# IIASA

**INTERIM REPORT**      IR-98-042 /June

# Practical Aspects of Value Efficiency Analysis

*Pekka Korhonen(korhonen@iiasa.ac.at)*
*Aapo Siljamäki(aapo.siljamaki@numplan.fi)*
*Margareta Soismaa(soismaa@iiasa.ac.at)*

**Approved by**
**Gordon MacDonald (macdon@iiasa.ac.at)**
**Director, *IIASA***

# Contents

## Abstract

In this paper, we consider practical aspects for measuring Value Efficiency in Data Envelopment Analysis. Value efficiency is an efficiency concept that takes into account the decision maker's preferences. It was developed by Halme, Joro, Korhonen, Salo, and Wallenius [1998]. The decision maker is assumed to compare alternatives using an implicitly known value function which reaches its maximum at the most preferred point on the efficient frontier. The unknown value function is assumed to be pseudoconcave and strictly increasing for outputs and strictly decreasing for inputs. The purpose of value efficiency analysis is to estimate a need to increase outputs and/or decrease inputs for reaching the indifference contour of the value function at the optimum. Because the value function is unknown, the indifference contours cannot be defined precisely. Value efficiency analysis never results in a more pessimistic evaluation than in the case of a known value function. To carry out value efficiency analysis, we have to locate the most preferred solution of the decision maker. In practice, this phase cannot be too complicated. We propose a few alternative ways to locate it and discuss the use of those ways in a practical application.

**Keywords**: Efficiency Analysis, Data Envelopment Analysis, Multiple Criteria Decision Making, Value Function, Practical Application

# Acknowledgments

## About the Authors

Pekka Korhonen is Project Leader of the Decision Analysis and Support Project at IIASA, and also Professor of Economics at the Department of Economics and Management Science, Helsinki School of Economics and Business Administration.

Aapo Siljamäki is a Ph.D. candidate at the Helsinki School of Economics and Business Administration.

Margareta Soismaa is a Research Scholar with the Decision Analysis and Support Project of IIASA.

# Practical Aspects of Value Efficiency Analysis

*Pekka Korhonen, Aapo Siljamäki, Margareta Soismaa*

## 1.    Introduction

*Data Envelopment Analysis* (DEA) was originally proposed by Charnes, Cooper and Rhodes [1978 and 1979] as a method for evaluating the *Relative (Technical) Efficiency* of *Decision Making Units* (DMUs) essentially performing the same task. Each of the units uses multiple inputs to produce multiple outputs. The purpose of DEA is to empirically estimate the so-called *Efficient Frontier* based on the set of available DMUs and to project all DMUs onto this frontier.  A DMU already lying on this frontier is referred to as an efficient unit, and a DMU for which there exists another (different) point on the efficient frontier which stands for an existing or virtual unit producing the same amount or more of outputs by consuming the same amount or less of inputs is inefficient.  DEA provides the user with information about the efficient and inefficient units, as well as the efficiency scores and reference sets for inefficient units. The results of the DEA analysis, especially the efficiency scores, are used in practical applications as performance indicators of DMUs.

The basic DEA is value-free in the sense that efficiency evaluation is based on the data available without taking into account the decision maker's (DM) preferences. But it is possible to incorporate into the analysis the DM's judgments or a priori knowledge (see e.g. Dyson and Thanassoulis [1988], Roll, Cook, and Golany [1991], Thompson et al. [1986], Wong and Beasley [1990], Charnes et al. [1989]). To incorporate the DM's preferences into efficiency analysis, Halme et al. [1998] developed a method called *Value Efficiency Analysis* (VEA).  Value efficiency analysis is based on the assumption that the DM compares alternatives using an implicitly known value function. The unknown value function is assumed to be pseudoconcave and strictly increasing for outputs and strictly decreasing for inputs. It reaches its maximum at a point on the efficient frontier. The point is called the *Most Preferred Solution* (MPS). The purpose of value efficiency analysis is to estimate a need to increase outputs and/or decrease inputs for reaching the indifference contour of the value function at the optimum. Because the value function is unknown, the indifference contour cannot be defined precisely. However, we can define the region consisting of the points surely less or equally preferred than the most preferred solution. This region is used in value efficiency analysis.  The analysis never results in a more pessimistic evaluation than in the case of a known value function.

To carry out value efficiency analysis, we have to explicitly locate the DM's MPS, i.e. the unit preferred to all other existing or virtual units available. Assuming that the DM would like to produce outputs as much as possible and use inputs as little as possible, the most preferred solution clearly lies on the efficient frontier. There are many ways to locate the MPS. We may, for instance, use an interactive Multiple Objective Linear Programming (MOLP) search procedure for this purpose.

Using the knowledge of the MPS and the above mentioned assumptions about the value function, the indifference contour of DM's (unknown) value function at the MPS can be approximated using a so-called tangent cone. This cone characterizes the region consisting of the points surely less or equally preferred to the most preferred point. The efficiency of each DMU is then determined with respect to this tangent cone. As a result we obtain scores that we call V*alue Efficiency Scores*, because the efficiency of each DMU is determined by means of an approximation of the indifference surface of an implicitly known value function at the MPS.

In this paper, our purpose is to consider the practical aspects of value efficiency analysis. We discuss various techniques to locate the MPS. In practice, a DM may even experience the use of an interactive multiple objective linear programming procedure too laborious for this purpose. Simpler methods are more preferable. He/she may be more comfortable with characterizing the MPS in terms of existing units. Furthermore, we will discuss different variations to carry out value efficiency analysis, with emphasis on the practical point of view.

The rest of this paper is organized as follows. In Section 2 we review the main principles of value efficiency analysis. In Section 3 we consider the use of value efficiency analysis in practice. Concluding remarks are presented in Section 4.

# 2. Value Efficiency Analysis

## 2.1 Basic DEA-models

Assume we have $n$ decision making units (DMU) each consuming $m$ inputs and producing $p$ outputs. Let $\mathbf{X} \in \Re_+^{m \times n}$ and $\mathbf{Y} \in \Re_+^{p \times n}$ be the matrices, consisting of nonnegative elements, containing the observed input and output measures for the DMUs. We further assume that there are no duplicated units in the data set. We denote by $\boldsymbol{x}_j$ (the jth column of $\mathbf{X}$) the vector of inputs consumed by DMU$_j$, and by $x_{ij}$ the quantity of input $i$ consumed by DMU$_j$. A similar notation is used for outputs. When it is not necessary to emphasize the different roles of inputs and outputs, we denote $\boldsymbol{u} = \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{-x} \end{bmatrix}$ and $\mathbf{U} = \begin{bmatrix} \mathbf{Y} \\ \mathbf{-X} \end{bmatrix}$. Furthermore, we denote $\boldsymbol{1} = [1, ..., 1]^{\mathrm{T}}$ and refer by $\mathbf{e}_i$ to the $i$[th] unit vector in $\Re^n$.

