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Equilibrium and Guaranteeing Solutions in Evolutionary Nonzero Sum Games

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Abstract

Advanced methods of theory of optimal guaranteeing control and techniques of generalized (viscosity, minimax) solutions of Hamilton-Jacobi equations are applied to nonzero sum game interaction between two large groups (coalitions) of agents (participants) arising in economic and biological evolutionary models. Random contacts of agents from different groups happen according to a control dynamical process which can be interpreted as Kolmogorov's differential equations in which coefficients describing flows are not fixed a priori and can be chosen on the feedback principle. Payoffs of coalitions are determined by the functionals of different types on infinite horizon. The notion of a dynamical Nash equilibrium is introduced in the class of control feedbacks. A solution based on feedbacks maximizing with the guarantee the own payoffs (guaranteeing feedbacks) is proposed. Guaranteeing feedbacks are constructed in the framework of the theory of generalized solutions of Hamilton-Jacobi equations. The analytical formulas are obtained for corresponding value functions. The equilibrium trajectory is generated by guaranteeing feedbacks and its properties are investigated. The considered approach provides new qualitative results for the equilibrium trajectory in evolutionary models. The first striking result consists in the fact that the designed equilibrium trajectory provides better (in some bimatrix games strictly better) index values for both coalitions than trajectories which converge to static Nash equilibria (as, for example, trajectories of classical models with the replicator dynamics). The second principle result implies both evolutionary and revolutionary properties of the equilibrium trajectory: evolution takes place in the characteristic domains of Hamilton-Jacobi equations and revolution at switching curves of guaranteeing feedbacks. The third specific feature of the proposed solution is the "positive" nature of guaranteeing feedbacks which maximize the own payoff unlike the "negative" nature of punishing feedbacks which minimize the opponent payoff and lead to static Nash equilibrium. The fourth concept takes into account the foreseeing principle in constructing feedbacks due to the multiterminal character of payoffs in which future states are also evaluated. The fifth idea deals with the venturous factor of the equilibrium trajectory and prescribes the risk barrier surrounding it. These results indicate promising applications of theory of guaranteeing control for constructing solutions in evolutionary models.

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Introduction

We consider a model of evolutionary nonzero sum game between two coalitions of participants in the framework of differential games theory (see [Krasovskii, Subbotin, 1988], [Krasovskii, 1985]) using, especially, some ideas of the approach for nonantagonistic problems (see [Kleimenov, 1993]), statements and methods of analysis of evolutionary games proposed in [Kryazhimskii, 1994]). We concentrate our attention on constructing dynamical Nash equilibria and guaranteeing feedbacks which maximize corresponding payoffs. We obtain resolving trajectories which give better results than solutions of classical models.

The dynamics of game interaction is related to differential (see [Isaacs, 1965]) and evolutionary game-theoretical models (see [Friedman, 1991], [Young, 1993], [Nelson, Winter, 1982], [Intriligator, 1971] [Hofbauer, Sigmund, 1988]), [Basar, Olsder, 1982] [Vorobyev, 1985], [Kaniovskii, Young, 1994]. Random contacts of participants are represented by a control dynamical process in which corresponding probabilities form the phase vector and informational signals play the role of control parameters. This dynamics can be interpreted as generalization of the well-known Kolmogorov's equations which arise in some stochastic models of mathematical economics and queueing theory. The generalization consists in introducing control parameters instead of fixed coefficients which describe incoming and outgoing flows within coalitions. The process evolves on the infinite interval of time. Payoffs of participants are specified by payoff matrixes. Payoffs of coalitions are defined as average payoffs of participants (payoff mean values). We consider different types of these mean values: terminal - for a fixed time; multiterminal - for a time interval; global terminal - for a limit on the infinite time interval; global integral - for a limit of integral payoffs on the infinite time interval. Note that non-zero sum games with discounting integral payoffs were analyzed in [Tarasyev, 1994]. The global functionals are connected with the foreseeing concept which takes into account not only local terminal interests of coalitions but is oriented also on the global future change.

We introduce the notion of a dynamical Nash equilibrium in the class of control feedbacks. Note that feedbacks generated by classical "punishing" solutions of static bimatrix games give the natural and elementary example of a Nash equilibrium in the dynamical sense. The nature of these feedbacks is antagonistic: they minimize the opponent payoff and don't maximize the own one.

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We propose another approach which is based on the "guaranteeing" concept and provides better results than classical solutions. This new solution is generated by constructions of the theory of positional differential games and involves guaranteeing feedbacks of auxiliary zero sum games (see [Krasovskii, Subbotin, 1988], [Kleimenov, 1993]). We consider these zero sum games in the framework of the theory of viscosity (minimax) solutions of Hamilton-Jacobi equations (see [Subbotin, 1980, 1991, 1995], [Crandall, Lions, 1983, 1984]). We construct analytically value functions and optimal guaranteeing feedbacks and verify the corresponding necessary and sufficient conditions for them formulated in terms of conjugate derivatives [Subbotin, Tarasyev, 1985]. The synthesis of guaranteeing feedbacks is determined by switching curves (generated by value functions) for control signals.

Stress once again that guaranteeing feedbacks maximize with the guarantee payoffs of coalitions in contrast to classical static strategies which punish each other by minimizing opponent payoffs. We generate equilibrium trajectories using these switching curves. Equilibrium trajectories have evolutionary properties as well as revolutionary one. An evolution takes place when equilibrium trajectories develop along the characteristics of Hamilton-Jacobi equations and a revolution happens on switching curves where coalitions change their behaviors. We consider the venturous factor of equilibrium trajectories and prescribe the risk barrier surrounding it.

The behavior of new equilibrium solutions generated by bang-bang control synthesis differs qualitatively from the evolution of trajectories of classical models with the replicator dynamics. Remind that these trajectories converge a fortiori to a static Nash equilibrium or circulate in its neighborhood (see [Young, 1993], [Nelson and Winter, 1982], [Hofbauer, Sigmund, 1988]). The new equilibrium solutions are disposed in the intersection of domains in which the payoffs values are better than the corresponding values calculated at a static Nash equilibrium. Examples of "almost antagonistic" games show that these trajectories converge to the points of intersection of switching curves - to the "new" points of equilibrium with better index values.

1 Nonzero Sum Evolutionary Game. Dynamical Nash Equilibrium

1.1 Dynamics, Payoff Functionals

Let us consider the system of differential equations which describes behavioral dynamics for two coalitions (populations)

$$\begin{aligned} \dot{x} &= -x + u \\ \dot{y} &= -y + v \end{aligned} \tag{1.1}$$

Assume that parameter x, $0 \leq x \leq 1$ is the probability of the fact that a randomly taken individual of the first coalition holds the first strategy (respectively, $(1-x)$ is the probability of playing the second one). Parameter y, $0 \leq y \leq 1$ is the probability of choosing the first strategy by an individual of the second coalition (respectively, $(1-y)$ is the probability of playing the second strategy). Control parameters u and v satisfy the restrictions $0 \leq u \leq 1$, $0 \leq v \leq 1$ and can be interpreted as signals for individuals to change their strategies. For example, the value $u = 0$ ($v = 0$) corresponds to the signal: "change the first strategy for the second one", the value $u = 1$ ($v = 1$) corresponds to the signal: "change the second strategy for the first one" and the value $u = x$ ($v = y$) corresponds to the signal: "keep previous strategies".

Let us note that the ground for dynamics (1.1) and its properties were developed in [Kryazhimskii, 1994], [Tarasyev, 1994]. In this dynamics we generalized Kolmogorov's differential equations assuming that coefficients of incoming and outgoing flows inside coalitions are not given a priori and can be designed in the control process on the feedback principle.

For interpretations of dynamics (1.1) let us consider the game interaction of two large group of firms (or their capital investments) on two markets. Let x be a part of facilities which firms of the first coalition (it may be a financial or industrial group) invest into the first market (it may be a market of currencies, goods or new technologies [Vorobyev, 1985], [Kaniovskii, Young, 1994]) and $(1-x)$ - into the second market respectively. Let y be a part of facilities which the second coalition invests into the first market and $(1-y)$ - into the second one. Assume that activity of coalitions on the markets can be regulated by managing councils. Managing councils using control parameters u and v can influence on the distribution of facilities x and y. The dynamics of this influence is described by the system (1.1) and provides some inertness (or independence) of firms with respect to control signals u, v since velocities $\dot{x}$, $\dot{y}$ of changing of distributions x, y are not proportional directly to signals but depend on distributions. For example, the first equation in (1.1) means that according to the signal $u = 0$ distribution x decreases with the diminishing velocity $\dot{x} = -x$.

We assume that the payoff of a participant from the first coalition is described by the payoff matrix $A = \{a_{ij}\}$, and the payoff of a participant from the second one - by the payoff matrix $B = \{b_{ij}\}$.

Specifically, for the game of firms on two markets we can consider the following situation. Assume that the first market is more profitable for investments than the second one and so the following relations hold for payoff matrixes A and B

$$a_{11} > a_{2j}, \quad a_{12} > a_{21}$$
$$b_{12} > b_{2j}, \quad b_{11} > b_{22}, \quad j = 1, 2$$

Let us suppose also that firms of the first coalition are stronger than firms of the second one. They try to conquer both markets and hence the payoff matrix A has the dominating main diagonal

$$a_{ii} > a_{ij}, \quad i \neq j, \quad i, j = 1, 2$$

Firms of the second coalition try to avoid interactions on the same market with firms of the first coalition and, therefore, the payoff matrix B has the dominating secondary diagonal

$$b_{ij} > b_{ii}, \quad i \neq j, \quad i, j = 1, 2$$

The terminal payoff functionals of coalitions are defined as mathematical expectations corresponding to payoff matrixes A, B and can be interpreted as "local" interests of coalitions (populations)

$$\begin{aligned} g_A(x(T), y(T)) &= a_{11}x(T)y(T) + a_{12}x(T)(1-y(T)) + \\ &\quad a_{21}(1-x(T))y(T) + a_{22}(1-x(T))(1-y(T)) = \\ &\quad C_A x(T)y(T) - \alpha_1 x(T) - \alpha_2 y(T) + a_{22} \end{aligned} \tag{1.2}$$

$$\begin{aligned} g_B(x(T), y(T)) &= b_{11}x(T)y(T) + b_{12}x(T)(1-y(T)) + \\ &\quad b_{21}(1-x(T))y(T) + b_{22}(1-x(T))(1-y(T)) = \\ &\quad C_B x(T)y(T) - \beta_1 x(T) - \beta_2 y(T) + b_{22} \end{aligned} \tag{1.3}$$

at a given instant T. Here parameters C_A, α_1, α_2 and C_B, β_1, β_2 are determined according to the classical theory of bimatrix games (see, for example, [Vorobyev, 1985])

$$\begin{aligned} C_A &= a_{11} - a_{12} - a_{21} + a_{22} \\ \alpha_1 &= a_{22} - a_{12} \\ \alpha_2 &= a_{22} - a_{21} \end{aligned} \tag{1.4}$$

$$\begin{aligned} C_B &= b_{11} - b_{12} - b_{21} + b_{22} \\ \beta_1 &= b_{22} - b_{12} \\ \beta_2 &= b_{22} - b_{21} \end{aligned} \tag{1.5}$$

We define "global" interests J_A^∞, J_B^∞ of coalitions (populations) as multifunctions generated by lower and upper limits of mean values

$$\begin{aligned} J_A^\infty &= [J_A^-, J_A^+] \qquad (1.6) \\ J_A^- &= J_A^-(x(\cdot), y(\cdot)) = \liminf_{t\to\infty} g_A(x(t), y(t)) \\ J_A^+ &= J_A^+(x(\cdot), y(\cdot)) = \limsup_{t\to\infty} g_A(x(t), y(t)) \end{aligned}$$

$$\begin{aligned} J_B^\infty &= [J_B^-, J_B^+] \qquad (1.7) \\ J_B^- &= J_B^-(x(\cdot), y(\cdot)) = \liminf_{t\to\infty} g_B(x(t), y(t)) \\ J_B^+ &= J_B^+(x(\cdot), y(\cdot)) = \limsup_{t\to\infty} g_B(x(t), y(t)) \end{aligned}$$

calculated on the trajectories $(x(\cdot), y(\cdot))$ of the system (1.1).

Consider an evolutionary nonzero sum game with the dynamics (1.1) and payoffs (1.6),(1.7). There is an approach (see, for example, [Krasovskii, Subbotin, 1988], [Kleimenov, 1993]) in the differential games theory for constructing equilibrium solutions in the class of feedback strategies $U = u(t, x, y, \varepsilon)$, $V = v(t, x, y, \varepsilon)$ for nonzero sum problems. This approach is based on solving auxiliary zero sum games. In connection with our statements of the problem (see functionals (1.6),(1.7)) we consider zero sum games with the functionals J_A^-, J_A^+, J_B^-, J_B^+. It is known that zero sum problems can be solved and resolving feedbacks can be constructed on the principle of dynamical programming. This principle requires finding value functions. Below we obtain corresponding solutions - value functions and optimal feedbacks for all considered problems in the framework of the theory of generalized (minimax, viscosity) solutions of Hamilton-Jacobi equations.

1.2 Definition of Dynamical Nash Equilibria

Following [Kleimenov, 1993], [Kryazhimskii, 1994] we introduce the notion of a dynamical Nash equilibrium in the class of closed-loop strategies (feedbacks) $U = u(t, x, y, \varepsilon)$, $V = v(t, x, y, \varepsilon)$ for the nonzero sum game with the dynamics (1.1) and multivalued payoff functionals (1.6),(1.7).

Definition 1.1 *Let $\varepsilon > 0$ and $(x_0, y_0) \in [0,1]\times[0,1]$. A pair of feedbacks $U^0 = u^0(t, x, y, \varepsilon)$, $V^0 = v^0(t, x, y, \varepsilon)$ is called a Nash equilibrium for an initial position (x_0, y_0) if for any other feedbacks $U = u(t, x, y, \varepsilon)$, $V = v(t, x, y, \varepsilon)$ the following condition holds: for all trajectories*

$$(x^0(\cdot), y^0(\cdot)) \in X(x_0, y_0, U^0, V^0), \quad (x_1(\cdot), y_1(\cdot)) \in X(x_0, y_0, U, V^0)$$
$$(x_2(\cdot), y_2(\cdot)) \in X(x_0, y_0, U^0, V)$$

the inequalities

$$J_A^-(x^0(\cdot), y^0(\cdot)) \geq J_A^+(x_1(\cdot), y_1(\cdot)) - \varepsilon \tag{1.8}$$

$$J_B^-(x^0(\cdot), y^0(\cdot)) \geq J_B^+(x_2(\cdot), y_2(\cdot)) - \varepsilon \tag{1.9}$$

are valid.

1.3 The Auxiliary Zero-Sum Games

To construct desired equilibrium feedbacks U^0, V^0 we use the approach of [Kleimenov, 1993]. According to this approach, we compose an equilibrium with the help of optimal feedbacks constructed for zero-sum differential games $\Gamma_A = \Gamma_A^- \cup \Gamma_A^+$ and $\Gamma_B = \Gamma_B^- \cup \Gamma_B^+$ with the payoffs J_A^∞ (1.6) and J_B^∞ (1.7). In the game Γ_A the first coalition maximizes with the guarantee the functional $J_A^-(x(\cdot), y(\cdot))$ using a feedback $U = u(t, x, y, \varepsilon)$, and the second coalition attempts, on the contrary, to minimize the functional $J_A^+(x(\cdot), y(\cdot))$ using a feedback $V = v(t, x, y, \varepsilon)$. Conversely, in the game Γ_B the second coalition maximizes with the guarantee the functional $J_B^-(x(\cdot), y(\cdot))$, and the first coalition minimizes the functional $J_B^+(x(\cdot), y(\cdot))$.

Let us introduce the following notations. By $u_A^0 = u_A^0(t, x, y, \varepsilon)$ and $v_B^0 = v_B^0(t, x, y, \varepsilon)$ denote feedbacks solving, respectively, the problem of guaranteeing maximization of the payoff functionals J_A^-, J_B^-. Note that these feedbacks perform guaranteeing maximization of the long term coalitions benefits and, therefore, can be called "positive" ones. By $u_B^0 = u_B^0(t, x, y, \varepsilon)$ and $v_A^0 = v_A^0(t, x, y, \varepsilon)$ we denote feedbacks mostly unfavorable for the opposite coalitions; namely, those minimizing the payoff functionals J_B^+, J_A^+ of the opposite coalitions respectively. These feedbacks can be called "punishment" feedbacks.

