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Convexity and Hamiltonian equations in differential games

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Abstract

We study a zero sum differential game under strong assumptions of convexity — the cost is convex for one player, and concave for the other. An explicit necessary and sufficient condition for a saddle point of the game is given in terms of convex analysis subgradients of the conjugate of the cost function. A generalized Hamiltonian equation is shown to describe saddle trajectories of the game.

About the Author

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Acknowledgment

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1   Introduction

A great variety of complex systems can be modelled by game theory. Whether it is the evolution of populations in biology or agents competing on a market, there are always players trying to maximize their profit — fitness of the species or income of an agent — at the cost of others. Differential games study competitive processes taking place over continuous time. Motivated by air combat at first, they now find wide applications in the analysis of economic systems. In this paper, we concentrate on zero sum differential games, in which the cost is convex for each player, and study their saddle values and saddle points. Convexity is an often present property of cost functions and, from the theoretical viewpoint, has greater implications than strong continuity or smoothness assumptions. Limiting our attention to open loop controls, we reduce the dynamic equilibrium question to a certain saddle point problem for each moment in time. This allows us to obtain a simple necessary and sufficient condition for saddle controls, expressed in terms of generalized derivatives of the saddle conjugate of the cost function. In contrast to more general sufficient conditions, for example in Leitmann and Stalford 1972, the one given here is constructive. Through the properties of subgradients, it allows us to conclude the existence of saddle controls for a wide class of saddle games. No reference to compactness arguments and saddle point properties in infinite dimensional Banach spaces — these were the arguments used by Bensoussan 1971 and Berkovitz 1971 — are needed. Another characterization of saddle points of the game is given in terms of a generalized Hamiltonian equation. Any solution of it will produce a trajectory generated by saddle controls of the game. On the technical side, our work relies on measurability and normal integrand theory in Rockafellar 1997, and convex analysis in Rockafellar 1970. The saddle point condition is expressed in terms of the subgradient $\partial \phi$ of a convex-concave function $\phi(\cdot, \cdot)$. We have

$$\partial \phi = \partial_1 \phi \times \partial_2 \phi$$

where
$$\partial_1 \phi(\bar{x}, \bar{y}) = \{p \mid \phi(x, \bar{y}) \geq \phi(\bar{x}, \bar{y}) + p \cdot (x - \bar{x})\}$$
is the (convex) subgradient in the sense of convex analysis of the convex function $\phi(\cdot, \bar{y})$, and
$$\partial_2 \phi(\bar{x}, \bar{y}) = \{q \mid \phi(\bar{x}, y) \leq \phi(\bar{x}, \bar{y}) + q \cdot (y - \bar{y})\}$$

is the (concave) subgradient of the concave function $\phi(\bar{x}, \cdot)$. Note that the subscripts also denote with respect to which variable the subgradient is taken — for this reason we allow ourselves to use the notation ∂_1 (∂_2) to denote the convex (concave) subgradient with respect to the first (second) variable of a convex (concave) function. Above, and in the sequel, $a \cdot b$ dentotes a scalar product of vectors a and b. The notation ϕ^* when used for a function ϕ will denote the conjugate function — whether in the convex, concave or saddle sense, and when used with matrices, M^* will denote the transpose of M.

2   Two player zero sum differential game

We analyze a system controlled by two players. Its state is denoted by x, and the dynamics are given by a differential equation

$$\dot{x}(t) = g(t, x(t), u(t), v(t)) \tag{1}$$

with an initial condition

$$x(\tau) = \xi \tag{2}$$

Controls $u(\cdot)$ and $v(\cdot)$ are chosen respectively by Player One and Player Two from some sets of admissible controls $\mathcal{U}$ and $\mathcal{V}$, subject to constraints

$$u(t) \in P(t) \qquad v(t) \in Q(t) \tag{3}$$

With every pair (τ, ξ) giving an initial condition and every choice of $u(\cdot)$ and $v(\cdot)$ we associate the resulting trajectory $x(\cdot)$ and the cost of that particular selection of controls:

$$\Phi(\tau, \xi, u(\cdot), v(\cdot)) = \int_\tau^T \phi(t, x(t), u(t), v(t)) dt + l(x(\tau)) \tag{4}$$

We later specify the conditions under which this cost is well defined, in particular, when the initial value problem (1), (2) has a unique solution for every pair of admissible controls. In the game $\mathcal{P}(\tau, \xi)$, Player One tries to minimize the cost, and his opponent, Player Two, tries to maximize it. The value of the game, if it exists, is the saddle value in the above problem. We will analyze this game with the help of two problems of optimal control. In problem $\mathcal{P}_1(\tau, \xi, \bar{v}(\cdot))$, some control $\bar{v}(\cdot)$ is fixed, and Player One chooses his control $u(\cdot)$ to minimize $\Phi(\tau, \xi, u(\cdot), \bar{v}(\cdot))$. Symmetrically, in problem $\mathcal{P}_2(\tau, \xi, \bar{u}(\cdot))$, some control $\bar{u}(\cdot)$ is fixed, and Player Two chooses his control $v(\cdot)$ to maximize the cost $\Phi(\tau, \xi, \bar{u}(\cdot), v(\cdot))$. In general, it may be impossible to find a saddle value and saddle controls of the game looking at these two problems, however, in our setting of convexity, saddle controls will correspond to solutions of control problems.

2.1   Controls: open loop vs. closed loop

In the game $\mathcal{P}(\tau, \xi)$ players can be allowed to choose their strategies from different classes of functions. Closed loop (or feedback) strategies are functions of both time

and space: $U : [0, \tau] \times I\!\!R^n \to I\!\!R^n$, $V : [0, \tau] \times I\!\!R^n \to I\!\!R^n$, and the controls at any given time are determined by

$$u(t) = U(t, x(t))$$

$$v(t) = V(t, x(t))$$

Open loop (or pure) strategies are just particular functions $u(\cdot)$ and $v(\cdot)$, dependent on time only. The admissible classes of closed loop controls can still vary, but they are usually general enough to include open loop controls. For example, if we allow $U(\cdot, \cdot)$ to be any function measurable in time and continuous in x, then every measurable function $u(\cdot)$ is such a function, since it is independent of x, so continuous in this variable. The value of the game, even though it exists in some class of feedback strategies, may not exist if the players are limited to pure strategies. Consider $x \in I\!\!R$, with the dynamics $\dot{x} = u + v$, an initial condition $x(0) = 0$ and the restrictions $u, v \in [-1, 1]$ (we will later explain how these conditions can be incorporated in the cost function f). The cost is given by $\Phi(u(\cdot), v(\cdot)) = |x(1)|$. We can check that there is no saddle value in the open loop controls, but it exists for a sufficiently wide class of (necessarily discontinuous in x) feedback controls. However, as will follow from a general property of saddle points, the existence of a saddle value or saddle controls in the open loop case implies the same for the closed loop setting.

