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International Institute for Applied Systems Analysis • A-2361 Laxenburg • Austria
Tel: +43 2236 807 • Fax: +43 2236 71313 • E-mail: info@iiasa.ac.at • Web: www.iiasa.ac.at

INTERIM REPORT IR-98-061 / September

Insurer's Portfolios of Risks: Approximating Infinite Horizon Stochastic Dynamic Optimization Problems

Lisa A. Korf (korf@math.ucdavis.edu)

Approved by
Joanne Linnerooth-Bayer (bayer@iiasa.ac.at)
Co-Leader, *Risk, Modeling and Policy Project*

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Abstract

Many optimal portfolio problems, due to uncertainties with rare occurrences and the need to bypass so-called "end of the world effects" require considering an infinite time horizon. Among these in particular are insurer's portfolios which may include catastrophic risks such as earthquakes, floods, etc. This paper sets up an approximation framework, and obtains bounds for a class of infinite horizon stochastic dynamic optimization problems with discounted cost criterion, in the framework of stochastic programming. The resulting framework is applied to an insurer's portfolio of risk contracts.

Keywords: infinite-horizon, stochastic programming, epi-convergence, portfolio selection, catastrophic risk, utility, premium, claim reserves

Acknowledgements

I appreciate the helpful comments received from Yuri Ermoliev, Tatiana Ermolieva and Georg Pflug. In addition, I am grateful to Roger Wets for introducing the topic to me, and for sharing his ideas. Thanks also to Joanne Linnerooth-Bayer, the IIASA staff and my fellow YSSP researchers for all of their support.

About the Author

Lisa A. Korf is receiving her Ph.D. in Mathematics at the University of California, Davis. She is a participant in the 1998 Young Scientists Summer Program at IIASA.

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1.1 Introduction

Optimal portfolios of insurers, in particular those that include rare events such as catastrophic risks, provide one of many examples of optimization problems in the presence of uncertainty, whose dynamic nature requires considering many, even an infinite number of time periods in order to have an accurate description of the problem. Multistage stochastic programs are well-suited for such problems, in particular when one needs to include various constraints (e.g. nonnegativity, limits on investments, etc.). The constraints, dynamics and uncertainty combine to make complex and ungainly problems. Stochastic programming methods which rely heavily on convexity and duality, problem structure, and decomposition techniques provide a possible means of approaching and eventually solving them.

Here we develop an approximation framework for *stationary infinite horizon stochastic dynamic optimation problems with discounted costs.* Stationarity means that the solution is independent of shifts in time, i.e. an action optimal in the present state will also be optimal in the same state at a future time period. In the insurance setting, gradual changes in the global environment over time (e.g. global warming) may render this an invalid assumption for the portfolio that depends on events linked to such changes. A further treatise on infinite horizon problems that are not necesarily stationary would therefore be of interest, but we restrict our attention to the stationary case here since it covers many problems that have not before been handled in this setting, and can provide the basis for further study.

"Infinite horizon" refers to a problem with an infinite number of stages, or time periods. This is an important consideration for the insurer who wants to optimize a portfolio of risks (contracts, regions, classes or...) for two reasons. One reason is that certain risks, such as earthquakes, floods, volcanic eruptions, etc. are extremely rare events that may occur only once in a few thousand years but with catastrophic effects. An extended time horizon is therefore essential in order to allow the magnitude of such events to affect the insurer's long-range objectives (e.g. of avoiding ruin) and hence the optimal portfolio appropriately.

The second reason an infinite time horizon is crucial in the description of the problem is to deal with what is known in the economics literature as "end of the world effects." Intuitively, a portfolio optimization problem with a finite time horizon will have a solution that uses up all resources in the final period. This has the effect of putting undue and unrealistic emphasis on the evaluation of the final stage of the problem. Anyway, a final period is often not in the insurer's interests, who likely wants to keep his company running indefinitely. The description of the problem via an infinite horizon circumvents this problem, as well as that of rare events, by taking into account the extended future.

In addition to the future, one must also model the underlying uncertainty of these problems. In the case of an insurer insuring risks and possibly borrowing and making investments, uncertainty comes in the form of claims and interest rates. Modeling these stochastic processes accurately is a formidable task, that warrants a separate development, cf. [**2**], [**3**], [**6**] and [**7**]. In the presentation here, it is assumed that one has the ability to simulate or approximate the claims and interest rates environments in a manner consistent with the problem, as in the above referenced situations.

The problems to be considered have an underlying natural dynamic structure of alternating states and decisions. This means that at the present state, a decision is made, then the world (uncertainty) is observed. A new state is obtained according to an equation governing the dynamics, from the previous state, the decision, and the world observations, and the process repeats. In terms of optimizing portfolios, the state might keep track of the current amounts insured, invested, borrowed, etc., while the decision at each time period might correspond to changes in each of the above sectors that adjust the system to optimize the objective.

Generally speaking this objective may be expressed as an expectation of an infinite sum of "utility functions" that is progressively discounted at each successive time period to take into account the greater importance of the "here and now" decision and the decreasing importance of future decisions. An important consideration is that the utility function may include more than one qualitative objective. In the insurer's portfolio problem, we will consider a utility function that maximizes the insurer's financial strength (measured in claim reserves).

Section 2 opens with a step-by-step development of the problem of optimizing a portfolio of catastrophic risk regions for an insurer. A discrete time formulation of the problem as a stationary infinite horizon stochastic dynamic optimization problem with discounted costs is presented. The insurer's attitude toward risk (utility), and constraints can naturally be modeled with certain quite flexible piecewise linear-quadratic monitoring functions. Such infinite horizon stochastic optimization problems arise naturally in many economic and financial planning as well as other important applications. In order to actually solve these problems, one must somehow approximate them by computationally more tractable ones. Finite horizon approximations are proposed which are validated in the remainder of the paper. The focus is on ways of analyzing and approximating a general class of convex *infinite-horizon stochastic dynamic optimization problems with discounted costs.*

Section 3 introduces a recursively defined *value function* associated with such a problem. This differs from the value function of the stochastic control literature [**1, 12**] in that here, infinite values are permissible (and identifiable with constraints). In addition, the eventual goal is not the pointwise evaluation of this value function as it would be in the control setting. Instead, the focus will be to approximate the value function so that it may serve as an "end term" for a finite-horizon approximation of the original problem that may eventually be solved using techniques of stochastic programming. Existence and optimality results are obtained that relate the value function to the original problem.

The major contributions of the fourth section are the approximation theorems. Here an iterative procedure is set up, and it is shown that one may approximate the value function via these iterations to obtain approximations including lower bounds that converge almost monotonically (see §4.1) to the value function. The convergence is shown to hold in the sense of epi-convergence, which in turn ensures the convergence of solutions to a solution of the original problem. A fixed point theorem is obtained when the domain of the value function is known a priori. Properties of the value function (e.g. convexity, lower semicontinuity, etc.) are also derived.

Section 5 is devoted to various finite-horizon approximations to the infinite-horizon problems considered. The focus is on *bounds*. The first technique introduces rough lower and upper bounds that do not take the extended future into consideration, but then proposes using the approximation theorems in §1.3 to obtain better and better bounds (in the epigraphical sense) that progressively take the future into account. The second technique extends the approximation methods of Grinold [**10**], and Flåm and Wets [**8, 9**] which take the future into account via taking convex combinations and averaging.

Section 6 is devoted to a particular class of infinite-horizon stochastic dynamic optimization problems in which the cost function is *piecewise linear-quadratic.* Such problems are quite flexible, yet have a highly exploitable structure. The main result here shows that approximation of an infinite-horizon problem with piecewise linear-quadratic costs via the approximation theorems preserves the piecewise linear-quadraticity of the problem. The theoretical implication is that one can keep the number of stages of a problem low, and still obtain *explicit* bounds as close as one would like to the original problem, though the end term may become increasingly more difficult to compute. All of the results in this paper are applicable to various problems, in particular to the problem we focus on here, of optimizing an insurer's portfolio of catastrophic risks.

2. Optimizing an insurer's portfolio of catastrophic risk regions

We consider a problem of an insurer who insures catastrophic risks in various geographic locations. The problem is one of optimizing a portfolio to determine the optimal amount of each region to insure based on the insurer's objectives. An expected utility approach over an infinite time horizon is taken, with the goal of maximizing financial strength. In addition to the risk contracts, the insurer may make investments in risky stocks and a riskless bond, and borrow at a fixed rate.

Let's suppose there are G regions to insure. To each region corresponds a fixed premium p^i, $i = 1, \ldots, G$. The claims rate process, $\xi_t = (\,\xi_t^1 \quad \ldots \quad \xi_t^G\,)^*$ describes the aggregate claims per period $t = 1, 2, \ldots$ in each of the regions $1, \ldots, G$. This will in practice be given by a simulation of the catastrophes in each region, which takes into account dependencies between geographic locations.

In addition to risk contracts, the insurer may also invest in S stocks. The return rate of the stocks is given by a random vector $\zeta_t = (\,\zeta_t^1 \quad \ldots \quad \zeta_t^S\,)^*$, which describes the gain (or loss) in stock price per period $t = 1, 2, \ldots$ for each of the stocks $1, \ldots, S$. The investor may also invest in riskless assets (bonds) at a fixed rate $r > 0$, and finance transactions at an interest rate $R \geq r$.

The insurer's objective is to maximize the total expected discounted utility of claim reserves over an infinite time horizon. The utility function $U : I\!\!R \to \overline{I\!\!R}$ should be nondecreasing, concave, continuous on its domain, $I\!\!R_+$, with $\lim_{c\to\infty} U'(c) = 0$. This indicates a preference for higher claim reserves, attitude towards risk, no jumps in utility, and that the importance of having more claim reserves decreases to zero as the claim reserves get arbitrarily high.

(i) *(Utility function).* Accordingly, we will use the following piecewise linear-quadratic utility function,

$$U(c) = \begin{cases} -c^2 + 2ac & \text{if } 0 \le c \le a \\ a^2 & \text{if } c > a \\ -\infty & \text{if } c < 0 \end{cases} .$$

Note that this choice is somewhat arbitrary, in that other piecewise-linear quadratic functions (for example, with more pieces) may have as easily been chosen. Also note that U is increasing, concave and continuous on its domain, and satisfies $\lim_{c\to\infty} U'(c) = 0$.

(ii) *(States, Controls, and Dynamics).* We set up the variables as follows. Let the state of the system be given by $x_t = (\, c_t \quad r_t^1 \quad \dots \quad r_t^G \quad s_t^1 \quad \dots \quad s_t^S \quad b_t \quad d_t \,)^*$, where

$$c_t = \text{ total amount of claim reserves at time } t \ ,$$

$$r_t^i = \text{ total amount (units) of region } i \text{ insured at time } t \ ,$$

$$s_t^i = \text{ total amount invested in stock } i \text{ at time } t \ ,$$

$$b_t = \text{ total amount invested in the bond at time } t \ ,$$

$$d_t = \text{ total amount borrowed (debt) at time t } .$$

Let the controls be given by

$$u_t = (\, \Delta r_t^1 \quad \dots \quad \Delta r_t^G \quad \Delta s_t^1 \quad \dots \quad \Delta s_t^S \quad \Delta b_t \quad \Delta d_t \,)^* \ ,$$

where

$$\Delta r_t^i = \text{ change in units of region } i \text{ insured at time } t \ ,$$

$$\Delta s_t^i = \text{ change in investment in stock } i \text{ at time } t \ ,$$

$$\Delta b_t = \text{ change in investment in the bond at time } t \ ,$$

$$\Delta d_t = \text{ change in the amount borrowed (debt) at time } t \ .$$

Then the dynamics become

$$x_t = A(\xi_t, \zeta_t) x_{t-1} + B u_t \quad P\text{-a.s.},$$

$$x_0 = x,$$

for $t = 1, 2, \ldots$, where

$$A(\xi,\zeta) = \begin{pmatrix} 1 & p^1-\xi_t^1 & \cdots & p^G-\xi_t^G & & & & & \\ & 1 & & & & & & & \\ & & \ddots & & & & & & \\ & & & 1 & & & & & \\ & & & & 1+\zeta_t^1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1+\zeta_t^S & & \\ & & & & & & & 1+r & \\ & & & & & & & & 1+R \end{pmatrix},$$

$$\text{and } B = \begin{pmatrix} & & & -1 & \cdots & -1 & -1 & 1 \\ 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix}.$$

(iii) *(Constraints).* The problem requires nonnegativity constraints for amounts (units) insured, investments, borrowing, and wealth, as well as upper bounds on the insurable units in each region. This amounts to requiring that $x_t \geq 0$, $\sum_{i=1}^{S} s_t^i + b_t - d_t + c_t \geq 0$, and $r_t^i \leq M^i$ almost surely, or in matrix form, $\bar{C}x_t + \bar{q} \geq 0$ almost surely, where $\bar{C} = \begin{pmatrix} 1 & & & & \\ & I & & & \\ & & I & & \\ & & & 1 & \\ & & & & 1 \\ 1 & 0 & e^* & 1 & -1 \\ & -I & & & \end{pmatrix}$, $\bar{q} = (0 \ \ \ldots \ \ 0 \ \ M^1 \ \ \ldots \ \ M^G)^*$, and $e = (1 \ \ \ldots \ \ 1)^*$. These constraints will be imposed as part of the objective function, which will take on the value $+\infty$ wherever the constraints are violated.