Let us note that inflexible solutions of the indicated problems can be obtained in the framework of the classical theory of bimatrix games. Really, assume for the definiteness that

$$C_A > 0, \quad C_B < 0$$
$$0 < x_A = \frac{\alpha_2}{C_A} < 1, \quad 0 < x_B = \frac{\beta_2}{C_B} < 1$$
$$0 < y_A = \frac{\alpha_1}{C_A} < 1, \quad 0 < y_B = \frac{\beta_1}{C_B} < 1$$

One can prove the following statement.

Proposition 1.1 *Differential games Γ_A^-, Γ_A^+ have equal values*

$$v_A^- = v_A^+ = v_A = \frac{a_{22}C_A - \alpha_1\alpha_2}{C_A} \tag{1.10}$$

and differential games Γ_B^-, Γ_B^+ have equal values

$$v_B^- = v_B^+ = v_B = \frac{b_{22}C_B - \beta_1\beta_2}{C_B} \tag{1.11}$$

for any initial position $(x_0, y_0) \in [0,1] \times [0,1]$. These values may be guaranteed, for example, by "positive" feedbacks u_A^{cl}, v_B^{cl} corresponding to classical static solutions x_A, y_B

$$u_A^0 = u_A^{cl} = u_A^{cl}(x, y) = \begin{cases} 0 & \text{if } x_A < x \leq 1 \\ 1 & \text{if } 0 \leq x < x_A \\ [0,1] & \text{if } x = x_A \end{cases} \tag{1.12}$$

$$v_B^0 = v_B^{cl} = v_B^{cl}(x,y) = \begin{cases} 0 & \text{if } y_B < y \leq 1 \\ 1 & \text{if } 0 \leq y < y_B \\ [0,1] & \text{if } y = y_B \end{cases} \tag{1.13}$$

"Punishment" feedbacks are determined by formulas

$$u_B^0 = u_B^{cl} = u_B^{cl}(x,y) = \begin{cases} 0 & \text{if } x_B < x \leq 1 \\ 1 & \text{if } 0 \leq x < x_B \\ [0,1] & \text{if } x = x_B \end{cases} \tag{1.14}$$

$$v_A^0 = v_A^{cl} = v_A^{cl}(x,y) = \begin{cases} 0 & \text{if } y_A < y \leq 1 \\ 1 & \text{if } 0 \leq y < y_A \\ [0,1] & \text{if } y = y_A \end{cases} \tag{1.15}$$

and correspond to classical static solutions x_B, y_A which generate a static Nash equilibrium $NE = (x_B, y_A)$.

Remark 1.1 *Note that "positive" feedbacks (1.12),(1.13) are rather inflexible because they are obtained in the static model of bimatrix games and don't take into account information about dynamics (1.1). Our main goal is to construct flexible "positive" feedbacks which essentially use information about dynamics.*

Remark 1.2 *Values of payoff functions $g_A(x,y)$, $g_B(x,y)$ coincide at points (x_A, y_B), (x_B, y_A)*

$$g_A(x_A, y_B) = g_A(x_B, y_A) = v_A, \quad g_B(x_A, y_B) = g_B(x_B, y_A) = v_B \tag{1.16}$$

But the point $NE = (x_B, y_A)$ is a "mutually punishing" Nash equilibrium and the point x_A, y_B does not have equilibrium properties in the corresponding static game.

1.4 Construction of Nash Equilibria

Let us construct now a Nash equilibrium pair of feedbacks by pasting together "positive" feedbacks u_A^0, v_B^0 and "punishment" feedbacks u_B^0, v_A^0.

Let us choose an initial position $(x_0, y_0) \in [0,1] \times [0,1]$ and an accuracy parameter $\varepsilon > 0$. Choose a trajectory $(x^0(\cdot), y^0(\cdot)) \in X(x_0, y_0, u_A^0(\cdot), v_B^0(\cdot))$ generated by "positive" feedbacks $u_A^0 = u_A^0(t,x,y,\varepsilon)$ and $v_B^0 = v_B^0(t,x,y,\varepsilon)$. Let $T_\varepsilon > 0$ be such that

$$\begin{aligned} g_A(x^0(t), y^0(t)) &> J_A^-(x^0(\cdot), y^0(\cdot)) - \varepsilon \\ g_B(x^0(t), y^0(t)) &> J_B^-(x^0(\cdot), y^0(\cdot)) - \varepsilon \\ t \in [T_\varepsilon, +\infty) & \end{aligned}$$

Denote by $u_A^\varepsilon(t) : [0, T_\varepsilon) \to [0,1]$, $v_B^\varepsilon(t) : [0, T_\varepsilon) \to [0,1]$ step-by-step realizations of strategies u_A^0, v_B^0 such that the corresponding step-by-step motion $(x_\varepsilon(\cdot), y_\varepsilon(\cdot))$ satisfies the condition

$$\max_{t \in [0, T_\varepsilon]} \|(x^0(t), y^0(t)) - (x_\varepsilon(t), y_\varepsilon(t))\| < \varepsilon$$

Using the approach of [Kleimenov, 1993] one can prove the following statement.

Proposition 1.2 *The pair of feedbacks $U^0 = u^0(t,x,y,\varepsilon)$, $V^0 = v^0(t,x,y,\varepsilon)$ pasting together "positive" feedbacks u_A^0, v_B^0 and "punishment" feedbacks u_B^0, v_A^0 in accordance with formulas*

$$U^0 = u^0(t,x,y,\varepsilon) = \begin{cases} u_A^\varepsilon(t) & \text{if} \quad \|(x,y) - (x_\varepsilon(t), y_\varepsilon(t))\| < \varepsilon \\ u_B^0(x,y) & \text{otherwise} \end{cases} \tag{1.17}$$

$$V^0 = v^0(t,x,y,\varepsilon) = \begin{cases} v_B^{\varepsilon}(t) & \textit{if} \quad \|(x,y)-(x_\varepsilon(t),y_\varepsilon(t))\| < \varepsilon \\ v_A^0(x,y) & \textit{otherwise} \end{cases} \tag{1.18}$$

is a dynamical Nash ε-equilibrium

Remark 1.3 *Let us note that the number ε can be interpreted as a parameter of "reliance" of coalitions to each other or a level of "risk" which the coalitions admit in the game. This parameter determines the risk barrier surrounding the equilibrium trajectory $(x_\varepsilon(\cdot), y_\varepsilon(\cdot))$. Coalitions either follow the equilibrium trajectory not leaving the prescribed risk barrier and then they obtain better index values or they violate it and then "punishment" strategies give worse results.*

Remark 1.4 *Consider trajectories which can be generated by a dynamical Nash equilibrium (1.17), (1.18) with inflexible classical feedbacks (1.12)-(1.15). If trajectory $(x^{pos}(\cdot), y^{pos}(\cdot))$ evolves according to "positive" strategies u_A^{cl}, v_B^{cl} (1.12), (1.13) then it converges to the new equilibrium point (x_A, y_B). If coalitions punish each other by strategies u_B^{cl}, v_A^{cl} (1.14),(1.15) then trajectory $(x^{pun}(\cdot), y^{pun}(\cdot))$ converges to a static Nash equilibrium (x_B, y_A). But values of functionals (1.6), (1.7) are equal in both cases (see Remark 1.3).*

$$\begin{aligned} J_A^\infty(x^{pos}(\cdot), y^{pos}(\cdot)) &= J_A^\infty(x^{pun}(\cdot), y^{pun}(\cdot)) = v_A \\ J_B^\infty(x^{pos}(\cdot), y^{pos}(\cdot)) &= J_B^\infty(x^{pun}(\cdot), y^{pun}(\cdot)) = v_B \end{aligned}$$

Below we construct flexible "positive" feedbacks which generate trajectories $(x^{fl}(\cdot), y^{fl}(\cdot))$ converging to the "better" positions than inflexible dynamical equilibria (x_B, y_A), (x_A, y_B) by both criteria $J_A^\infty(x^{fl}(\cdot), y^{fl}(\cdot)) \geq v_A$, $J_B^\infty(x^{fl}(\cdot), y^{fl}(\cdot)) \geq v_B$. To this end in Section 2 we consider auxiliary zero sum games with terminal payoffs and obtain for them complete solutions. In Section 3 we construct lower envelopes of value functions of terminal games and deduce solutions for auxiliary multiterminal games. In Section 4 we derive flexible "positive" feedbacks from the structure of value functions of multiterminal games.

2 The Analytical Solution of the Differential Game with the Terminal Functional

2.1 Value Functions and Generalized Solutions of Hamilton-Jacobi Equations

In this section we consider auxiliary zero sum terminal differential games with dynamics (1.1) and payoff functionals (1.2) and (1.3) respectively. Further solutions of terminal differential games will be used for constructing foreseeing feedbacks by calculating low envelopes for multiterminal functionals. The value functions $w_i(T,t,x,y)$, $i = 1,2$ of terminal games are determined as values of corresponding maximin (minimax) operations

$$\begin{aligned} w_1(T,t_0,x_0,y_0) &= \max_{u(t,x,y)} \min_{(x_1(\cdot),y_1(\cdot))} g_A(x_1(T),y_1(T)) = \\ & \min_{v(t,x,y)} \max_{(x_2(\cdot),y_2(\cdot))} g_A(x_2(T),y_2(T)) \end{aligned} \tag{2.1}$$

$$\begin{aligned} w_2(T,t_0,x_0,y_0) &= \max_{v(t,x,y)} \min_{(x_2(\cdot),y_2(\cdot))} g_B(x_2(T),y_2(T)) = \\ & \min_{u(t,x,y)} \max_{(x_1(\cdot),y_1(\cdot))} g_B(x_1(T),y_1(T)) \end{aligned} \tag{2.2}$$

for every initial position (t_0, x_0, y_0). Here trajectories $(x_1(\cdot), y_1(\cdot))$ are generated by feedback controls $u(t,x,y,\varepsilon)$ and arbitrary behaviors $v(t)$, trajectories $(x_2(\cdot), y_2(\cdot))$ are generated by feedback controls $v(t,x,y,\varepsilon)$ and arbitrary behaviors $u(t)$ from the initial position (t_0, x_0, y_0).

The value functions $w_i(T,t,x,y)$, $i=1,2$ satisfy the principle of dynamical programming which implies the existence of nondecreasing and nonincreasing directions accessible for the dynamical system at every current position (the so-called properties of u and v stability of the value function). At points where the value functions are differentiable these properties turn into the first order partial differential equations of the Hamilton-Jacobi type

$$\frac{\partial w_1}{\partial t} - \frac{\partial w_1}{\partial x}x - \frac{\partial w_1}{\partial y}y + \max_{0\le u\le 1}\frac{\partial w_1}{\partial x}u + \min_{0\le v\le 1}\frac{\partial w_1}{\partial y}v = 0 \tag{2.3}$$

$$\frac{\partial w_2}{\partial t} - \frac{\partial w_2}{\partial x}x - \frac{\partial w_2}{\partial y}y + \min_{0\le u\le 1}\frac{\partial w_2}{\partial x}u + \max_{0\le v\le 1}\frac{\partial w_2}{\partial y}v = 0 \tag{2.4}$$

The value functions $w_i(T,t,x,y)$, $i=1,2$ satisfy also the terminal boundary condition when $t=T$

$$w_1(T,T,x,y) = C_A xy - \alpha_1 x - \alpha_2 y + a_{22} = g_A(x,y) \tag{2.5}$$

$$w_2(T,T,x,y) = C_B xy - \beta_1 x - \beta_2 y + b_{22} = g_B(x,y) \tag{2.6}$$

We consider now the terminal boundary-value problems (2.3),(2.5) and (2.4),(2.6) for value functions $w_1(T,t,x,y)$, $w_2(T,t,x,y)$. We turn our attention to the first problem (2.3),(2.5). It is known (see [Crandall, Lions, 1983, 1985], [Subbotin, 1980, 1991]) that the value function $w_1(T,t,x,y)$ coincides with the generalized (minimax, viscosity) solution of this problem which exists, is unique and determined by the terminal boundary value condition (2.5) and the pair of differential inequalities for conjugate derivatives D^*w_1 and D_*w_1 corresponding to the Hamilton-Jacobi equation (2.3)

$$D^*w_1(T,t,x,y)|(s) \ge H(x,y,s) \tag{2.7}$$

$$D_*w_1(T,t,x,y)|(s) \le H(x,y,s) \tag{2.8}$$

$$(t,x,y) \in [t_0,T) \times (0,1) \times (0,1), \quad s=(s_1,s_2) \in R^2$$

Conjugate derivatives D^*w_1 and D_*w_1 and the Hamiltonian H are given by formulas (see [Subbotin, Tarasyev, 1985])

$$D^*w_1(T,t,x,y)|(s) = \sup_{h\in R^2}(\langle s,h\rangle - \partial_- w_1(T,t,x,y)|(1,h)) \tag{2.9}$$

$$D_*w_1(T,t,x,y)|(s) = \inf_{h\in R^2}(\langle s,h\rangle - \partial_+ w_1(T,t,x,y)|(1,h)) \tag{2.10}$$

$$H(x,y,s) = -s_1 x - s_2 y + \max_{0\le u\le 1} s_1 u + \min_{0\le v\le 1} s_2 v \tag{2.11}$$

Here symbol $\langle s,h\rangle$ denotes the usual inner product of vectors s and h, symbols $\partial_- w_1(T,t,x,y)|(1,h), \partial_+ w_1(T,t,x,y)|(1,h)$ denote Dini directional derivatives of the value function w_1 at a point (t,x,y) in a direction $(1,h), h=(h_1,h_2) \in R^2$

$$\partial_- w_1(T,t,x,y)|(1,h) = \liminf_{\delta\downarrow 0}\frac{w_1(T,t+\delta,x+\delta h_1,y+\delta h_2) - w_1(T,t,x,y)}{\delta} \tag{2.12}$$

$$\partial_+ w_1(T,t,x,y)|(1,h) = \limsup_{\delta\downarrow 0}\frac{w_1(T,t+\delta,x+\delta h_1,y+\delta h_2) - w_1(T,t,x,y)}{\delta} \tag{2.13}$$

For the piecewise smooth value function w_1 directional derivatives and conjugate derivatives can be calculated in the framework of nonsmooth and convex analysis. Let us assume that in some neighborhood $O_\varepsilon(t_*, x_*, y_*)$ of a point $(t_*, x_*, y_*) \in [t_0, T) \times (0,1) \times (0,1)$ the function w_1 is given by formulas

$$w_1(T,t,x,y) = \min_{i\in I}\max_{j\in J}\varphi_{ij}(T,t,x,y) = \max_{j\in J}\min_{i\in I}\varphi_{ij}(T,t,x,y) \tag{2.14}$$

$$w_1(T,t_*,x_*,y_*) = \varphi_{ij}(T,t_*,x_*,y_*), \quad i \in I, \quad j \in J$$

Directional derivatives are determined in this case by relations

$$\partial_- w_1(T,t_*,x_*,y_*)|(1,h) = \partial_+ w_1(T,t_*,x_*,y_*)|(1,h) = \partial w_1(T,t_*,x_*,y_*)|(h) =$$
$$\min_{i\in I}\max_{j\in J}(a_{ij} + \langle b_{ij}, h\rangle) = \max_{j\in J}\min_{i\in I}(a_{ij} + \langle b_{ij}, h\rangle) \tag{2.15}$$

$$a_{ij} = \frac{\partial\varphi_{ij}}{\partial t}$$
$$b_{ij} = \left(\frac{\partial\varphi_{ij}}{\partial x}, \frac{\partial\varphi_{ij}}{\partial y}\right)$$

Assume

$$C = \bigcap_{i\in I} B_i, \quad B_i = co\{b_{ij} : j \in J\}$$
$$D = \bigcap_{j\in J} B_j, \quad B_j = co\{b_{ij} : i \in I\}$$

Conjugate derivatives are determined by relations

$$D^* w_1(T,t_*,x_*,y_*)|(s) = \begin{cases} \max_{i\in I}\min\{-\sum_{j\in J}\lambda_j(s)a_{ij}\} & \text{if } s \in C \\ +\infty & \text{otherwise} \end{cases} \tag{2.16}$$

$$D_* w_1(T,t_*,x_*,y_*)|(s) = \begin{cases} \min_{j\in J}\max\{-\sum_{i\in I}\lambda_i(s)a_{ij}\} & \text{if } s \in D \\ -\infty & \text{otherwise} \end{cases} \tag{2.17}$$