Theorem 1 *Assume that $\psi : \mathcal{X} \times \mathcal{Y} \to \overline{I\!\!R}$ and some subsets $X \subset \mathcal{X}$, $Y \subset \mathcal{Y}$ have the property that for any $\bar{x} \in X$ and any $\bar{y} \in Y$,*

$$\inf_{x \in \mathcal{X}} \psi(x, \bar{y}) = \inf_{x \in X} \psi(x, \bar{y}) \tag{5}$$

$$\sup_{y \in \mathcal{Y}} \psi(\bar{x}, y) = \sup_{y \in Y} \psi(\bar{x}, y) \tag{6}$$

If the saddle value of ψ over $X \times Y$ exists, that is

$$\inf_{x \in X} \sup_{y \in Y} \psi(x, y) = \sup_{y \in Y} \inf_{x \in X} \psi(x, y)$$

then it exists over $\mathcal{X} \times \mathcal{Y}$

$$\inf_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} \psi(x, y) = \sup_{y \in \mathcal{Y}} \inf_{x \in \mathcal{X}} \psi(x, y)$$

and the above saddle values are equal. If in addition a saddle point for ψ on $X \times Y$ exists, then it is also a saddle point on $\mathcal{X} \times \mathcal{Y}$.

Proof. We have

$$\inf_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} \psi(x, y) = \inf_{x \in \mathcal{X}} \sup_{y \in Y} \psi(x, y) \leq \inf_{x \in X} \sup_{y \in Y} \psi(x, y)$$

where the inequality follows from $X \subset \mathcal{X}$, and the equality holds since the "inside" supremum is taken with fixed x. Similarly,

$$\sup_{y \in Y} \inf_{x \in X} \psi(x, y) \leq \sup_{y \in \mathcal{Y}} \inf_{x \in X} \psi(x, y) = \sup_{y \in \mathcal{Y}} \inf_{x \in \mathcal{X}} \psi(x, y)$$

We assumed that the saddle value on $X \times Y$ exists, so, combining this with the above inequalities, we get

$$\inf_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} \psi(x, y) \leq \sup_{y \in \mathcal{Y}} \inf_{x \in \mathcal{X}} \psi(x, y)$$

and since the reverse inequality is always true, the saddle value on $\mathcal{X} \times \mathcal{Y}$ exists. Now assume that $(\bar{x}, \bar{y})$ is a saddle point for ψ on $X \times Y$. Then

$$\inf_{x \in X} \psi(x, \bar{y}) = \psi(\bar{x}, \bar{y}) = \sup_{y \in Y} \psi(\bar{x}, y)$$

which by assumption implies

$$\inf_{x \in \mathcal{X}} \psi(x, \bar{y}) = \psi(\bar{x}, \bar{y}) = \sup_{y \in \mathcal{Y}} \psi(\bar{x}, y)$$

and this is exactly the saddle point condition on $\mathcal{X} \times \mathcal{Y}$. $\qquad\qquad\square$

If either player has already chosen a strategy — even a closed loop one — it is sufficient for the other player to look for a open loop control. Indeed, assume that Player Two strategy is fixed to be $V(t, x)$, and that the best reply to this for Player One is $U(t, x)$. Let $\bar{x}$ be the arc determined by these two controls. The same payoff could be achieved by Player Two by the following open loop control: $\bar{u}(t) = U(t, \bar{x}(t))$, also, this control paired with $V(t, x)$, generates the same arc $\bar{x}$. This means that the condition (5) in Theorem 1 is satisfied in our game setting, where X is the class of open loop controls, and $\mathcal{X}$ are closed loop controls. Actually, this condition holds for all x in $\mathcal{X}$, not just in X. A symmetric argument can be made when Player One fixes his control.

Corollary 2 *If the game $\mathcal{P}(\tau, \xi)$ has a saddle value in the class of open loop controls, then it has the same saddle value over closed loop controls. If $\bar{u}(\cdot)$ and $\bar{v}(\cdot)$ are open loop saddle controls, then the feedback strategies*

$$U(t, x) = \bar{u}(t)$$
$$V(t, x) = \bar{v}(t)$$

for all x are saddle controls for the game $\mathcal{P}'$.

Note that the players can restrict their attention to open loop controls only if the other player has fixed his strategy. It is not true that the existence of a saddle value in closed loop strategies implies the existence of this value for open loop ones.

3   Open loop convex-concave game

In this section, we study games for which the cost functional $\Phi(u(\cdot), v(\cdot))$ is convex in the control $u(\cdot)$ for any fixed control $v(\cdot)$, and concave in the control $v(\cdot)$ for any fixed control $u(\cdot)$. The dynamics are linear:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + C(t)v(t) \tag{7}$$

Assumption 1 *The matrices $A(t) \in \mathbb{R}^{n \times n}$, $B(t) \in \mathbb{R}^{n \times k}$ and $C(t) \in \mathbb{R}^{n \times l}$ depend continuously on t on $(-\infty, T]$, the control spaces $\mathcal{U}$ and $\mathcal{V}$ are $L^1(-\infty, T]$ spaces of appropriate dimensions, and the constraint sets $P(t) \subset \mathbb{R}^k$, $Q(t) \subset \mathbb{R}^l$ are nonempty, closed, convex, and depend measurably on t.*