(iv) *(Objective function).* The objective is to maximize the total expected discounted utility of claim reserves, which in the discrete setting may be written as

$$\max_{u_t} E \sum_{t=1}^{\infty} \delta^{t-1} U(c_t)$$

subject to the aforementioned dynamics and constraints. To write this in the desired piecewise linear-quadratic form, we need the utility function to include the constraints and to have the form $\rho_{V,Q}(q - Cx - Du)$, where $\rho_{V,Q}(x) = \sup_{v \in V}\{x \cdot v - \frac{1}{2}v \cdot Qv\}$. Since we work in the setting of minimization, we will actually obtain the negative of the utility, plus a constant which is superfluous in the problem since it will not affect the solution. We begin by letting

$$V = \mathbb{R}_- \times \mathbb{R}_+^{2G+S+4}, \quad Q = \begin{pmatrix} \frac{1}{2} & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} -1 & 0 \cdots & 0 \\ & \bar{C} & \end{pmatrix}, \text{ and } D = 0.$$

Now, observe that

$$\begin{aligned} \rho_{V,Q}(q - Cx - Du) &= \rho_{\boldsymbol{R}_-, \frac{1}{2}}(-a + c) + \rho_{\boldsymbol{R}_+^{2G+S+4}, 0}(\bar{q} - \bar{C}x) \\ &= \rho_{\boldsymbol{R}_-, \frac{1}{2}}(-a + c) + \delta_{\boldsymbol{R}_+^{2G+S+4}}(-\bar{q} + \bar{C}x). \end{aligned}$$

The last term, $\delta_{\boldsymbol{R}_+^{2G+S+4}}(-\bar{q} + \bar{C}x)$, is the indicator function of the set $\{x \,|\, -\bar{q} + \bar{C}x \geq 0\}$, and gives all the constraints of the system, so we need only check that $\rho_{\boldsymbol{R}_-, \frac{1}{2}}(-a + c) = -U(c) +$ a constant. We compute:

$$\rho_{\boldsymbol{R}_-, \frac{1}{2}}(-a + c) = \sup_{v \in \boldsymbol{R}_-} \{-av + cv - \frac{1}{4}v^2\}.$$

The optimality conditions for this problem are

$$0 \in \nabla(-av + cv - \frac{1}{4}v^2) - N_{\boldsymbol{R}_-}(v),$$

where $N_{\boldsymbol{R}_-}(v)$ is the normal cone to $\mathbb{R}_-$ at the point v, which is given by

$$N_{\boldsymbol{R}_-}(v) = \begin{cases} \mathbb{R}^+ & \text{if } v = 0 \\ 0 & \text{if } v < 0 \end{cases}.$$

So we arrive at the conditions $v = -2a + 2c < 0$ and $v = 0 \geq a - c$, whereby

$$\begin{aligned} \rho_{\boldsymbol{R}_-, \frac{1}{2}}(-a + c) &= \begin{cases} -2ac + a^2 + c^2 & \text{if } c < a \\ 0 & \text{if } c \geq a \end{cases} \\ &= -U(c) + a^2, \end{aligned}$$

as claimed.

Hence the problem now becomes

$$\begin{aligned} &\text{minimize } E\sum_{t=1}^{\infty}\delta^{t-1}\rho_{V,Q}(q-Cx_{t-1}-Du_t)\\ &\text{subject to } x_t = A(\xi_t)x_{t-1}+Bu_t \quad P\text{-a.s. for } t=1,2,\dots\\ &\qquad x_0 = x,\\ &\qquad u_t \quad \mathcal{G}_t \text{ measurable for } t=1,2,\dots \end{aligned}$$

which is of the form $P(x)$ to be presented in the following sections.

(v) *(Finite-horizon approximations).* We are now ready to derive a finite horizon approximation to the problem. With $c(x,u)=\rho_{V,Q}(q-Cx-Du)$ as our starting point, we may apply the techniques of this chapter, and obtain the finite-horizon problem in piecewise linear-quadratic form,

$$\begin{aligned} &\text{minimize } E\sum_{t=1}^{T}\delta^{t-1}\rho_{V,Q}(q-Cx_{t-1}-Du_t)+\frac{\delta^T}{1-\delta}\rho_{V,Q}(q-Cx_T-Du_{T+1})\\ &\text{subject to } x_t = A(\xi_t)x_{t-1}+B(\xi_t)u_t+b_t\, P\text{-a.s. for } t=1,\dots,T-1\\ &\qquad \big(I-\delta EA(\xi)\big)x_T-\delta EB(\xi)u_{T+1}\\ &\qquad\quad = (1-\delta)\big(A(\xi_T)x_{T-1}+B(\xi_T)u_T+b(\xi_T)\big)+\delta Eb(\xi)\\ &\qquad u_t \quad \mathcal{G}_t \text{ measurable for } t=1,\dots,T, \end{aligned}$$

from the results of §5.2. An alternative, if the assumption in (4) is satisfied, relies on the approximation theorems to obtain

$$\begin{aligned} &\text{minimize } E\sum_{t=1}^{T}\delta^{t-1}\rho_{V,Q}(q-Cx_{t-1}-Du_t)+E\delta^T\rho_{V_T,Q_T}(q_T-C_Tx_T)\\ &\text{subject to } x_t = A(\xi_t)x_{t-1}+B(\xi_t)u_t \quad P\text{-a.s. for } t=1,\dots,T,\\ &\qquad x_0 = x,\\ &\qquad u_t \quad \mathcal{G}_t \text{ measurable for } t=1,\dots,T. \end{aligned}$$

where $\rho_{V_T,Q_T}(q_T-C_Tx_T)$ is derived from Theorem 6.4.

Problems of this form are highly decomposable, and therefore amenable to highly parallelizable stochastic programming techniques, c.f. [**14**], [**13**], [**11**]. We now investigate the details and justification for approximating the problem in these two ways.

3. The value function

We work in the following setting. Let $c : I\!R^s \times I\!R^n \to \overline{I\!R}$ be a convex, proper ($\neq -\infty$, $\not\equiv +\infty$), lower semicontinuous (lsc) function, bounded on its domain (i.e. where it is finite-valued), $\delta \in (0,1)$ a discount factor, $(\Omega, \mathcal{F}, P)$ a probability space, $\xi : \Omega \to \Xi \subset I\!R^d$ a random vector and ξ_t, $t \in I\!N$, i.i.d. copies of ξ that represent a sequence of observations. Let $\mathcal{G}_t = \sigma\text{-}(\xi_1, \ldots, \xi_{t-1})$, i.e. the σ-field generated by the first $t-1$ observations. Then the *stationary infinite-horizon stochastic dynamic optimization problem with discounted cost* is given by

$$\begin{aligned}
&\text{minimize } E\sum_{t=1}^{\infty} \delta^{t-1} c(x_{t-1}, u_t) \\
&\text{subject to } x_t = A(\xi_t)x_{t-1} + B(\xi_t)u_t + b(\xi_t) \quad P\text{-a.s. for } t = 1, 2, \ldots \\
&\qquad x_0 = x \\
&\qquad u_t \quad \mathcal{G}_t \text{ measurable for } t = 1, 2, \ldots
\end{aligned} \qquad P(x)$$

Here $E\sum_{t=1}^{\infty} \delta^{t-1} c(x_{t-1}, u_t)$ is understood to mean $\lim_{T\to\infty} E\sum_{t=1}^{T} \delta^{t-1} c(x_{t-1}, u_t)$, which will always exist (possibly $= +\infty$) by the assumption that c is bounded on its domain, and the expectation is taken with respect to the sequence space, $(\Omega^\infty, \mathcal{F}^\infty, P^\infty)$. A, B, and b are all mappings on Ξ with matrix values of appropriate dimensions, and such that $EA(\xi)$, $EB(\xi)$ and $Eb(\xi)$ all exist and are finite. We can think of the u_t's as the primary decisions, or controls, at each time period, while the x_t's keep track of the evolution of the state of the system. The x_t's may be thought of in tandem both as problem variables and as a tracking mechanism for the dynamics of the system. We let $\min P(x)$ denote the optimal value of the problem $P(x)$. Similarly, we let feas$P$ denote the set of feasible states, or the set of $x \in I\!R^s$ such that $\min P(x) < \infty$.

A solution $(u_1, u_2, \ldots)$ to $P(x)$ is *stationary* with respect to shifts in time, if for any corresponding trajectory $(x_0, x_1, \ldots)$,

$$u_t(\xi_1, \ldots, \xi_{t-1}) = u_s(\xi_1, \ldots, \xi_{s-1})$$

whenever

$$x_{t-1}(\xi_1, \ldots \xi_{t-1}) = x_{s-1}(\xi_1, \ldots, \xi_{s-1})\, P\text{-a.s.}$$

for any $s, t \in I\!N$. That stationary solutions exist when P is feasible will follow straightforwardly from the assumption that the ξ_t's are i.i.d. and because c does not depend on time. An *optimal policy* for $P(\cdot)$ is then a function u :feas$P \to I\!R^n$ such that any sequence $(u_1, u_2, \ldots)$ defined by

$$\begin{aligned} x_0 &= x, & u_1 &= u(x_0) \\ x_t &= A(\xi_t)x_{t-1} + B(\xi_t)u_t + b(\xi_t) \quad P\text{-a.s.} & u_{t+1} &= u(x_t) \end{aligned} \tag{1}$$

for any such trajectory $(x_0, x_1, \ldots)$, solves $P(x)$ for every $x \in I\!R^s$. Note that such a solution $(u_1, u_2, \ldots)$ is stationary.

Now let us consider the following recursively defined function,

$$Q(x) = \inf_u \left\{ c(x,u) + \delta E Q\big(A(\xi)x + B(\xi)u + b(\xi)\big) \right\}.$$

Here the expectation is taken with respect to $(\Xi, \mathcal{F}, P)$. This looks similar to the "value function" of the optimal control literature, c.f. [**1, 12**], the primary distinction being that Q has possibly infinite values. Note also that no smoothness assumptions have been imposed on c. Our first goal is to set up the correspondence between P and Q, in the process verifying the existence of Q.

Theorem 3.1 (existence of recursive value function)**.** *For each $x \in I\!R^s$, let $Q(x) = \min P(x)$, the value of the problem $P(x)$ at optimality (note $Q(x)$ could be $+\infty$). Then $Q(x) = \inf_u \left\{ c(x,u) + \delta E Q\big(A(\xi)x + B(\xi)u + b(\xi)\big) \right\}$.*

Proof. We can first express $\inf_u \left\{ c(x,u) + \delta E Q\big(A(\xi)x + B(\xi)u + b(\xi)\big) \right\}$ as the optimal value of the problem

$$\begin{aligned} &\text{minimize } c(x_0, u) + \delta E Q(x_1) \\ &\text{subject to } x_1 = A(\xi)x_0 + B(\xi)u + b(\xi) \quad P\text{-a.s.} \\ &\qquad x_0 = x. \end{aligned}$$

Then it suffices to show that this problem is equivalent to $P(x)$. It may again be rewritten as

$$\begin{aligned} &\text{minimize } c(x_0, u_1) + \delta E_{\xi_1} \min_{u_t, t \geq 2} \lim_{T \to \infty} E_{\xi_2, \xi_3, \ldots} \sum_{t=2}^{\infty} \delta^{t-2} c(x_{t-1}, u_t) \\ &\text{subject to } x_t = A(\xi_t)x_{t-1} + B(\xi_t)u_t + b(\xi_t)\, P\text{-a.s. for } t = 1, 2, \ldots \\ &\qquad x_0 = x \\ &\qquad u_t \quad \mathcal{G}_t \text{ measurable for } t = 1, 2, \ldots \end{aligned}$$

By a straightforward exchange of the expectation and the minimization, and the bounded convergence theorem [**4**], this problem is equivalent to $P(x)$, hence it's optimal value is $Q(x)$. □

We have established the existence of a particular recursively defined function Q which we will from now on refer to as the *value function* for P. The next theorem establishes the equivalence between optimal policies of P and functions u that "solve" Q.