Here coefficients $\lambda_j(s)$ and $\lambda_i(s)$ satisfy the relations

$$\sum_{j\in J}\lambda_j(s)b_{ij} = s, \quad \lambda_j(s) \geq 0, \quad \sum_{j\in J}\lambda_j(s) = 1$$

$$\sum_{i\in I}\lambda_i(s)b_{ij} = s, \quad \lambda_i(s) \geq 0, \quad \sum_{i\in I}\lambda_i(s) = 1$$

2.2 The Description of the Analytical Solution for the Terminal Boundary Value Problem

The terminal boundary value problem (2.3), (2.5) has the analytic solution. The corresponding value function $w_1(T,t,x,y)$ is piecewise smooth and consists of five smooth functions $\varphi_k(T,t,x,y)$, $k = 1,...,5$ which are pasted by operations of maximum and minimum. Analytic formulas for smooth components $\varphi_k(T,t,x,y)$, $k = 1,...,5$ can be obtained via methods of characteristics for corresponding linear Hamilton-Jacobi equations which arise from the nonlinear one (2.3) by substituting different combinations of extremal values 0 and 1 into *max* and *min* expressions. Let us give formulas for these functions

$$\begin{aligned}
\varphi_1(T,t,x,y) &= C_A e^{2(t-T)}xy - \alpha_1 e^{(t-T)}x - \alpha_2 e^{(t-T)}y + a_{22} && (2.18)\\
\varphi_2(T,t,x,y) &= C_A e^{2(t-T)}xy - \alpha_1 e^{(t-T)}x - \\
&\quad (C_A e^{2(t-T)} + (\alpha_2 - C_A)e^{(t-T)})y + \alpha_1 e^{(t-T)} + a_{12} && (2.19)\\
\varphi_3(T,t,x,y) &= C_A e^{2(t-T)}xy - (C_A e^{2(t-T)} + (\alpha_1 - C_A)e^{(t-T)})x - \\
&\quad (C_A e^{2(t-T)} + (\alpha_2 - C_A)e^{(t-T)})y + \\
&\quad C_A e^{2(t-T)} + (\alpha_1 + \alpha_2 - 2C_A)e^{(t-T)} + a_{11} && (2.20)\\
\varphi_4(T,t,x,y) &= C_A e^{2(t-T)}xy - (C_A e^{2(t-T)} + (\alpha_1 - C_A)e^{(t-T)})x - \\
&\quad -\alpha_2 e^{(t-T)}y + \alpha_2 e^{(t-T)} + a_{21} && (2.21)\\
\varphi_5(T,t,x,y) &= \frac{a_{22}C_A - \alpha_1\alpha_2}{C_A} = \frac{a_{11}a_{22} - a_{12}a_{21}}{C_A} = \frac{D_A}{C_A} = v_A && (2.22)
\end{aligned}$$

Here v_A is the value of the static matrix game. Functions φ_k, $k = 1, ..., 5$ are continuously pasted together on four lines $L_m = L_m(T,t)$, $m = 1, ..., 4$

$$L_1 = \{(x,y): \quad x_1(T,t) \le x \le 1, \quad y = y_2(T,t)\} \tag{2.23}$$

$$L_2 = \{(x,y): \quad x = x_1(T,t), \quad y_1(T,t) \le y \le 1\} \tag{2.24}$$

$$L_3 = \{(x,y): \quad 0 \le x \le x_2(T,t), \quad y = y_1(T,t)\} \tag{2.25}$$

$$L_4 = \{(x,y): \quad x = x_2(T,t), \quad 0 \le y \le y_2(T,t)\} \tag{2.26}$$

Here

$$\begin{aligned}
x_1(T,t) &= \max\{0, 1 - (1 - \frac{\alpha_2}{C_A})e^{(T-t)}\}\\
x_2(T,t) &= \min\{1, \frac{\alpha_2}{C_A}e^{(T-t)}\}\\
y_1(T,t) &= \max\{0, 1 - (1 - \frac{\alpha_1}{C_A})e^{(T-t)}\}\\
y_2(T,t) &= \min\{1, \frac{\alpha_1}{C_A}e^{(T-t)}\} && (2.27)
\end{aligned}$$

Let us give the description of the value function w_1.

Proposition 2.1 *The value function $w_1(T,t,x,y)$ is determined by the formula*

$$w_1(T,t,x,y) = \varphi_k(T,t,x,y), \quad \textit{if} \quad (x,y) \in D_k(T,t), \quad k = 1, ..., 5 \tag{2.28}$$

Here domains $D_k = D_k(T,t)$, $k = 1, ..., 5$ are given by inequalities

$$\begin{aligned}
D_1(T,t) &= \{(x,y) \in [0,1]\times[0,1]: \quad x_2(T,t) \le x \le 1, \quad 0 \le y \le y_2(T,t)\}\\
D_2(T,t) &= \{(x,y) \in [0,1]\times[0,1]: \quad x_1(T,t) \le x \le 1, \quad y_2(T,t) \le y \le 1\}\\
D_3(T,t) &= \{(x,y) \in [0,1]\times[0,1]: \quad 0 \le x \le x_1(T,t), \quad y_1(T,t) \le y \le 1\}\\
D_4(T,t) &= \{(x,y) \in [0,1]\times[0,1]: \quad 0 \le x \le x_2(T,t), \quad 0 \le y \le y_1(T,t)\}\\
D_5(T,t) &= \{(x,y) \in [0,1]\times[0,1]: \quad x_1(T,t) \le x \le x_2(T,t),\\
&\quad y_1(T,t) \le y \le y_2(T,t)\} && (2.29)
\end{aligned}$$

Remark 2.1 *Some of domains D_k, $k = 1, ..., 5$ can be empty. In the case when*

$$0 < x_A = \frac{\alpha_2}{C_A} < 1, \quad 0 < y_A = \frac{\alpha_1}{C_A} < 1 \tag{2.30}$$

all domains D_k, $k = 1, ..., 5$ have nonempty interior during the finite interval of time $(T_f, T]$,

$$\begin{aligned} T_f &= \max\{t_{x_1}, t_{x_2}, t_{y_1}, t_{y_2}\} \\ t_{x_1} &= \max\{t : x_1(T,t) = 0\} \\ t_{x_2} &= \max\{t : x_2(T,t) = 1\} \\ t_{y_1} &= \max\{t : y_1(T,t) = 0\} \\ t_{y_2} &= \max\{t : y_2(T,t) = 1\} \end{aligned} \tag{2.31}$$

Then interior parts of domains D_1, D_2, D_3, D_4 disappear at the corresponding moments t_{x_2}, t_{y_2}, t_{x_1}, t_{y_1} and the value function $w_1(T,t,x,y)$ becomes equal to the constant

$$w_1(T,t,x,y) = \varphi_5 = \frac{D_A}{C_A} = v_A \tag{2.32}$$

$$\left(\quad t \le T_d, \quad (x,y) \in [0,1] \times [0,1] \quad \right)$$

$$T_d = \min\{t_{x_1}, t_{x_2}, t_{y_1}, t_{y_2}\} \tag{2.33}$$

On Fig.1 lines L_m, $m = 1, ..., 4$ and domains D_k, $k = 1, ..., 5$ are shown for the case when $x_A = 0.6$, $y_A = 0.4$, $e^{(T-t)} = 1.5$.

2.3 Verification of Differential Inequalities in the Terminal Boundary Value Problem

Let us prove that the necessary and sufficient conditions (2.5), (2.7), (2.8) are valid for the function $w_1(T,t,x,y)$ determined by formulas (2.28), (2.29).

Proof.

It is obvious that the terminal boundary value condition (2.5) is fulfilled for the function $w_1(T,t,x,y)$. Let us verify that the function $w_1(T,t,x,y)$ satisfies differential inequalities (2.7), (2.8). It is not difficult to convince oneself that functions $\varphi_k(T,t,x,y)$, $k = 1, ..., 5$ satisfy Hamilton-Jacobi equation (2.3) at all internal points $D_k^0 = \text{int}\{D_k\}$ of domains D_k, $k = 1, ..., 5$. It remains to verify inequalities (2.7), (2.8) at points of boundaries ∂D_k of these domains, more precisely, at points of lines $L_m(T,t)$, $m = 1, ..., 4$. Let us consider, for example, a part

$$L_1^{12} = L_1^{12}(T,t) = \{(x,y) \in [0,1] \times [0,1] : \quad x_2(T,t) < x \le 1, \quad y = y_2(T,t)\}$$

of the line L_1 at points of which functions φ_1 and φ_2 are pasted together (see Fig.1). We calculate derivatives $\partial\varphi_i/\partial t$, $\partial\varphi_i/\partial x$, $\partial\varphi_i/\partial y$, $i = 1, 2$ on L_1^{12}

$$\begin{aligned} \frac{\partial\varphi_1}{\partial t} &= 2C_A e^{2(t-T)} xy - \alpha_1 e^{(t-T)} x - \alpha_2 e^{(t-T)} y = \alpha_1 e^{(t-T)} x - \frac{\alpha_1\alpha_2}{C_A} \\ \frac{\partial\varphi_1}{\partial x} &= C_A e^{2(t-T)} y - \alpha_1 e^{(t-T)} = 0 \\ \frac{\partial\varphi_1}{\partial y} &= C_A e^{2(t-T)} x - \alpha_2 e^{(t-T)} \end{aligned}$$

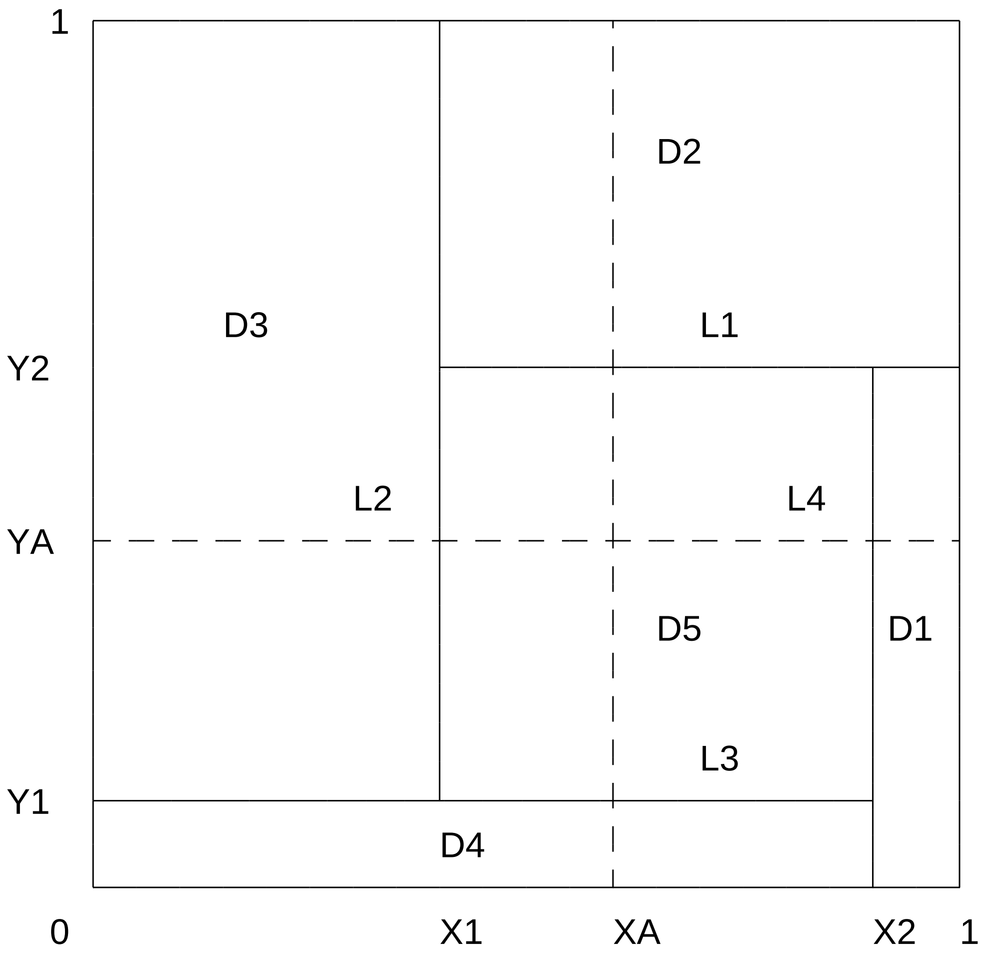

Figure 1: The structure of the value function w_1 in the terminal problem.

$$
\begin{aligned}
\frac{\partial \varphi_2}{\partial t} &= 2C_A e^{2(t-T)}xy - \alpha_1 e^{(t-T)}x - 2C_A e^{2(t-T)}y - (\alpha_2 - C_A)e^{(t-T)}y \\
&\quad + \alpha_1 e^{(t-T)} = \alpha_1 e^{(t-T)}x - \frac{\alpha_1\alpha_2}{C_A} - \alpha_1(e^{(t-T)} - 1) \\
\frac{\partial \varphi_2}{\partial x} &= C_A e^{2(t-T)}y - \alpha_1 e^{(t-T)} = 0 \\
\frac{\partial \varphi_2}{\partial y} &= C_A e^{2(t-T)}x - (C_A e^{2(t-T)} + (\alpha_2 - C_A)e^{(t-T)})
\end{aligned}
$$

One can see that partial derivatives $\partial\varphi_1/\partial y$ and $\partial\varphi_2/\partial y$ are connected at points of the set L_1^{12} by inequalities

$$0 \le \frac{\partial \varphi_1}{\partial y} \le \frac{\partial \varphi_2}{\partial y}$$

Therefore, functions φ_1 and φ_2 are pasted on the line L_1^{12} by the operation of the maximum type. Thus, for function $w_1(T,t,x,y)$ the following relation

$$w_1(T,t,x,y) = \max\{\varphi_1(T,t,x,y), \varphi_2(T,t,x,y)\}$$

is valid in some neighborhood of the set L_1^{12}. Besides that, one can obtain the equalities

$$\frac{\partial \varphi_1}{\partial x} = \frac{\partial \varphi_2}{\partial x} = 0$$

Hence, the directional derivative in a direction $(1,h) = (1,h_1,h_2)$ is determined by the formula

$$\partial w_1(T,t,x,y)|(1,h) = \max\{\frac{\partial \varphi_1}{\partial t} + \frac{\partial \varphi_1}{\partial y}h_2, \frac{\partial \varphi_2}{\partial t} + \frac{\partial \varphi_2}{\partial y}h_2\}$$

Then for the conjugate derivatives we obtain relations

$$D^* w_1(T,t,x,y)|(s) = \begin{cases} -\lambda a_1 - (1-\lambda)a_2 & \text{if } s_1 = 0 \text{ and } s_2 = \lambda b_1 + (1-\lambda)b_2 \\ +\infty & \text{otherwise} \end{cases}$$

$$D_* w_1(T,t,x,y)|(s) = -\infty$$

Here $0 \le \lambda \le 1$, $a_i = \partial\varphi_i/\partial t$, $b_i = \partial\varphi_i/\partial y$, $i = 1,2$.

Let us remind that we need to verify the pair of conditions

$$D^* w_1(T,t,x,y)|(s) \ge H(x,y,s)$$

$$D_* w_1(T,t,x,y)|(s) \le H(x,y,s)$$

It is obvious that the second inequality is fulfilled. Let us verify the first one. For the vectors

$$s = (s_1, s_2), \quad s_1 = 0, \quad s_2 = \lambda\frac{\partial \varphi_1}{\partial y} + (1-\lambda)\frac{\partial \varphi_2}{\partial y}$$

we have relations

$$D^* w_1(T,t,x,y)|(s) = (1-\lambda)\alpha_1(e^{(t-T)} - 1) - \alpha_1 e^{(t-T)}x + \frac{\alpha_1\alpha_2}{C_A}$$

$$
\begin{aligned}
&H(x,y,s) = \max_{0\le u\le 1} s_1(-x+u) + \min_{0\le v\le 1} s_2(-y+v) = -s_2 y = \\
&(1-\lambda)\alpha_1(e^{(t-T)} - 1) - \alpha_1 e^{(t-T)}x + \frac{\alpha_1\alpha_2}{C_A}
\end{aligned}
$$

Hence, we obtain

$$D^*w_1(T,t,x,y)|(s) = H(x,y,s)$$

for $s \in \mathrm{dom}\{D^*w_1\}$. Thus, differential inequalities (2.7),(2.8) are valid for the function w_1 on the set L_1^{12}.