We consider the set T = { $\boldsymbol{u}$ $|$ $\boldsymbol{u} = \mathbf{U}\lambda, \lambda \in \Lambda$}, where $\Lambda = \{ \lambda \mid \lambda \in \Re_+^n$ and $\mathbf{A}\lambda \leq \boldsymbol{b}\}$, $\mathbf{e}_i \in \Lambda$, $i = 1,..., n$, and $\mathbf{A} \in \Re^{k \times n}$ and vector $\boldsymbol{b} \in \Re^k$. Matrix $\mathbf{A} \in \Re^{k \times n}$ and vector $\boldsymbol{b} \in \Re^k$ are used to specify the feasible values of $\lambda$-variables. All efficient DMUs lie on the

efficient frontier, which is defined as a subset of points of set T satisfying the efficiency condition defined below:

**Definition 1.** A solution $(\mathbf{Y}\lambda^*, \mathbf{X}\lambda^*) = (y^*, x^*)$, $\lambda^* \in \Lambda$, is *efficient* iff there does not exist another $(y, x) \in$ T such that $y \geq y^*$, $x \leq x^*$, and $(y, x) \neq (y^*, x^*)$.

**Definition 2.** A point $(y^*, x^*) \in$ T is *weakly efficient* iff there does not exist another $(y, x) \in \mathbf{T}$ such that $y > y^*$ and $x < x^*$.

The original DEA-models as introduced by Charnes et al. [1978] are constant return to scale models, and later on they are called CCR-models. They are fractional linear programs which can easily be formulated and solved as linear programs. Later Banker, Charnes and Cooper [1984] developed the so-called BCC models with variable returns to scale. The CCR- and BCC-models are the basic model types in DEA. To unify the presentation we formulate a general model (for short, GEN) which includes CCR- and BCC-models as special cases. Note that following Charnes and Cooper, the original primal formulation is currently called the dual and vice versa.

| **General Formulation for DEA-models (Primal)** | **General Formulation for DEA-models (Dual)** |
|---|---|
| $max \quad Z = \sigma + \varepsilon(\mathbf{1}^T s^+ + \mathbf{1}^T s^-)$ <br><br> s.t. $\hspace{3cm}$ (2.1a) <br><br> $\mathbf{Y}\lambda - \sigma w^y - s^+ = g^y$ <br> $\mathbf{X}\lambda + \sigma w^x + s^- = g^x$ <br> $\lambda \in \Lambda$ <br> $s^-, s^+ \geq \mathbf{0}$ <br> $\varepsilon > 0$ ("Non-Archimedean") [1)] | $min \quad W = v^T g^x - \mu^T g^y + \xi^T b$ <br><br> s.t. $\hspace{3cm}$ (2.1b) <br><br> $-\mu^T \mathbf{Y} + v^T \mathbf{X} + \xi^T \mathbf{A} \geq \mathbf{0}^T$ <br> $\mu^T w^y + v^T w^x = 1$ <br> $\mu, v \geq \varepsilon \mathbf{1}$ <br> $\xi \geq \mathbf{0}$ <br> $\varepsilon > 0$ |

First, we show how all basic DEA-models can be introduced from models (2.1a,b). We next consider the primal models. We refer to the unit under consideration by superscript '0'. Thus the different models for measuring the efficiency of unit '0' are obtained from model (2.1a) as shown in Table 1 (Korhonen [1997]).

---

[1)] For details, see Arnold, Bardhan, Cooper, and Gallegos [1997].

**Table 1:** Modifications of Model (2.1a) for Different (Primal) DEA-Models.

| # | Model Type | $w^x$ | $g^x$ | $w^y$ | $g^y$ | $\Lambda$ |
|---|---|---|---|---|---|---|
| 1 | **Output-Oriented CCR-model** (Charnes et al. [1978]) | $0$ | $x^0$ | $y^0$ | $0$ | $\Re_+^n$ |
| 2 | **Input-Oriented CCR-model** (Charnes et al. [1978]).[2] | $x^0$ | $0$ | $0$ | $y^0$ | $\Re_+^n$ |
| 3 | **Combined CCR-model** (Joro et al. [1998]) | $x^0$ | $x^0$ | $y^0$ | $y^0$ | $\Re_+^n$ |
| 4 | **Output-Oriented BCC-model** (Banker et al. [1984]) | $0$ | $x^0$ | $y^0$ | $0$ | $\{\lambda \mid \lambda \in \Re_+^n \text{ and } \mathbf{1}^T\lambda = 1\}$ |
| 5 | **Input-Oriented BCC-model** (Banker et al. [1984]) [2] | $x^0$ | $0$ | $0$ | $y^0$ | $\{\lambda \mid \lambda \in \Re_+^n \text{ and } \mathbf{1}^T\lambda = 1\}$ |
| 6 | **Combined BCC-model** (Joro et al. [1998]) | $x^0$ | $x^0$ | $y^0$ | $y^0$ | $\{\lambda \mid \lambda \in \Re_+^n \text{ and } \mathbf{1}^T\lambda = 1\}$ |
| 7 | **General Combined model** | - | $x^0$ | - | $y^0$ | - |

Each model 1-7 produces an efficient solution corresponding to the given $g = \begin{pmatrix} g^y \\ g^x \end{pmatrix}$. For simplicity, we assume that $g \in T$. Which efficient solution is obtained depends on the model type (= constraint set), the weighting vector $w = \begin{pmatrix} w^y \\ w^x \end{pmatrix} > 0$, the value of $\varepsilon$, and the possible weights used for the slack variables in the objective functions.