Theorem 3.2 (equivalence between P and Q)**.** *u is an optimal policy for P if and only if for all $x \in$feas$P$, $u(x) \in \operatorname*{argmin}_u \big\{c(x,u) + \delta EQ\big(A(\xi)x + B(\xi)u + b(\xi)\big)\big\}$.*

Proof. Suppose first that u is an optimal policy for P. For fixed x, consider a sequence $(u_1, u_2, \ldots)$ defined by u via

$$\begin{aligned} &x_0 = x, && u_1 = u(x_0) \\ &x_t = A(\xi_t)x_{t-1} + B(\xi_t)u_t + b(\xi_t) \quad P\text{-a.s.} && u_{t+1} = u(x_t) \end{aligned}$$

Then, using the fact that Q is the value function for P, i.e. $Q(x) = \min P(x)$, we obtain,

$$\inf_u \big\{c(x,u) + \delta EQ\big(A(\xi)x + B(\xi)u + b(\xi)\big)\big\} = Q(x)$$

$$= E\sum_{t=1}^{\infty} \delta^{t-1} c(x_{t-1}, u_t) = c(x_0, u_1) + \delta EQ(x_1)$$

$$= c(x, u(x)) + \delta EQ\big(A(\xi)x + B(\xi)u(x) + b(\xi)\big),$$

whereby $u(x) \in \operatorname*{argmin}_u \big\{c(x,u) + \delta EQ\big(A(\xi)x + B(\xi)u + b(\xi)\big)\big\}$.

To proceed in the other direction, assuming now that $u(x) \in \operatorname*{argmin}_u \big\{c(x,u) + \delta EQ\big(A(\xi)x + B(\xi)u + b(\xi)\big)\big\}$, and letting $(u_1, u_2, \ldots)$ be a sequence obtained by the same identifications as above, we have that

$$\min P(x) = Q(x) = c(x_0, u_1) + \delta EQ\big(A(\xi)x_0 + B(\xi)u_1 + b(\xi)\big)$$

$$= E\sum_{t=1}^{\infty} \delta^{t-1} c(x_{t-1}, u_t),$$

whereby u is an optimal policy for P. ◻

The next theorem establishes the existence of optimal policies (and therefore stationary solutions) when P is feasible.

Theorem 3.3 (optimal policies from solutions)**.** *Suppose that for each $x \in$feas$P$, $P(x)$ has an optimal solution $(u_1^x, u_2^x, \ldots)$, with an associated trajectory $(x_0^x, x_2^x, \ldots)$. Let u :feas$P \to I\!R^n$ be defined by $u(x) = u_1^x$. Then u is an optimal policy for P.*

Proof. The proof relies on the previous development by observing that the function u minimizes

$$c\big(x, u(x)\big) + \delta EQ\big(A(\xi)x + B(\xi)u(x) + b(\xi)\big),$$

for all $x \in$feas$P$, through the fact that $u(x) = u_1^x$. Therefore u is an optimal policy for P by Theorem 3.2. □

This theorem brings up the important question of when optimal policies (or equivalently solutions) for P exist. We next address an important criterion that will guarantee that P will have a solution.

Definition 3.4 (uniform level-boundedness)**.** *A function $f : I\!R^s \times I\!R^n \to \overline{I\!R}$ with values $f(x,u)$ is level-bounded in u locally uniformly in x if for each $\bar{x} \in I\!R^s$ and $\alpha \in I\!R$ there is a neighborhood V of $\bar{x}$ along with a bounded set $B \subset I\!R^s$ such that $\big\{u \,\big|\, f(x,u) \le \alpha\big\} \subset B$ for all $x \in V$; or equivalently, there is a neighborhood V of $\bar{x}$ such that the set $\big\{(x,u) \,\big|\, x \in V,\ f(x,u) \le \alpha\big\}$ is bounded in $I\!R^s \times I\!R^n$.*

We make use of the following Theorem from [**15**].

Theorem 3.5 (parametric minimization)**.** *Consider*

$$p(x) := \inf_u f(x,u), \qquad U(x) := \operatorname{argmin}_u f(x,u),$$

in the case of a proper, lsc function $f : I\!R^s \times I\!R^n \to \overline{I\!R}$ such that $f(x,u)$ is level-bounded in u locally uniformly in x. Then the function p is proper and lsc on $I\!R^s$, and for each $x \in \operatorname{dom} p$ the set $U(x)$ is nonempty and compact, whereas $U(x) = \emptyset$ when $x \notin \operatorname{dom} p$.

Lemma 3.6 (boundedness of value function)**.** *If the cost function , $c : I\!R^s \times I\!R^n \to \overline{I\!R}$ is bounded on its domain so that $\sup_{(x,u)\in \operatorname{dom} c} |c(x,u)| \le K$, then the value function $Q : I\!R^s \to \overline{I\!R}$ is also bounded on its domain; in particular*

$$\sup_{x \in \operatorname{dom} Q} |Q(x)| \le \frac{K}{1-\delta}.$$

Proof. Using the fact that Q is the value function for P,

$$|Q(x)| = |\min P(x)| \le \sum_{t=1}^{\infty} \delta^{t-1} \sup_{(x,u)\in \operatorname{dom} c} |c(x,u)| \le \frac{K}{1-\delta},$$

which provides the desired bound. □

Theorem 3.7 (attainment of minimum)**.** *Suppose c is level bounded in u locally uniformly in x, feas$P \neq \emptyset$, and Q is lsc. Then there exists an optimal policy $u :$feas$P \to I\!R^n$.*

Proof. This applies Theorem 3.5 to the function $g : I\!R^s \times I\!R^n \to \overline{I\!R}$ defined by

$$g(x,u) = c(x,u) + \delta EQ\big(A(\xi)x + B(\xi)u + b(\xi)\big),$$

once we can show that g is uniformly level-bounded, lsc and proper. The lower semicontinuity comes out of that of c and Q via Fatou's Lemma. The properness comes out of the observation that for any u such that $c(x,u) < \infty$, $\operatorname{dom} c(\cdot,u) \supset \operatorname{dom} Q =$feas$P$. For the uniform level-boundedness, fix $\bar{x} \in I\!R^s$, $\alpha \in I\!R$, and let V be a neighborhood of $\bar{x}$, B a bounded set, such that

$$\big\{u \,\big|\, c(x,u) \le \alpha + \frac{K}{1-\delta}\big\} \subset B \text{ for all } x \in V,$$

which is possible by the uniform level-boundedness of c. Next, observe through Lemma 3.6 that

$$\begin{aligned}\big\{u \,\big|\, g(x,u) \le \alpha\big\} &= \big\{u \,\big|\, c(x,u) + \delta EQ\big(A(\xi)x + B(\xi)u + b(\xi)\big) \le \alpha\big\} \\ &\subset \big\{u \,\big|\, c(x,u) \le \alpha + \frac{K}{1-\delta}\big\} \\ &\subset B.\end{aligned}$$

We have shown that g is lsc, proper, and uniformly level-bounded. Hence it satisfies the assumptions of Theorem 3.5, which implies that an optimal policy for P exists by the fact that $\operatorname*{argmin}_u g(x,u)$ is nonempty (and compact) for each x in $\operatorname{dom} Q$. □

4. Approximation theorems

Now that we have established the existence of the value function Q and its relation to P, we may proceed with approximation theorems for Q. In particular, our interest is in obtaining approximations and lower and upper bounds for Q to aid in the development of finite horizon approximations of P. Some properties of Q will come out of this development that are of interest by themselves.

The results of Section 3 have shown that the finite-horizon problem,

$$\begin{aligned} &\text{minimize } E\sum_{t=1}^{T}\delta^{t-1}c(x_{t-1},u_t)+E\delta^T h(x_T) \\ &\text{subject to } x_t = A(\xi_t)x_{t-1}+B(\xi_t)u_t+b(\xi_t) \quad P\text{-a.s. for } t=1,\dots,T \qquad P_h^T(x) \\ &\qquad\qquad x_0 = x \\ &\qquad\qquad u_t \quad \mathcal{G}_t \text{ measurable for } t=1,\dots,T \end{aligned}$$

is equivalent to P when $h=Q$ for any T (in particular $T=1$) in the sense that $\min P(x)$ is equal to $\min P_Q^T(x)$, and an optimal policy for P also solves P_Q^T, i.e. if u is an optimal policy for P, and we let

$$\begin{aligned} &x_0 = x, && u_1 = u(x_0) \\ &x_t = A(\xi_t)x_{t-1}+B(\xi_t)u_t+b(\xi_t) \quad P\text{-a.s.} && u_{t+1} = u(x_t) \end{aligned}$$

for $t=1,\dots,T$, then $(u_1,\dots,u_T)$ with trajectory $(x_0,\dots,x_T)$ solves $P_Q^T(x)$. So, we have an exact finite-horizon representation of P that theoretically could be amenable to computational schemes. The only problem is that we have no explicit representation for Q. If we could obtain an explicit function Q^a that approximates Q in the right sense, to obtain the problem $P_{Q^a}^T$, we would be set. This is precisely the motivation for the approximation results set forth in the remainder of this chapter.

4.1 Epi-convergence

When referring to "approximation" for a minimization problem, the appropriate notion of convergence is *epi-convergence*, which ensures the convergence of infima and solutions to those of the original problem. A sequence of functions, $f^\nu : I\!R^n \to \overline{I\!R}$ is said to *epi-converge* to $f : I\!R^n \to \overline{I\!R}$, written $f^\nu \xrightarrow{e} f$, if

i) $\forall\, x^\nu \to x$, $\liminf_\nu f^\nu(x^\nu) \geq f(x)$,
ii) $\exists\, x^\nu \to x$, $\limsup_\nu f^\nu(x^\nu) \leq f(x)$.

Epi-convergence is so-named because it corresponds to the set-convergence of the *epigraphs* of sequences of functions. A basic theorem relating epi-convergence to the convergence of infima and solutions is given below.

Theorem 4.1 (epi-convergence in minimization)**.** *Let* $\left\{f, f^\nu : I\!R^n \to \overline{I\!R} \,\middle|\, \nu \in I\!N\right\}$ *be such that* $f^\nu \xrightarrow{e} f$*. Then*

$$\limsup_\nu (\inf f^\nu) \leq \inf f.$$

Moreover, if there exists $x^k \to x$ *and a subsequence* $\left\{x^{\nu_k}\right\}_{k \in \boldsymbol{N}}$ *such that* $x^{\nu_k} \in \operatorname{argmin} f^{\nu_k}$, $k \in I\!N$*, then*

$$x \in \operatorname{argmin} f \;\; \textit{and} \;\; \inf f^{\nu_k} \to \inf f.$$

These results are well-known. For a proof one could consult [**15**]. We begin with some useful properties of epi-convergence, the proofs of which can also be found in [**15**].

Theorem 4.2 (properties of epi-limits)**.** *The following properties hold for any sequence* $\{f^\nu\}_{\nu \in \boldsymbol{N}}$ *of functions on* $I\!R^n$*.*

(a) *The functions* $\text{e-}\liminf_\nu f^\nu$ *and* $\text{e-}\limsup_\nu f^\nu$ *are lower semicontinuous, and so too is* $\text{e-}\lim_\nu f^\nu$ *when it exists.*

(b) *The functions* $\text{e-}\liminf_\nu f^\nu$ *and* $\text{e-}\limsup_\nu f^\nu$ *depend only on the sequence* $\{\operatorname{cl} f^\nu\}_{\nu \in \boldsymbol{N}}$*; thus, if* $\operatorname{cl} g^\nu = \operatorname{cl} f^\nu$ *for all* ν*, one has both* $\text{e-}\liminf_\nu g^\nu = \text{e-}\liminf_\nu f^\nu$ *and* $\text{e-}\limsup_\nu g^\nu = \text{e-}\limsup_\nu f^\nu$*.*

(c) *If the sequence* $\{f^\nu\}_{\nu \in \boldsymbol{N}}$ *is nonincreasing (*$f^\nu \geq f^{\nu+1}$*), then* $\text{e-}\lim_\nu f^\nu$ *exists and equals* $\operatorname{cl}[\inf_\nu f^\nu]$*;*

(d) *If the sequence* $\{f^\nu\}_{\nu \in \boldsymbol{N}}$ *is nondecreasing (*$f^\nu \leq f^{\nu+1}$*), then* $\text{e-}\lim_\nu f^\nu$ *exists and equals* $\sup_\nu [\operatorname{cl} f^\nu]$ *(rather than* $\operatorname{cl}[\sup_\nu f^\nu]$*).*

Theorem 4.3 (epi-limits of convex functions). *For any sequence $\{f^\nu\}_{\nu\in\mathbb{N}}$ of convex functions on $\mathbb{R}^n$, the function* $\text{e-lim sup}_\nu f^\nu$ *is convex, and so too is the function* $\text{e-lim}_\nu f^\nu$ *when it exists.*

Moreover, under the assumption that f is a convex, lsc function on $\mathbb{R}^n$ such that $\operatorname{dom} f$ *has nonempty interior, the following are equivalent:*

(a) $f = \text{e-lim}_\nu f^\nu$;

(b) *there is a dense subset D of $\mathbb{R}^n$ such that $f^\nu(x) \to f(x)$ for all x in D;*

(c) *f^ν converges uniformly to f on every compact set C that does not contain a boundary point of* $\operatorname{dom} f$.

