n : y = \sum_{j=0}^n \alpha_j y_j, \quad y_j \in GR(t), \quad \alpha_j \geq 0, \quad \sum_{j=0}^n \alpha_j = 1\}$$

according to the given simplicial partition Ω .

Practically we introduce operator $y \rightarrow G^*(t, \Delta, u)(y) : D_t^* \rightarrow R$

$$\begin{aligned} G^*(t, \Delta, u)(y) &= \sum_{j=0}^n \alpha_j G(t, \Delta, u)(y_j) \\ y \in D_t^*, \quad y &= \sum_{j=0}^n \alpha_j y_j, \quad y_j \in GR(t), \quad \alpha_j \geq 0, \quad \sum_{j=0}^n \alpha_j = 1 \end{aligned} \quad (7.15)$$

Here numbers $\alpha_j = \alpha_j(\Omega)$ and nodes $y_j = y_j(\Omega)$ are determined by partition Ω .

Let us consider the approximation scheme with operator G^* (7.15) for partition Γ of interval T with step Δ

$$\begin{aligned} u_{\Gamma}^*(\vartheta, y) &= \sigma^*(y) = \sum_{j=0}^n \alpha_j \sigma(y_j), \quad y \in D_{\vartheta}^*, \quad y = \sum_{j=0}^n \alpha_j y_j \\ \sum_{j=0}^n \alpha_j &= 1, \quad \alpha_j = \alpha_j(\Omega) \geq 0, \quad y_j = y_j(\Omega) \in GR(\vartheta), \quad j = 0, \dots, n \\ u_{\Gamma}^*(t, x) &= G^*(t, t_{i+1} - t, u_{\Gamma}^*(t_{i+1}, \cdot))(x) \\ t &\in [t_i, t_{i+1}), \quad x \in D_t^*, \quad i = 0, \dots, N-1 \end{aligned} \quad (7.16)$$

Assume that the approximate function u_{Γ}^* is calculated at all points (t, y) , $y \in D_t^*$, $t \in \Gamma$. Let us determine at first values of the optimal feedback $U^* = U^*(t, x)$ at nodes x of grid $GR(t)$, $t \in \Gamma$ using operator $G(t, \Delta, u_{\Gamma}^*(t + \Delta, \cdot))(x) = G^*(t, \Delta, u_{\Gamma}^*(t + \Delta, \cdot))(x)$

$$\begin{aligned} U^* &= U^*(t, x) = \arg \min_{p \in P} \langle s^*, B(t, x)p \rangle \\ s^* &= s^*(t, x) = \arg \min_{s \in Dg(y^*)} \{ \Delta H(t, x, s) + g(y^*) - \langle s, y^* - x \rangle \} \\ y^* &= y^*(t, x) = \arg \min_{y \in \bar{O}(x, K\Delta)} \min_{s \in Dg(y)} \{ \Delta H(t, x, s) + g(y) - \langle s, y - x \rangle \} \end{aligned} \quad (7.17)$$

We define the optimal feedback $U^C(t, y)$ in domain $y \in D_t^*$, $t \in \Gamma$ by the piecewise constant interpolation of values $\{U^*(t, x)\}$ (7.17) calculated at nodes $x \in GR(t)$, $t \in \Gamma$ neighboring to points y

$$U^C(t, y) = U^*(t, x), \quad x = x(y) = \arg \min_{z \in GR(t)} \|y - z\| \quad (7.18)$$

Let us introduce the Euler trajectory

$$y(\cdot) = \{y(t, t_*, y_*, U^C, q(\cdot)) : t \in \Gamma \cap [t_*, \vartheta]\} \quad (7.19)$$

generated by feedback U^C (7.17), (7.18) and an arbitrary Lebesgue measurable control $\tau \rightarrow q(\tau) : [t_*, \vartheta] \rightarrow Q$

$$\begin{aligned} y(t_{i+1}) &= y(t_i + \Delta) = y(t_i) + \Delta(h(t_i, y(t_i)) + B(t_i, y(t_i))U^C) + \\ &\int_{t_i}^{t_i+\Delta} C(\tau, y(t_i))q(\tau)d\tau, \quad y(t_*) = y_*, \quad t_i, t_{i+1} \in \Gamma \cap [t_*, \vartheta] \end{aligned} \quad (7.20)$$