Refering to the value of the objective function, we can define that unit $DMU_0$ with $u^0 = \begin{bmatrix} y^0 \\ -x^0 \end{bmatrix}$ is efficient iff

$$Z^* = W^* = \begin{cases} 1, & \text{for models 1 and 4} \\ 0, & \text{for models 3, 6 and 7} \\ -1, & \text{for models 2 and 5} \end{cases} \quad ;$$

otherwise it is inefficient (Charnes et al., 1994). Then all slack variables $s^-$, $s^+$ equal zero. All efficient DMUs lie on the efficient frontier, which is defined as a subset of points of set T satisfying the efficiency condition above. Note that the efficient frontier depends on the model type used in the analysis.

## 2.2 A Value Efficiency Model

The idea of value efficiency analysis is to incorporate the DM's preference information regarding a desirable combination of inputs and outputs into the analysis. This is in contrast with traditional DEA, which assumes that no single output or input is more important than another one. As explained in Halme et al. (1998), the preference information is incorporated via the MPS, i.e. a (virtual or existing) DMU on the efficient frontier having the most desirable values of inputs and outputs.

---

[2] The input oriented models are usually in DEA solved as a minimization problem by writing $w^x = -x^0$ and modifying the objective function accordingly.

The purpose of value efficiency analysis is to evaluate efficiency of each unit in relation to the indifference contour of a value function passing through the most preferred solution. The evaluation could be done easily, if we explicitly knew the DM's value function. In practice, this assumption is not generally realistic. Because the value function is unknown, we cannot characterize the indifference curve precisely. Our aim is to find a region containing all vectors $(y, x) \in \mathfrak{R}^{p+m}$ *surely less or equally preferred* than the most preferred unit. To do this, it is necessary to make some assumptions. Halme et al. [1998] developed the requisite theory by assuming that the DM's value function $v(\boldsymbol{u})$, $\boldsymbol{u} = \begin{bmatrix} \boldsymbol{y} \\ -\boldsymbol{x} \end{bmatrix} \in \mathfrak{R}^{p+m}$ is pseudoconcave, and strictly increasing in $\boldsymbol{u}$ (i.e. strictly increasing in $\boldsymbol{y}$ and strictly decreasing in $\boldsymbol{x}$) and with a (local) maximal value $v(\boldsymbol{u}^*)$, $\boldsymbol{u}^* = \begin{bmatrix} \boldsymbol{y}^* \\ -\boldsymbol{x}^* \end{bmatrix} \in \mathrm{T}$, at the most preferred solution $\boldsymbol{u}^*$. Because the value function is pseudoconcave, then the region containing all vectors $\boldsymbol{u} \in \mathfrak{R}^{p+m}$ surely less or equally preferred than the most preferred unit can be characterized by using all possible tangent hyperplanes as explained in Halme et al. [1998]. Those hyperplanes define a new 'Efficiency Frontier' and in relation to this frontier, efficiency is then defined using a standard DEA-technique. Mathematically this reduces to a straightforward application of linear programming. The resulting scores are called value efficiency scores. Because of using an approximation described above for the indifference contour of the value function, the resulting value efficiency scores are always optimistic approximations of the true scores.

The basic idea of value efficiency analysis is illustrated in Figure1. We have five units (A, B, C, D, E), which produce two outputs and use the same amount of one input. In Figure 1, the problem has been described in the output space. Clearly all units but unit B are efficient. The efficiency measure for unit B in standard DEA is the ratio: $\dfrac{\mathrm{OB}}{\mathrm{OB}^1}$. In value efficiency analysis, we would like to evaluate the ratio: $\dfrac{\mathrm{OB}}{\mathrm{OB}^4}$, but because the value function is unknown, we are not able to do it. If we could approximate the indifference contour by a tangent, then we could use the ratio: $\dfrac{\mathrm{OB}}{\mathrm{OB}^3}$. However, we cannot even make an assumption that the tangent is known. That's why we have to consider all possible tangents of the contour. This idea leads to the use of the ratio: $\dfrac{\mathrm{OB}}{\mathrm{OB}^2}$ as an approximation to the (true) value efficiency score. Because this approximation is the best we can get, we will call this ratio simply value efficiency score.
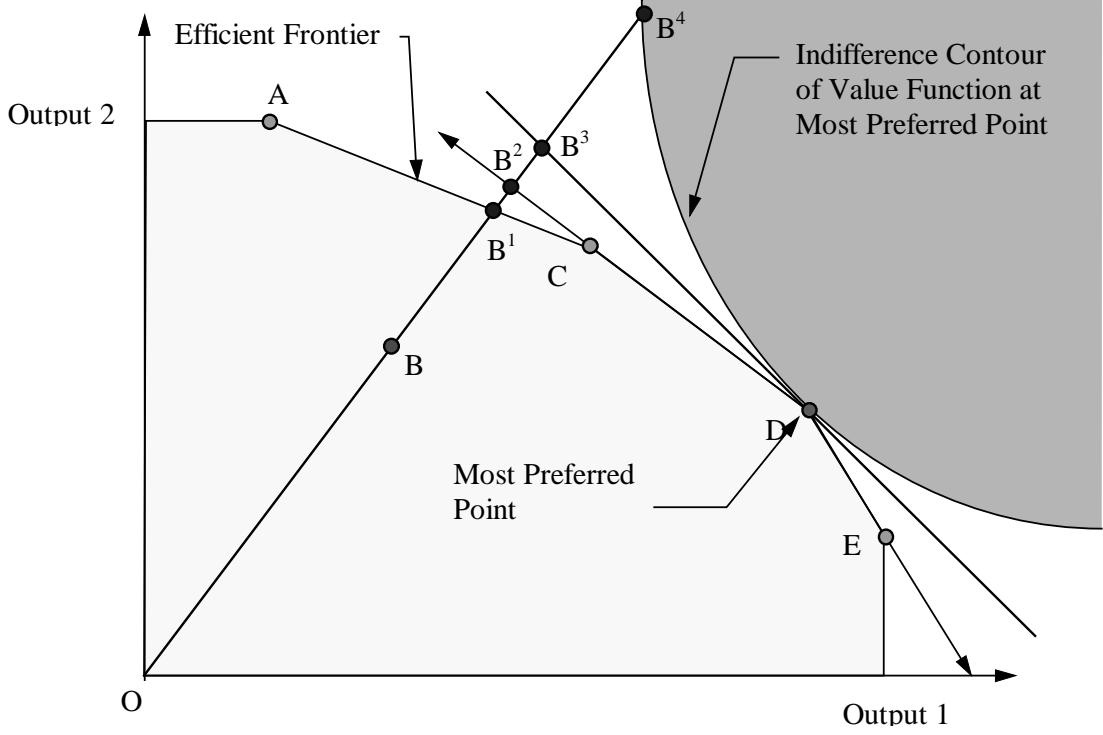
**Figure 1:** Illustration of Value Efficiency Analysis.