Theorem 4.4 (epi-limits of sums of functions). *For sequences of functions f_1^ν and f_2^ν on $\mathbb{R}^n$ one has*

$$\text{e-lim inf}_\nu\, f_1^\nu + \text{e-lim inf}_\nu\, f_2^\nu \;\le\; \text{e-lim inf}_\nu (f_1^\nu + f_2^\nu).$$

When $f_1^\nu \xrightarrow{e} f_1$ and $f_2^\nu \xrightarrow{e} f_2$, either one of the following conditions is sufficient to ensure that $f_1^\nu + f_2^\nu \xrightarrow{e} f_1 + f_2$:

(a) *$f_1^\nu \to f_1$ pointwise and $f_2^\nu \to f_2$ pointwise;*

(b) *one of the two sequences converges continuously.*

The result presented next is new, and provides a test for epi-convergence when a sequence of functions is *almost monotonic.* A sequence of functions $f^\nu : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *almost nonincreasing* if there exists a nonnegative sequence $\{\alpha^\nu\}_{\nu\in\mathbb{N}}$ such that $\sum_{k=1}^\infty \alpha^k < \infty$, and for all $\nu \in \mathbb{N}$, $f^\nu \ge f^{\nu+1} - \alpha^\nu$. A sequence of functions $f^\nu : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *almost nondecreasing* if there exists a nonnegative sequence $\{\alpha^\nu\}_{\nu\in\mathbb{N}}$ such that $\sum_{k=1}^\infty \alpha^k < \infty$, and for all $\nu \in \mathbb{N}$, $f^\nu \le f^{\nu+1} + \alpha^\nu$.

Theorem 4.5 (epi-limits of almost monotonic functions). *Let $f^\nu : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a sequence of lsc functions that converges pointwise to $f : \mathbb{R}^n \to \overline{\mathbb{R}}$.*

(a) *If $\{f^\nu\}_{\nu\in\mathbb{N}}$ is almost nonincreasing, and f is lsc, then $f^\nu \xrightarrow{e} f$.*

(b) *If $\{f^\nu\}_{\nu\in\mathbb{N}}$ is almost nondecreasing, then f is lsc and $f^\nu \xrightarrow{e} f$.*

Proof. For part (a), let $g^\nu = f^\nu - \sum_{k=1}^{\nu-1} \alpha^k$, and $g = f - \sum_{k=1}^\infty \alpha^k$. Then g^ν is nonincreasing since

$$g^\nu = f^\nu - \sum_{k=1}^{\nu-1} \alpha^k \ge f^{\nu+1} - \sum_{k=1}^{\nu} \alpha^k = g^{\nu+1}.$$

By Theorem 4.2 (c) and the lower semicontinuity of f, $\text{e-lim}_\nu\, g^\nu$ exists and

$$\operatorname*{e-lim}_\nu g^\nu = \operatorname{cl}\big[\inf_\nu g^\nu\big] = f - \sum_{k=1}^\infty \alpha^k = g.$$

Now observe that $f^\nu = g^\nu + \sum_{k=1}^{\nu-1} \alpha^k$ and $f = g + \sum_{k=1}^{\infty} \alpha^k$, where now g^ν converges to g both epigraphically and pointwise, and $\sum_{k=1}^{\nu-1} \alpha^k \to \sum_{k=1}^{\infty} \alpha^k$ (both epigraphically and pointwise when considered as constant functions). Applying Theorem 4.4 (a) for epi-limits of sums of functions gives us that $f^\nu \xrightarrow{e} f$.

In part (b), the approach is similar. Let $g^\nu = f^\nu + \sum_{k=1}^{\nu-1} \alpha^k$, and $g = f + \sum_{k=1}^{\infty} \alpha^k$. Then g^ν is nondecreasing since

$$g^\nu = f^\nu + \sum_{k=1}^{\nu-1} \alpha^k \leq f^{\nu+1} + \sum_{k=1}^{\nu} \alpha^k = g^{\nu+1}.$$

Theorem 4.2 (d) says that e-$\lim_\nu g^\nu$ exists and equals $\sup_\nu g^\nu = f + \sum_{k=1}^{\infty} \alpha^k = g$. We have that $f^\nu = g^\nu - \sum_{k=1}^{\nu-1} \alpha^k$ and $f = g - \sum_{k=1}^{\infty} \alpha^k$, and g^ν converges to g both pointwise and epigraphically. Also, because $-\sum_{k=1}^{\nu-1} \alpha^k \to -\sum_{k=1}^{\infty} \alpha^k$, applying Theorem 4.4 (a) gives us that $f^\nu \xrightarrow{e} f$, and Theorem 4.2 (a) implies that f is lsc.□

4.2 Approximation theorems I: domain of Q known a priori

With these tools in hand, we are ready to investigate approximations to Q. The first instance we consider is when $\operatorname{dom} Q$ is known a priori. This might happen, for example, if there are no *induced constraints*, i.e. implicit constraints on u_1 that if violated show up in later stages in the form of future infeasible decisions and trajectories. In this case a standard fixed-point approach is possible. We begin by establishing a *complete* space of functions to which the fixed-point theorem will apply.

For a given problem P with cost c, let B denote the space of functions $h : \mathbb{R}^s \to \overline{\mathbb{R}}$ such that $\operatorname{dom} h = \operatorname{dom} Q$ and $\sup_{x \in \operatorname{dom} Q} |h(x)| \leq \frac{K}{1-\delta}$, where K satisfies $\sup_{(x,u) \in \operatorname{dom} c} |c(x,u)| \leq K$. We know $Q \in B$ by Lemma 3.6. Equip B with the sup norm, i.e. $\|h\| = \sup_{x \in \operatorname{dom} Q} |h(x)|$.

Lemma 4.6. *B is a complete metric space.*

Proof. Let $\{h^\nu \in B \,|\, \nu \in \mathbb{N}\}$ be a Cauchy sequence, i.e. for all $\varepsilon > 0$, there exists an N such that $\mu, \nu > N$ implies $|h^\nu(x) - h^\mu(x)| < \varepsilon$ for all $x \in \operatorname{dom} Q$. First we show that the pointwise limit exists and is in B, which will then necessarily be the uniform limit. Suppose the pointwise limit does not exist. Then there is some $x \in \operatorname{dom} Q$, $\gamma > 0$ such that $\liminf_\nu h^\nu(x) + \gamma < \limsup_\nu h^\nu(x)$. Find N such that for all $\mu, \nu > N$, $\sup_{x \in \operatorname{dom} Q} |h^\nu(x) - h^\mu(x)| < \gamma$. Then for our particular x, we also have that for all $\mu, \nu > N$, $|h^\nu(x) - h^\mu(x)| < \gamma$. This implies that

$$|\limsup_\nu h^\nu(x) - \liminf_\nu h^\nu(x)| \leq \gamma,$$

and this is a contradiction. That the pointwise limit h is in B follows from

$$|h(x)| \le \lim_\nu |h^\nu(x)| \le \frac{K}{1-\delta},$$

for any $x \in \operatorname{dom} Q$.

To show that the pointwise limit h is also a uniform limit, fix $\varepsilon > 0$ and choose N such that $\mu, \nu > N$ implies $|h^\nu(x) - h^\mu(x)| < \frac{\varepsilon}{2}$ for all $x \in \operatorname{dom} Q$. For each $x \in \operatorname{dom} Q$, find $\mu(x) > N$ such that $|h^{\mu(x)}(x) - h(x)| < \frac{\varepsilon}{2}$. Then for $\nu > N$, for any $x \in \operatorname{dom} Q$,

$$\begin{aligned}|h^\nu(x) - h(x)| &\le |h^\nu(x) - h^{\mu(x)}(x)| + |h^{\mu(x)}(x) - h(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

whereby $h^\nu \to h$ uniformly on $\operatorname{dom} Q$. Therefore, B is a complete space. □

Next we define a mapping on B and show that it is a contraction mapping; a mapping $T : X \to X$ on a metric space (X, d) is a *contraction mapping* if for all f, g in X, $d(Tf, Tg) < \alpha d(f, g)$ for some $\alpha \in (0, 1)$. Let $T : B \to B$ be defined for $h \in B$ by

$$Th(x) = \begin{cases} \inf_u \left\{ c(x,u) + \delta E h\big(A(\xi)x + B(\xi)u + b(\xi)\big) \right\} & \text{if } x \in \operatorname{dom} Q \\ +\infty & \text{otherwise.} \end{cases}$$

T maps B into itself since for any $x \in \operatorname{dom} Q$,

$$\begin{aligned}|Th(x)| &\le \sup_{(x,u)\in \operatorname{dom} c} |c(x,u)| + \delta \sup_{x \in \operatorname{dom} Q} |h(x)| \\ &\le K + \frac{\delta K}{1-\delta} \\ &= \frac{K}{1-\delta}.\end{aligned}$$

We will also need to extend the notion of uniform convergence to take into account functions with values equal to $+\infty$. For any function $f : I\!R^n \to \overline{I\!R}$ and any $\rho \in (0, \infty)$, the *ρ-truncation* of f is the function $f_{\wedge\rho}$ defined by

$$f_{\wedge\rho}(x) = \begin{cases} -\rho & \text{if } f(x) \in (-\infty, -\rho), \\ f(x) & \text{if } f(x) \in [-\rho, \rho], \\ \rho & \text{if } f(x) \in (\rho, \infty). \end{cases}$$

A sequence of functions f^ν will be said to *converge uniformly* to f on a set $X \subset I\!R^n$ if, for every $\rho > 0$, their truncations $f^\nu_{\wedge\rho}$ converge uniformly to $f_{\wedge\rho}$ on X in the bounded sense.

Theorem 4.7 (fixed point theorem). *T is a contraction mapping on B. Consequently, Q is the unique fixed point of T in B, and $T^\nu h \to Q$ uniformly.*

Proof. Let $g, h \in B$. Then for fixed $x \in \operatorname{dom} Q$, $\varepsilon > 0$, suppose without loss of generality that $Th(x) \geq Tg(x)$, and let $\bar{u} \in I\!R^n$ satisfy

$$c(x, \bar{u}) + \delta E g\big(A(\xi)x + B(\xi)\bar{u} + b(\xi)\big) \leq Tg(x) + \varepsilon,$$

which will always be possible by the definition of T ($\bar{u}$ is just an approximate minimizer). Then

$$\begin{aligned}|Th(x) - Tg(x)| &\leq |\delta E h\big(A(\xi)x + B(\xi)\bar{u} + b(\xi)\big) - \delta E g\big(A(\xi)x + B(\xi)\bar{u} + b(\xi)\big)| \\ &\leq \delta E|h\big(A(\xi)x + B(\xi)\bar{u} + b(\xi)\big) - g\big(A(\xi)x + B(\xi)\bar{u} + b(\xi)\big)| + \varepsilon.\end{aligned}$$

This implies that

$$\begin{aligned}&\sup_{x \in \operatorname{dom} Q} |Th(x) - Tg(x)| \\ &\leq \sup_{x \in \operatorname{dom} Q} \delta E|h\big(A(\xi)x + B(\xi)\bar{u} + b(\xi)\big) - g\big(A(\xi)x + B(\xi)\bar{u} + b(\xi)\big)| + \varepsilon \\ &\leq \delta \sup_{x \in \operatorname{dom} Q} |h(x) - g(x)| + \varepsilon.\end{aligned}$$

Since ε was arbitrary, and $\delta \in (0,1)$, T is a contraction mapping.

It is well-known that a contraction mapping on a complete metric space has a unique fixed point, and that repeated applications of the mapping to any point in the space will converge to this fixed point. In this case, T has a unique fixed point which must therefore be Q, and also $T^\nu h \to Q$ uniformly on $\operatorname{dom} Q$. Since the approximations are equal to $+\infty$ outside $\operatorname{dom} Q$, it follows from the extended definition of uniform convergence that $T^\nu h \to Q$ uniformly on all of $I\!R^s$. □

This gives a starting point for approximations to Q. If $\operatorname{dom} Q$ is known a priori, and $T^\nu h$ is computable, then $T^\nu h$ may serve as the end term of a finite-horizon problem $P_{T^\nu h}$, as proposed at the beginning of §4. We next derive the epi-convergence of $T^\nu h$ to Q. We begin this development with some results about convexity.