For trajectory $y(\cdot)$ (7.19) we define accompanying points

$$(x_-(t_i), x_+(t_{i+1})) \quad (7.21)$$

by relations

$$x_-(t_i) = x(y(t_i)) = \arg \min_{z \in GR(t)} \|y(t_i) - z\| \quad (7.22)$$

$$\begin{aligned} x_+(t_{i+1}) &= x_-(t_i) + \Delta(h(t_i, x_-(t_i)) + B(t_i, x_-(t_i))U^C) + \\ &\int_{t_i}^{t_i+\Delta} C(\tau, x_-(t_i))q(\tau)d\tau, \quad t_i, t_{i+1} \in \Gamma \cap [t_*, \vartheta] \end{aligned} \quad (7.23)$$

Lemma 7.3 For trajectory $y(\cdot)$ (7.19) and accompanying points $\{(x_-(t_i), x_+(t_{i+1}))\}$ (7.21) the estimate

$$\|y(t_{i+1}) - x_+(t_{i+1})\| \leq (1 + L_1\Delta)\|y(t_i) - x_-(t_i)\| + 2L_2K\Delta^2 \quad (7.24)$$

is valid.

Lemma 7.4 *Let parameter γ of grid $GR(t)$, $t \in \Gamma$ be an infinitesimal value with respect to step Δ*

$$\gamma = \varepsilon(\Delta), \quad \lim_{\Delta \rightarrow 0} \varepsilon(\Delta) = 0$$

for example,

$$\gamma = \rho\Delta^a, \quad a > 0, \quad \rho > 0 \quad (7.25)$$

and, hence, step h of the spatial grid $GR(t)$ is a high order infinitesimal value

$$h = \rho\Delta^{1+a} \quad (7.26)$$

with respect to the time step Δ .

Then the basic inequality for function u_Γ^* is valid

$$u_\Gamma^*(t_i, y(t_i)) \geq u_\Gamma^*(t_{i+1}, y(t_{i+1})) - L_w \left(\frac{\sqrt{n}}{2} (2 + L_1\Delta) \rho\Delta^a + 3L_2K\Delta \right) \Delta \quad (7.27)$$

Proof.

The Lipschitz continuity of function u_Γ^* and relations (7.9), (7.24) imply inequalities

$$\begin{aligned} u_\Gamma^*(t_i, y(t_i)) &\geq u_\Gamma^*(t_i, x_-(t_i)) - L_w \|y(t_i) - x_-(t_i)\| \geq \\ &u_\Gamma^*(t_{i+1}, x_+(t_{i+1})) - L_w L_2 K \Delta^2 - L_w \|y(t_i) - x_-(t_i)\| \geq \\ &u_\Gamma^*(t_{i+1}, y(t_{i+1})) - L_w (2 + L_1\Delta) \|y(t_i) - x_-(t_i)\| - 3L_2K\Delta^2 \end{aligned}$$

Relation (7.25) provides the inequality

$$\|y(t_i) - x_-(t_i)\| \leq \frac{\sqrt{n}}{2} \rho\Delta^{1+a}$$

for trajectory $y(\cdot)$ and accompanying points $x_-(t_i)$.

The last two inequalities give estimate (7.27). \square

Using estimate (7.27) one can prove the following proposition for trajectory $y(\cdot)$ (7.19) generated by the optimal feedback U^C (7.17), (7.18).

Theorem 7.2 *For any partition Γ , grid $GR(t)$, $t \in \Gamma$ with high order infinitesimal parameters (7.26), initial position (t_*, y_*) and Lebesgue measurable control $\tau \rightarrow q(\tau) : [t_*, \vartheta] \rightarrow Q$ the trajectory $y(\cdot)$ (7.19) generated by the optimal feedback U^C (7.17), (7.18) with the piecewise constant interpolation satisfies the estimate*

$$\begin{aligned} \sigma(y(\vartheta)) &\leq u_\Gamma^*(t_*, y_*) + \varphi(\Delta) \quad (7.28) \\ \varphi(\Delta) &= (\vartheta - t_*) L_w \left(\frac{\sqrt{n}}{2} (2 + L_1\Delta) \rho\Delta^a + 3L_2K\Delta \right), \quad \lim_{\Delta \rightarrow 0} \varphi(\Delta) = 0 \end{aligned}$$

and, hence, by Theorem 3.4

$$\sigma(y(\vartheta)) \leq w(t_*, y_*) + C^* \Delta^{1/2} + \varphi(\Delta) \quad (7.29)$$

Fixing an arbitrary number $\varepsilon > 0$ one can indicate step Δ of partition Γ which provides the estimate

$$\sigma(y(\vartheta)) < w(t_*, y_*) + \varepsilon$$

Proof.