These assumptions imply the existence and uniqueness of solutions of (7). We look at the cost functions $\phi : (-\infty, T] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^l \to \overline{\mathbb{R}}$ having the following properties: $\phi(t, \cdot, \cdot, v)$ is convex for every v, and $\phi(t, \cdot, u, \cdot)$ is concave for every u. The endpoint cost is linear, that is $r(x(\tau)) = d \cdot x(\tau)$, where d is a vector in $\mathbb{R}^n$. Notice that ϕ must actually be of the form

$$\alpha(t) \cdot x + f(t, u, v)$$

for some function f, convex in u and concave in v. Indeed, f is both convex and concave in x, so it must be affine in x:

$$\alpha(t, u, v) \cdot x + f(t, u, v)$$

Clearly, $f(t, \cdot, \cdot)$ is a saddle function. The function $\alpha(t, u, v) \cdot x$ needs to be convex in u and concave in v and, since we can use $-x$ in place of x, it needs to be affine in u and v. But expressions of the type $u \cdot Qx$ (or $v \cdot Rx$) are jointly convex in x and u (or x and v) only if they are 0. To see this, we write the function $w = (x, u) \to u \cdot Qx$ as $w \to w \cdot Dw$ for a uniquely determined symmetric matrix D. This matrix is positive semidefinite if and only if $u \cdot Qx$ is convex. After substituting $(x, -u)$ for w, we see that D must be zero. Since $\alpha(\cdot)$ can depend only on time, we can look only at problems with the cost functional

$$\int_\tau^T \alpha(t) \cdot x(t) + f(t, u(t), v(t)) dt + d \cdot x(T)$$

The above problem can be reduced to one with the cost funtion independent of x. Indeed, add another linear equation to the dynamics:

$$\dot{w}(t) = \alpha(t) \cdot x(t)$$

with initial condition $w(\tau) = 0$. Then the above cost functional can be rewritten as

$$\int_\tau^T f(t, u(t), v(t)) dt + d \cdot x(T) + w(T)$$

in which the running cost is independent of x and the endpoint cost is still linear in the state variable. Therefore, from now on, we concentrate on the cost of the form

$$\Phi(\tau, \xi, u, v) = \int_\tau^T f(t, u(t), v(t)) dt + d \cdot x(T) \tag{8}$$

We now present the assumptions on function f. The reasons for defining it on the whole space $\mathbb{R}^k \times \mathbb{R}^n$ will become apparent in further sections.

Assumption 2 *The function $f : (-\infty, T] \times \mathbb{R}^k \times \mathbb{R}^l \to \overline{\mathbb{R}}$ has the following properties: $f(t, u, v)$ is measurable in t for every (u, v), convex in u for every (t, v) and concave in v for every (t, u). The set where $f(t, \cdot, \cdot)$ is finite valued is $P(t) \times Q(t)$.*

Since f is convex, we must have $f(t, u, v) = +\infty$ for $u \notin P(t)$, $v \in Q(t)$, and $f(t, u, v) = -\infty$ for $u \in P(t)$, $v \notin Q(t)$, and there are several possibilities for the values of $f(t, u, v)$ when $u \notin P(t)$, $v \notin Q(t)$ — different combinations of $\pm\infty$ can be used. The fact that $f(t, u(t), v(t))$ is measurable for properly chosen controls $u(\cdot)$ and $v(\cdot)$, will follow from lemma 4. The following assumption can now be stated:

Assumption 3 *For any $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$ with $u(t) \in P(t)$ and $v(t) \in Q(t)$ almost everywhere, $\int_\tau^T f(t, u(t), v(t))dt$ is finite.*

3.1 Constraints and infinite values

In optimal control theory, the use of infinite values of integrands is used to impose constraints on controls, without mentioning them explicitly. Direct attempt to use f without constraints on controls in our differential game would lead to ambiguities in integrating f: $f(t, u(t), v(t))$ can have both ∞ and $-\infty$ as values. We now propose a new game $\mathcal{P}'(\tau, \xi)$. It is not a zero sum game any more, but the outcome of it agrees with the outcome of $\mathcal{P}(\tau, \xi)$ except pathological situations. The cost $\Phi_1(\tau, \xi, u(\cdot), v(\cdot))$ for Player One is defined to be $+\infty$ if $f(t, u(t), v(t)) = +\infty$ on a positive measure set, otherwise it is equal to $\Phi(\tau, \xi, u(\cdot), v(\cdot))$. Similarly, the cost for Player Two, $\Phi_2(\tau, \xi, u(\cdot), v(\cdot))$ equals $-\infty$ if $f(t, u(t), v(t)) = -\infty$ on a positive measure set, otherwise it is equal to $\Phi(\tau, \xi, u(\cdot), v(\cdot))$. Player One is minimizing his cost, while Player Two is maximizing his — to state this problem more in the convention of non-zero sum games, we could reverse the sign of Φ_2, and then both players would be minimizing their costs. In their decissions, players use the worst case analysis — Player One looks to minimize $\sup_{v(\cdot)} \Phi(\tau, \xi, u(\cdot), v(\cdot))$ over $u(\cdot) \in \mathcal{U}$. If $u(t) \notin P(t)$ on a positive measure set, and $v(t) \in Q(t)$ almost everywhere, then, by our convention and the form of f, $\Phi(\tau, \xi, u(\cdot), v(\cdot)) = \infty$, so also $\sup_{v(\cdot)} \Phi(\tau, \xi, u(\cdot), v(\cdot)) = \infty$. Assuming some rationality of Player One, we can see that the constraint $u(t) \in P(t)$ almost everywhere is in effect. Similar argument can be made for Player Two. We can actually show that the solutions of games $\mathcal{P}'(\tau, \xi)$ and $\mathcal{P}(\tau, \xi)$ are equivalent. By a solution of a nonzero sum games we understand the Nash equilibrium - a generalization of the saddle point.

Lemma 3 *If $(\bar{u}(\cdot), \bar{v}(\cdot))$ is a Nash equilibrium for $\mathcal{P}'(\tau, \xi)$, then $\bar{u}(t) \in P(t)$, $\bar{v}(t) \in Q(t)$ almost everywhere.*

Proof. For any $u(\cdot)$ and $v(\cdot)$ we have

$$\Phi_1(\bar{u}(\cdot), \bar{v}(\cdot)) \le \Phi_1(u(\cdot), \bar{v}(\cdot)) \qquad \Phi_2(\bar{u}(\cdot), v(\cdot)) \le \Phi_2(u(\cdot), \bar{v}(\cdot))$$