Let us verify conditions (2.7),(2.8) at points of one more typical pasting line. Consider the set

$$L_4^{15} = L_4^{15}(T,t) = \{(x,y) \in [0,1] \times [0,1] : \quad x = x_2(T,t), \quad y_1(T,t) \le y \le y_2(T,t)\}$$

where functions φ_1 and φ_5 are pasted together (see Fig.1). It is obvious that partial derivatives of the function φ_5 are equal to zero

$$\frac{\partial \varphi_5}{\partial t} = \frac{\partial \varphi_5}{\partial x} = \frac{\partial \varphi_5}{\partial y} = 0$$

As is easily seen the following inequality takes place

$$\frac{\partial \varphi_1}{\partial x} = C_A e^{2(t-T)} y - \alpha_1 e^{(t-T)} \le 0 = \frac{\partial \varphi_5}{\partial x}$$

on the set L_4^{15} since $y \le (\alpha_1 e^{(T-t)})/C_A$ for points $(x,y) \in L_4^{15}$. Therefore, functions φ_1 and φ_5 are pasted with the help of the minimum type operation

$$w_1(T,t,x,y) = \min\{\varphi_1(T,t,x,y), \varphi_5(T,t,x,y)\}$$

on the set L_4^{15}. Taking into account that

$$\partial \varphi_1/\partial y = \partial \varphi_5/\partial y = 0$$

we obtain the following relations for the directional derivative on the line L_4^{15}

$$\partial w_1(T,t,x,y)|(1,h) = \min\{0, \frac{\partial \varphi_1}{\partial t} + \frac{\partial \varphi_2}{\partial x} h_1\}$$

The conjugate derivatives are determined by formulas

$$D^*w_1(T,t,x,y)|(s) = +\infty$$

$$D_*w_1(T,t,x,y)|(s) = \begin{cases} -\lambda a_1 - (1-\lambda) a_5 & \text{if } s_2 = 0 \text{ and } s_1 = \lambda b_1 + (1-\lambda) b_5 \\ -\infty & \text{otherwise} \end{cases}$$

Here $a_i = \partial \varphi_i/\partial t$, $b_i = \partial \varphi_i/\partial x$, $i = 1, 5$.

It is obvious that

$$D^*w_1(T,t,x,y)|(s) \ge H(x,y,s), \quad s \in R^2$$

Let us calculate the lower conjugate derivative $D_*w_1(T,t,x,y)|(s)$ and the Hamiltonian $H(x,y,s)$ on vectors

$$s = (s_1, s_2), \quad s_2 = 0, \quad s_1 = \lambda \frac{\partial \varphi_1}{\partial x} + (1-\lambda) \frac{\partial \varphi_5}{\partial x}$$

We have relations

$$D_*w_1(T,t,x,y)|(s) = -\lambda \alpha_2 e^{(t-T)} y + \lambda \frac{\alpha_1 \alpha_2}{C_A}$$

$$H(x,y,s) = \max_{0\le u\le 1} s_1(-x+u) + \min_{0\le v\le 1} s_2(-y+v) = -s_1 x = -\lambda\alpha_2 e^{(t-T)}y + \lambda\frac{\alpha_1\alpha_2}{C_A}$$

It is obvious that

$$D_* w_1(T,t,x,y)|(s) = H(x,y,s)$$

for $s \in \mathrm{dom}\{D_* w_1\}$. Thus, differential inequalities (2.7), (2.8) are also fulfilled for the function w_1 on the set L_4^{15}.

The conditions (2.7),(2.8) on other parts of lines L_m, $m = 1, ..., 4$ can be verified analogously. Thus, it is proved that the function $w_1(T,t,x,y)$ determined by formulas (2.28), (2.29) is the generalized (viscosity, minimax) solution of the terminal boundary value problem (2.3), (2.5) and, hence, it coincides with the value function of the corresponding differential game (1.1),(1.2).

3 The Lower Envelope of Terminal Value Functions and the Value Function of the Game with the Multiterminal Payoff Functional

3.1 The Differential Game with the Multiterminal Functional

In the previous section we have obtained the solution for the auxiliary terminal boundary value problem (2.3),(2.5). The solution of this problem (the value function) $w_1(T,t,x,y)$ depends on the terminal instant T. Of course, such a solution is not appropriate in the evolutionary sense because we obtain a "good" result at a single moment T but not at other times including infinity. Therefore, in this section we will construct the value function for the differential game with the multiterminal payoff functional

$$G_A(x(\cdot),y(\cdot)) = \inf_{t_0\le t<+\infty} g_A(x(t),y(t)) \tag{3.1}$$

The functional (3.1) determines the foreseeing principle since it takes into account future states $g_A(x(t),y(t))$ from time t_0 till infinity $+\infty$.

Using results of differential games theory (see [Krasovskii, Subbotin, 1988]) and viability theory (see [Aubin, 1990]) one can prove that the zero-sum differential game with the dynamics (1.1) and the payoff (3.1) has the value

Theorem 3.1 *There exists the saddle point determining the stationary value function*

$$\begin{aligned}
&\sup_{u(t,x,y,\varepsilon)} \inf_{(x_1(\cdot),y_1(\cdot))} \inf_{s\in[t_0,+\infty)} g_A(x_1(s),y_1(s)) = \\
&\inf_{v(t,x,y,\varepsilon)} \sup_{(x_2(\cdot),y_2(\cdot))} \inf_{s\in[t_0,+\infty)} g_A(x_2(s),y_2(s)) = \\
&\lim_{T\to+\infty} \min_{v(t,x,y,\varepsilon)} \max_{(x_2(\cdot),y_2(\cdot))} \min_{s\in[t_0,T]} g_A(x_2(s),y_2(s)) = \\
&\lim_{T\to+\infty} \max_{u(t,x,y,\varepsilon)} \min_{(x_1(\cdot),y_1(\cdot))} \min_{s\in[t_0,T]} g_A(x_1(s),y_1(s)) = \\
&w_A(t_0,x_0,y_0) = w_A(x_0,y_0)
\end{aligned} \tag{3.2}$$

Here trajectories $(x_1(\cdot),y_1(\cdot))$, $(x_2(\cdot),y_2(\cdot))$ are generated from the initial position (t_0,x_0,y_0) by feedback controls $u(t,x,y,\varepsilon)$, $v(t,x,y,\varepsilon)$ of maximizing and minimizing players respectively and arbitrary controls of their opponents.

Proof.

The proof follows from the theorem on alternative (see [Krasovskii, Subbotin, 1988]), stationary property of dynamics (1.1), finiteness of values of functional G_A (3.1) and can be carried out via the notion of viability kernel (see [Aubin, 1990], [Sonnevend, 1981]). The scheme of the proof is the following.

In the general case inequalities take place

$$\begin{aligned}
&\sup_{u(t,x,y,\varepsilon)} \inf_{(x_1(\cdot),y_1(\cdot))} \inf_{s\in[t_0,+\infty)} g_A(x_1(s),y_1(s)) \leq \\
&\inf_{v(t,x,y,\varepsilon)} \sup_{(x_2(\cdot),y_2(\cdot))} \inf_{s\in[t_0,+\infty)} g_A(x_2(s),y_2(s)) \leq \\
&\lim_{T\to+\infty} \min_{v(t,x,y,\varepsilon)} \max_{(x_2(\cdot),y_2(\cdot))} \min_{s\in[t_0,T]} g_A(x_2(s),y_2(s)) = \\
&\lim_{T\to+\infty} \max_{u(t,x,y,\varepsilon)} \min_{(x_1(\cdot),y_1(\cdot))} \min_{s\in[t_0,T]} g_A(x_1(s),y_1(s)) = \\
&w_A(t,x,y)
\end{aligned} \tag{3.3}$$

One can verify the following properties of the function $w_A(t,x,y)$ (3.3).

Proposition 3.1 *The function w_A is a stationary one (it does not depend on t). More precisely,*

$$w_A(t,x,y) = w_A(s,x,y) = w_A(x,y) \tag{3.4}$$

for all $(x,y) \in [0,1]\times[0,1]$, $t \in R$, $s \in R$.

Proposition 3.2 *The function w_A satisfies the Lipschitz condition*

$$|w_A(x_1,y_1) - w_A(x_2,y_2)| \leq K(|x_1-x_2| + |y_1-y_2|) \tag{3.5}$$

for all $(x_i,y_i) \in [0,1]\times[0,1]$, $i=1,2$.

Proposition 3.3 *The function w_A is majorized by the payoff g_A*

$$w_A(x,y) \leq g_A(x,y), \quad (\quad (x,y) \in [0,1]\times[0,1] \quad) \tag{3.6}$$

Proposition 3.4 *The function w_A is the maximal function which satisfies inequality (3.6) and is subjected to the principle of dynamical programming. The corresponding properties of u-stability and v-stability can be represented in the infinitesimal form as*

$$\min_{0\leq v\leq 1} \max_{0\leq u\leq 1} \partial_+ w_A(x,y)|(-x+u,-y+v) \geq 0 \tag{3.7}$$

for all $(x,y) \in (0,1)\times(0,1)$, and

$$\max_{0\leq u\leq 1} \min_{0\leq v\leq 1} \partial_- w_A(x,y)|(-x+u,-y+v) \leq 0 \tag{3.8}$$

for all $(x,y) \in (0,1)\times(0,1)$ where $w_A(x,y) < g_A(x,y)$. Here the directional derivatives $\partial_- w_A(x,y)|(h_1,h_2)$, $\partial_+ w_A(x,y)|(h_1,h_2)$ of the function w_A at a point $(x,y) \in (0,1)\times(0,1)$ along a direction $h=(h_1,h_2)$ are determined by

$$\partial_- w_A(x,y)|(h_1,h_2) = \liminf_{\delta\downarrow 0} \frac{w_A(x+\delta h_1, y+\delta h_2) - w_A(x,y)}{\delta} \tag{3.9}$$

$$\partial_+ w_A(x,y)|(h_1,h_2) = \limsup_{\delta\downarrow 0} \frac{w_A(x+\delta h_1, y+\delta h_2) - w_A(x,y)}{\delta} \tag{3.10}$$

Proposition 3.5 *Properties of u-stability (3.7) and v-stability (3.8) can be rewritten (see, for example, [Subbotin, Tarasyev, 1985]) in terms of conjugate derivatives*

$$D_* w_A(x,y)|(s) \leq H(x,y,s) \tag{3.11}$$

$$(\quad (x,y) \in (0,1)\times(0,1), \quad s=(s_1,s_2) \in R^2 \quad)$$

$$D^* w_A(x,y)|(s) \geq H(x,y,s) \tag{3.12}$$

$$(\quad (x,y) \in (0,1)\times(0,1), \quad w_A(x,y) < g_A(x,y), \quad s=(s_1,s_2) \in R^2 \quad)$$

Here the conjugate derivatives $D^* w_A$, $D_* w_A$ and the Hamiltonian H are determined by

$$D^* w_A(x,y)|(s) = \sup_{h\in R^2} (\langle s,h\rangle - \partial_- w_A(x,y)|(h)) \tag{3.13}$$

$$D_* w_A(x,y)|(s) = \inf_{h\in R^2} (\langle s,h\rangle - \partial_+ w_A(x,y)|(h)) \tag{3.14}$$

$$H(x,y,s) = -s_1 x - s_2 y + \max\{0,s_1\} + \min\{0,s_2\} \tag{3.15}$$

Taking into account stability (viability) properties and using the concept of "extremal shift" strategy (see [Krasovskii, 1985]) one can prove that corresponding trajectories provide the value of functional G_A (3.1) on $[t_0,+\infty)$ equal to the value of function w_A (3.2). Hence, all inequalities in (3.3) turn into equalities and this fact proves Theorem 3.1.

The inverse result is also valid.

Proposition 3.6 *Differential inequalities (3.7),(3.8) (or, equivalently, (3.11),(3.12)) together with the boundary condition (3.6) determine the value function w_A uniquely. More precisely, there exists the single maximal function which satisfies these conditions, and this function coincides with w_A.*

Thus, either the set of Propositions 3.3,3.4,3.6 or the set of Propositions 3.3,3.5,3.6 provide necessary and sufficient conditions for the value function w_A.

Stability properties (3.7),(3.8) (or, equivalently, (3.11),(3.12)) are connected with the stationary Hamilton-Jacobi equation.

Proposition 3.7 *At points $(x,y) \in (0,1)\times(0,1)$ where $w_A(x,y) < g_A(x,y)$ and w_A is differentiable, inequalities (3.7),(3.8) (or (3.11),(3.12)) turn into the stationary Hamilton-Jacobi equation*

$$-\frac{\partial w_A}{\partial x}x - \frac{\partial w_A}{\partial y}y + \max\{0, \frac{\partial w_A}{\partial x}\} + \min\{0, \frac{\partial w_A}{\partial y}\} = 0 \tag{3.16}$$

For piecewise smooth functions we have the following stationary version of stability properties.

Proposition 3.8 *If the value function w_A is piecewise smooth then conjugate derivatives (3.13),(3.14) are determined by*

$$D^* w_A(x_*,y_*)|(s) = \begin{cases} 0 & \textit{if } s \in C \\ +\infty & \textit{otherwise} \end{cases} \tag{3.17}$$

$$D_* w_A(x_*,y_*)|(s) = \begin{cases} 0 & \textit{if } s \in D \\ -\infty & \textit{otherwise} \end{cases} \tag{3.18}$$

Here

$$C = \bigcap_{i \in I} B_i, \quad B_i = co\{b_{ij} : j \in J\}$$
$$D = \bigcap_{j \in J} B_j, \quad B_j = co\{b_{ij} : i \in I\}$$

$$b_{ij} = \left(\frac{\partial \varphi_{ij}}{\partial x}, \frac{\partial \varphi_{ij}}{\partial y} \right)$$
$$w_A(x,y) = \min_{i \in I} \max_{j \in J} \varphi_{ij}(x,y) = \max_{j \in J} \min_{i \in I} \varphi_{ij}(x,y)$$
$$w_A(x_*, y_*) = \varphi_{ij}(x_*, y_*), \quad i \in I, \quad j \in J$$
$$(x,y) \in O_\varepsilon(x_*, y_*)$$

And differential inequalities (3.11),(3.12) turn into formulas

$$0 \leq -s_1 x - s_2 y + \max\{0, s_1\} + \min\{0, s_2\}$$
$$(\quad (x,y) \in (0,1) \times (0,1), \quad s = (s_1, s_2) \in D \quad)$$
$$0 \geq -s_1 x - s_2 y + \max\{0, s_1\} + \min\{0, s_2\}$$
$$(\quad (x,y) \in (0,1) \times (0,1), \quad w_A(x,y) < g_A(x,y), \quad s = (s_1, s_2) \in C \quad)$$

3.2 Description of a Solution to the Game with the Multiterminal Functional

Before describing an analytic solution of the game in question, we introduce the envelopes of the smooth components φ_1, φ_3 (2.18),(2.20) of the value function w_1 (2.28) parametrized by $s = t - T$. Low envelopes provide considering of multiterminal interests and introducing foreseeing principle for designing feedbacks. To construct the envelope ψ_A^1 of φ_1 it is necessary to calculate the derivative with respect to s, set it equal to zero, find the root of the obtained equation, and substitute the root into φ_1. Namely, we have

$$\begin{aligned} \frac{\partial \varphi_1}{\partial s} &= 2C_A e^{2s} xy - \alpha_1 e^s x - \alpha_2 e^s y = 0 \\ e^s &= \frac{\alpha_1 x + \alpha_2 y}{2C_A xy} \\ \psi_A^1(x,y) &= \varphi_1(s,x,y) = a_{22} - \frac{(\alpha_1 x + \alpha_2 y)^2}{4C_A xy} \end{aligned} \tag{3.19}$$

Making the same with φ_3 we obtain its lower envelope ψ_A^2 with respect to s

$$\begin{aligned} \frac{\partial \varphi_3}{\partial s} &= 2C_A e^{2s} xy - \\ & (C_A e^{2s} + (\alpha_1 - C_A)e^s)x - (C_A e^{2s} + (\alpha_2 - C_A)e^s)y + \\ & C_A e^{2s} + (\alpha_1 + \alpha_2 - 2C_A)e^s = 0 \\ e^s &= \frac{(\alpha_1 - C_A)(x-1) + (\alpha_2 - C_A)(y-1)}{2C_A(x-1)(y-1)} \\ \psi_A^2(x,y) &= \varphi_3(s,x,y) = a_{11} - \frac{((C_A - \alpha_1)(1-x) + (C_A - \alpha_2)(1-y))^2}{4C_A(1-x)(1-y)} \end{aligned} \tag{3.20}$$

Similarly we define the lower envelopes ψ_A^3, ψ_A^4 of smooth components ϕ_2, ϕ_4 (2.19),(2.21), respectively

$$\begin{aligned} \psi_A^3(x,y) &= C_A xy - \alpha_1 x - \alpha_2 y + a_{22} && (3.21) \\ \psi_A^4(x,y) &= \frac{a_{22}C_A - \alpha_1\alpha_2}{C_A} = v_A && (3.22) \end{aligned}$$

The smooth functions ψ_A^i, $i = 1,...,4$ are continuously pasted together along the following lines K_A^j, $j = 1,...,5$

$$\begin{aligned} K_A^1 &= \{(x,y) \in [0,1]\times[0,1] : \quad x = \frac{\alpha_2}{C_A}, \quad 0 \le y \le 1\} \\ K_A^2 &= \{(x,y) \in [0,1]\times[0,1] : \quad \frac{\alpha_2}{C_A} \le x \le 1, \\ & \frac{\alpha_1}{C_A} \le y \le 1, \quad y = \frac{\alpha_1}{\alpha_2}x\} \\ K_A^3 &= \{(x,y) \in [0,1]\times[0,1] : \quad 0 \le x \le \frac{\alpha_2}{C_A}, \\ & 0 \le y \le \frac{\alpha_1}{C_A}, \quad y = -\frac{(C_A-\alpha_1)}{(C_A-\alpha_2)}(1-x)+1\} \\ K_A^4 &= \{(x,y) \in [0,1]\times[0,1] : \quad \frac{\alpha_2}{C_A} \le x \le 1, \\ & 0 \le y \le \frac{\alpha_1}{C_A}, \quad y = \frac{\alpha_1 x}{2C_A x - \alpha_2}\} \\ K_A^5 &= \{(x,y) \in [0,1]\times[0,1] : \quad 0 \le x \le \frac{\alpha_2}{C_A}, \\ & \frac{\alpha_1}{C_A} \le y \le 1, \quad y = -\frac{(C_A-\alpha_1)(1-x)}{2C_A(1-x)-(C_A-\alpha_2)}+1\} \end{aligned}$$

Let us pass to the analytic description of the value function w_A.