Theorem 4.8. $\operatorname{dom} Q$ *is convex.*

Proof. $\operatorname{dom} Q$ coincides with $\operatorname{feas} P$, which is convex by the convexity of c and the affine dynamic equations: Given two feasible points $x^1, x^2 \in \operatorname{feas} P$, and $\alpha \in (0,1)$, let $x^\alpha = (1-\alpha)x^1 + \alpha x^2$. For $i = 1, 2$, there exist $X_i = (x_0^i, x_1^i, \ldots)$, $U_i = (u_1^i, u_2^i, \ldots)$ such that

$$\begin{aligned} x_t^i &= A(\xi_t)x_{t-1}^i + B(\xi_t)u_t^i + b(\xi_t), \\ x_0^i &= x^i, \\ u_t^i &\quad \mathcal{G}_t \text{ measurable }, \\ \text{and } E\sum_{t=1}^{\infty} & c(x_{t-1}^i, u_t^i) < \infty. \end{aligned}$$

Letting $X_\alpha = (1-\alpha)X_1 + \alpha X_2$, and $U_\alpha = (1-\alpha)U_1 + \alpha U_2$, observe that they satisfy

$$\begin{aligned} x_t^\alpha &= A(\xi_t)x_{t-1}^\alpha + B(\xi_t)u_t^\alpha + b(\xi_t), \\ x_0^\alpha &= x^\alpha, \\ u_t^\alpha &\quad \mathcal{G}_t \text{ measurable }, \end{aligned}$$

and

$$E\sum_{t=1}^{\infty} c(x_{t-1}^\alpha, u_t^\alpha) \le (1-\alpha)E\sum_{t=1}^{\infty} \delta^{t-1} c(x_{t-1}^1, u_{t-1}^1) + \alpha E\sum_{t=1}^{\infty} \delta^{t-1} c(x_{t-1}^2, u_t^2) < \infty,$$

whereby $x^\alpha \in \text{feas}P$, which shows that feas$P$ (hence dom Q) is convex. □

Lemma 4.9. *If $h : \mathbb{R}^s \to \overline{\mathbb{R}}$ in B is convex, then $T^\nu h : \mathbb{R}^s \to \overline{\mathbb{R}}$ is convex.*

Proof. By induction, it suffices to show that Th is convex. Let $x^1, x^2 \in \text{dom}\, Q$, $\alpha \in (0,1)$, and let $x^\alpha = (1-\alpha)x^1 + \alpha x^2$ which is also in dom Q by Theorem 4.8. Then

$$\begin{aligned} Th(x^\alpha) &= \inf_u \big\{c(x^\alpha, u) + \delta E h\big(A(\xi)x^\alpha + B(\xi)u + b(\xi)\big)\big\} \\ &\le \inf_{u^1,u^2} \big\{c\big(x^\alpha, (1-\alpha)u^1 + \alpha u^2\big) \\ &+\delta E h\big((1-\alpha)\big(A(\xi)x^1 + B(\xi)u^1 + b(\xi)\big) + \alpha\big(A(\xi)x^2 + B(\xi)u^2 + b(\xi)\big)\big)\big\} \\ &\le \inf_{u^1,u^2} \big\{(1-\alpha)c(x^1, u^1) + \alpha c(x^2, u^2) \\ &+(1-\alpha)\delta E h\big(A(\xi)x^1 + B(\xi)u^1 + b(\xi)\big) + \alpha\delta E h\big(A(\xi)x^2 + B(\xi)u^2 + b(\xi)\big)\big\} \\ &= (1-\alpha)\inf_{u^1} \big\{c(x^1, u^1) + \delta E h\big(A(\xi)x^1 + B(\xi)u^1 + b(\xi)\big)\big\} \\ &+\alpha \inf_{u^2} \big\{c(x^2, u^2) + \delta E h\big(A(\xi)x^2 + B(\xi)u^2 + b(\xi)\big)\big\} \\ &= (1-\alpha)Th(x^1) + \alpha Th(x^2), \end{aligned}$$

which completes the proof. □

Corollary 4.10. *Q is convex.*

Proof. This comes out of the uniform convergence in Theorem 4.7, and the convexity results that follow in Theorem 4.8 and Lemma 4.9, since limits of convex functions are convex. □

Corollary 4.11 (epi-convergence of iterates)**.** *Let $h \in B$ be convex. If Q is lsc and* $\operatorname{dom} Q$ *has nonempty interior, then $T^\nu h$ epi-converges to Q.*

Proof. The epi-convergence of $T^\nu h$ to Q just applies Theorem 4.3 to the uniform convergence result of Theorem 4.7, through the fact that Q is convex and Lemma 4.9 which provides the convexity of $T^\nu h$. □

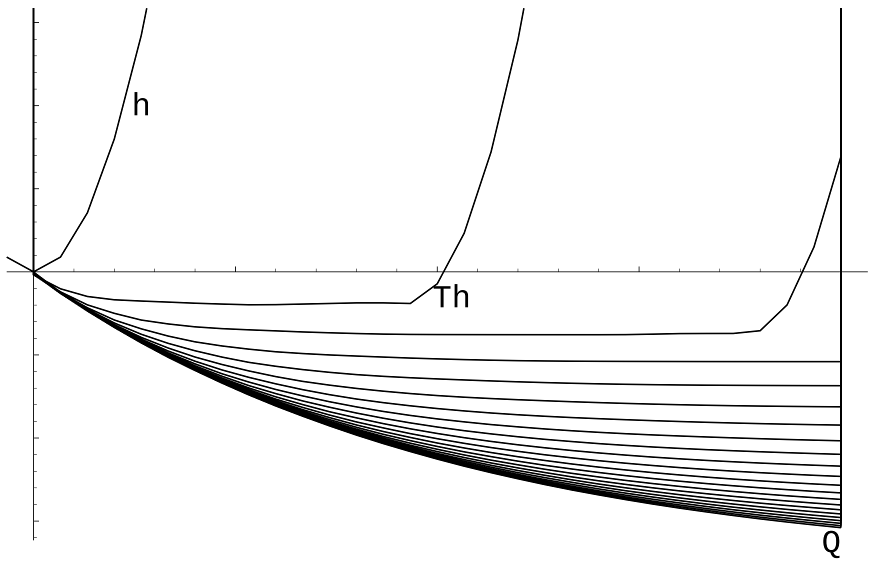

Fig. 1. Epi-convergence of iterates to Q

Conditions under which Q is lsc will be established a bit later in this section. For now, we keep it in the theorem statements as an assumption. The next goal is to obtain the convergence of optimal policies for P. Equivalently, what is needed is that solutions $u^\nu(x)$ of

$$\text{minimize } c(x,u) + \delta E T^\nu h\big(A(\xi)x + B(\xi)u + b(\xi)\big) \qquad P^1_{T^\nu h}(x)$$

converge to a solution $u(x)$ of

$$\text{minimize } c(x,u) + \delta E Q\big(A(\xi)x + B(\xi)u + b(\xi)\big) \qquad P^1_Q(x)$$

for every $x \in \operatorname{dom} Q$. And for this we will appeal once again to epi-convergence.

Theorem 4.12 (convergence of optimal policies)**.** *Suppose that Q is lsc,* $\operatorname{dom} Q$ *has nonempty interior, and the matrices, $B(\xi)$ have full row rank for all $\xi \in \Xi$. For each $x \in \operatorname{dom} Q$, let $g_x, g_x^\nu : \mathbb{R}^n \to \overline{\mathbb{R}}$ be defined by*

$$g_x^\nu(u) = c(x,u) + \delta E T^\nu h\big(A(\xi)x + B(\xi)u + b(\xi)\big),$$

$$g_x(u) = c(x,u) + \delta E Q\big(A(\xi)x + B(\xi)u + b(\xi)\big).$$

Then $g_x^\nu \xrightarrow{e} g_x$. In particular, the conclusions of Theorem 4.1 are valid.

Proof. Fix $x \in \operatorname{dom} Q$. First let's examine the terms

$$f_x^\nu(u) = E T^\nu h\big(A(\xi)x + B(\xi)u + b(\xi)\big),$$

$$f_x(u) = E Q\big(A(\xi)x + B(\xi)u + b(\xi)\big).$$

The bounded convergence theorem [**4**] along with the uniform convergence of $T^\nu h$ to Q implies that $f_x^\nu \to f_x$ pointwise. Let $u \in \mathbb{R}^n$, and let $u^\nu \to u$. Then for all $\xi \in \Xi$, $A(\xi)x + B(\xi)u^\nu + b(\xi) \to A(\xi)x + B(\xi)u + b(\xi)$. By Corollary 4.11 and Fatou's Lemma we obtain

$$\begin{aligned}\liminf_\nu f_x^\nu(u^\nu) &= \liminf_\nu E T^\nu h\big(A(\xi)x + B(\xi)u^\nu + b(\xi)\big)\\ &\geq E \liminf_\nu T^\nu h\big(A(\xi)x + B(\xi)u^\nu + b(\xi)\big)\\ &\geq E Q\big(A(\xi)x + B(\xi)u + b(\xi)\big)\\ &= f_x(u).\end{aligned}$$

For the lim sup direction, there is a sequence $x^\nu \to EA(\xi)x + EB(\xi)u + Eb(\xi)$ such that

$$\begin{aligned}\limsup_\nu T^\nu h\big(x^\nu(\xi)\big) &\leq Q\big(EA(\xi)x + EB(\xi)u + Eb(\xi)\big)\\ &\leq EQ\big(A(\xi)x + B(\xi)u + b(\xi)\big)\end{aligned}$$

by Corollary 4.11 and Jensen's inequality. Since $B(\xi)$ has full row rank for all $\xi \in \Xi$, we can find a sequence of integrable $u^\nu : \Xi \to \mathbb{R}^n$ that satisfies $B(\xi)u^\nu(\xi) = x^\nu - A(\xi)x - b(\xi)$. Let $u^\nu = Eu^\nu(\xi)$. Then applying first Jensen's inequality followed by Corollary 4.11 we obtain

$$\begin{aligned}\limsup_\nu f_x^\nu(u^\nu) &= \limsup_\nu E T^\nu h\big(A(\xi)x + B(\xi)u^\nu + b(\xi)\big)\\ &\leq \limsup_\nu E T^\nu h\big(A(\xi)x + B(\xi)u^\nu(\xi) + b(\xi)\big)\\ &= \limsup_\nu T^\nu h\big(x^\nu\big)\\ &\leq EQ\big(A(\xi)x + B(\xi)u + b(\xi)\big)\\ &= f_x(u)\end{aligned}$$

Thus we have that $f_x^\nu \xrightarrow{e} f_x$. For fixed x, $g_x^\nu = c(x, \cdot) + \delta f_x^\nu$ and $g_x = c(x, \cdot) + \delta f_x$. Theorem 4.4 (a) may now be applied to obtain that $g_x^\nu \xrightarrow{e} g_x$, which completes the proof. ◻

Observe in this theorem that g_x^ν are the objective functions for the problems $P^1_{T^\nu h}(x)$ and their epi-limit g_x is the objective function for the problem $P^1_Q(x)$. We have already shown that the optimal policies of P^1_Q (as well as P^T_Q for any $T \in I\!N$) coincide with those of P. Thus, this theorem sets up a pointwise (in x) approximation framework for optimal policies of P by appealing to the epi-convergence in minimization properties set forth in Theorem 4.1.

4.3 Approximation theorems II: domain of Q unknown

Thus far, we have restricted our attention to the case when $\operatorname{dom} Q$ is known ahead of time. Many problems are not so simple however, and the subject of determining the domain a priori is an important area to investigate in its own right. We proceed now to develop approximation theorems for P and Q which do not depend on knowing the set of feasible initial points. We restrict our attention to approximations from below, with an emphasis on drawing out the almost monotonic convergence (see §4.1 for the definition) that is inherent in the approximations.

Our setting is the same as in §4.2 except that instead of considering the complete space B which depended on knowing $\operatorname{dom} Q$, we work in the space C of functions $h : \mathbb{R}^s \to \overline{\mathbb{R}}$ that are bounded by $\frac{K}{1-\delta}$ on their domains, and also bounded above by Q (i.e. $\sup_{x \in \operatorname{dom} h} |h(x)| \leq \frac{K}{1-\delta}$ and $h \leq Q$). Define the operator W for $h \in C$ by

$$Wh(x) = \inf_u \big\{ c(x,u) + \delta E h\big(A(\xi)x + B(\xi)u + b(\xi)\big)\big\}.$$

If we begin with a given function $h \in C$, every iteration $W^\nu h$ will be a lower bound of Q. In addition, we can obtain the almost monotonicity of these iterates.