By induction inequality (7.27) implies the estimate

$$\begin{aligned} u_{\Gamma}^*(t_i, y(t_i, t_*, y_*, U^C, q(\cdot))) &\leq u_{\Gamma}^*(t_*, y_*) + \\ (t_i - t_*)\Delta^{-1}L_w\left(\frac{\sqrt{n}}{2}(2 + L_1\Delta)\rho\Delta^a + 3L_2K\Delta\right)\Delta \end{aligned}$$

which provides relation (7.28) when $t_i = \vartheta$.

Inequality (7.29) follows directly from relations (3.20), (7.28). \square

Remark 7.4 *One can use other piecewise constant interpolations of feedback $U^* = U^*(t, x)$ (7.17). For example, the optimal feedback $U^E(t, y)$ can be interpolated into domain $y \in D_t^*$, $t \in \Gamma$ by controls $U^*(t, x)$ calculated at nodes $x \in GR(t)$ with least values of the approximate function u_{Γ}^* in the neighboring simplex*

$$\begin{aligned} U^E(t, y) &= U^*(t, x), \quad x = x(y) = \arg \min_{y_j} u_{\Gamma}^*(y_j) \quad (7.30) \\ y &= \sum_{j=0}^n \alpha_j y_j, \quad y_j = y_j(\Omega) \in GR(t), \quad \sum_{j=0}^n \alpha_j = 1, \quad \alpha_j = \alpha_j(\Omega) \geq 0 \end{aligned}$$

In this case estimates (7.27) and (7.29) can be rewritten in the following way

$$u_{\Gamma}^*(t_i, y(t_i)) \geq u_{\Gamma}^*(t_{i+1}, y(t_{i+1})) - L_w\left(\frac{\sqrt{n}}{2}(1 + L_1\Delta)\rho\Delta^a + 3L_2K\Delta\right)\Delta \quad (7.31)$$

$$\sigma(y(\vartheta)) \leq w(t_*, y_*) + C^*\Delta^{1/2} + \varphi^*(\Delta) \quad (7.32)$$

$$\varphi^*(\Delta) = (\vartheta - t_*)L_w\left(\frac{\sqrt{n}}{2}(1 + L_1\Delta)\rho\Delta^a + 3L_2K\Delta\right)$$

Remark 7.5 *For trajectory $y(\cdot)$ generated by feedback U^P (7.33) which is determined at nodes x of grid $GR(t)$ by minimax operator PMM^* (7.2)*

$$U^P = U^P(t, x) = \arg \min_{p \in P} \max_{q \in Q} u_{\Gamma}^*(t + \Delta, y(t, x, \Delta, p, q)) \quad (7.33)$$

and interpolated into domain D_t^* by piecewise constant rules (7.18) or (7.30) one can obtain estimates (7.27), (7.29) or (7.31), (7.32).

Let us consider the case of quasiconvex approximate functions $y \rightarrow u_{\Gamma}^*(t, y) : D_t^* \rightarrow R$, $t \in \Gamma$ for grids $GR(t)$ with the linear dependence of space-time steps h, Δ

$$h = \gamma\Delta \quad (7.34)$$

Here γ is a fixed constant.

Let us formulate the property of quasiconvexity for functions $y \rightarrow u_{\Gamma}^*(t, y)$.

Conjecture 7.1 *Approximate functions $y \rightarrow u_{\Gamma}^*(t, y) : D_t^* \rightarrow R$, $t \in \Gamma$ satisfy the convexity condition up to the infinitesimal value $\mu\Delta^{1+b}$, $b > 0$, $\mu > 0$ in domains with radius $\nu\Delta$ - the quasiconvexity condition*

$$\begin{aligned} \sum_{j=0}^n \alpha_j u_{\Gamma}^*(t, z_j) + \mu\Delta^{1+b} &\geq u_{\Gamma}^*(t, \sum_{j=0}^n \alpha_j z_j) \quad (7.35) \\ \sum_{j=0}^n \alpha_j &= 1, \quad \alpha_j \geq 0, \quad j = 0, \dots, n \\ \|z_k - z_l\| &\leq \nu\Delta, \quad z_k, z_l \in D_t^*, \quad k, l = 0, \dots, n \end{aligned}$$