If $\bar{v}(t) \in Q(t)$ almost everywhere, then for any $u(\cdot)$ such that $u(t) \in P(t)$ almost everywhere, $\Phi_1(u(\cdot), \bar{v}(\cdot))$ is finite, so $\Phi_1(\bar{u}(\cdot), \bar{v}(\cdot)) < \infty$. But this implies that $\bar{u}(\cdot) \in P(t)$ almost everywhere. This, and a symmetric argument, shows that if one of the Nash controls satisfies the corresponding constraint, the other one also has this property. Now suppose both Nash controls violate the constraints. Then, for any $u(\cdot)$ such that $u(t) \in P(t)$ almost everywhere, $\Phi_1(u(\cdot), \bar{v}(\cdot)) = -\infty$, so $\Phi_1(\bar{u}(\cdot), \bar{v}(\cdot)) = -\infty$. This implies $f(t, \bar{u}(t), \bar{v}(t)) = -\infty$ on a positive measure set, so we must have $\Phi_2(\bar{u}(\cdot), \bar{v}(\cdot)) = -\infty$ and this implies $\Phi_2(\bar{u}(\cdot), v(\cdot)) = -\infty$ for any

$v(\cdot)$. But for any $v(\cdot)$ such that $v(t) \in Q(t)$ almost everywhere, $\Phi_2(\bar{u}(\cdot), v(\cdot)) = \infty$. This is a contradiction. $\qquad\blacksquare$

This lemma implies that if a Nash equilibrium exists for the game $\mathcal{P}'(\tau, \xi)$, the Nash pair must be a saddle point for $\mathcal{P}(\tau, \xi)$. It is easy to check that the converse also holds — any saddle point of $\mathcal{P}(\tau, \xi)$ is a Nash equilibrium for $\mathcal{P}'(\tau, \xi)$.

Lemma 4 *Under assumption A2, the function $f(\cdot, \cdot, v(\cdot))$ is a normal integrand for any measurable $v(\cdot)$ such that $v(t) \in Q(t)$ almost everywhere in $[\tau, T]$. Symmetically, $-f(\cdot, u(\cdot), \cdot)$ is a normal integrand for any measurable $u(\cdot)$ such that $u(t) \in P(t)$ almost everywhere in $[\tau, T]$.*

Proof. First assume that $v(t) \in Q(t)$ almost everywhere in $[\tau, T]$. Then $f(\cdot, \cdot, v(\cdot)) = \tilde{f}(\cdot, \cdot, v(\cdot))$ where $\tilde{f}(t, u, v) = f(t, u, v)$ when $v \in Q(t)$ and $\tilde{f}(t, u, v) = +\infty$ elsewhere. We can view $\tilde{f}$ as a sum of a Caratheodory integrand $\hat{f}$ and an indicator of $P(t) \times Q(t)$, so according to 14.32 in Rockafellar 1997, $\tilde{f}$ is a normal integrand. The mentioned $\hat{f}$ can be, for example

$$\hat{f}(t, u, v) = f\left(t, P_{P(t) \times Q(t)}\left((u, v)\right)\right)$$

where P_S is the projection onto the set S. By 14.17 in Rockafellar 1997, the expression $P_{P(t) \times Q(t)}\left((u, v)\right)$ is measurable in t, so also $\hat{f}$ is measurable in t for fixed (u, v). For a fixed time t, the projection is continuous in (u, v), so the same property holds for $\hat{f}$. Thus $\hat{f}$ is a Caratheodory integrand. The proof of the second part of the lemma is symmetric. $\qquad\blacksquare$

We see that under assumptions A1, 2, 3, the cost $\Phi(u(\cdot), v(\cdot))$ is well defined for controls satisfying 3. From now on, let assumptions A1, 2, 3 hold.

3.2 Reduction to running cost only

Given the controls $u(\cdot)$ and $v(\cdot)$, and the initial condition $x(\tau) = \xi$, we obtain, for $t \in [\tau, T]$, the trajectory

$$x(t) = \mathcal{A}(t, \tau)\xi + \int_\tau^t \mathcal{A}(t, s)(Bu(s) + Cv(s))ds \tag{9}$$

where $\mathcal{A}(t, \tau)$ is the fundamental solution of $\dot{w}(t) = A(t)w(t)$ with $\mathcal{A}(\tau, \tau)$ being the identity matrix. We can rewrite the cost $\Phi(\tau, \xi, u(\cdot), v(\cdot))$ as

$$\int_\tau^T f(t, u(t), v(t))dt + d \cdot \left(\int_\tau^T \mathcal{A}(T, s)(Bu(s) + Cv(s))ds + \mathcal{A}(T, \tau)\xi\right)$$

$$= \int_\tau^T \left[f(t, u(t), v(t)) + d \cdot \mathcal{A}(T, t)(Bu(t) + Cv(t))\right]dt + d \cdot \mathcal{A}(T, \tau)\xi$$

The last term in the last expression is independent of the controls, and the integrand does not depend on $x(t)$. We now relate the saddle points of the integrand to the saddle points of the integral functional. Assumption of decomposability is satisfied in particular by the space of all measurable functions, and by L^p spaces. For details, see Rockafellar 1998, chapter 14.

Lemma 5 *Let $\gamma : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ be a function such that $t \to g(t, u(t), v(t))$ is measurable for any $u(\cdot) \in \mathcal{U}$, $v(\cdot) \in \mathcal{V}$, where $\mathcal{U}$ and $\mathcal{V}$ are some sets of measurable functions. Define $\Gamma(u(\cdot), v(\cdot)) = \int_a^b \gamma(t, u(t), v(t))dt$.*

    a. *Let $\mathcal{U}(t) = \{u(t) \mid u \in \mathcal{U}\}$, $\mathcal{V}(t) = \{v(t) \mid v \in \mathcal{V}\}$. If $\bar{u}(\cdot) \in \mathcal{U}$ and $\bar{v}(\cdot) \in \mathcal{V}$ are such that $(\bar{u}(t), \bar{v}(t))$ is a saddle point of $\gamma(t, \cdot, \cdot)$ over $\mathcal{U}(t) \times \mathcal{V}(t)$ for almost all $t \in [a, b]$, then $(\bar{u}(\cdot), \bar{v}(\cdot))$ is a saddle point for $\Gamma(\cdot, \cdot)$ over $\mathcal{U} \times \mathcal{V}$.*

Assume additionally that $\mathcal{U}$ and $\mathcal{V}$ are decomposable, and that $\gamma(\cdot, \cdot, v(\cdot))$ and $-\gamma(\cdot, u(\cdot), \cdot)$ are normal integrands for any $u(\cdot) \in \mathcal{U}$, $v(\cdot) \in \mathcal{V}$.