Value efficiency analysis can be carried out as easily as standard DEA using linear programming. Point (unit) $g = \begin{bmatrix} g^y \\ g^x \end{bmatrix} \in$ T is value inefficient with respect to any strictly increasing pseudoconcave value function $v(u)$, $u = \begin{bmatrix} y \\ -x \end{bmatrix}$ with a maximum at point $u^*$, if the optimum value $Z^*$ of the following problem is strictly positive:

$$max \quad Z = \sigma + \varepsilon(\mathbf{1}^T s^+ + \mathbf{1}^T s^-)$$

s.t. (2.2)

$$\mathbf{Y}\lambda - \sigma w^y - s^+ = g^y,$$
$$\mathbf{X}\lambda + \sigma w^x + s^- = g^x,$$
$$\mathbf{A}\lambda + \mu = b,$$
$$s^-, s^+ \geq 0,$$
$$\varepsilon > 0, \quad \text{(``Non-Archimedean'')}$$
$$\lambda_j \geq 0, \text{ if } \lambda_j^* = 0, \ j = 1,2, ..., n$$
$$\mu_j \geq 0, \text{ if } \mu_j^* = 0, \ j = 1,2, ..., k$$

where $\lambda^*$ and $\mu^*$ correspond to the MPS:

$$y^* = \mathbf{Y}\lambda^*$$
$$x^* = \mathbf{X}\lambda^*.$$

Note that model (2.2) corresponds to model (2.1a) with one exception. Nonnegativity constraints referring to the basic $\lambda$- and $\mu$-variables with positive values corresponding to defining the MPS are relaxed. It is important that the MPS really lies on the efficient frontier, otherwise the solution is unbounded (for a more detailed discussion, see Halme et al., 1998). The model used to locate the MPS and the model used for value efficiency analysis must have the same constraint set $\Lambda$; otherwise efficiency of the MPS cannot be guaranteed.

# 3.    Determining the Most Preferred Unit in Practice

To make considerations more illustrative, throughout this section we will consider BCC-models, i.e. $\Lambda = \{\lambda \mid \lambda \in \Re_+^n$ and $I^T\lambda = 1\}$, and calculations are made with output-oriented BCC-models.

The first task in value efficiency analysis is to locate the MPS. It means that we have to solve the following multiple objective programming problem (MOLP):

$$Max \quad \mathbf{Y}\lambda$$

$$Min \quad \mathbf{X}\lambda$$

$$s.t. \tag{3.1}$$

$$\lambda \in \Lambda.$$

The solution of the MOLP model (3.1) is a point on the efficient frontier which coincides with the efficient frontier of the corresponding BCC model (see, for example Joro et al. [1998]). This can be found e.g. by using some available interactive multiple objective programming method like Pareto Race (Korhonen and Wallenius [1988]).

In practice, a DM is not necessarily willing to make a free search on the efficient frontier to locate the MPS. He/she may be more comfortable with some other (simpler) method. For instance, the DM may

- prefer to point out a real unit instead of trying to find an abstract unit on the efficient frontier, or

- like to name a set of real units as the "MPS", or

- not be willing to point one "MPS" to all units, but to point different "MPSs" corresponding e.g. to different levels of input values.

In this section, we will consider in more detail various ways to determine the MPS. We first consider the use of Pareto Race to locate a MPS, and then deal with other approaches which may be more preferable in practice, before the DMs learn to take full benefit of interactive multiple objective programming methods.

Throughout this section, we will consider a data set consisting of 25 firms situated in Finland belonging to the same group. The original problem is simplified by taking into consideration only two output variables (Sales, Net Profit) and two input variables (Man Hour, Sales Space). "Man Hour" refers to labor force available within a certain period and "Sales Space" is the total sales area of the firm.

**Table 2:** The Values of Output and Input Variables of 25 Firms.

|  | Outputs | | Inputs | |
|---|---|---|---|---|
|  | **Sales** | **NetProfit** | **Man Hour** | **SalesSpace** |
| FIRM1 | 115.266 | 1.708 | 79.056 | 4.986 |
| FIRM2 | 75.191 | 1.811 | 60.096 | 3.3 |
| FIRM3 | 225.454 | 10.393 | 126.699 | 8.117 |
| FIRM4 | 185.581 | 10.417 | 153.857 | 6.695 |
| FIRM5 | 84.52 | 2.357 | 65.684 | 4.735 |
| FIRM6 | 103.328 | 4.347 | 76.83 | 4.083 |
| FIRM7 | 78.755 | 0.162 | 50.157 | 2.531 |
| FIRM8 | 59.327 | 1.299 | 44.771 | 2.47 |
| FIRM9 | 65.718 | 1.485 | 48.058 | 2.324 |
| FIRM10 | 163.178 | 6.261 | 89.702 | 4.911 |
| FIRM11 | 70.679 | 2.802 | 56.923 | 2.24 |
| FIRM12 | 142.648 | 2.745 | 112.637 | 5.42 |
| FIRM13 | 127.767 | 2.701 | 106.869 | 6.281 |
| FIRM14 | 62.383 | 1.418 | 54.932 | 3.135 |
| FIRM15 | 55.225 | 1.375 | 48.809 | 4.43 |
| FIRM16 | 95.925 | 0.742 | 59.188 | 3.979 |
| FIRM17 | 121.604 | 3.059 | 74.514 | 5.318 |
| FIRM18 | 107.019 | 2.983 | 94.596 | 3.691 |
| FIRM19 | 65.402 | 0.618 | 47.042 | 3.001 |
| FIRM20 | 70.982 | 0.005 | 54.645 | 3.865 |
| FIRM21 | 81.175 | 5.121 | 90.116 | 3.31 |
| FIRM22 | 128.303 | 3.887 | 95.241 | 4.245 |
| FIRM23 | 134.989 | 4.728 | 80.079 | 3.786 |
| FIRM24 | 98.931 | 1.861 | 68.703 | 2.985 |
| FIRM25 | 66.743 | 7.409 | 62.282 | 3.1 |

Applying model (3.1) to the data in Table 2 results in the MOLP model of Table 3. This is used to search the efficient frontier.