$$\nu > 2K + \sqrt{n}(1 + L_1\Delta)\gamma + 2L_2K\Delta \quad (7.36)$$

Let us define feedback $U^L = U^L(t, y)$, $y \in D_t^*$, $t \in \Gamma$ by the linear interpolation of control values $\{U^*(t, x)\}$ (7.17) calculated at nodes $x = y_j(\Omega)$ of grid $GR(t)$

$$U^L = U^L(t, y) = \sum_{j=0}^n \alpha_j U_j^*, \quad U_j^* = U^*(t, y_j) \quad (7.37)$$

$$y = \sum_{j=0}^n \alpha_j y_j, \quad y_j = y_j(\Omega) \in GR(t), \quad \sum_{j=0}^n \alpha_j = 1, \quad \alpha_j = \alpha_j(\Omega) \geq 0, \quad j = 0, \dots, n$$

Let us introduce the Euler trajectory

$$z(\cdot) = \{z(t, t_*, y_*, U^L, q(\cdot)) : t \in \Gamma \cap [t_*, \vartheta]\} \quad (7.38)$$

generated by feedback U^L (7.37) and an arbitrary Lebesgue measurable control $\tau \rightarrow q(\tau) : [t_*, \vartheta] \rightarrow Q$

$$z(t_{i+1}) = z(t_i + \Delta) = z(t_i) + \Delta(h(t_i, z(t_i)) + B(t_i, z(t_i))U^L) + \int_{t_i}^{t_i+\Delta} C(\tau, z(t_i))q(\tau)d\tau, \quad z(t_*) = z_*, \quad t_i, t_{i+1} \in \Gamma \cap [t_*, \vartheta] \quad (7.39)$$

Lemma 7.5 *The basic estimate*

$$u_\Gamma^*(t_i, z(t_i)) \geq u_\Gamma^*(t_{i+1}, z(t_{i+1})) - \mu\Delta^{1+b} - L_w(\sqrt{n}L_1\gamma + 3L_2K)\Delta^2 \quad (7.40)$$

is valid for trajectory $z(\cdot)$ (7.38), (7.39).

Proof.

According to the property of quasiconvexity (7.35), the Lipschitz continuity of function u_Γ^* and definition of feedback U^* (7.17) we obtain the necessary chain of inequalities

$$\begin{aligned} u_\Gamma^*(t_{i+1}, z(t_{i+1})) &= u_\Gamma^*(t_i + \Delta, z(t_i) + \Delta(h(t_i, z(t_i)) + B(t_i, z(t_i))U^L) + \\ &\int_{t_i}^{t_i+\Delta} C(\tau, z(t_i))q(\tau)d\tau) = u_\Gamma^*(t_i + \Delta, \sum_{j=0}^n \alpha_j z_j + \Delta \sum_{j=0}^n \alpha_j (h(t_i, z(t_i)) - h(t_i, y_j)) + \\ &\Delta \sum_{j=0}^n \alpha_j (B(t_i, z(t_i)) - B(t_i, y_j))U_j^* + \int_{t_i}^{t_i+\Delta} \sum_{j=0}^n \alpha_j (C(\tau, z(t_i)) - C(\tau, y_j))q(\tau)d\tau) \leq \\ &\sum_{j=0}^n \alpha_j u_\Gamma^*(t_i + \Delta, z_j) + \mu\Delta^{1+b} + L_w(\sqrt{n}L_1\gamma + 2L_2K)\Delta^2 \leq \\ &\sum_{j=0}^n \alpha_j u_\Gamma^*(t_i, y_j) + \mu\Delta^{1+b} + L_w(\sqrt{n}L_1\gamma + 3L_2K)\Delta^2 = \\ &u_\Gamma^*(t_i, z(t_i)) + \mu\Delta^{1+b} + L_w(\sqrt{n}L_1\gamma + 3L_2K)\Delta^2 \end{aligned}$$

Here points $z(t_i)$, y_j , z_j are connected by relations

$$\begin{aligned} z(t_i) &= \sum_{j=0}^n \alpha_j y_j, \quad \sum_{j=0}^n \alpha_j = 1, \quad \alpha_j \geq 0, \quad j = 0, \dots, n \\ z_j &= y_j + \Delta(h(t_i, y_j) + B(t_i, y_j)U_j^*) + \int_{t_i}^{t_i+\Delta} C(\tau, y_j)q(\tau)d\tau \end{aligned}$$

and points z_j are disposed in the ball of radius $\nu\Delta$

$$\begin{aligned} \|z_k - z_l\| &\leq (1 + L_1\Delta)\|y_k - y_l\| + 2L_2K\Delta^2 + \\ \Delta\|B(t_i, y_k)U_k^*\| + \Delta\|B(t_i, y_l)U_l^*\| &< \nu\Delta, \quad k, l = 0, \dots, n \quad \square \end{aligned}$$

The basic estimate (7.40) implies the following proposition.