    b. *If $(\bar{u}(\cdot), \bar{v}(\cdot))$ is a saddle point for $\Gamma(\cdot, \cdot)$ over $\mathcal{U} \times \mathcal{V}$, and the saddle value is finite, then $(\bar{u}(t), \bar{v}(t))$ are a saddle point for $\gamma(t, \cdot, \cdot)$ over $\mathbb{R}^n \times \mathbb{R}^m$ for almost all $t \in [a, b]$.*

Proof. Part **a** is direct. For any controls $u(\cdot) \in \mathcal{U}$, $v(\cdot) \in \mathcal{V}$ we have

$$\gamma(t, \bar{u}(t), v(t)) \leq \gamma(t, \bar{u}(t), \bar{v}(t)) \leq \gamma(t, u(t), \bar{v}(t))$$

for almost all $t \in [a, b]$, since $u(t) \in \mathcal{U}(t)$ and $v(t) \in \mathcal{V}(t)$. We integrate the above to get the claim. For part **b** we apply theorem 14.60 in Rockafellar 1997. The fact that $\bar{u}(\cdot)$ minimizes $\Gamma(\cdot, \bar{v}(\cdot))$ implies $\bar{u}(t)$ minimizes $\gamma(t, \cdot, v(t))$. Similarly, from $\bar{v}(\cdot)$ minimizing $-\Gamma(\bar{u}(\cdot), \cdot)$ — this is why we assume normality of $-\gamma(\cdot, u(\cdot), \cdot)$ — we imply that $\bar{v}(t)$ maximizes $\gamma(t, u(t), \cdot)$. $\qquad\square$

The problem of finding the saddle point of $\Phi(\tau, \xi, u(\cdot), v(\cdot))$ now reduces to finding $u(t)$ and $v(t)$ as the saddle point of

$$f(t, u, v) + d \cdot \mathcal{A}(T, t)(Bu + Cv) \tag{10}$$

Now $(u(t), v(t))$ are a saddle point of the above if and only if

$$(0, 0) \in \partial f(t, u(t), v(t)) + (B^* \mathcal{A}^*(T, t)d, C^* \mathcal{A}^*(T, t)d)$$

where $\mathcal{A}^*$ is the transpose of $\mathcal{A}$. This is equivalent to either of the expressions

$$(-B^* \mathcal{A}^*(T, t)d, -C^* \mathcal{A}^*(T, t)d) \in \partial f(t, u(t), v(t)) \tag{11}$$

$$(u(t), v(t)) \in \partial f^*(t, B^* \mathcal{A}^*(T, t)d, C^* \mathcal{A}^*(T, t)d) \tag{12}$$

The last formula gives explicit conditions on saddle controls.

Theorem 6 *Any pair of controls $(\bar{u}(\cdot), \bar{v}(\cdot)) \in \mathcal{U} \times \mathcal{V}$ satisfying (12) for almost all $t \in [\tau, T]$ is a saddle point for the game $\mathcal{P}(\tau, \xi)$. Conversely, if $\mathcal{U}$ and $\mathcal{V}$ are decomposable and if saddle controls exist, they satisfy (12)*

The theorem enables us to state conditions guaranteeing the existence of saddle controls. For example, if we know that the right side of the inclusion 12 — which is always closed — is measurable in t and nonempty almost everywhere, then we can conclude the existence of measurable controls satisfying 12. These controls automatically have to satisfy the constraints 3. If also the controls are in $\mathcal{U} \times \mathcal{V}$, the saddle point of the game exists and the saddle value is finite.

3.3 Hamiltonian analysis of the saddle game

To simplify the notation, we supress the time dependence of matrices A, B and C where it is irrelevant to the results.

Assumption 4

$$\sup_u \inf_v \{p \cdot u + q \cdot v - f(t, u, v)\} = \inf_v \sup_u \{p \cdot u + q \cdot v - f(t, u, v)\} \qquad (13)$$

and the common value, denoted by $f^*(t, u, v)$, is finite.

$f^*(t, \cdot, \cdot)$ is the conjugate saddle function to $f(t, \cdot, \cdot)$, for details see Rockafellar 1970. By 35.1 and 35.8.1 in the mentioned reference, $f^*(t, \cdot, \cdot)$ is continuous, locally Lipschitz and, by Rademacher's theorem, differentiable almost everywhere. The Hamiltonian for the convex-concave game is given by:

$$H(t, x, y) = \sup_u \inf_v \{y \cdot (Ax + Bu + Cv) - f(t, u, v)\} = y \cdot Ax + f^*(t, B^*y, C^*y) \quad (14)$$

The order of taking inf and sup can be reversed, by assumption 4. Note that $H(t, \cdot, y)$ is an affine function for any fixed t and y, but we can not conclude much about convexity properties of $H(t, x, \cdot)$. The lack of convexity prohibits us from using convex analysis subgradients to write Hamiltonian conditions, but, since $f^*(t, \cdot, \cdot)$ — so also $H(t, \cdot, \cdot)$ — is locally Lipschitz, we can use Clarke subgradient. The Hamiltonian condition is

$$\begin{bmatrix} -\dot{y}(t) \\ \dot{x}(t) \end{bmatrix} \in \partial^c H(t, x(t), y(t)) \qquad (15)$$

We can calculate $\partial^c H(t, \cdot, \cdot)$ directly from 2.5.1 in Clarke 1990.

$$\partial^c H(t, x, y) = \text{con} \left\{ (p, q) \mid \exists (x^\nu, y^\nu) \to (x, y) \text{ with } \nabla H(t, x^\nu, y^\nu) \to (p, q) \right\}$$

$$\nabla H(t, x^\nu, y^\nu) = (A^*y^\nu, Ax^\nu + B\nabla_1 f^*(t, B^*y^\nu, C^*y^\nu) + C\nabla_2 f^*(t, B^*y^\nu, C^*y^\nu))$$

The expressions A^*y^ν and Ax^ν converge to A^*y and Ax whenever $(x^\nu, y^\nu) \to (x, y)$, so we can look only at those (p, q) for which $p = A^*y$ and q such that there exists $y^\nu \to y$ with

$$Ax + B\nabla_1 f^*(t, B^*y^\nu, C^*y^\nu) + C\nabla_2 f^*(t, B^*y^\nu, C^*y^\nu) \to q$$