**Table 3:** The Structure of the Multiple Objective Linear Programming Model for Searching the Most Preferred Solution of the BCC-model.

|  | FIRM1 | FIRM2 | … | FIRM23 | FIRM24 | FIRM25 |  |  |
|---|---|---|---|---|---|---|---|---|
| **Sales** | 115.266 | 75.191 |  | 134.989 | 98.931 | 66.743 | $\rightarrow$ | max |
| **NetProfit** | 1.708 | 1.811 | … | 4.728 | 1.861 | 7.409 | $\rightarrow$ | max |
| **ManHour** | 79.056 | 60.096 |  | 80.079 | 68.703 | 62.282 | $\rightarrow$ | min |
| **SalesSpace** | 4.986 | 3.3 |  | 3.786 | 2.985 | 3.1 | $\rightarrow$ | min |
| $\lambda$-constr. | 1 | 1 | … | 1 | 1 | 1 | = | 1 |

## 3.1  Pareto Race

To perform the search on the efficient frontier, reflecting on our own bias we recommend the use of the VIG software (Korhonen [1987]). VIG implements Pareto Race (see, Figure 2), a dynamic and visual free search type of interactive procedure for multiple objective linear programming. It enables a DM to freely search any part of the efficient frontier by controlling the speed and direction of motion. The objective function values are represented in numeric form and as bar graphs on the computer screen.

The theoretical foundations of Pareto Race are based on the reference direction approach developed by Korhonen and Laakso [1986]. In the reference direction approach, any direction $r$ specified by the DM is projected onto the efficient frontier using the following model:

$$max \qquad \sigma + \varepsilon(\mathbf{1}^T\mathbf{s}^+ + \mathbf{1}^T\mathbf{s}^-)$$

$$\text{s.t.} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.2)$$

$$\mathbf{Y}\lambda - \sigma\mathbf{w}^y - \mathbf{s}^+ = \mathbf{g}^y + t\mathbf{r}^y$$

$$\mathbf{X}\lambda + \sigma\mathbf{w}^x + \mathbf{s}^- = \mathbf{g}^x + t\mathbf{r}^x$$

$$\lambda \in \Lambda$$

$$\mathbf{s}^-, \mathbf{s}^+ \geq \mathbf{0}$$

$$\varepsilon > 0$$

when t: $0 \to \infty$.

The solution of model (3.2) is an efficient path starting from the current solution and traversing through the efficient frontier until it ends up at some corner point of the efficient frontier.

Pareto Race is the implementation of the dynamic version of the reference direction approach as proposed by Korhonen and Wallenius [1988]. In Pareto Race, a reference direction is determined by the system on the basis of preference information received from the DM. By pressing number keys corresponding to the ordinal numbers of the objectives, the DM expresses which objectives he/she would like to improve and how strongly. In this way he/she implicitly specifies a reference direction. Figure 2 provides the Pareto Race interface for the search, embedded in the VIG software (Korhonen and Wallenius [1988]).

```
┌─────────────────────────────────────────────────────────────┐
│                         Pareto Race                          │
│                                                              │
│                                                              │
│ Goal  1 (max ): Sales  ==>                                   │
│ ■■■■■■■■■■■■■■■■■■■■■■■■■■■■■■■■■■■■■■■ 180.989                │
│  Goal  2 (max ): Profit  ==>                                 │
│ ■■■■■■■■■■■■■■■■■■■■■■■■■■■■■■■ 7.4427                         │
│   Goal  3 (min ): Working H   ==>                            │
│ ■■■■■■■■■■■■■■■■■■■■■■■■■■■■■■■■■■■■ 100.283                   │
│ Goal  4 (min ): Size   ==>                                   │
│ ■■■■■■■■■■■■■■■■■■■■■■■■■■■■■■■■■ 5.8279                       │
│                                                              │
│                                                              │
│ Bar:Accelerator  F1:Gears (B)  F3:Fix     num:Turn           │
│  F5:Brakes        F2:Gears (F)  F4:Relax  F10:Exit           │
└─────────────────────────────────────────────────────────────┘
```

**Figure 2:** Searching for the Most Preferred Values for Inputs and Outputs.

To implement these features, Pareto Race uses certain control mechanisms, which are controlled by the specific keys (shown at the bottom of the screen in Figure 2).

Figure 2 displays one solution to the model in Table 3.

## 3.2   Practical Approaches to Locate one Most Preferred Unit [3]

In practice, the DM may be more willing to name one existing unit as the most preferred unit (MPU) than to make a search on the efficient frontier. Existing units are often more concrete to him/her than a "virtual unit" on the efficient frontier. Moreover, he/she can also consider it as a target to other units. The DM may be willing to pick this unit directly from the original data set. However, this approach has one drawback. Without any help the DM may be unable to pick out an efficient unit. A simple solution to this problem is to first project the unit (referred by superscript 'a') onto the efficient frontier by using the DEA-model under consideration. Because in these considerations, we use the output-oriented BCC-model, the problem formulation should be as follows:

$$max \quad Z = \sigma + \varepsilon(\mathbf{1}^T s^+ + \mathbf{1}^T s^-)$$

$$\text{s.t.} \tag{3.3}$$

$$\mathbf{Y}\lambda - \sigma y^a - s^+ = y^a$$
$$\mathbf{X}\lambda + s^- = x^a$$
$$\lambda \in \Lambda$$
$$s^-, s^+ \geq \mathbf{0}$$
$$\varepsilon > 0,$$

---

[3] When referring to an existing decision making unit on the efficient frontier we use the term *unit*. As a general term for a point on the efficient frontier we use the term *solution*.

where vector $\begin{bmatrix} y^a \\ -x^a \end{bmatrix}$ is the point, the DM chose as the most preferred one.

The solution vector $\lambda* \in \Lambda$ corresponding to the optimal solution of model (3.3) determines a proper most preferred solution: $y* = \mathbf{Y}\lambda*$ and $x* = \mathbf{X}\lambda*$, and in value efficiency analysis nonnegativity constraints are set up only for those components of vector $\lambda$ for which $\lambda_j* = 0$.

Another simple way to avoid the selection of an inefficient alternative is to first compute the technical efficiency of the units, and then to make the choice from the set of efficient units. If the number of efficient units is big, the DM may need some help to pick his/her MPU from this set. For this purpose, he/she may use some existing discrete MCDM-method like VIMDA (Korhonen [1988]).