Proposition 3.9 *In the case when $C_A > 0$ the value function $(x,y) \to w_A(x,y)$ is determined by*

$$w_A(x,y) = \psi_A^i(x,y), \quad \text{if} \quad (x,y) \in E_A^i, \quad i = 1,...,4 \qquad (3.23)$$

Here the domains E_A^i, $i = 1,...,4$ are defined as follows

$$\begin{aligned} E_A^1 &= \{(x,y) \in [0,1]\times[0,1] : \quad \frac{\alpha_2}{C_A} \le x \le 1, \\ & \frac{\alpha_1 x}{2C_A x - \alpha_2} \le y \le \frac{\alpha_1}{\alpha_2}x\} \\ E_A^2 &= \{(x,y) \in [0,1]\times[0,1] : \quad 0 \le x \le \frac{\alpha_2}{C_A}, \\ & -\frac{(C_A-\alpha_1)}{(C_A-\alpha_2)}(1-x)+1 \le y \le -\frac{(C_A-\alpha_1)(1-x)}{2C_A(1-x)-(C_A-\alpha_2)}+1\} \\ E_A^3 &= E_A^{31} \cup E_A^{32} \\ E_A^{31} &= \{(x,y) \in [0,1]\times[0,1] : \quad \frac{\alpha_2}{C_A} \le x \le 1, \\ & 0 \le y \le \frac{\alpha_1 x}{2C_A x - \alpha_2}\} \\ E_A^{32} &= \{(x,y) \in [0,1]\times[0,1] : \quad 0 \le x \le \frac{\alpha_2}{C_A}, \\ & -\frac{(C_A-\alpha_1)(1-x)}{2C_A(1-x)-(C_A-\alpha_2)}+1 \le y \le 1\} \end{aligned}$$

$$
\begin{aligned}
E_A^4 &= E_A^{41} \cup E_A^{42} \\
E_A^{41} &= \{(x,y) \in [0,1] \times [0,1] : \quad \frac{\alpha_2}{C_A} \le x \le 1, \\
&\quad \frac{\alpha_1}{\alpha_2} x \le y \le 1\} \\
E_A^{42} &= \{(x,y) \in [0,1] \times [0,1] : \quad 0 \le x \le \frac{\alpha_2}{C_A}, \\
&\quad 0 \le y \le -\frac{(C_A - \alpha_1)}{(C_A - \alpha_2)}(1-x) + 1\} \qquad (3.24)
\end{aligned}
$$

On Fig.2 lines K_A^j, $j = 1, ..., 5$ and domains E_A^i, $i = 1, ..., 4$ are shown for the case when $C_A = 5$, $\alpha_1 = 2$, $\alpha_2 = 3$.

Remark 3.1 *In the domain E_A^4 the following relations hold*

$$
g_A(x,y) \ge w_A(x,y) = v_A \qquad (3.25)
$$

Remark 3.2 *The positional strategy $U_A^0 = u_A^0(x,y)$ corresponding to the value function w_A (see below relations (4.1)) provides viability of trajectories $(x(\cdot), y(\cdot))$ of the system (1.1) in the domain E_A^4.*

Remark 3.3 *In the case when $C_A < 0$ the value function $(x,y) \to w_A(x,y)$ is determined by relations*

$$
\begin{aligned}
w_A(x,y) &= \psi_A^i(x,y), \quad \text{if} \quad (x,y) \in E_A^i, \quad i = 1, ..., 4 \qquad (3.26) \\
\psi_A^1(x,y) &= a_{21} + \frac{((C_A - \alpha_1)x + \alpha_2(1-y))^2}{4C_A x(1-y)} \\
\psi_A^2(x,y) &= a_{12} + \frac{(\alpha_1(1-x) + (C_A - \alpha_2)y)^2}{4C_A(1-x)y} \\
\psi_A^3(x,y) &= C_A xy - \alpha_1 x - \alpha_2 y + a_{22} \\
\psi_A^4(x,y) &= v_A
\end{aligned}
$$

Here the domains E_A^i, $i = 1, ..., 4$ are defined as follows

$$
\begin{aligned}
E_A^1 &= \{(x,y) \in [0,1] \times [0,1] : \quad \frac{\alpha_2}{C_A} \le x \le 1, \\
&\quad -\frac{(C_A - \alpha_1)}{\alpha_2} x + 1 \le y \le -\frac{(C_A - \alpha_1)x}{2C_A x - \alpha_2} + 1\} \\
E_A^2 &= \{(x,y) \in [0,1] \times [0,1] : \quad 0 \le x \le \frac{\alpha_2}{C_A}, \\
&\quad \frac{\alpha_1(1-x)}{2C_A(1-x) - (C_A - \alpha_2)} \le y \le \frac{\alpha_1}{(C_A - \alpha_2)}(1-x)\} \\
E_A^3 &= E_A^{31} \cup E_A^{32} \\
E_A^{31} &= \{(x,y) \in [0,1] \times [0,1] : \quad \frac{\alpha_2}{C_A} \le x \le 1, \\
&\quad -\frac{(C_A - \alpha_1)x}{2C_A x - \alpha_2} + 1\} \le y \le 1 \\
E_A^{32} &= \{(x,y) \in [0,1] \times [0,1] : \quad 0 \le x \le \frac{\alpha_2}{C_A},
\end{aligned}
$$

$$
\begin{aligned}
& 0 \le y \le \frac{\alpha_1(1-x)}{2C_A(1-x)-(C_A-\alpha_2)}+1\} \\
E_A^4 &= E_A^{41} \cup E_A^{42} \\
E_A^{41} &= \{(x,y) \in [0,1]\times[0,1]: \quad \frac{\alpha_2}{C_A} \le x \le 1, \\
& 0 \le y \le -\frac{(C_A-\alpha_1)}{\alpha_2}x+1\} \\
E_A^{42} &= \{(x,y) \in [0,1]\times[0,1]: \quad 0 \le x \le \frac{\alpha_2}{C_A}, \\
& \frac{\alpha_1}{(C_A-\alpha_2)}(1-x) \le y \le 1\}
\end{aligned} \tag{3.27}
$$

Remark 3.4 *For the matrix B the value function w_B can be determined analogously. In the case when $C_B > 0$ the value function $(x,y) \to w_B(x,y)$ is determined by relations*

$$
w_B(x,y) = \psi_B^i(x,y), \quad \text{if} \quad (x,y) \in E_B^i, \quad i = 1,...,4 \tag{3.28}
$$

$$
\begin{aligned}
\psi_B^1(x,y) &= b_{22} - \frac{(\beta_1 x + \beta_2 y)^2}{4C_B xy} \\
\psi_B^2(x,y) &= b_{11} - \frac{((C_B-\beta_1)(1-x)+(C_B-\beta_2)(1-y))^2}{4C_B(1-x)(1-y)} \\
\psi_B^3(x,y) &= C_B xy - \beta_1 x - \beta_2 y + b_{22} \\
\psi_B^4(x,y) &= v_B = \frac{b_{22}C_B - \beta_1\beta_2}{C_B}
\end{aligned}
$$

Here domains E_B^i, $i = 1,...,4$ are defined as follows

$$
\begin{aligned}
E_B^1 &= \{(x,y) \in [0,1]\times[0,1]: \quad \frac{\beta_1}{C_B} \le y \le 1, \\
& \frac{\beta_2 y}{2C_B y - \beta_1}\} \le x \le \frac{\beta_2}{\beta_1}y\} \\
E_B^2 &= \{(x,y) \in [0,1]\times[0,1]: \quad 0 \le y \le \frac{\beta_1}{C_B}, \\
& -\frac{(C_B-\beta_2)}{(C_B-\beta_1)}(1-y)+1 \le x \le -\frac{(C_B-\beta_2)(1-y)}{2C_B(1-y)-(C_B-\beta_1)}+1\} \\
E_B^3 &= E_B^{31} \cup E_B^{32} \\
E_B^{31} &= \{(x,y) \in [0,1]\times[0,1]: \quad \frac{\beta_1}{C_B} \le y \le 1, \\
& 0 \le x \le \frac{\beta_2 y}{2C_B y - \beta_1}\} \\
E_B^{32} &= \{(x,y) \in [0,1]\times[0,1]: \quad 0 \le y \le \frac{\beta_1}{C_B}, \\
& -\frac{(C_B-\beta_2)(1-y)}{2C_B(1-y)-(C_B-\beta_1)}+1 \le x \le 1\} \\
E_B^4 &= E_B^{41} \cup E_B^{42} \\
E_B^{41} &= \{(x,y) \in [0,1]\times[0,1]: \quad \frac{\beta_1}{C_B} \le y \le 1, \\
& \frac{\beta_2}{\beta_1}y \le x \le 1\}
\end{aligned}
$$

$$
\begin{aligned}
E_B^{42} &= \{(x,y)\in[0,1]\times[0,1]: \quad 0\le y\le \frac{\beta_1}{C_B},\\
&\quad 0\le x\le -\frac{(C_B-\beta_2)}{(C_B-\beta_1)}(1-y)+1\}
\end{aligned} \tag{3.29}
$$

In the case when $C_B<0$ the value function $(x,y)\to w_B(x,y)$ is determined by relations

$$
\begin{aligned}
w_B(x,y) &= \psi_B^i(x,y), \quad \textit{if} \quad (x,y)\in E_B^i, \quad i=1,...,4 \qquad (3.30)\\
\psi_B^1(x,y) &= b_{12}+\frac{(\beta_1(1-x)+(C_B-\beta_2)y)^2}{4C_B(1-x)y}\\
\psi_B^2(x,y) &= b_{21}+\frac{((C_B-\beta_1)x+\beta_2(1-y))^2}{4C_Bx(1-y)}\\
\psi_B^3(x,y) &= C_Bxy-\beta_1x-\beta_2y+b_{22}\\
\psi_B^4(x,y) &= v_B
\end{aligned}
$$

Here domains E_B^i, $i=1,...,4$ are defined as follows

$$
\begin{aligned}
E_B^1 &= \{(x,y)\in[0,1]\times[0,1]: \quad \frac{\beta_1}{C_B}\le y\le 1,\\
&\quad -\frac{(C_B-\beta_2)}{\beta_1}y+1\le x\le -\frac{(C_B-\beta_2)y}{2C_By-\beta_1}+1\}\\
E_B^2 &= \{(x,y)\in[0,1]\times[0,1]: \quad 0\le y\le \frac{\beta_1}{C_B},\\
&\quad \frac{\beta_2(1-y)}{2C_B(1-y)-(C_B-\beta_1)}\le x\le \frac{\beta_2}{(C_B-\beta_1)}(1-y)\}\\
E_B^3 &= E_B^{31}\cup E_B^{32}\\
E_B^{31} &= \{(x,y)\in[0,1]\times[0,1]: \quad \frac{\beta_1}{C_B}\le y\le 1,\\
&\quad -\frac{(C_B-\beta_2)y}{2C_By-\beta_1}+1\le x\le 1\}\\
E_B^{32} &= \{(x,y)\in[0,1]\times[0,1]: \quad 0\le y\le \frac{\beta_1}{C_B},\\
&\quad 0\le x\le \frac{\beta_2(1-y)}{2C_B(1-y)-(C_B-\beta_1)}\}\\
E_B^4 &= E_B^{41}\cup E_B^{42}\\
E_B^{41} &= \{(x,y)\in[0,1]\times[0,1]: \quad \frac{\beta_1}{C_B}\le y\le 1,\\
&\quad 0\le x\le -\frac{(C_B-\beta_2)}{\beta_1}y+1\}\\
E_B^{42} &= \{(x,y)\in[0,1]\times[0,1]: \quad 0\le y\le \frac{\beta_1}{C_B},\\
&\quad \frac{\beta_2}{(C_B-\beta_1)}(1-y)\le x\le 1\}
\end{aligned} \tag{3.31}
$$

3.3 Testing $u-$ and $v-$ Stability for the Multiterminal Game

In this section we prove that for function w_A defined by (3.23),(3.24) the necessary and sufficient conditions for being the value of the multiterminal game, i.e. the boundary condition (3.6) and the differential inequalities (3.11),(3.12), are fulfilled.

Proof.

The boundary condition is obviously fulfilled because functions ψ_A^i, $i = 1, ..., 4$ are the lower envelopes of the terminal solution $w_1(T, t, x, y)$ and, hence,

$$\psi_A^i(x, y) \le \phi_i(t, t, x, y) \le g_A(x, y), \quad (\quad i = 1, ..., 4, \quad (x, y) \in [0, 1] \times [0, 1] \quad)$$

Let us verify now that differential inequalities (3.11),(3.12) are valid for the function w_A. It is not difficult to prove that functions ψ_A^i, $i = 1, 2, 4$, (3.19),(3.20),(3.22) satisfy the Hamilton-Jacobi equation (3.16) at internal points of domains E_A^i, $i = 1, 2, 4$. One can verify that function ψ_A^3 (3.21) coincides with the boundary function g_A and satisfies the inequality

$$-\frac{\partial \psi_A^3}{\partial x} x - \frac{\partial \psi_A^3}{\partial y} y + \max\{0, \frac{\partial \psi_A^3}{\partial x}\} + \min\{0, \frac{\partial \psi_A^3}{\partial y}\} \ge 0$$

at internal points of the domain E_A^3.

It remains to check differential inequalities (3.11),(3.12) on lines K_A^j, $j = 1, ..., 5$. Let us do it on lines K_A^2, K_A^3 (see Fig.2). At points of the line K_A^2 functions ψ_A^1 and ψ_A^4 are continuously pasted together. Let us calculate partial derivatives of these functions

$$\begin{aligned} \frac{\partial \psi_A^1}{\partial x} &= \frac{\alpha_1^2 x^2 - \alpha_2^2 y^2}{4 C_A x^2 y} \\ \frac{\partial \psi_A^1}{\partial y} &= \frac{\alpha_1^2 x^2 - \alpha_2^2 y^2}{4 C_A x y^2} \\ \frac{\partial \psi_A^4}{\partial x} &= 0 \\ \frac{\partial \psi_A^4}{\partial y} &= 0 \end{aligned}$$

As is easily seen these derivatives are equal to zero on line K_A^2

$$\frac{\partial \psi_A^1}{\partial x} = \frac{\partial \psi_A^4}{\partial x} = 0, \quad \frac{\partial \psi_A^1}{\partial y} = \frac{\partial \psi_A^4}{\partial y} = 0$$

In other words functions ψ_A^1 and ψ_A^4 are pasted together smoothly and continuously. Hence, relations (3.11),(3.12) on K_A^2 are fulfilled as equality (3.16) (see Proposition (3.7)). Analogously one can prove smooth pasting of functions ψ_A^2, ψ_A^4 which leads to (3.11),(3.12) in form (3.16) on line K_A^3.