Theorem 7.3 *Assume that Conjecture 7.1 is fulfilled for approximate functions u_Γ^* in schemes with operator G^* (7.15) and the linear dependence (7.34) of space-time steps.*

Then for any initial position (t_, y_*) and Lebesgue measurable control $\tau \rightarrow q(\tau) : [t_*, \vartheta] \rightarrow Q$ the trajectory $z(\cdot)$ (7.38) generated by the optimal feedback U^L (7.17), (7.37) with the linear interpolation satisfies the estimate*

$$\begin{aligned} \sigma(z(\vartheta)) &\leq u_\Gamma^*(t_*, z_*) + \psi(\Delta) \\ \psi(\Delta) &= (\vartheta - t_*)(\mu\Delta^b + L_w(\sqrt{n}L_1\gamma + 3L_2K)\Delta), \quad \lim_{\Delta \rightarrow 0} \psi(\Delta) = 0 \end{aligned} \quad (7.41)$$

and, hence, by Theorem 3.4

$$\sigma(z(\vartheta)) \leq w(t_*, z_*) + C^* \Delta^{1/2} + \psi(\Delta) \quad (7.42)$$

Fixing an arbitrary number $\varepsilon > 0$ one can indicate step Δ of partition Γ which provides the estimate

$$\sigma(z(\vartheta)) < w(t_*, z_*) + \varepsilon$$

Proof.

By induction inequality (7.40) implies relations

$$\begin{aligned} u_\Gamma^*(t_i, z(t_i, t_*, z_*, U^L, q(\cdot))) &\leq u_\Gamma^*(t_*, z_*) + \\ (t_i - t_*)\Delta^{-1}(\mu\Delta^{1+b} + L_w(\sqrt{n}L_1\gamma + 3L_2K)\Delta^2) & \end{aligned}$$

which leads to the estimate (7.41) when $t_i = \vartheta$.

Inequality (7.42) follows directly from relations (3.20), (7.41). \square

Remark 7.6 *Let Conjecture 7.1 be fulfilled for approximate functions u_Γ^* in schemes with operator PMM^* (7.2). Then trajectories $z(\cdot)$ generated by feedback U^P (7.33) which is defined at nodes x of grid $GR(t)$ by minimax operator PMM^* (7.2) and interpolated linearly (7.37) into domain D_t^* satisfy estimates (7.40), (7.42).*

Remark 7.7 *Assume that Conjecture 7.1 is true for approximate functions u_Γ^* in schemes with operator LA (6.29). Consider trajectories $z(\cdot)$ generated by feedback U^A (7.43) which is defined at nodes x of grid $GR(t)$ by operator LA (6.29) - the minimax formula on local linear hulls*

$$U^A = U^A(t, x) = \arg \min_{p \in P} \max_{q \in Q} L(y(t, x, \Delta, p, q)) = \arg \min_{p \in P} \langle A, B(t, x)p \rangle \quad (7.43)$$

and interpolated linearly (7.37) into domain D_t^* . Then estimates (7.40), (7.42) are valid.

Proof.

Conditions of quasiconvexity (7.35), (7.36) imply inequalities

$$\begin{aligned} L(y) &\geq u_\Gamma^*(t, y) - \mu\Delta^{1+b}, \quad L(y) \leq g(y) \\ y &\in \overline{O}(x, K\Delta), \quad x \in GR(t), \quad t \in \Gamma \end{aligned}$$

and, therefore, provide estimates (7.40), (7.42). \square

Conclusion

The unique grid scheme for constructing value functions and control synthesis is proposed for solving optimal guaranteed control problems which arise in mechanics, mathematical economics, evolutionary biology. Finite difference operators based on constructions of non-smooth analysis - subdifferentials of local convex hulls, superdifferentials of local concave hulls and their modifications, are elaborated for local approximations of nondifferentiable value functions and its generalized gradients (dual vectors). Control synthesis is obtained by the method of extremal shift in the direction of generalized gradients. Convergence of approximation schemes is proved by using methods of the theory of Hamilton-Jacobi equations. Properties of space-time grids providing optimality of designed trajectories are examined.

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