In Table 4, we compute value efficiency scores assuming that alternatives #3, #8, and #10 one at a time have been pointed out by the DM as his/her MPUs. Note that value efficiency scores are never higher than efficiency scores.

## 3.3 Practical Approaches to Locate Several Most Preferred Units

There are situations when the DM prefers to name a set of efficient units $(y^i, x^i)$, $i \in M$, as his/her "MPU". We call this set the most preferred set. Set M consists of indices of units belonging to the named set. This seems to happen for two reasons:

1) he/she is simply unable to point out only one MPU. A group "represents" better than one point the MPU he/she has in mind, or

2) he/she thinks that "good performance" means different things.

Let us consider the first reason. The MPS is clearly a combination of named units. How to find out what the DM has really in his/her mind, requires the use of an analytic tool. For instance, to find the "best" linear combination of the units to represent the MPS, we may use e.g. the Analytic Hierarchy Process (AHP) (developed by Saaty [1980]). However, in practice, the DM may feel that this approach is too time consuming. A simple way is to use the unweighted mean for this purpose:

$$(y^m, x^m) = \frac{1}{|M|} \sum_{i \in M} (y^i, x^i),$$

where $|M|$ is referring to the number of elements in M.

All units $(y^i, x^i)$, $i \in M$, may be efficient and on the same facet, efficient but from different facets, or some of them may be inefficient. Only in the case when the efficient units are picked from the same facet, is point $(y^m, x^m)$ efficient. In other cases, it is generally inefficient. When just picking out the units from the original data sets or on the basis of efficiency scores, we can never be sure if the units are on the same facet. That's why in the case of several units, we always have to test the efficiency of the unit

($y^m$, $x^m$) by projecting it onto the efficient frontier by using model (3.3). The solution vector $\lambda*$ determines, which nonnegativity constraints are relaxed.

Table 4 illustrates two cases. First, we assume that the DM names units #3 and #10 as his/her most preferred set. Because these units lie on the same facet, so does the mean of those units, and in value efficiency analysis nonnegativity constraints corresponding to those units are relaxed. Note that in this case value efficiency score is never higher than the smallest value efficiency score computed for units #3 and # 10 separately. The situation will change if the units are not on the same facet. That situation is demonstrated by assuming that the DM has chosen units #3 and #8 to belong to the most preferred set. However, these units are not on the same facet and the mean ($y^m$, $x^m$) is therefore not efficient. When it is projected with model (3.3) onto the efficient facet, we will get the facet characterized by units #8, #10, and #25. This means that the strictly positive components of $\lambda*$-vectors correspond to those units. In value efficiency analysis, the nonnegativity constraints of the components of the $\lambda*$-vector corresponding to those units are relaxed, not the components corresponding to units #3 and #8. In this case, value efficiency score may be higher than the smallest one corresponding to units #3 and #8.

In the case when the DM is thinking that "good performance" means different things, and more than one MPU is needed to illustrate the pluralistic nature of good performance, we may analyze the problem as follows. We perform value efficiency analysis for each different MPU. Each observation is associated with each MPU a priori or a posteriori. A priori association means that we select in advance the MPU used as an example to a unit. The selection can be made according to some specific values of input and output variables, or some uncontrollable characteristics of units. A posteriori association means that the classification of units is made afterwards. Each unit is associated to the MPU having the highest value efficiency score.

**Table 4**: Efficiency and Value Efficiency Scores.

| | Efficiency | Value Efficiency Score | | | | |
|---|---|---|---|---|---|---|
| | | Firm # 3 | Firm # 8 | Firm # 10 | Firms # 3 & 10 | Firms # 3 & 8 $\Rightarrow$ # 8 & 10 & 25 |
| FIRM1 | 0.821 | 0.794 | 0.768 | 0.821 | 0.794 | 0.720 |
| FIRM2 | 0.772 | 0.663 | 0.772 | 0.772 | 0.663 | 0.758 |
| FIRM3 | 1.000 | 1.000 | 0.931 | 1.000 | 1.000 | 0.931 |
| FIRM4 | 1.000 | 1.000 | 0.661 | 1.000 | 1.000 | 0.661 |
| FIRM5 | 0.769 | 0.689 | 0.769 | 0.769 | 0.689 | 0.757 |
| FIRM6 | 0.806 | 0.780 | 0.805 | 0.806 | 0.780 | 0.805 |
| FIRM7 | 1.000 | 0.815 | 1.000 | 1.000 | 0.815 | 0.953 |
| FIRM8 | 1.000 | 0.731 | 1.000 | 1.000 | 0.731 | 1.000 |
| FIRM9 | 1.000 | 0.743 | 1.000 | 1.000 | 0.743 | 1.000 |
| FIRM10 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| FIRM11 | 1.000 | 0.749 | 0.913 | 0.966 | 0.749 | 0.913 |
| FIRM12 | 0.824 | 0.824 | 0.613 | 0.824 | 0.824 | 0.592 |
| FIRM13 | 0.673 | 0.673 | 0.588 | 0.673 | 0.673 | 0.558 |
| FIRM14 | 0.736 | 0.596 | 0.736 | 0.736 | 0.596 | 0.722 |
| FIRM15 | 0.803 | 0.625 | 0.803 | 0.803 | 0.625 | 0.802 |
| FIRM16 | 0.978 | 0.858 | 0.948 | 0.978 | 0.858 | 0.881 |
| FIRM17 | 0.930 | 0.884 | 0.914 | 0.930 | 0.884 | 0.887 |
| FIRM18 | 0.817 | 0.767 | 0.613 | 0.807 | 0.767 | 0.605 |
| FIRM19 | 0.969 | 0.716 | 0.969 | 0.959 | 0.716 | 0.915 |
| FIRM20 | 0.804 | 0.681 | 0.774 | 0.804 | 0.681 | 0.703 |
| FIRM21 | 0.858 | 0.793 | 0.600 | 0.846 | 0.793 | 0.600 |
| FIRM22 | 0.876 | 0.854 | 0.724 | 0.876 | 0.854 | 0.716 |
| FIRM23 | 1.000 | 0.960 | 0.973 | 1.000 | 0.960 | 0.973 |
| FIRM24 | 0.973 | 0.787 | 0.838 | 0.907 | 0.787 | 0.819 |
| FIRM25 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

To illustrate the above considerations, we select two MPUs out of nine possible (efficient) units. Firm #3 with the highest value of Sales represents the units with high value on Sales and firm #8 with the lowest value of Sales (among the efficient ones) those with low value on Sales. The units are then classified accordingly. The results are presented in Table 5, where the units are ranked according to Sales.