Theorem 4.13 (almost nondecreasing iterates)**.** *For any $h \in C$, we have $Wh \in C$ and $\big\{W^\nu h\big\}_{\nu \in \mathbb{N}}$ is almost nondecreasing; specifically, for $\alpha^\nu = \delta^\nu \frac{2K}{1-\delta}$, $W^\nu h \leq W^{\nu+1} h + \alpha^\nu$.*

Proof. We first demonstrate that $W : C \to C$. W maps C into itself since for any $x \in \operatorname{dom} Wh$, there exists a $u \in \mathbb{R}^n$ such that $x \in \operatorname{dom} c(\cdot, u)$ and $Eh\big(A(\xi)x + B(\xi)u + b(\xi)\big) < \infty$, so that

$$\begin{aligned} |Wh(x)| &\leq \sup_{(x,u) \in \operatorname{dom} c} |c(x,u)| + \delta \sup_{x \in \operatorname{dom} h} |h(x)| \\ &\leq K + \frac{\delta K}{1-\delta} \\ &= \frac{K}{1-\delta}. \end{aligned}$$

And additionally,

$$\begin{aligned} Wh(x) &= \inf_u \big\{ c(x,u) + \delta E h\big(A(\xi)x + B(\xi)u + b(\xi)\big)\big\} \\ &\leq \inf_u \big\{ c(x,u) + \delta E Q\big(A(\xi)x + B(\xi)u + b(\xi)\big)\big\} \\ &= Q(x). \end{aligned}$$

To show that $\big\{W^\nu h\big\}_{\nu \in \mathbb{N}}$ is almost nondecreasing, we will make use of the fact that for all $\nu \in \mathbb{N}$, for all $x \in \mathbb{R}^s$, $W^\nu h(x) = \min P_h^\nu(x)$. Fix $x \in \mathbb{R}^s$. If there is no feasible point for $P_h^{\nu+1}(x)$, then $W^{\nu+1}h(x) = +\infty$ and then trivially $W^\nu h(x) \le W^{\nu+1}h(x) + \alpha^\nu$. Suppose there is a feasible point $(u_1, \ldots, u_{\nu+1})$, with trajectory $(x_0, \ldots, x_{\nu+1})$ for $P_h^{\nu+1}(x)$. then $(u_1, \ldots, u_\nu)$ and $(x_0, \ldots, x_\nu)$ are feasible for $P_h^\nu(x)$, and

$$\begin{aligned} Eh(x_\nu) &\le \frac{K}{1-\delta} + \big(Ec(x_\nu, u_{\nu+1}) + K\big) + \big(\delta Eh(x_{\nu+1}) + \frac{\delta K}{1-\delta}\big) \\ &= Ec(x_\nu, u_{\nu+1}) + \delta Eh(x_{\nu+1}) + \frac{2K}{1-\delta}. \end{aligned}$$

This implies that

$$E\sum_{t=1}^{\nu} \delta^{t-1} c(x_{t-1}, u_t) + E\delta^\nu h(x_\nu) \le E\sum_{t=1}^{\nu+1} \delta^{t-1} c(x_{t-1}, u_t) + E\delta^{\nu+1} h(x_{\nu+1}) + \alpha^\nu.$$

Since this is true of any feasible point of $P_h^{\nu+1}(x)$, it follows that

$$\min P_h^\nu(x) \le \min P_h^{\nu+1}(x) + \alpha^\nu,$$

which translates to $W^\nu h(x) \le W^{\nu+1}h(x) + \alpha^\nu$ as claimed. Next, observe that

$$\sum_{t=1}^{\infty} \alpha^t = \sum_{t=1}^{\infty} \delta^t \frac{2K}{1-\delta} = \frac{2K}{(1-\delta)^2} < \infty,$$

which implies that $\{W^\nu h\}_{\nu \in \mathbb{N}}$ is almost nondecreasing. □

Lemma 4.14 (pointwise convergence of iterates). *$W^\nu h$ converges to Q pointwise.*

Proof. Fix $x \in \mathbb{R}^s$ and let $(u_1, u_2, \ldots)$ be any sequence such that u_t is $\mathcal{G}_t$ measurable and $(x_0, x_1, \ldots)$ the corresponding trajectory. By the boundedness of h on its domain we have that

$$\liminf_\nu \delta^\nu Eh(x_\nu) + \sum_{k=1}^{\nu-1} \alpha^k \ge \sum_{k=1}^{\infty} \alpha^k.$$

This implies that

$$\liminf_\nu \min P_h^\nu(x) + \sum_{k=1}^{\nu-1} \alpha^k \ge \min P(x) + \sum_{k=1}^{\infty} \alpha^k.$$

Using the facts that $W^\nu h(x) = \min P_h^\nu(x)$ and $Q(x) = \min P(x)$, and that $W^\nu h + \sum_{k=1}^{\nu-1} \alpha^k$ is nondecreasing, yields

$$\lim_\nu W^\nu h(x) + \sum_{k=1}^{\nu-1} \alpha^k = Q(x) + \sum_{k=1}^{\infty} \alpha^k,$$

whereby $\lim_\nu W^\nu h(x) = Q(x)$. □

Lemma 4.15 (lower semi-continuity of $W^\nu h$ under uniform level-boundedness)**.** *If c is level-bounded in u locally uniformly in x, and $h \in C$ is proper and lsc, then $W^\nu h$ is proper, lsc, and level-bounded in u locally uniformly in x.*

Proof. This follows from Theorem 3.5 via the same arguments used in the proof of Theorem 3.7, applied first to

$$g(x,u) = c(x,u) + \delta E h\big(A(\xi)x + B(\xi)u + b(\xi)\big),$$

and then relying on induction since $Wh \in C$ whenever $h \in C$. □

Theorem 4.16 (epi-convergence of iterates)**.** *If $W^\nu h$ is lsc for all $\nu \in I\!N$, in particular when c is uniformly level-bounded and h is proper and lsc, then $W^\nu h$ epi-converges to Q and Q is lsc.*

Proof. This applies Theorem 4.5 (b) about the epiconvergence of almost nondecreasing functions to $W^\nu h$ through Theorem 4.14. Lemma 4.15 supplies the lower semi-continuity of $W^\nu h$ when the uniform level-boundedness condition is satisfied. The lower semi-continuity of Q comes from Theorem 4.2 (a). □

It is not in general true that if h is lsc then $W^\nu h$ is lsc. The uniform level-boundedness of c is one possible sufficient condition for this. In §6 we concentrate on a broad class of piecewise linear-quadratic functions for which lower semi-continuity of the iterates holds even when the uniform level-boundedness assumption may fail. We conclude this section by establishing the convergence of optimal policies for P.

Theorem 4.17 (convergence of optimal policies)**.** *Suppose that $h : I\!R^s \to \overline{I\!R}$ is in C, and that for all $\nu \in I\!N$, $W^\nu h$ is lsc (e.g. if c is uniformly level-bounded and h is proper and lsc). Suppose that the matrices $B(\xi)$ have full row rank for all $\xi \in \Xi$. For each $x \in \operatorname{dom} Q$ let $g_x, g_x^\nu : I\!R^n \to \overline{I\!R}$ be defined by*

$$g_x^\nu(u) = c(x,u) + \delta E W^\nu h\big(A(\xi)x + B(\xi)u + b(\xi)\big),$$

$$g_x(u) = c(x,u) + \delta E Q\big(A(\xi)x + B(\xi)u + b(\xi)\big).$$

Then $g_x^\nu \xrightarrow{e} g_x$. In particular, the conclusions of Theorem 4.1 are valid.

Proof. For each $x \in \operatorname{dom} Q$, let

$$f_x^\nu(u) = E W^\nu h\big(A(\xi)x + B(\xi)u + b(\xi)\big),$$

$$f_x(u) = EQ\big(A(\xi)x + B(\xi)u + b(\xi)\big).$$

The bounded convergence theorem [**4**] along with the pointwise convergence of $W^\nu h$ to Q implies that $f_x^\nu \to f_x$ pointwise. Let $u \in I\!R^n$, and let $u^\nu \to u$. Then for all $\xi \in \Xi$, $A(\xi)x+B(\xi)u^\nu+b(\xi) \to A(\xi)x+B(\xi)u+b(\xi)$. By Theorem 4.16 and Fatou's Lemma we obtain

$$\begin{aligned}
\liminf_\nu f_x^\nu(u^\nu) &= \liminf_\nu EW^\nu h\big(A(\xi)x + B(\xi)u^\nu + b(\xi)\big)\\
&\geq E\liminf_\nu W^\nu h\big(A(\xi)x + B(\xi)u^\nu + b(\xi)\big)\\
&\geq EQ\big(A(\xi)x + B(\xi)u + b(\xi)\big)\\
&= f_x(u).
\end{aligned}$$

For the lim sup direction, there is a sequence $x^\nu \to EA(\xi)x + EB(\xi)u + Eb(\xi)$ such that

$$\begin{aligned}
\limsup_\nu W^\nu h\big(x^\nu(\xi)\big) &\leq Q\big(EA(\xi)x + EB(\xi)u + Eb(\xi)\big)\\
&\leq EQ\big(A(\xi)x + B(\xi)u + b(\xi)\big)
\end{aligned}$$

by Theorem 4.16 and Jensen's inequality. Since $B(\xi)$ has full row rank for all $\xi \in \Xi$, we can find a sequence of integrable $u^\nu : \Xi \to I\!R^n$ that satisfies $B(\xi)u^\nu(\xi) = x^\nu - A(\xi)x - b(\xi)$. Let $u^\nu = Eu^\nu(\xi)$. Then applying first Jensen's inequality followed by Theorem 4.16 we obtain

$$\begin{aligned}
\limsup_\nu f_x^\nu(u^\nu) &= \limsup_\nu EW^\nu h\big(A(\xi)x + B(\xi)u^\nu + b(\xi)\big)\\
&\leq \limsup_\nu EW^\nu h\big(A(\xi)x + B(\xi)u^\nu(\xi) + b(\xi)\big)\\
&= \limsup_\nu W^\nu h\big(x^\nu\big)\\
&\leq EQ\big(A(\xi)x + B(\xi)u + b(\xi)\big)\\
&= f_x(u)
\end{aligned}$$

Thus we have that $f_x^\nu \xrightarrow{e} f_x$. For fixed x, $g_x^\nu = c(x,\cdot) + \delta f_x^\nu$ and $g_x = c(x,\cdot) + \delta f_x$. Theorem 4.4 (a) may now be applied to obtain that $g_x^\nu \xrightarrow{e} g_x$, which completes the proof. □

5. Finite-horizon approximations

The goal of this section is to come up with good approximations to the problem P, using two different methods. The first relies on the approximation theorems for Q that we have been analyzing in the previous sections. The aim is to come up with a good function h that approximates the future of the system, so that P_h^T is a good approximation of the original problem. Lower and upper bounds are considered. The second method concentrates on approximations from below, by combining a terminal cost function with terminal dynamics that in some way average the future. This adds more information to the problem, but removes the possibility of using the iterated methods of the approximation theorems to obtain better bounds. The idea of this method stems from the work of Grinold [**10**], and Flåm and Wets [**8, 9**].

5.1 Bounds via approximation theorems

We begin with a fairly simple lower bound which is significant in that it satisfies the assumptions which allow iterated convergence to Q, as discussed in the previous section. Such iterations will be especially useful when some explicit representation is available for them. This is the case when c is piecewise linear-quadratic, and we investigate that case in particular in §6.

Recall that the approximating problem takes the form

$$\begin{aligned} &\text{minimize } E\sum_{t=1}^{T} \delta^{t-1} c(x_{t-1}, u_t) + \delta^T E h(x_T) \\ &\text{subject to } x_t = A x_{t-1} + B u_t + b \text{ for } t = 1, \dots, T \qquad\qquad P_h^T(x) \\ &x_0 = x \\ &u_t \quad \mathcal{G}_t \text{ measurable for } t = 1, \dots, T, \end{aligned}$$

where h in our context will be a function that provides a lower (or upper) bound for Q. For each x, given an optimal solution $(u_1^x, \dots, u_T^x)$, we will call a function u :feas$P_h^T \to I\!R^n$ an *optimal first stage policy* for P_h^T if $u(x) = u_1^x$ for each $x \in$feasP_h^T. The main theorems of this section provide some very intuitive bounds for Q.