These q's are included in the set of q' for which there exists $(y_1^\nu, y_2^\nu) \to (B^*y, C^*y)$ with

$$Ax + B\nabla_1 f^*(t, y_1^\nu, y_2^\nu) + C\nabla_2 f^*(t, y_1^\nu, y_2^\nu) \to q'$$

which is exactly the condition in the definition of $\partial^c f^*(t, \cdot, \cdot)$. Therefore

$$\partial^c H(t, x, y) \subset \begin{bmatrix} A^*y \\ Ax + \begin{pmatrix} B \\ C \end{pmatrix} \partial^c f^*(t, B^*y, C^*y) \end{bmatrix} \qquad (16)$$

The Hamiltonian condition (15) now reduces to

$$-\dot{y}(t) = A^*y(t) \qquad (17)$$

$$\dot{x}(t) \in Ax(t) + \begin{pmatrix} B \\ C \end{pmatrix} \partial^c f^*(t, B^*y(t), C^*y(t)) \tag{18}$$

We concentrate on the second equation. Clarkes subgradient $\partial^c f^*(t, \cdot, \cdot)$ is equal to $\mathrm{con}\, \partial f^*(t, \cdot, \cdot)$ for the subgradient in the sense of Rockafellar 1997. For fixed p and q, $f^*(t, p, q)$ is measurable in t. This follows from the fact that conjugacy preserves measurability in time — applying this twice to $f(\cdot, u, v)$ gives us measurability of $f^*(\cdot, p, q)$. This, and continuity of $f^*(t, \cdot, \cdot)$ implies that $f^*(\cdot, \cdot, \cdot)$ is a Caratheodory integrand. By 14.56, $\partial f^*(t, B^T(t)y(t), C^T(t)y(t))$ is measurable, so by 14.12, also $\partial^c f^*(t, B^T(t)y(t), C^T(t)y(t))$. The mapping $w \to E(t)w$ is a Caratheodory mapping — here $E(t) = \begin{pmatrix} B(t) \\ C(t) \end{pmatrix}$. For any $t \in [\tau, T]$, there exists a $w \in \partial^c f^*(t, B^T(t)y(t), C^T(t)y(t))$ such that $E(t)w \in \dot{x}(t) - A(t)x(t)$, and the mapping on the right side of the inclusion is single, so closed, valued and measurable. This is the setting of theorem 14.16. Therefore, there exist a measurable $w(\cdot) \in \partial^c f^*(t, B^T(t)y(t), C^T(t)y(t))$ on $[\tau, T]$ such that

$$\dot{x}(t) = A(t)x(t) + E(t)w(t)$$

By 6.1 in Rockafellar 1998, Clarke subgradient of a saddle function agrees with the convex-concave subgradient $(\partial_1 \times \partial_2) f^*(t, \cdot, \cdot)$. We can then write $w(t)$ as $(u(t), v(t))$, where $(u(t), v(t)) \in \partial_1 \times \partial_2 f^*(t, \cdot, \cdot)$.

Theorem 7 *Hamiltonian equation* (15) *implies that*

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + C(t)v(t)$$

for some controls $u(\cdot)$ *and* $v(\cdot)$ *satisfying*

$$u(t) \in \partial_1 f^*(t, B^*(t)y(t), C^*(t)y(t)) \qquad v(t) \in \partial_2 f^*(t, B^*(t)y(t), C^*(t)y(t))$$

Corollary 8 *Hamiltonian equation* (15) *paired with the transversality condition* $y(t) = -d$ *implies that* $x(\cdot)$ *is generated by saddle controls satisfying* 12.

Arguments given in the proof of theorem 7 show in particuar that the right side of 12 is measurable, so, under assumption 4, measurable controls satisfying condition 12 always exist. Various conditions can be now stated to guarantee that these controls would actually be in feasible sets $\mathcal{U}$ and $\mathcal{V}$. Epi/hypo-continuity of f^* in t (in view of finiteness, it is equivalent to continuity) implies the continuity of $\mathrm{gph}\, \partial f^*(t, \cdot, \cdot)$ in t, so also local boundedness of the right side of 12. This guarantees that the controls satisfying this saddle condition will actually be in L^∞.

Corollary 9 *If* f^* *is continuous, the game* $\mathcal{P}(\tau, \xi)$ *has saddle points and a finite saddle value.*

Continuity assumption covers the cases of time independent f, linear-quadratic cost function with coefficients continuous in time, and, in particular, the example discussed in section 3.5.

We now turn to study the Hamiltonian equations for problems $\mathcal{P}_1(\bar{v}(\cdot))$ and $\mathcal{P}_2(\bar{u}(\cdot))$. Hamiltonians are given by

$$
\begin{aligned}
H_1^{\bar{v}}(t,x,y) &= \sup_u\{y\cdot(Ax+Bu+C\bar{v}(t))-f(t,u,\bar{v}(t))\} \qquad (19)\\
&= y\cdot Ax+y\cdot C\bar{v}(t)+f^{*1}(t,B^*y,\bar{v}(t))
\end{aligned}
$$

$$
\begin{aligned}
H_2^{\bar{u}}(t,x,y) &= \inf_v\{y\cdot(Ax+B\bar{u}(t)+Cv)-f(t,\bar{u}(t),v)\} \qquad (20)\\
&= y\cdot Ax+y\cdot B\bar{u}(t)+f^{*2}(t,\bar{u}(t),C^*y)
\end{aligned}
$$

where

$$
f^{*1}(t,p,v)=\sup_u\{p\cdot u-f(t,u,v)\}
$$

$$
f^{*2}(t,u,q)=\inf_v\{q\cdot v-f(t,u,v)\}
$$

are the partial convex and concave conjugates of the saddle function $f(t,\cdot,\cdot)$. Both $H_1(t,\cdot,y)$ and $H_2(t,\cdot,y)$ are affine, with $H_1(t,x,\cdot)$ convex and $H_2(t,x,\cdot)$ concave. In particular, H_1 is concave-convex for fixed t, and H_2 is convex-concave. We now write the Hamiltonian and transversality conditions for $\mathcal{P}_1(\bar{v}(\cdot))$ and $\mathcal{P}_2(\bar{u}(\cdot))$. The chain rule has the following form:

$$
\partial_1\left(f^{*1}(t,B^*y,q)\right)(\cdot)\supset B\partial_1 f^{*1}(t,B^*\cdot,q)
$$