Consider now line K_A^4 (see Fig.2) where functions ψ_A^1 and ψ_A^3 are pasted together. It can be verified that pasting is smooth. For partial derivatives calculated on line K_A^4 we have

$$\begin{aligned} \frac{\partial \psi_A^1}{\partial x} &= \frac{\partial \psi_A^3}{\partial x} = \frac{\alpha_1(\alpha_2 - C_A x)}{2 C_A x - \alpha_2} \\ \frac{\partial \psi_A^1}{\partial y} &= \frac{\partial \psi_A^3}{\partial y} = C_A x - \alpha_2 \end{aligned}$$

Smooth pasting means that function w_A satisfies Hamilton-Jacobi equation on line K_A^4. One can similarly verify smoothness of function w_A on line K_A^5.

Along the last line K_A^1 (see Fig.2) functions ψ_A^3 and ψ_A^4 are pasted together. Their derivatives on K_A^1 are determined by

$$\begin{aligned} \frac{\partial \psi_A^3}{\partial x} &= C_A y - \alpha_1, \quad \frac{\partial \psi_A^4}{\partial x} = 0 \\ \frac{\partial \psi_A^3}{\partial y} &= \frac{\partial \psi_A^4}{\partial y} = 0 \end{aligned}$$

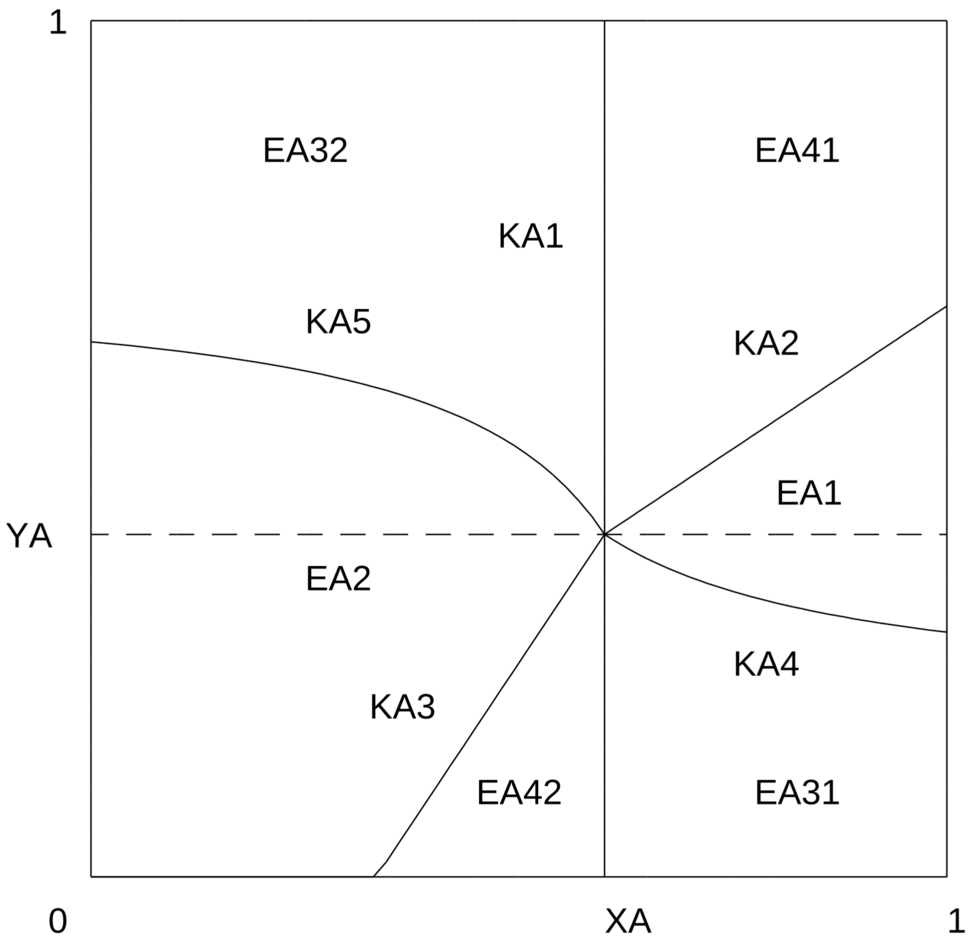

Figure 2: The structure of the value function w_A in the multiterminal problem.

Note that since $w_A = \psi_A^3 = \psi_A^4 = g_A$ on line K_A^1, we need to verify only condition (3.11). As it is easily seen, in a small neighborhood of line K_A^1 we have

$$w_A(x,y) = \min\{\psi_A^3(x,y), \psi_A^4(x,y)\}$$

Hence for points $(x,y) \in K_A^1$ we obtain (see (3.10),(3.14))

$$\begin{aligned} \partial w_A(x,y)|(h_1,h_2) &= \min\{0, (C_A y - \alpha_1)h_1\} \\ D_* w_A(x,y)|(s_1,s_2) &= \begin{cases} 0 & \text{if } s_1 = \lambda(C_A y - \alpha_1) \text{ and } s_2 = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned} \tag{3.32}$$

Here $0 \le \lambda \le 1$. For points $(x,y) \in K_A^1$ and vectors $s = (s_1, s_2)$, $s_1 = \lambda(C_A y - \alpha_1)$, $s_2 = 0$ the Hamiltonian $H(x,y,s)$ is determined by

$$\begin{aligned} H(x,y,s) &= -s_1 x + \max\{0, s_1\} = \\ & \begin{cases} -s_1 x & \text{if } s_1 \le 0 \\ s_1(1-x) & \text{otherwise} \end{cases} \end{aligned}$$

It is clear that for these values the Hamiltonian is larger or equal than the lower conjugate derivative (3.32). Hence, inequality (3.11) on line K_A^1 is proved.

We have verified conditions (3.11),(3.12) for the function w_A at all points of the square $[0,1] \times [0,1]$. Thus, we have proved that the function w_A (3.23),(3.24) is the value function in the game with the multiterminal functional.

4 Flexible "Positive" Feedback Controls Generated by Value Functions of Multiterminal Games

4.1 Description of Optimal Feedback Controls

Let us give the description of the flexible "positive" feedback control $u_A^0 = u_A^{fl} = u_A^{fl}(x,y)$ which solves the problem of guaranteeing maximization for the multiterminal functional $G_A(x_1(\cdot), y_1(\cdot))$ (3.1) on the trajectories $(x_1(\cdot), y_1(\cdot))$ of the system (1.1). This maximizing feedback is constructed via "extremal shift" principle in the direction of the gradient (generalized gradient) of the value function w_A and takes into account its foreseeing nature.

It is not difficult to verify that partial derivative $\partial w_A / \partial x$ of the value function w_A changes its sign on lines K_A^2 and K_A^3 (see Fig.2). Therefore, the optimal feedback control u_A^0 has the following structure (see, for example, [Krasovskii, Subbotin, 1988]). The control parameter $u_A^{fl} = u_A^{fl}(x,y)$ is equal to zero if the current position $(x,y) = (x_1(t), y_1(t))$ lies on the right from the line $K_A = K_A^2 \cup K_A^3$, it is equal to unit if the current position lies on the left from this line and it can take arbitrary values at points of line K_A. Namely, if $C_A > 0$ then

$$u_A^0 = u_A^{fl} = u_A^{fl}(x,y) = \begin{cases} 0 & \text{if } (x,y) \in D_A^1 \\ 1 & \text{if } (x,y) \in D_A^2 \\ \in [0,1] & \text{if } (x,y) \in K_A \end{cases} \tag{4.1}$$

$$\begin{aligned} D_A^1 &= D_A^{11} \cup D_A^{12} \\ D_A^{11} &= \{(x,y) \in [0,1] \times [0,1] : \quad y < \frac{\alpha_1}{\alpha_2} x, \quad y \ge \frac{\alpha_1}{C_A}\} \\ D_A^{12} &= \{(x,y) \in [0,1] \times [0,1] : \quad y < -\frac{(C_A - \alpha_1)}{(C_A - \alpha_2)}(1-x) + 1, \quad y \le \frac{\alpha_1}{C_A}\} \end{aligned}$$

$$\begin{aligned}
D_A^2 &= D_A^{21} \cup D_A^{22} \\
D_A^{21} &= \{(x,y) \in [0,1]\times[0,1]: \quad y > \frac{\alpha_1}{\alpha_2}x, \quad y \geq \frac{\alpha_1}{C_A}\} \\
D_A^{22} &= \{(x,y) \in [0,1]\times[0,1]: \quad y > -\frac{(C_A-\alpha_1)}{(C_A-\alpha_2)}(1-x)+1, \quad y \leq \frac{\alpha_1}{C_A}\} \\
K_A &= K_A^2 \cup K_A^3 \\
K_A^2 &= \{(x,y) \in [0,1]\times[0,1]: \quad y = \frac{\alpha_1}{\alpha_2}x, \quad y \geq \frac{\alpha_1}{C_A}\} \\
K_A^3 &= \{(x,y) \in [0,1]\times[0,1]: \quad y = -\frac{(C_A-\alpha_1)}{(C_A-\alpha_2)}(1-x)+1, \quad y \leq \frac{\alpha_1}{C_A}\} \qquad (4.2)
\end{aligned}$$

On Fig.3 the switch line K_A and domains D_A^1, D_A^2 are depicted. Directions of velocity $\dot{x}$ in domains D_A^1, D_A^2 are shown by arrows.

If $C_A < 0$ then the flexible "positive" feedback u_A^{fl} has the structure (4.1) where the switching line K_A and the domains D_A^1, D_A^2 are determined by

$$\begin{aligned}
D_A^1 &= D_A^{11} \cup D_A^{12} \\
D_A^{11} &= \{(x,y) \in [0,1]\times[0,1]: \quad y > \frac{\alpha_1}{(C_A-\alpha_2)}(1-x), \quad y \geq \frac{\alpha_1}{C_A}\} \\
D_A^{12} &= \{(x,y) \in [0,1]\times[0,1]: \quad y > -\frac{(C_A-\alpha_1)}{\alpha_2}x+1, \quad y \leq \frac{\alpha_1}{C_A}\} \\
D_A^2 &= D_A^{21} \cup D_A^{22} \\
D_A^{21} &= \{(x,y) \in [0,1]\times[0,1]: \quad y < \frac{\alpha_1}{(C_A-\alpha_2)}(1-x), \quad y \geq \frac{\alpha_1}{C_A}\} \\
D_A^{22} &= \{(x,y) \in [0,1]\times[0,1]: \quad y < -\frac{(C_A-\alpha_1)}{\alpha_2}x+1, \quad y \leq \frac{\alpha_1}{C_A}\} \\
K_A &= K_A^2 \cup K_A^3 \\
K_A^2 &= \{(x,y) \in [0,1]\times[0,1]: \quad y = \frac{\alpha_1}{(C_A-\alpha_2)}(1-x), \quad y \geq \frac{\alpha_1}{C_A}\} \\
K_A^3 &= \{(x,y) \in [0,1]\times[0,1]: \quad y = -\frac{(C_A-\alpha_1)}{\alpha_2}x+1, \quad y \leq \frac{\alpha_1}{C_A}\} \qquad (4.3)
\end{aligned}$$

The problem of guaranteeing optimization of the multiterminal functional $G_B(x_2(\cdot), y_2(\cdot))$ for the second coalition is solved analogously. The flexible "positive" feedback control $v_B^0 = v_B^{fl} = v_B^{fl}(x,y)$ has the structure similar to (4.1). Namely, if $C_B > 0$ then the optimal feedback v_B^{fl} is described by

$$v_B^0 = v_B^{fl} = v_B^{fl}(x,y) \begin{cases} 0 & \text{if } (x,y) \in D_B^1 \\ 1 & \text{if } (x,y) \in D_B^2 \\ \in [0,1] & \text{if } (x,y) \in K_B \end{cases} \qquad (4.4)$$

$$\begin{aligned}
D_B^1 &= D_B^{11} \cup D_B^{12} \\
D_B^{11} &= \{(x,y) \in [0,1]\times[0,1]: \quad y > \frac{\beta_1}{\beta_2}x, \quad x \geq \frac{\beta_2}{C_B}\} \\
D_B^{12} &= \{(x,y) \in [0,1]\times[0,1]: \quad y > -\frac{(C_B-\beta_1)}{(C_B-\beta_2)}(1-x)+1, \quad x \leq \frac{\beta_2}{C_B}\} \\
D_B^2 &= D_B^{21} \cup D_B^{22} \\
D_B^{21} &= \{(x,y) \in [0,1]\times[0,1]: \quad y < \frac{\beta_1}{\beta_2}x, \quad x \geq \frac{\beta_2}{C_B}\}
\end{aligned}$$

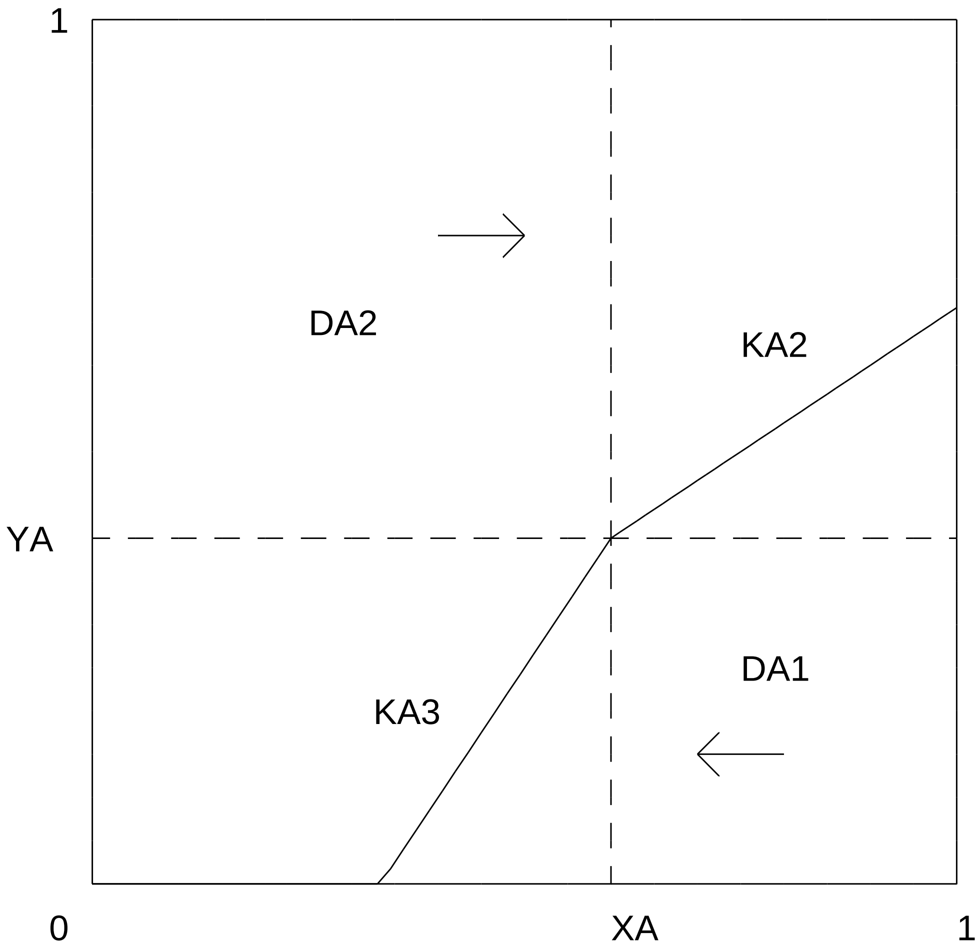

Figure 3: The synthesis of the flexible "positive" feedback control u_A^{fl}.

$$
\begin{aligned}
D_B^{22} &= \{(x,y)\in[0,1]\times[0,1]: \quad y < -\frac{(C_B-\beta_1)}{(C_B-\beta_2)}(1-x)+1, \quad x \le \frac{\beta_2}{C_B}\} \\
K_B &= K_B^2 \cup K_B^3 \\
K_B^2 &= \{(x,y)\in[0,1]\times[0,1]: \quad y = \frac{\beta_1}{\beta_2}x, \quad x \ge \frac{\beta_2}{C_B}\} \\
K_B^3 &= \{(x,y)\in[0,1]\times[0,1]: \quad y = -\frac{(C_B-\beta_1)}{(C_B-\beta_2)}(1-x)+1, \quad x \le \frac{\beta_2}{C_B}\} \qquad (4.5)
\end{aligned}
$$