Theorem 5.1 (lower bounds when dom$\,Q$ known a priori)**.** *Let* $h : I\!R^s \to \overline{I\!R}$ *be defined by*

$$\begin{cases} h(x) = \frac{1}{1-\delta} \inf_u c(x,u) & \text{if } x \in \operatorname{dom} Q \\ +\infty & \text{otherwise.} \end{cases}$$

Then,

(a) *$h \in B$ and $h \le Q$, $T^\nu h \le Q$ and $T^\nu h \to Q$ uniformly on $I\!R^s$. In addition, if Q is lsc and* $\operatorname{dom} Q$ *has nonempty interior, then $T^\nu h \xrightarrow{e} Q$ on $I\!R^s$.*

(b) *If Q is lsc,* $\operatorname{dom} Q$ *has nonempty interior, and the matrices $B(\xi)$ have full row rank for all $\xi \in \Xi$, then for any $T \in I\!N$, the optimal first stage policies of $P^T_{T^\nu h}$ converge pointwise in x to the optimal policies of P as $\nu \to \infty$ in the sense of Theorem 4.1.*

Proof. That $h \in B$ and $h \le Q$ follows from the expansion of h as

$$h(x) = \begin{cases} \inf_u \sum_{t=1}^{\infty} \delta^{t-1} c(x,u) & \text{if } x \in \operatorname{dom} Q \\ +\infty & \text{otherwise.} \end{cases}$$

The fact that $h \le Q$ implies that

$$\begin{aligned} Th(x) &= \begin{cases} \inf_u \left\{ c(x,u) + \delta E h\big(A(\xi)x + B(\xi)u + b(\xi)\big)\right\} & \text{if } x \in \operatorname{dom} Q \\ +\infty & \text{otherwise} \end{cases} \\ &\le \inf_u \left\{ c(x,u) + \delta E Q\big(A(\xi)x + B(\xi)u + b(\xi)\big)\right\} \\ &= Q(x), \end{aligned}$$

so by induction $T^\nu h \le Q$ for all $\nu \in I\!N$. The remaining facts in (a) follow from Theorem 4.7, Theorem 4.8 and Corollary 4.11 after observing that h is convex through the convexity of c. Part (b) just applies Theorem 4.12. □

So far we have concentrated only on lower bounds for P. This gives us some finite-horizon approximates from which approximate solutions may be obtained. But up to now, we have no way of testing how good these approximate solutions are. One way of doing this is to obtain finite-horizon approximates that are upper bounds for P, which may be used to test for optimality. For example, if $P^T_{\underline{h}}$ provides a lower bound and $P^{T'}_{\overline{h}}$ provides an upper bound, and $\bar{u}$ is an optimal first stage policy for $P^T_{\underline{h}}$, then for a given x we may evaluate the value of $P^T_{\underline{h}}(x)$ and compare it to the optimal value of $P^{T'}_{\overline{h}}(x)$. If the difference is small, we can be sure that we are close to a solution of $P(x)$ since

$$\min P^T_{\underline{h}} \le \min P \le \min P^{T'}_{\overline{h}}.$$

To obtain an upper approximation, what is needed initially is a function $h : I\!R^s \to \overline{I\!R}$ that is an upper bound of Q, i.e. $h \ge Q$. If $\operatorname{dom} Q$ is known a priori, we may set $h = +\infty$ outside $\operatorname{dom} Q$. If not, we need to find a set $H \subset \operatorname{dom} Q$, and let $\operatorname{dom} h = H$, or somehow otherwise be assured that $h \ge Q$. We concentrate on the case in which $\operatorname{dom} Q$ is known a priori.

Theorem 5.2 (upper bounds when dom Q known a priori)**.** *Let $h \in B$ be convex such that $h \geq Q$. Then*

(a) *$T^\nu h \geq Q$ and $T^\nu h \to Q$ uniformly. In addition, if Q is lsc and* dom Q *has nonempty interior, then $T^\nu h \xrightarrow{e} Q$ on $I\!R^s$.*

(b) *If Q is lsc,* dom Q *has nonempty interior, and the matrices $B(\xi)$ have full row rank for all $\xi \in \Xi$, then for any $T \in I\!N$, at each $x \in$feas$P$, the optimal policies of $P^T_{T^\nu h}$ converge pointwise in x to the optimal policies of P in the sense of Theorem 4.1.*

Proof. $T^\nu h \geq Q$ follows by induction since for any $h \in B$ such that $h \geq Q$,

$$\begin{aligned} Th(x) &= \begin{cases} \inf_u \{c(x,u) + \delta E h\big(A(\xi)x + B(\xi)u + b(\xi)\big)\} & \text{if } x \in \operatorname{dom} Q \\ +\infty & \text{otherwise} \end{cases} \\ &\geq \inf_u \{c(x,u) + \delta E Q\big(A(\xi)x + B(\xi)u + b(\xi)\big)\} \\ &= Q(x). \end{aligned}$$

By the convexity of h, part (a) follows from Theorem 4.7 and Corollary 4.11, and part (b) applies Theorem 4.12. □

Next we obtain lower bounds when the domain of Q is not known ahead of time.

Theorem 5.3 (lower bounds when dom Q unknown)**.** *Let $h : I\!R^s \to \overline{I\!R}$ be defined by*

$$h(x) = \frac{1}{1-\delta} \inf_u c(x,u).$$

Then,

(a) *$h \in C$, $h \leq Q$, $W^\nu h \leq Q$, and $W^\nu h \to Q$ pointwise. If $W^\nu h$ is lsc for all $\nu \in I\!N$, in particular if c is level-bounded in u locally uniformly in x, then $W^\nu h \xrightarrow{e} Q$ on $I\!R^s$ and Q is lsc.*

(b) *If $W^\nu h$ is lsc for all $\nu \in I\!N$, in particular if c is uniformly level-bounded, and the matrices $B(\xi)$ have full row rank for all $\xi \in \Xi$, then for any $T \in I\!N$, for fixed $x \in$feas$P$, any sequence of optimal policies of $P^T_{W^\nu h}$ converges to an optimal policy of P in the sense of Theorem 4.1.*

Proof. That $h \in C$ follows from the expansion of h as

$$h(x) = \inf_u \sum_{t=1}^{\infty} \delta^{t-1} c(x,u).$$

The remaining facts in (a) come from a direct application of Lemma 4.14 and Theorem 4.16, observing that h is proper and lsc via Theorem 3.5 in the case that c is uniformly level-bounded. Part (b) is derived from the results of Theorem 4.17. □

These results are going to be significant for computations only when the terms $T^\nu h$ or $W^\nu h$ are easily computable or explicitly available. The next situation we deal with may be useful when this is not necessarily the case; i.e., we still want a good approximation of the problem P that does not necessarily rely on "iterations."

5.2 Lower bounds via "averaging the future"

For a general convex cost function c, this section examines a possible lower bound for P in a manner similar to that considered by Grinold [**10**], Flåm and Wets [**8, 9**] for deterministic and stochastic infinite-horizon optimization problems. In the paper dealing with the stochastic case, Flåm and Wets assume that the (convexified) problem is convex jointly in x, u, and the ξ variable. We do not make that assumption here, but rely on the affine structure of the dynamics to compensate for the lack of joint convexity between (x,u) and ξ. This section consists of results that give lower bounds for P that rely on "averaging" the future.

We remain interested in optimal policies, so the epi-convergence results we seek will once again be in u for fixed x. The finite-horizon problems we propose have the form

$$\begin{aligned}
&\text{minimize } E\sum_{t=1}^{T}\delta^{t-1}c(x_{t-1},u_t)+\frac{\delta^T}{1-\delta}c(x_T,u_{T+1})\\
&\text{subject to } x_t=A(\xi_t)x_{t-1}+B(\xi_t)u_t+b_t\ P\text{-a.s. for } t=1,\dots,T-1\\
&\qquad\big(I-\delta EA(\xi)\big)x_T-\delta EB(\xi)u_{T+1}\\
&\qquad=(1-\delta)\big(A(\xi_T)x_{T-1}+B(\xi_T)u_T+b(\xi_T)\big)+\delta Eb(\xi)\\
&\qquad\qquad u_t\quad \mathcal{G}_t \text{ measurable for } t=1,\dots,T.
\end{aligned}\qquad P_F^T(x)$$

The unintuitive terminal dynamics will be justified shortly. Through the developments of the previous sections, the objective functions in u, (inclusive now of all dynamics and constraints) then have the form, for fixed $x\in\mathbb{R}^s$,

$$g_x^T(u)=c(x,u)+\delta E\min P_F^T\big(A(\xi)x+B(\xi)u+b(\xi)\big),$$

$$g_x(u)=c(x,u)+\delta EQ\big(A(\xi)x+B(\xi)u+b(\xi)\big).$$

We make the assumption throughout that a policy u of P satisfies

$$E\sum_{t=1}^{\infty}\delta^{t-1}|u_t|<\infty \text{ and } E\sum_{t=1}^{\infty}\delta^{t-1}|x_{t-1}|<\infty, \tag{2}$$

under the identifications given in (1) for a solution given a policy, where $|\cdot|$ is taken componentwise. A policy $u : \mathbb{R}^n \to \mathbb{R}^s$ is called a *feasible policy* for P if for any $x \in$feas$P$, the associated objective is finite, i.e. $g_x\big(u(x)\big) < \infty$.

Theorem 5.4 (feasibility). *If $P(x)$ is feasible, then $P_F^T(x)$ is feasible.*

Proof. Suppose $x \in$feas$P$. Let u be a feasible policy for P such that the corresponding solution $(u_1, u_2, \ldots)$ and trajectory $(x_0, x_1, \ldots)$ is feasible for initial state x. For fixed $T \in \mathbb{N}$, let

$$\bar{x}_T = E^{\mathcal{G}_{T+1}}(1-\delta)\sum_{t=T}^{\infty}\delta^{t-T}x_t,$$
$$\bar{u}_{T+1} = E^{\mathcal{G}_{T+1}}(1-\delta)\sum_{t=T}^{\infty}\delta^{t-T}u_t,$$

which exist by the assumption (2). What we will now show is that $(u_1, \ldots, u_T, \bar{u}_{T+1})$ and $(x_0, \ldots, x_{T-1}, \bar{x}_T)$ are feasible for $P_F^T(x)$. For this it suffices to show that the terminal dynamics are satisfied. Observe that

$$\begin{aligned}
\bar{x}_T &= E^{\mathcal{G}_{T+1}}(1-\delta)\sum_{t=T}^{\infty}\delta^{t-T}x_t \\
&= E^{\mathcal{G}_{T+1}}(1-\delta)\sum_{t=T}^{\infty}\delta^{t-T}\big(A(\xi_t)x_{t-1}+B(\xi_t)u_t+b(\xi_t)\big) \\
&= (1-\delta)\big(A(\xi_t)x_{t-1}+B(\xi_t)u_t+b(\xi_t)\big) \\
&+EA(\xi)(1-\delta)\sum_{t=T+1}^{\infty}\delta^{t-T}E^{\mathcal{G}_{T+1}}x_{t-1} \\
&+EB(\xi)(1-\delta)\big(\sum_{t=T+1}^{\infty}\delta^{t-T}E^{\mathcal{G}_{T+1}}u_t \\
&+\delta Eb(\xi) \\
&= (1-\delta)A(\xi_T)x_{T-1}+\delta EA(\xi)\bar{x}_T+(1-\delta)B(\xi_T)u_T+\delta EB(\xi)\bar{u}_{T+1} \\
&+(1-\delta)b(\xi_T)+\delta Eb(\xi).
\end{aligned}$$

This is just a rearrangement of the terminal dynamics given for $P_F^T(x)$, whereby $x \in$feasP_F^T. ☐

Next, we show that the objective functions are monotonically increasing, and provide lower bounds for P.

Theorem 5.5 (monotonically increasing lower bounds)**.** *For any* $x \in \mathbb{R}^s$, $T \in \mathbb{N}$, $g_x^T \le g_x^{T+1} \le g_x$.

Proof. It suffices to show from the definition of the objectives that $\min P_F^T \le \min P_F^{T+1} \le Q$. Let u be a feasible policy for P. Fix $x \in \mathbb{R}^s$ and let $(u_1, u_2, \ldots)$ and $(x_0, x_1, \ldots)$ be a corresponding solution with $\bar{x}_T$ and $\bar{u}_{T+1}$ as in Theorem 5.4. By Jensen's inequality for conditional expectations, and a convexity argument based on the fact that $(1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} = 1$, we obtain

$$\begin{aligned}
&\frac{1}{1-\delta} Ec(\bar{x}_T, \bar{u}_{T+1}) \\
&= \frac{1}{1-\delta} Ec\big(E^{\mathcal{G}_{T+1}}(1-\delta) \sum_{t=T+1}^{\infty} \delta^{t-T-1} x_{t-1}, E^{\mathcal{G}_{T+1}}(1-\delta) \sum_{t=T+1}^{\infty} \delta^{t-T-1} u_t\big) \\
&= \frac{1}{1-\delta} Ec\big((1-\delta) x_T + \delta(1-\delta) \sum_{t=T+1}^{\infty} \delta^{t-T-1} E^{\mathcal{G}_{T+2}} x_t, \\
&\qquad\qquad (1-\delta) u_{T+1} + \delta(1-\delta) \sum_{t=T+1}^{\infty} \delta^{t-T-1} E^{\mathcal{G}_{T+2}} u_{t+1}\big) \\
&\le Ec(x_T, u_{T+1}) + \frac{\delta}{1-\delta} Ec(\bar{x}_{T+1}, \bar{u}_{T+2}).
\end{aligned}$$

This immediately implies that $g_x^T \le g_x^{T+1}$. In the same manner,

$$\begin{aligned}
&\frac{1}{1-\delta} Ec(\bar{x}_T, \bar{u}_{T+1}) \\
&= \frac{1}{1-\delta} Ec\big(E^{\mathcal{G}_{T+1}}(1-\delta) \sum_{t=T+1}^{\infty} \delta^{t-T-1} x_{t-1}, E^{\mathcal{G}_{T+1}}(1-\delta) \sum_{t=T+1}^{\infty} \delta^{t-T-1} u_t\big) \\
&\le Ec\big((1-\delta) \sum_{t=T+1}^{\infty} \delta^{t-T-1} x_{t-1}, (1-\delta) \sum_{t=T+1}^{\infty} \delta^{t-T-1} u_t\big) \\
&\le \sum_{t=T+1}^{\infty} \delta^{t-T-1} Ec(x_{t-1}, u_t),
\end{aligned}$$

which shows that for any $T \in \mathbb{N}$, $\min P_F^T(x) \le Q(x)$. So we may conclude that $g_x^T \le g_x^{T+1} \le g_x$. □

We conclude this section by showing that the objective functions epi-converge as $T \to \infty$, which in turn ensures the convergence of optimal policies to an optimal policy of P. This will follow directly from the results in §4.