$$
\partial_2\left(f^{*2}(t,p,C^*y)\right)(\cdot)\supset C\partial_2 f^{*2}(t,p,C^*\cdot)
$$

with equalities holding when the range of B (or C) contains a point of the relative interior of dom $f^{*1}(t,\cdot,q)$ (or dom $f^{*2}(t,p,\cdot)$). This holds in particular when either $B(t)$ is of full rank or $f^{*1}(t,\cdot,v)$ is finite for any $v\in Q(t)$, and, symmetrically, either $C(t)$ is of full rank or $f^{*2}(t,u,\cdot)$ is finite for any $u\in P(t)$. Because of the inclusions, the following conditions are actually stronger than needed: For $\mathcal{P}_1(v(\cdot))$, we have

$$
\dot{y}(t)=A^*y(t),\qquad \dot{x}(t)\in Ax(t)+Cv(t)+B\partial_1 f^{*1}(t,B^*y(t),v(t)) \qquad (21)
$$

with $y(\tau)=-d$, and for $\mathcal{P}_2 u(\cdot)$ we get

$$
\dot{z}(t)=A^*y(t),\qquad \dot{x}(t)\in Ax(t)+Bu(t)+C\partial_2 f^{*2}(t,u(t),-C^*z(t)) \qquad (22)
$$

with $z(\tau)=d$. The above conditions, because of the linear dynamics, are equivalent to: for $\mathcal{P}_1(v(\cdot))$ (with $y(t)=-\mathcal{A}^*(T,t)d$):

$$
u(t)\in\partial_1 f^{*1}(t,B^*y(t),v(t))
$$

which is equivalent to

$$
B^*y(t)\in\partial_1 f(t,u(t),v(t))
$$

and the Hamiltonian condition for $\mathcal{P}_2 u(\cdot)$), for $z(t)=A^*(T,t)d$:

$$
v(t)\in\partial_2 f^{*2}(t,u(t),-C^*z(t))
$$

which is equivalent to

$$
-C^*z(t)\in\partial_2 f(t,u(t),v(t))
$$

The two above inclusions for $B^*y(t)$ and $C^*y(t)$ are equivalent to (11), since $z(t)=-y(t)$, so also to (12). We have shown:

Lemma 10 *Controls $\bar{u}(\cdot)$ and $\bar{v}(\cdot)$ satisfy (21), (22) — the (stronger) Hamiltonian conditions for $\mathcal{P}_1(\tau,\xi,\bar{v}(\cdot))$ and $\mathcal{P}_2(\tau,\xi,\bar{u}(\cdot))$ — if and only if they satisfy (12) - the saddle point condition for $\mathcal{P}(\tau,\xi)$.*

3.4 Hamilton-Jacobi properties of the value funtion.

The previous section shows that the convex-concave game has a saddle value

$$W(\tau, \xi) = \int_\tau^T f(t, \bar{u}(t), \bar{v}(t))dt + d^*\bar{x}(T)$$

for any initial condition $x(\tau) = \xi$. An arc $\bar{x}(\cdot)$ is given by

$$\bar{x}(t) = \mathcal{A}(t, \tau)\xi + \int_\tau^t \mathcal{A}(t, s)(B\bar{u}(s) + C\bar{v}(s))ds$$

We can easily check that $W(\tau, \xi)$ is differentiable, and

$$W_\tau(\tau, \xi) = -f(\tau, \bar{u}(\tau), \bar{v}(\tau)) - d \cdot \mathcal{A}(T, \tau)A\xi - d \cdot \mathcal{A}(T, \tau)B\bar{u}(\tau) - d \cdot \mathcal{A}(T, \tau)C\bar{v}(\tau)$$

$$W_\xi(\tau, \xi) = \mathcal{A}^*(T, \tau)d$$

Lemma 11 *The value function in the convex-concave game satisfies the Hamilton-Jacobi equation:*
$$-W_\tau(\tau, \xi) + H(\tau, \xi, -W_\xi(\tau, \xi)) = 0$$

Proof.

$$H(\tau, \xi, -W_\xi(\tau, \xi)) = -d \cdot \mathcal{A}(T, \tau)A\xi + f^*(\tau, -B^*\mathcal{A}^*(T, \tau)d, -C^*\mathcal{A}^*(T, \tau)d)$$

so

$$-W_\tau(\tau, \xi) + H(\tau, \xi, -W_\xi(\tau, \xi))$$
$$= f(\tau, \bar{u}(\tau), \bar{v}(\tau)) + d \cdot \mathcal{A}(T, \tau)B\bar{u}(\tau)$$
$$+d \cdot \mathcal{A}(T, \tau)C\bar{v}(\tau) + f^*(\tau, -B^*\mathcal{A}^T(T, \tau)d, -C^*\mathcal{A}^*(T, \tau)d)$$

Now

$$f(\tau, \bar{u}(\tau), \bar{v}(\tau)) + d \cdot \mathcal{A}(T, \tau)B\bar{u}(\tau) = -f^{*1}(\tau, -B^*\mathcal{A}^T(T, \tau)d, \bar{v}(\tau))$$

since

$$\bar{u}(\tau) \in \partial_1 f^{*1}(t, -B^*\mathcal{A}(T, \tau)d, \bar{v}(\tau))$$

and

$$f^*(\tau, -B^*\mathcal{A}^*(T, \tau)d, -C^*\mathcal{A}^*(T, \tau)d) + d^*\mathcal{A}(T, \tau)C\bar{v}(\tau)$$
$$= -f^{*1}(\tau, -B^*\mathcal{A}^*(T, \tau)d, \bar{v}(\tau))$$

since

$$\bar{v}(\tau) \in \partial_2 f^*(t, -B^*\mathcal{A}(T, \tau)d, -C^*\mathcal{A}(T, \tau)d)$$

we see that

$$-W_\tau(\tau, \xi) + H(\tau, \xi, -W_\xi(\tau, \xi)) = 0$$

■

Let us look at the general case for a while. Fix some strategies $\bar{u}(\cdot)$ and $\bar{v}(\cdot)$. Define

the value functions $W_1^{\bar{v}}(\tau,\xi)$ and $W_2^{\bar{u}}(\tau,\xi)$ as the optimal values in problems $\mathcal{P}_1(\bar{v}(\cdot))$ and $\mathcal{P}_2(\bar{u}(\cdot))$. Note that then we have the following inequality

$$W_1^{\bar{v}}(\tau,\xi) \leq W(\tau,\xi) \leq W_2^{\bar{u}}(\tau,\xi)$$

If $\bar{u}(\cdot)$ and $\bar{v}(\cdot)$ happen to be the saddle controls for the problem defining $W(\tau',\xi')$, we also have

$$W_1^{\bar{v}}(\tau',\xi') = W(\tau',\xi') = W_2^{\bar{u}}(\tau',\xi')$$

We see that the functions $W_1^{\bar{v}}(\cdot,\cdot)$ and $W_2^{\bar{u}}(\cdot,\cdot)$ control the growth of $W(\cdot,\cdot)$ at (τ',ξ'). There seems to be a possibility of exploring this in the Hamilton-Jacobi analysis of W.