If $C_B < 0$ then the optimal feedback v_B^{fl} is determined by (4.4) where the switching line K_B and the domains D_B^1, D_B^2 are given by

$$
\begin{aligned}
D_B^1 &= D_B^{11} \cup D_B^{12} \\
D_B^{11} &= \{(x,y)\in[0,1]\times[0,1]: \quad y > -\frac{(C_B-\beta_1)}{\beta_2}x+1, \quad x \ge \frac{\beta_2}{C_B}\} \\
D_B^{12} &= \{(x,y)\in[0,1]\times[0,1]: \quad y > \frac{\beta_1}{(C_B-\beta_2)}(1-x), \quad x \le \frac{\beta_2}{C_B}\} \\
D_B^2 &= D_B^{21} \cup D_B^{22} \\
D_B^{21} &= \{(x,y)\in[0,1]\times[0,1]: \quad y < -\frac{(C_B-\beta_1)}{\beta_2}x+1, \quad x \ge \frac{\beta_2}{C_B}\} \\
D_B^{22} &= \{(x,y)\in[0,1]\times[0,1]: \quad y < \frac{\beta_1}{(C_B-\beta_2)}(1-x), \quad x \le \frac{\beta_2}{C_B}\} \\
K_B &= K_B^2 \cup K_B^3 \\
K_B^2 &= \{(x,y)\in[0,1]\times[0,1]: \quad y = -\frac{(C_B-\beta_1)}{\beta_2}x+1, \quad x \ge \frac{\beta_2}{C_B}\} \\
K_B^3 &= \{(x,y)\in[0,1]\times[0,1]: \quad y = \frac{\beta_1}{(C_B-\beta_2)}(1-x), \quad x \le \frac{\beta_2}{C_B}\} \qquad (4.6)
\end{aligned}
$$

4.2 The Value Provided by Optimal Feedbacks for Multiterminal Payoffs

The optimal feedback $u_A^{fl}(x,y)$ (4.1) guarantees that the current payoff to the first coalition becomes in the long term no worse than the value $v_A = D_A/C_A$, $C_A > 0$ of the zero-sum matrix game for the matrix A (call it the A-matrix game). Note that this result is not obviously worse than the value of the function w_A

$$v_A \ge w_A(x,y), \quad \forall (x,y)\in[0,1]\times[0,1]$$

More precisely, the following statement is true.

Proposition 4.1 *For any initial position $(x_0,y_0)\in[0,1]\times[0,1]$ and any trajectory*

$$(x_1(\cdot),y_1(\cdot)) \in X(x_0,y_0,u_A^{fl}), \quad x_1(t_0)=x_0, \quad y_1(t_0)=y_0, \quad t_0=0$$

generated by the optimal feedback control $u_A^{fl} = u_A^{fl}(x,y)$ there exists a finite time $t_s \in [0,T_A]$

$$T_A = \ln(\max\{\frac{C_A}{\alpha_2}, \frac{C_A}{C_A-\alpha_2}\})$$

such that at this time the trajectory $(x_1(\cdot),y_1(\cdot))$ enters into the domain E_A^4 (see (3.24) or (3.27))

$$(x_1(t_s),y_1(t_s)) \in E_A^4$$

where the value function w_A is equal to the the value v_A of the A-matrix game

$$w_A(x_1(t_s), y_1(t_s)) = v_A,$$

and remains in the domain E_A^4 on the time interval $[t_s, +\infty)$ (and, hence, on the time interval $[T_A, +\infty)$) (see (3.23),(3.24) and Remark (3.2)). Therefore, according to the definition (3.2) of the value function w_A the following inequality takes place

$$g_A(x_1(t), y_1(t)) \geq v_A, \quad (\quad t \geq t_s \quad) \tag{4.7}$$

and, in particular,

$$\liminf_{t \to +\infty} g_A(x_1(t), y_1(t)) \geq v_A \tag{4.8}$$

Proof. Note that the domain E_A^4 (see (3.24) or (3.27) has nonempty intersections with all lines L_λ, $0 \leq \lambda \leq 1$

$$L_\lambda = \{(x, y) \in (0, 1) \times (0, 1) : \quad y = \lambda\}$$

Hence, any possible trajectory $(x_1(\cdot), y_1(\cdot))$ generated by the optimal feedback u_A^{fl} (the values of which are equal to zero or to unit) crosses this domain E_A^4 because the projection of velocity on lines L_λ for such trajectories is not equal to zero and conserves the sign till the moment of intersection of the trajectory with the domain E_A^4.

The analogous statement can be formulated for B-matrix game.

Proposition 4.2 *For any initial position $(x_0, y_0) \in [0, 1] \times [0, 1]$ and any trajectory*

$$(x_2(\cdot), y_2(\cdot)) \in X(x_0, y_0, v_B^{fl}), \quad x_2(t_0) = x_0, \quad y_2(t_0) = y_0, \quad t_0 = 0$$

generated by the optimal feedback control $v_B^{fl} = v_B^{fl}(x, y)$ (4.4) there exists a finite time $t_s \in [0, T_B]$

$$T_B = \ln(\max\{\frac{C_B}{\beta_1}, \frac{C_B}{C_B - \beta_1}\})$$

such that at this time the trajectory $(x_2(\cdot), y_2(\cdot))$ enters into the domain E_B^4 (see (3.29) or (3.31))

$$(x_2(t_s), y_2(t_s)) \in E_B^4$$

where the value function w_B is equal to the the value v_B of the B-matrix game

$$w_B(x_2(t_s), y_2(t_s)) = v_B,$$

and remains in the domain E_B^4 on the time interval $[t_s, +\infty)$ (and, hence, on the time interval $[T_B, +\infty)$). Therefore, according to the definition of the value function w_B the following inequality takes place

$$g_B(x_2(t), y_2(t)) \geq v_B, \quad (\quad t \geq t_s \quad) \tag{4.9}$$

and, in particular,

$$\liminf_{t \to +\infty} g_B(x_2(t), y_2(t)) \geq v_B \tag{4.10}$$

Propositions 4.1, 4.2 imply the following statement.

Proposition 4.3 *The intersection E^0 of sets E_B^4 and E_B^4 is not empty*

$$E^0 = E_A^4 \cap E_B^4 \neq \emptyset \tag{4.11}$$

and, hence, optimal strategies u_A^{fl}, v_B^{fl} generate the trajectory $(x^{fl}(\cdot), y^{fl}(\cdot))$ which enters the intersection E^0 and remains in it on the time interval $[T^0, +\infty)$, $T^0 = \max\{T_A, T_B\}$. In the set E^0 the inequalities

$$g_A(x^{fl}(t), y^{fl}(t)) \geq v_A, \quad g_B(x^{fl}(t), y(^{fl}t)) \geq v_B, \quad t \in [T^0, +\infty) \tag{4.12}$$

are fulfilled. Therefore, the set E^0 can be called the favorable domain for both coalitions.

On Fig.4 domains E_A^4, E_B^4, E^0 are shown for the game with the following payoff matrixes

$$A = \begin{pmatrix} 5 & 3 \\ 2 & 5 \end{pmatrix} \tag{4.13}$$

$$C_A = a_{11} - a_{12} - a_{21} + a_{22} = 5$$

$$\alpha_1 = a_{22} - a_{12} = 2$$

$$\alpha_2 = a_{22} - a_{21} = 3$$

$$x_A = \frac{\alpha_2}{C_A} = 0.6, \quad y_A = \frac{\alpha_1}{C_A} = 0.4$$

$$B = \begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix} \tag{4.14}$$

$$C_B = b_{11} - b_{12} - b_{21} + b_{22} = -5$$

$$\beta_1 = b_{22} - b_{12} = -4$$

$$\beta_2 = b_{22} - b_{21} = -2$$

$$x_B = \frac{\beta_2}{C_B} = 0.4, \quad y_B = \frac{\beta_1}{C_B} = 0.8$$

Directions of velocity $\dot{x}$ for the first coalition are depicted by horizontal arrows and directions of velocity $\dot{y}$ for the second one - by vertical arrows.

Note that matrixes A (4.13) and B (4.14) correspond to the game of firms interacting on two markets which was described in Section 1.

5 A Nash Equilibrium with Flexible "Positive" Feedbacks of Multiterminal Games

5.1 The Structure of a Nash Equilibrium

Let us construct now a Nash equilibrium pair of feedbacks by pasting together flexible "positive" feedbacks $u_A^0 = u_A^{fl}$, $v_B^0 = v_B^{fl}$ (4.1), (4.4) and "punishment" feedbacks $u_B^0 = u_B^{cl}$, $v_A^0 = v_A^{cl}$ (1.14), (1.15). Let us choose an initial position $(x_0, y_0) \in [0,1] \times [0,1]$ and an accuracy parameter $\varepsilon > 0$. Choose a trajectory $(x^{fl}(\cdot), y^{fl}(\cdot)) \in X(x_0, y_0, u_A^{fl}(\cdot), v_B^{fl}(\cdot))$ generated by flexible "positive" feedbacks u_A^{fl} and v_B^{fl}. Let $T_\varepsilon > 0$ be such that

$$\begin{aligned} g_A(x^{fl}(t), y^{fl}(t)) &> J_A^-(x^{fl}(\cdot), y^{fl}(\cdot)) - \varepsilon \\ g_B(x^{fl}(t), y^{fl}(t)) &> J_B^-(x^{fl}(\cdot), y^{fl}(\cdot)) - \varepsilon \\ t \in [T_\varepsilon, +\infty) & \end{aligned}$$

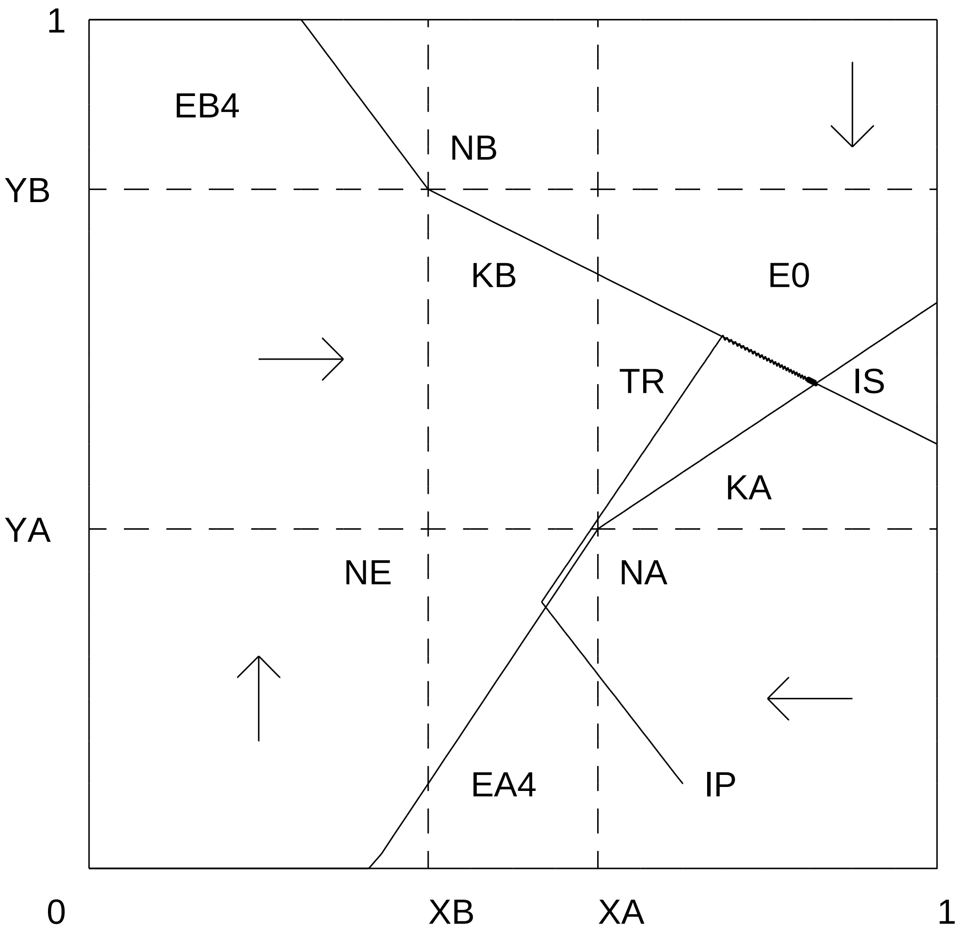

Figure 4: The portrait of guaranteeing feedbacks u_A^{fl}, v_B^{fl} and the Nash equilibrium trajectory $TR = (x^{fl}(\cdot), y^{fl}(\cdot))$.

Denote by $u_A^{fl,\varepsilon}(t) : [0, T_\varepsilon) \to [0, 1]$, $v_B^{fl,\varepsilon}(t) : [0, T_\varepsilon) \to [0, 1]$ step-by-step realizations of strategies u_A^{fl}, v_B^{fl} such that the corresponding step-by-step motion $(x_\varepsilon^{fl}(\cdot), y_\varepsilon^{fl}(\cdot))$ satisfies the condition

$$\max_{t\in[0,T_\varepsilon]} \|(x^{fl}(t), y^{fl}(t)) - (x_\varepsilon^{fl}(t), y_\varepsilon^{fl}(t))\| < \varepsilon$$

Proposition 5.1 *A pair of feedbacks $U^0 = u^0(t, x, y, \varepsilon)$, $V^0 = v^0(t, x, y, \varepsilon)$ pasting together flexible "positive" feedbacks u_A^{fl}, v_B^{fl} (4.1), (4.4) and "punishment" feedbacks u_B^{cl}, v_A^{cl} (1.14), (1.15) in accordance with Proposition 1.2*

$$U^0 = u^0(t, x, y, \varepsilon) = \begin{cases} u_A^{fl,\varepsilon}(t) & \text{if} \quad \|(x, y) - (x_\varepsilon^{fl}(t), y_\varepsilon^{fl}(t))\| < \varepsilon \\ u_B^{cl}(x, y) & \text{otherwise} \end{cases} \tag{5.1}$$

$$V^0 = v^0(t, x, y, \varepsilon) = \begin{cases} v_B^{fl,\varepsilon}(t) & \text{if} \quad \|(x, y) - (x_\varepsilon^{fl}(t), y_\varepsilon^{fl}(t))\| < \varepsilon \\ v_A^{cl}(x, y) & \text{otherwise} \end{cases} \tag{5.2}$$

is a dynamical Nash ε-equilibrium.

Let us remind that the trajectory $(x_\varepsilon^{fl}(\cdot), y_\varepsilon^{fl}(\cdot))$ constitutes the core of the dynamical Nash equilibrium. Therefore, it can be called the equilibrium trajectory. It is generated by guaranteeing feedbacks u_A^{fl} and v_B^{fl} and so provides better index values than at a static Nash equilibrium with the guarantee. This trajectory is surrounded by the risk ε-barrier which gives the possibility for trusting and venturing.

5.2 Trajectories Generated by Flexible "Positive" Feedbacks

The qualitative behavior of trajectories generated by flexible "positive" feedbacks which form the basis of the dynamical Nash equilibrium (5.1),(5.2) is the question of great interest. The complete classification of possible limit points, attractors, cycles or chaotic circulation could form the theme of the future analysis. Preliminary, we can propose the following statement.

Proposition 5.2 *The values of the payoff functionals J_A^-, J_B^- on an arbitrary trajectory $(x^{fl}(\cdot), y^{fl}(\cdot))$ generated by flexible "positive" feedback controls u_A^{fl}, v_B^{fl} (4.1), (4.4) are no worse than the values of these functionals on any trajectory converging to the static Nash equilibrium $(x_B, y_A) = (\beta_2/C_B, \alpha_1/C_A)$ whose components have the sense of coalitions' distributions mostly unfavorable for the opposite coalition. According to Propositions 4.1-4.3 trajectories $(x^{fl}(\cdot), y^{fl}(\cdot))$ enter the favorable domain E^0 and stay in it on the infinite time interval. There are the following possible qualitative behaviors of the trajectory $(x^{fl}(\cdot), y^{fl}(\cdot))$ in the favorable domain E^0: - it can converge to an intersection point of switching lines K_A, K_B;*
- it can tend to points situated on the boundary of the square (in the case when the intersection of lines K_A, K_B is empty);
- it can tend to nonantagonistic static Nash equilibria (in the case when such equilibria exist);
- it can simply circulate in the favorable domain E^0.