Theorem 5.6 (convergence of optimal policies). *If* $\min P_F^T(\cdot)$ *are lsc for all* $T \in \mathbb{N}$, *in particular if* c *is uniformly level-bounded and* $\min P_F^1(\cdot)$ *is proper and lsc, and the matrices* $B(\xi)$ *have full row rank for all* $\xi \in \Xi$, *then* $g_x^T \xrightarrow{e} g_x$, *and the conclusions of Theorem 4.1 are satisfied.*

Proof. This applies Theorem 4.17 through the observation that $\min P_F^1(\cdot)$ is in C, and $\min P_F^T = W \min P_F^{T-1}$. □

6. Piecewise linear-quadratic costs

Let's now consider the case in which the cost function has the form

$$c(x,u) = p \cdot u + \tfrac{1}{2} u \cdot Pu + \rho_{V,Q}(q - Cx - Du), \tag{3}$$

where P and Q are (symmetric) positive semidefinite, V is polyhedral convex, and

$$\rho_{V,Q}(x) := \sup_{v \in V} \{ v \cdot x - \tfrac{1}{2} v \cdot Qv \}.$$

Note that c is convex, lsc (possibly infinite-valued), and piecewise linear-quadratic on its domain. This is an important class of problems, due to their flexibility and exploitable structure, c.f. [**11**].

Our objective is to investigate the form of the functions h and $W^\nu h$ obtained in §5 when the cost has the piecewise linear-quadratic form of (3). We will make use of some facts about symmetric positive semidefinite matrices. For a matrix P that is positive semidefinite of rank r, we have that $P = S\Lambda S^*$ where S is orthogonal and Λ is a diagonal matrix of eigenvalues whose first r entries are nonzero (positive). Let S_2 denote the matrix consisting of the last $n - r$ columns of S, and P' the pseudo-inverse of P. We make use of the following lemma, whose proof relies on extended linear-quadratic programmming duality, detailed in [**11**].

Lemma 6.1. *Let* V *be a convex polyhedral subst of* $\mathbb{R}^d$, *and let* $J : \mathbb{R}^s \times V \to \mathbb{R}$ *be defined by*

$$J(u,v) = p \cdot u + \tfrac{1}{2} u \cdot Pu + q \cdot v - \tfrac{1}{2} v \cdot Qv - Du \cdot v,$$

where P *and* Q *are symmetric positive semidefinite matrices. Suppose*

$$0 \in V \ \text{ and } \ p \in \operatorname{col} P. \tag{4}$$

Then

$$\inf_u \sup_{v\in V} J(u,v) = \sup_{v\in V} \inf_u J(u,v).$$

Proof. Observe that $\sup_{v\in V} \inf_u J(u,v)$ is finite since for $v=0$,

$$\inf_u J(u,0) = \inf_u p\cdot u + \tfrac{1}{2}u\cdot Pu = -\tfrac{1}{2}p\cdot P'p > -\infty,$$

by virtue of $p \in \operatorname{col} P$. Applying extended linear-quadratic programming duality to the fact that $\sup_{v\in V} \inf_u J(u,v)$ is finite-valued implies that

$$\inf_u \sup_{v\in V} J(u,v) = \sup_{v\in V} \inf_u J(u,v),$$

which completes the proof. □

This leads us to the first main result, which shows that piecewise linear-quadraticity in the form of (3) is preserved under inf-projection.

Theorem 6.2 (preservation of piecewise linear-quadraticity)**.** *With c as above and $g(x) = \inf_u c(x,u)$, if assumption (4) is satisfied for the p, P, and V in the definition of c in (3), then*

$$g(x) = \rho_{V^0,Q^0}(q^0 - C^0 x) + a^0,$$

where $V^0 = \left\{v \in V \,\middle|\, S_2^* D^* v - S_2^* p = 0\right\}$ is polyhedral convex with $0 \in V^0$, $Q^0 := DP'D^* + Q$ is symmetric positive semidefinite, $q^0 = q + DP'p$, $C^0 = C$, and $a^0 = -\frac{1}{2}p\cdot P'p$.

Proof. We begin by letting $J_x(u,v) = p\cdot u + \frac{1}{2}u\cdot Pu + v\cdot(q - Cx) - v\cdot Du - \frac{1}{2}v\cdot Qv$. Observe that

$$\begin{aligned} g(x) &= \inf_u \left\{p\cdot u + \tfrac{1}{2}u\cdot Pu + \sup_{v\in V}\{v\cdot(q-Cx) - v\cdot Du - \tfrac{1}{2}v\cdot Qv\}\right\} \\ &= \inf_u \sup_{v\in V} \left\{p\cdot u + \tfrac{1}{2}u\cdot Pu + v\cdot(q-Cx) - v\cdot Du - \tfrac{1}{2}v\cdot Qv\right\}. \end{aligned}$$

The assumption in (4) along with the conclusions of Lemma 6.1 allow the limits to be interchanged, so that

$$g(x) = \sup_{v\in V} \inf_u J_x(u,v).$$

Setting $\nabla_u J_x(\cdot, v) = 0$ results in the equation $Pu = D^*v - p$.

If there are no solutions to this equation, then by the symmetry of P, for any nonzero $u \in \ker P$, $u\cdot(D^*v - p) \neq 0$. Picking $u \in \ker P$ such that $u\cdot(D^*v-p) > 0$, it can be seen that $L(\nu u, v; x) \to -\infty$ as $\nu \to \infty$. Otherwise, for P' the pseudoinverse of P (which exists since P is symmetric, hence diagonalizable), whenever $D^*v - p \in \operatorname{col} P$ (or equivalently $S_2^*D^*v - S_2^*p = 0$), we have that $u = P'(D^*v - p)$, is a particular solution of $Pu = D^*v - p$. This yields

$$
\begin{aligned}
g(x) &= \begin{cases} \sup\limits_{v\in V}\Big\{ -\tfrac{1}{2}v\cdot DP'D^*v + v\cdot DP'p \\ \quad -\tfrac{1}{2}p\cdot P'p + v\cdot(q-Cx) - \tfrac{1}{2}v\cdot Qv\Big\} & \text{if } \exists v\in V,\ S_2^*D^*v - S_2^*p = 0 \\ \quad -\infty & \text{otherwise} \end{cases} \\
&= \sup_{v\in V^0}\Big\{ -\tfrac{1}{2}v\cdot DP'D^*v + v\cdot DP'p - \tfrac{1}{2}p\cdot P'p + v\cdot(q-Cx) - \tfrac{1}{2}v\cdot Qv\Big\} \\
&= \rho_{V^0,Q^0}\big(DP'p + q - Cx\big) - \tfrac{1}{2}p\cdot P'p,
\end{aligned}
$$

as claimed. Q^0 is symmetric positive semidefinite since both Q and P' are, the symmetry and positive semidefiniteness of P' coming from that of P. V^0 is polyhedral convex since it is the intersection of V with $\{v \,|\, S_2^*D^*v - S_2^*p = 0\}$, an affine set. $0 \in V^0$ follows immediately from $0 \in V$ and $p \in \operatorname{col} P$ since then $S_2^*p = 0$. □

The next proposition gives an explicit formula for the lower bound h obtained in Theorem 5.3.

Proposition 6.3 (piecewise linear-quadraticity of end term)**.** *Let $h : I\!R^s \to \overline{I\!R}$ be defined as in Theorem 5.3 by $h(x) = \frac{1}{1-\delta}\inf_u c(x,u)$. With c piecewise linear-quadratic of the form (3), if assumption (4) is satisfied, then*

$$h(x) = \rho_{V^0,Q^0}(q^0 - C^0x) + a^0,$$

where $V^0 = \big\{v \in V \,\big|\, S_2^*D^*v - S_2^*p = 0\big\}$ is polyhedral convex with $0 \in V^0$, $Q^0 := \frac{1}{1-\delta}(DP'D^* + Q)$ is symmetric positive semidefinite, $q^0 = \frac{1}{1-\delta}(q + DP'p)$, $C^0 = \frac{1}{1-\delta}C$, and $a^0 = -\frac{1}{2(1-\delta)}p \cdot P'p$.

Proof. This is a direct consequence of Theorem 6.2, after the observation that for any $\alpha > 0$,

$$\alpha\rho_{V,Q}(z) = \rho_{V,\alpha Q}(\alpha z),$$

which follows directly from the definition of $\rho_{V,Q}$. □

What we have established so far is a rough lower bound for Q that is expressible as a piecewise linear-quadratic function. But this may be improved upon by performing the successive iterations,

$$Wh(x) := \inf_u \big\{c(x,u) + \delta Eh(Ax + Bu + b)\big\},$$

$$W^\nu h(x) := \inf_u \{c(x,u) + \delta E W^{\nu-1} h(Ax + Bu + b)\},$$

each a better lower bound approximation to Q than its predecessor, and each expressible explicitly in piecewise linear-quadratic form. The result that follows proves the piecewise linear-quadraticity for the iterates.

Proposition 6.4 (piecewise linear-quadraticity of iterates)**.** *Let $\Xi = \{1, \dots, L\}$, and let p_i be the associated probabilities for $i = 1, \dots, L$. Let $c : I\!R^s \times I\!R^n \to \overline{I\!R}$ be piecewise linear-quadratic as in (3). Suppose $W^{\nu-1}h(x) = \rho_{V^{\nu-1},Q^{\nu-1}}(q^{\nu-1} - C^{\nu-1}x) + a^{\nu-1}$, and that $0 \in V^{\nu-1}$. If assumption (4) is satisfied, then $W^\nu h(x) = \rho_{V^\nu,Q^\nu}(q^\nu - C^\nu x) + a^\nu$, where*

$$V^\nu = \left\{ v \in V \times (V^{\nu-1})^L : S_2^* \tilde{D}^* v - S_2^* p = 0 \right\}, \quad \tilde{D} = \begin{pmatrix} D \\ \delta p_1 C^{\nu-1} B_1 \\ \vdots \\ \delta p_L C^{\nu-1} B_L \end{pmatrix},$$

$$Q^\nu = \tilde{D} P' \tilde{D}^* + \begin{pmatrix} Q & & & \\ & \delta p_1 Q^{\nu-1} & & \\ & & \ddots & \\ & & & \delta p_L Q^{\nu-1} \end{pmatrix},$$

$$q^\nu = \tilde{D} P' p + \begin{pmatrix} q \\ \delta p_1 (q^{\nu-1} - C^{\nu-1} b_1) \\ \vdots \\ \delta p_L (q^{\nu-1} - C^{\nu-1} b_L) \end{pmatrix}, \quad C^\nu = \begin{pmatrix} C \\ \delta C^{\nu-1} A_1 \\ \vdots \\ \delta C^{\nu-1} A_L \end{pmatrix},$$

and $a^\nu = -\frac{1}{2} p \cdot P' p + \delta a^{\nu-1}$. V^ν is polyhedral convex with $0 \in V^\nu$, and Q^ν is symmetric positive semi-definite.

Proof. Observe that

$$\begin{aligned} W^\nu h(x) &= \inf_u \Big\{ p \cdot u + \tfrac{1}{2} u \cdot Pu + \rho_{V,Q}(q - Cx - Du) \\ &\qquad + \sum_{i=1}^{L} \delta p_i \rho_{V^{\nu-1},Q^{\nu-1}} \big(q^{\nu-1} - C^{\nu-1}(A_i x + B_i u + b_i) \big) \Big\} + \delta a^{\nu-1} \\ &= \inf_u \Big\{ p \cdot u + \tfrac{1}{2} u \cdot Pu + \rho_{V^\nu,\tilde{Q}}(\tilde{q} - C^\nu x - \tilde{D} u) \Big\} + \delta a^{\nu-1}, \end{aligned}$$

where $\tilde{Q} = \begin{pmatrix} Q & & & \\ & \delta p_1 Q^{\nu-1} & & \\ & & \ddots & \\ & & & \delta p_L Q^{\nu-1} \end{pmatrix}$, and $\tilde{q} = \begin{pmatrix} q \\ \delta p_1 (q^{\nu-1} - C^{\nu-1} b_1) \\ \vdots \\ \delta p_L (q^{\nu-1} - C^{\nu-1} b_L) \end{pmatrix}$.

Now apply Theorem 6.2 in this setting to obtain the result. ∎

These results considered in the framework of the approximation theorems and finite horizon approximates in §4 and §5, allow one to approximate an infinite-horizon problem with piecewise linear-quadratic cost with a sequence of almost nondecreasing finite horizon extended linear-quadratic problems with explicit end term. These are highly decomposable problems with a fully developed strong duality theory and flexible structure. For more on extended linear-quadratic programming for dynamic and stochastic dynamic problems, see [**11**], [**13**] and [**14**].

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