Lemma 12 *Assume that $\bar{u}(\cdot)$ and $\bar{v}(\cdot)$ are a saddle point for the problem defining $W(\tau',\xi')$. Also assume that $W_1^{\bar{v}}(\cdot,\cdot)$ and $W_2^{\bar{u}}(\cdot,\cdot)$ are viscosity solutions of the Hamilton-Jacobi equations for H_1 and H_2, respectively, at (τ',ξ'). Then $W(\cdot,\cdot)$ is a viscosity solution of the Hamilton-Jacobi equation for H at (τ',ξ').*

Proof.

$$\partial_+ W(\tau',\xi') \subset \partial_+ W_1^{\bar{v}}(\tau',\xi')$$

$$\partial_- W(\tau',\xi') \subset \partial_- W_2^{\bar{u}}(\tau',\xi')$$

Since $W_1^{\bar{v}}$ is a viscosity solution of the Hamilton-Jacobi equation for $H_1^{\bar{v}}$, we have

$$-\alpha + H_1(\tau,\xi',-\beta) \leq 0$$

for any $(\alpha,\beta) \in \partial_+ W(\tau',\xi')$ and by the definitions of H and H_1

$$-\alpha + H(\tau,\xi',-\beta) \leq 0$$

Symmetrically we show that for any $(\gamma,\delta) \in \partial_- W_-(\tau',\xi')$

$$-\gamma + H(\tau,\xi',-\delta) \geq 0$$

The above two inequalities are equivalent to the thesis. Note that in a similar fashion, if $W_1^{\bar{v}}$ and $W_2^{\bar{u}}$ are differentiable and solve corresponding HJ equations, we can conclude that W is differentiable and solves HJ equation. $\qquad\square$

3.5 A particular convex-concave game.

We now present a case of a convex-concave game where the cost function is separable in controls u and v. For simplicity of presentation we forget about the possible time dependence of the cost function. We let

$$\Phi(u(\cdot),v(\cdot)) = \int_0^\tau g(u(t)) - h(v(t))dt + d^T x(\tau)$$

where $g : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ are strictly convex coercive functions. Then

$$H(x,y) = y \cdot Ax + g^*(B^*y) + h^*(C^*y)$$

where g^* is the convex conjugate of g given by

$$g^*(z) = \sup_u \{z \cdot Tu - g(u)\}$$

and h^* is the concave conjugate of $-h$ given by

$$h^*(z) = \inf_v \{z \cdot Tv + h(v)\}.$$

The order of taking sup and inf in the definition of the Hamiltonian is clearly not important. Coercivity of g and h is equivalent to finiteness of g^* and h^*, so, in particular, the Hamiltonian is finite. Assumption of strict convexity of g and h is now equivalent to differentiability of g^* and h^*. The saddle controls $\bar{u}$ and $\bar{v}$ are

$$\bar{u}(t) = \nabla g^*(B^* \bar{y}(t))$$

$$\bar{v}(t) = \nabla h^*(C^* \bar{y}(t))$$

where

$$\bar{y}(t) = -e^{A^*(T-t)} d$$

Note that not only do the controls $\bar{u}(\cdot)$ and $\bar{v}(\cdot)$ not depend on either the initial conditions or current position of the system, controls for each player are totally independent of the data of the other player. That is, Player One needs not to know the control matrix C.

3.6 Initial position adjustment

We now add an extra feature to our game. We assume that given the position ξ players can change it momentarily as the game starts, with the use of controls u_τ and v_τ. To be precise, the initial condition (2) is now given by

$$x(\tau) = g_\tau(\xi, u_\tau, v_\tau) \tag{23}$$

for some function g_τ. The cost of this adjustment is given by $\phi(u_\tau, v_\tau)$ for some function ϕ, and so the cost for players is

$$\phi(u_\tau, v_\tau) + \Phi(\tau, x(\tau), u(\cdot), v(\cdot)) \tag{24}$$

where Φ is given by (3). We look for the saddle point and value of (24) among pairs of admissible controls $((u_\tau, u), (v_\tau, v))$. The initial adjustment cost $\phi(u_\tau, v_\tau)$ is independent of controls $u(\cdot)$ and $v(\cdot)$, so we can concentrate on saddle points and value of

$$\phi(u_\tau, v_\tau) + W(\tau, g_0(\xi, u_\tau, v_\tau)) \tag{25}$$

where $W(\cdot, \cdot)$ is the value function for the fixed initial point game. Assume we are back in the saddle setting, the initial cost $\phi(\cdot, \cdot)$ is convex-concave, and the "initial dynamics" are linear:

$$x(\tau) = \xi + B_\tau u_\tau + C_\tau v_\tau$$

$W(\tau, \cdot)$ is affine, so $(u_\tau, v_\tau) \to W(\tau, A_\tau u_\tau + B_\tau v_\tau)$ is also affine — in particular, convex-concave. Then (25) is a saddle function, and (u_τ, v_τ) are a saddle point if and only if

$$-\left(B_\tau^* \mathcal{A}(T, \tau) d, C_\tau^* \mathcal{A}(T, \tau) d\right) \in \partial \phi(u_\tau, v_\tau)$$

which is equivalent to

$$(u_\tau, v_\tau) \in \partial \phi^* \left(-B_\tau^* \mathcal{A}(T, \tau) d, -C_\tau^* \mathcal{A}^*(T, \tau) d \right) \tag{26}$$

Note the similarity of this condition to 12, and the presence of the "adjoint arc" $-\mathcal{A}(T, t)d$.

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