**Table 5:** Value Efficiency Scores with Different MPUs and Classification of Firms Based on High and Low Values of Sales.

|  | Value Efficiency Score Firm # 3 | Value Efficiency Score Firm # 8 | Sales | Net Profit | Man Hour | Sales Space |
|---|---|---|---|---|---|---|
| FIRM3 | **1.000** | 0.931 | 225.454 | 10.393 | 126.699 | 4.986 |
| FIRM4 | **1.000** | 0.661 | 185.581 | 10.417 | 153.857 | 6.695 |
| FIRM10 | **1.000** | 1.000 | 163.178 | 6.261 | 89.702 | 4.911 |
| FIRM12 | **0.824** | 0.613 | 142.648 | 2.745 | 112.637 | 5.42 |
| FIRM23 | **0.960** | 0.973 | 134.989 | 4.728 | 80.079 | 3.786 |
| FIRM22 | **0.854** | 0.724 | 128.303 | 3.887 | 95.241 | 4.245 |
| FIRM13 | **0.673** | 0.588 | 127.767 | 2.701 | 106.869 | 6.281 |
| FIRM17 | **0.884** | 0.914 | 121.604 | 3.059 | 74.514 | 5.318 |
| FIRM1 | **0.794** | 0.768 | 115.266 | 1.708 | 79.056 | 4.986 |
| FIRM18 | **0.767** | 0.613 | 107.019 | 2.983 | 94.596 | 3.691 |
| FIRM6 | **0.780** | 0.805 | 103.328 | 4.347 | 76.83 | 4.083 |
| FIRM24 | **0.787** | 0.838 | 98.931 | 1.861 | 68.703 | 2.985 |
| FIRM16 | **0.858** | 0.948 | 95.925 | 0.742 | 59.188 | 3.979 |
| FIRM5 | 0.689 | **0.769** | 84.52 | 2.357 | 65.684 | 4.735 |
| FIRM21 | 0.793 | **0.600** | 81.175 | 5.121 | 90.116 | 3.31 |
| FIRM7 | 0.815 | **1.000** | 78.755 | 0.162 | 50.157 | 2.531 |
| FIRM2 | 0.663 | **0.772** | 75.191 | 1.811 | 60.096 | 3.3 |
| FIRM20 | 0.681 | **0.774** | 70.982 | 0.005 | 54.645 | 3.865 |
| FIRM11 | 0.749 | **0.913** | 70.679 | 2.802 | 56.923 | 2.24 |
| FIRM25 | 1.000 | **1.000** | 66.743 | 7.409 | 62.282 | 3.1 |
| FIRM9 | 0.743 | **1.000** | 65.718 | 1.485 | 48.058 | 2.324 |
| FIRM19 | 0.716 | **0.969** | 65.402 | 0.618 | 47.042 | 3.001 |
| FIRM14 | 0.596 | **0.736** | 62.383 | 1.418 | 54.932 | 3.135 |
| FIRM8 | 0.731 | **1.000** | 59.327 | 1.299 | 44.771 | 2.47 |
| FIRM15 | 0.625 | **0.803** | 55.225 | 1.375 | 48.809 | 4.43 |

It is interesting to notice from Table 5 that for firms with high sales there are five cases out of 13 that would have obtained a higher value efficiency score if the MPU had been firm #8 instead of firm #3. On the other hand, among firms with low values of sales there is only one case where firm #3 would have been a "better" MPU than firm #8. A plausible explanation for this is the mix of output and input values which resembles more that of the other MPU.

In Table 6, we classify the units afterwards. From all nine efficient units we select as MPUs the ones with the best values for the output and input variables. Additionally we select one efficient unit representing "average" units. The bases for the selection can be seen on the last row of Table 6. For each firm we then determine the highest score of value efficiency, which is highlighted in Table 6. The DM can then use the results and select one or more MPUs that correspond to her/his definition or opinion of "good performance".

**Table 6:** Value Efficiency Scores with Different MPUs.

| | Efficiency | Value Efficiency Score | | | | | Max |
|---|---|---|---|---|---|---|---|
| | | Firm # 3 | Firm # 4 | Firm # 8 | Firm # 11 | Firm # 23 | |
| FIRM1 | 0.821 | **0.794** | 0.613 | 0.768 | 0.668 | 0.766 | 0.794 |
| FIRM2 | 0.772 | 0.663 | 0.562 | **0.772** | 0.729 | 0.757 | 0.772 |
| FIRM3 | 1.000 | **1.000** | **1.000** | 0.931 | 0.848 | 0.945 | 1.000 |
| FIRM4 | 1.000 | **1.000** | **1.000** | 0.661 | 0.865 | **1.000** | 1.000 |
| FIRM5 | 0.769 | 0.689 | 0.501 | **0.769** | 0.619 | 0.671 | 0.769 |
| FIRM6 | 0.806 | 0.780 | 0.767 | 0.805 | 0.793 | **0.806** | 0.806 |
| FIRM7 | 1.000 | 0.815 | 0.578 | **1.000** | **1.000** | **1.000** | 1.000 |
| FIRM8 | 1.000 | 0.731 | 0.545 | **1.000** | 0.933 | 0.958 | 1.000 |
| FIRM9 | 1.000 | 0.743 | 0.637 | **1.000** | **1.000** | **1.000** | 1.000 |
| FIRM10 | 1.000 | **1.000** | **1.000** | **1.000** | 0.980 | **1.000** | 1.000 |
| FIRM11 | 1.000 | 0.749 | 0.833 | 0.913 | **1.000** | **1.000** | 1.000 |
| FIRM12 | 0.824 | **0.824** | 0.735 | 0.613 | 0.703 | 0.811 | 0.824 |
| FIRM13 | 0.673 | **0.673** | 0.600 | 0.588 | 0.544 | 0.647 | 0.673 |
| FIRM14 | 0.736 | 0.596 | 0.480 | **0.736** | 0.674 | 0.705 | 0.736 |
| FIRM15 | 0.803 | 0.625 | 0.337 | **0.803** | 0.518 | 0.589 | 0.803 |
| FIRM16 | 0.978 | 0.858 | 0.565 | **0.948** | 0.764 | 0.867 | 0.948 |
| FIRM17 | 0.930 | 0.884 | 0.657 | **0.914** | 0.741 | 0.815 | 0.914 |
| FIRM18 | 0.817 | 0.767 | 0.758 | 0.613 | **0.817** | **0.817** | 0.817 |
| FIRM19 | 0.969 | 0.716 | 0.460 | **0.969** | 0.776 | 0.851 | 0.969 |
| FIRM20 | 0.804 | 0.681 | 0.405 | **0.774** | 0.592 | 0.688 | 0.774 |
| FIRM21 | 0.858 | 0.793 | 0.850 | 0.600 | **0.858** | **0.858** | 0.858 |
| FIRM22 | 0.876 | 0.854 | 0.828 | 0.724 | 0.833 | **0.876** | 0.876 |
| FIRM23 | 1.000 | 0.960 | **1.000** | 0.973 | **1.000** | **1.000** | 1.000 |
| FIRM24 | 0.973 | 0.787 | 0.760 | 0.838 | **0.973** | **0.973** | 0.973 |
| FIRM25 | 1.000 | **1.000** | **1.000** | **1.000** | **1.000** | **1.000** | 1.000 |
| Selection Criteria | | Sales | Net Profit | Man Hour | Sales Space | "Average" | |

We see from Table 6 that firm #25 is value efficient whichever firm from the proposed five is selected as the MPU. It is important to notice that the degree of association between any unit and a MPU depends on the compatibility of the mixes of output and input values, not their absolute values.

# 4. Concluding Remarks

In this paper, we have considered the use of value efficiency analysis in practice. A key issue for value efficiency analysis is to locate the most preferred solution or the most preferred unit on the efficient frontier. In practice, the decision maker often seems to be more willing just to pick out his/her most preferred solution or most preferred unit without trying to find it using an interactive multiple objective linear programming

technique. In this paper, we have proposed different ways to deal with this kind of an approach.

We have illustrated the use of value efficiency analysis with the data set that was extracted from a real application. In the numerical considerations we have consistently employed the output-oriented BCC model. It is of course possible to perform a general value efficiency analysis and use any of data envelopment analysis models.

Value efficiency analysis gives a response to many practical problems. Here we have considered a few and showed how to use value efficiency analysis in those problems. Value efficiency analysis provides a possibility to understand more deeply the nature of performance at the units under considerations. In addition, value efficiency analysis is as easy to carry out as DEA. It does not need any special models. The only difference to a DEA model is the relaxation of the nonnegativity constraints of the strictly positive variables determining the most preferred solution. People working in practice have found efficiency analysis in general and value efficient analysis in particular useful.

## References

**Arnold V., Bardhan, I., Cooper, W.W., Gallegos, A.** (1997), "Primal and Dual Optimality in Computer Codes Using Two-Stage Solution Procedures in DEA", (Forthcoming in Aronson, J. and S. Zionts (Eds.): <u>Operations Research: Models, Methods and Applications,</u> Kluwer, Norwell. (A Volume in honor of G.L. Thompson)).

**Banker, R.D., Charnes, A. and Cooper, W.W.** (1984), "Some Models for Estimating Technical and Scale Inefficiencies in Data Envelopment Analysis", <u>Management Science</u> 30, 1078-1092.

**Charnes, A., Cooper, W., Lewin, A.Y. and Seiford, L.M.** (1994) <u>Data Envelopment Analysis: Theory, Methodology and Applications,</u> Kluwer Academic Publishers, Norwell.

**Charnes, A., Cooper, W.W. and Rhodes, E.** (1978), "Measuring Efficiency of Decision Making Units", <u>European Journal of Operational Research</u> 2, 429-444.

**Charnes, A., Cooper, W.W. and Rhodes, E.** (1979), "Short Communication: Measuring Efficiency of Decision Making Units", <u>European Journal of Operational Research</u> 3, 339.

**Charnes, A., Cooper, W.W., Wei, Q.L., and Huang, Z. M.** (1989), "Cone Ratio Data Envelopment Analysis and Multi-Objective Programming", <u>International Journal of Systems Science</u>, 20, pp. 1099-1118.

**Dyson, R.G. and E. Thanassoulis** (1988), "Reducing Weight Flexibility in Data Envelopment Analysis", <u>Journal of Operational Research Society</u> 6, 563-576.

**Halme, M., Joro, T., Korhonen, P., Salo, S., and Wallenius, J.** (1998): A Value Efficiency Approach to Incorporating Preference Information in Data Envelopment Analysis", (Forthcoming in <u>Management Science</u>).

**Joro, T., Korhonen, P. and Wallenius, J.** (1998): "Structural Comparison of Data Envelopment Analysis and Multiple Objective Linear Programming", (Forthcoming in <u>Management Science</u>, July 1998).

**Korhonen, P.** (1987): "VIG - A Visual Interactive Support System for Multiple Criteria Decision Making", <u>Belgian Journal of Operations  Research, Statistics and Computer Science</u>, Vol. 27, N:o 1,   pp. 3-15.

**Korhonen, P.** (1988): "A Visual Reference Direction Approach to Solving Discrete Multiple Criteria Problems",  <u>European Journal of Operational Research</u>, Vol. 34, No 2, pp. 152-159.

**Korhonen, P.** (1997), "Searching the Efficient Frontier in Data Envelopment Analysis", Interim Report, IR-97-79, IIASA.

**Korhonen, P., and Laakso, J.** (1986). "A Visual Interactive Method for Solving the Multiple Criteria Problem", <u>European Journal of Operational Research</u>  24,  pp. 277-287.

**Korhonen, P. and Wallenius, J.** (1988), "A Pareto Race", <u>Naval Research Logistics</u> 35, 615-623.

**Roll, Y., Cook, W., and Golany, B.** (1991), "Controlling Factor Weights in Data Envelopment Analysis", <u>IEEE Transactions</u>, 23, pp. 2-9.

**Saaty, T.** (1980), <u>The Analytic Hierarchy Process</u>, McGraw-Hill, New York.

**Thompson, R., Singleton, F., Thrall, R., Smith, B.** (1986), "Comparative Site Evaluations for Locating a High-Energy Physics Labs in Texas", <u>Interfaces</u>, 16, pp. 35-49.

**Wong, Y. and Beasley, J. E.** (1990), "Restricting Weight Flexibility in Data Envelopment Analysis", <u>Journal of Operational Research Society</u>, 41, pp. 829-835.