For example, on Fig.4 one of the cases mentioned in Proposition 5.2 is shown for the game with the payoff matrixes A (4.13) and B (4.14). The trajectory $TR = (x^{fl}(\cdot), y^{fl}(\cdot))$ generated by flexible "positive" feedbacks u_A^{fl}, v_B^{fl} starts from the initial position $IP = (0.7, 0.1)$ and moves at first along the straight line to the corner $(0, 1)$ of the unit square $[0, 1] \times [0, 1]$. Then it contacts the switching line K_A. Next it moves along the line K_A and further to the corner $(1, 1)$ till the meeting with the switching line K_B. After that it evolves along the line K_B on the border of the domain E^0 altering velocity directions from the corner $(1,1)$ to the corner $(1,0)$ till arriving to the point $IS = K_A \bigcap K_B$ of intersection of the switching lines K_A, K_B. Note that here we give description of the trajectories behavior in the domain E^0 up to the value of the accuracy parameter ε.

Remark 5.1 *Note that the "punishment" feedbacks u_B^{cl}, v_A^{cl} (1.14),(1.15) being components of the dynamical Nash equilibrium (5.1),(5.2) lead trajectories to the unfavorable static Nash equilibrium (x_B, y_A).*

Remark 5.2 *Using constructions of flexible "positive" feedbacks we obtain the unexpected striking result: the equilibrium trajectory $TR = (x^{fl}(\cdot), y^{fl}(\cdot))$ provides better (in the given example strictly better) index values for both coalitions than index values at static Nash equilibrium (x_B, y_A). Hence, the equilibrium trajectory has better properties than trajectories of classical models with replicator and best response dynamics which converge or circulate around static Nash equilibrium (x_B, y_A).*

6 Quasioptimal Feedbacks in Games with Integral Payoff Functionals

6.1 Nash Equilibria in the Nonzero-Sum Differential Game with the Averaged Integral Functionals

In this section we consider the nonzero sum differential game of two coalitions with the averaged integral payoff functionals

$$\begin{aligned} JI_A^{\infty} &= [JI_A^-, JI_A^+] \\ JI_A^- &= JI_A^-(x(\cdot), y(\cdot)) = \liminf_{T\to\infty} \frac{1}{(T-t_*)} \int_{t_*}^{T} g_A(x(t), y(t))dt \\ JI_A^+ &= JI_A^+(x(\cdot), y(\cdot)) = \limsup_{T\to\infty} \frac{1}{(T-t_*)} \int_{t_*}^{T} g_A(x(t), y(t))dt \end{aligned} \tag{6.1}$$

$$\begin{aligned} JI_B^{\infty} &= [JI_B^-, JI_B^+] \\ JI_B^- &= JI_B^-(x(\cdot), y(\cdot)) = \liminf_{T\to\infty} \frac{1}{(T-t_*)} \int_{t_*}^{T} g_B(x(t), y(t))dt \\ JI_B^+ &= JI_B^+(x(\cdot), y(\cdot)) = \limsup_{T\to\infty} \frac{1}{(T-t_*)} \int_{t_*}^{T} g_B(x(t), y(t))dt \end{aligned} \tag{6.2}$$

determined on the trajectories $(x(\cdot), y(\cdot))$ of the system (1.1).

The averaged integral functionals (6.1), (6.2) are traditional for problems of evolutionary economics and biological evolution. Unlike the multiterminal payoffs (3.1) they allow to loose on some intervals in order to win on others and to obtain the better summarized result. This property provides another character of switching curves in guaranteeing feedbacks which give the opportunity for longer staying in profitable domains where $g_A(x,y) > v_A$, $g_B(x,y) > v_B$.

We introduce the notion of dynamical Nash equilibrium for the evolutionary game with the integral functionals (6.1), (6.2) in the same context as for the game with functionals (1.6), (1.7)

Definition 6.1 *A dynamical Nash equilibria* (U^0, V^0), $U^0 = u^0(t, x, y, \varepsilon)$, $V^0 = v^0(t, x, y, \varepsilon)$ *in the class of feedback controls* $U = u(t, x, y, \varepsilon)$, $V = v(t, x, y, \varepsilon)$ *for this problem are defined analogously to Definition 1.1 by inequalities*

$$JI_A^-(x^0(\cdot), y^0(\cdot)) \geq JI_A^+(x_1(\cdot), y_1(\cdot)) - \varepsilon \tag{6.3}$$

$$JI_B^-(x^0(\cdot), y^0(\cdot)) \geq JI_B^+(x_2(\cdot), y_2(\cdot)) - \varepsilon \tag{6.4}$$

$$(x^0(\cdot), y^0(\cdot)) \in X(x_0, y_0, U^0, V^0), \quad (x_1(\cdot), y_1(\cdot)) \in X(x_0, y_0, U, V^0)$$
$$(x_2(\cdot), y_2(\cdot)) \in X(x_0, y_0, U^0, V)$$

Proposition 6.1 *A dynamical Nash equilibrium* (U^0, V^0) *can be constructed according to Proposition 1.2. Namely, in Sections 6.2, 6.3 we find solutions of auxiliary two-step optimal control problems and obtain switching curves* M_A *(6.20),(6.21),* M_B *(6.23),(6.24) for "positive" dual-step feedbacks* $u_A^0 = u_A^{ds}(x, y)$, $v_B^0 = v_B^{ds}(x, y)$ *maximizing integral payoff functionals (6.1),(6.2). Then we paste these "positive" feedbacks* u_A^{ds}, v_B^{ds} *with "punishment" feedbacks* u_B^{cl}, v_A^{cl} *(1.14),(1.15) according to the scheme (1.17),(1.18) and obtain a Nash equilibrium* (U^0, V^0).

6.2 Two-step Optimal Control Problems

In order to construct "positive" feedbacks $u_A^0 = u_A^{ds}(x, y)$, $v_B^0 = v_B^{ds}(x, y)$ we consider in this section the auxiliary two-step optimal control problem with the integral payoff functional for the first coalition when actions of the second coalition are the most unfavorable. More precisely, we analyze the optimal control problem for the dynamical system (1.1)

$$\begin{aligned} \dot{x} &= -x + u, \quad x(0) = x_0 \\ \dot{y} &= -y + v, \quad y(0) = y_0 \end{aligned}$$

with the payoff functional

$$J_A^f = \int_0^{T_f} g_A(x(t), y(t))dt \tag{6.5}$$

We determine the terminal instant $T_f = T_f(x_0, y_0)$ later.

Let us assume for definiteness that the following conditions hold

$$\begin{aligned} &C_A > 0, \quad v_A = \frac{D_A}{C_A} = 0 \\ &0 < x_A = \frac{\alpha_2}{C_A} < 1, \quad 0 < y_A = \frac{\alpha_1}{C_A} < 1 \end{aligned} \tag{6.6}$$

We consider the case when initial conditions (x_0, y_0) for the system (1.1) satisfy relations

$$x_0 = x_A, \quad y_0 > y_A \tag{6.7}$$

Let us suppose that actions of the second coalition are the most unfavorable for the first coalition. For the trajectories of the system (1.1) starting from the initial positions (x_0, y_0) (6.7) these actions $v_A^0 = v_A^{cl}(x, y)$ are rather simple

$$v_A^{cl}(x, y) \equiv 0 \tag{6.8}$$

The optimal actions $u_A^0 = u_A^{ds}(x, y)$ of the first coalition with respect to the payoff functional J_A^f in this situation can be represented as the two-step bang-bang control: it is equal to unit from the initial instant $t_0 = 0$ till the optimal switch instant s and then it is equal to zero till the final instant T_f

$$u_A^0(t) = u_A^{ds}(x(t), y(t)) = \begin{cases} 1 & \text{if } t_0 \le t < s \\ 0 & \text{if } s \le t < T_f \end{cases} \tag{6.9}$$

The parameter s here is the parameter of optimization. The final instant T_f is determined by the condition that a trajectory $(x(\cdot), y(\cdot))$ of the system (1.1) starting from the line where $x(t_0) = x_A$ returns to this line $x(T_f) = x_A$.

Thus, we consider two aggregates of characteristics. The first one is described by the system of differential equations

$$\begin{aligned} \dot{x} &= -x + 1 \\ \dot{y} &= -y \end{aligned} \tag{6.10}$$

solutions of which are determined by the Cauchy formula

$$\begin{aligned} x(t) &= (x_0 - 1)e^{-t} + 1 \\ y(t) &= y_0 e^{-t} \end{aligned} \tag{6.11}$$

where initial positions (x_0, y_0) satisfy conditions (6.7) and time parameter t satisfies inequality $0 \le t < s$.

The second aggregate of characteristics is given by the system of differential equations

$$\begin{aligned} \dot{x} &= -x \\ \dot{y} &= -y \end{aligned} \tag{6.12}$$

solutions of which are determined by the Cauchy formula

$$\begin{aligned} x(t) &= x_1 e^{-t} \\ y(t) &= y_1 e^{-t} \end{aligned} \tag{6.13}$$

where initial positions $(x_1, y_1) = (x_1(s), y_1(s))$ are determined by relations

$$\begin{aligned} x_1 = x_1(s) &= (x_0 - 1)e^{-s} + 1 \\ y_1 = y_1(s) &= y_0 e^{-s} \end{aligned} \tag{6.14}$$

and time parameter t satisfies inequality $0 \le t < p$. Here the final moment of time $p = p(s)$ and the final position $(x_2, y_2) = (x_2(s), y_2(s))$ of the whole trajectory $(x(\cdot), y(\cdot))$ are given by formulas

$$\begin{aligned} x_1 e^{-p} &= x_A, \quad p = p(s) = \ln \frac{x_1(s)}{x_A} \\ x_2 &= x_A \\ y_2 &= y_1 e^{-p} \end{aligned} \tag{6.15}$$

The optimal control problem consists in finding such moment of time s and corresponding switching point $(x_1, y_1) = (x_1(s), y_1(s))$ on the trajectory $(x(\cdot), y(\cdot))$ that the integral $I = I(s)$ attains the maximum value

$$I(s) = I_1(s) + I_2(s) \tag{6.16}$$

$$\begin{aligned} I_1(s) = \int_0^s &(C_A((x_0 - 1)e^{-t} + 1)y_0 e^{-t} - \\ &\alpha_1((x_0 - 1)e^{-t} + 1) - \alpha_2 y_0 e^{-t} + a_{22})dt \end{aligned} \tag{6.17}$$

$$\begin{aligned} I_2(s) = \int_0^{p(s)} &(C_A x_1(s) y_1(s) e^{-2t} - \\ &\alpha_1 x_1(s) e^{-t} - \alpha_2 y_1(s) e^{-t} + a_{22})dt \end{aligned} \tag{6.18}$$

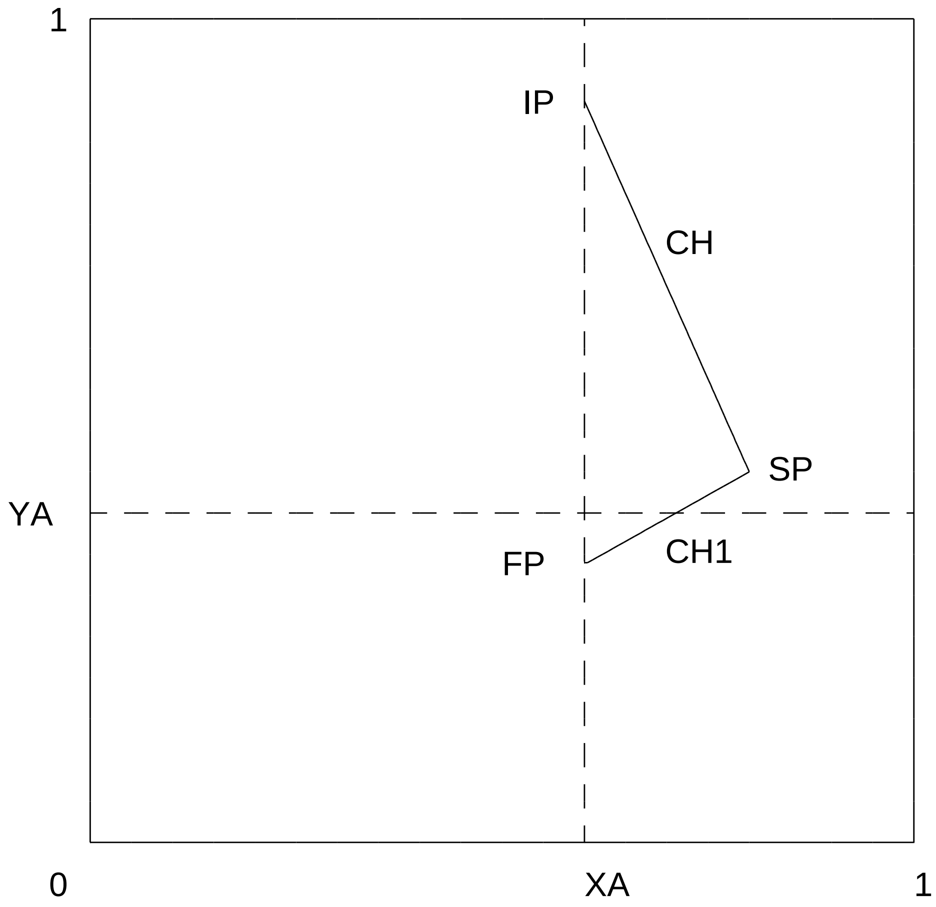

Figure 5: Two families of characteristics $CH = (x(\cdot), y(\cdot))$, $CH1 = (x_1(\cdot), y_1(\cdot))$ in the dual-step problem.

On Fig.5 initial position $IP = (x_0, y_0) = (0.6, 0.9)$, characteristics $CH = (x(\cdot), y(\cdot))$, $CH1 = (x_1(\cdot), y_1(\cdot))$, switching point $SP = (x_1, y_1)$ and final point $FP = (x_2, y_2)$ are shown.

6.3 The Solution of the Two-step Optimal Control Problem

We obtain the solution of the two-step optimal control problem (6.10)-(6.18) if calculate the derivative dI/ds, express it as the function of the optimal switching points $(x, y) = (x_1, y_1)$, equate this derivative to zero $dI/ds = 0$ and find the equation $F(x, y) = 0$ for the curve consisting of optimal switching points (x, y).

Let us calculate at first the integrals I_1, I_2

$$\begin{aligned} I_1 = I_1(s) \quad &= \quad C_A(x_0 - 1)y_0 \frac{(1 - e^{-2s})}{2} + C_A y_0 (1 - e^{-s}) - \\ & \alpha_1((x_0 - 1)(1 - e^{-s}) + s) - \alpha_2 y_0 (1 - e^{-s}) + a_{22} s \end{aligned}$$

$$I_2 = I_2(s) = C_A x_1(s) y_1(s) \frac{(1-e^{-2p(s)})}{2} - \alpha_1 x_1(s)(1-e^{-p(s)}) - \alpha_2 y_1(s)(1-e^{-p(s)}) + a_{22} p(s)$$

Let us calculate derivatives dI_1/ds, dI_2/ds and express them as functions of optimal switching points $(x,y) = (x_1, y_1)$

$$\begin{aligned}\frac{dI_1}{ds} &= C_A(x_0-1)y_0 e^{-2s} + C_A y_0 e^{-s} - \\ &\quad \alpha_1((x_0-1)e^{-s}+1) - \alpha_2 y_0 e^{-s} + a_{22} = \\ &\quad C_A xy - \alpha_1 x - \alpha_2 y + a_{22}\end{aligned}$$

$$\begin{aligned}\frac{dI_2}{ds} &= C_A(\frac{dx}{ds} y \frac{(1-e^{-2p})}{2} + \\ &\quad x\frac{dy}{ds}\frac{(1-e^{-2p})}{2} + xye^{-2p}\frac{dp}{ds}) - \\ &\quad \alpha_1 \frac{dx}{ds}(1-e^{-p}) - \alpha_1 x e^{-p}\frac{dp}{ds} - \\ &\quad \alpha_2 \frac{dy}{ds}(1-e^{-p}) - \alpha_2 y e^{-p}\frac{dp}{ds} + a_{22}\frac{dp}{ds} = \\ &\quad (C_A^2 x^2 y - \alpha_2^2 y - 2C_A^2 x^3 y - \\ &\quad 2\alpha_1 C_A x^2 + 2\alpha_1 C_A x^3 + 2\alpha_2 C_A x^2 y + \\ &\quad 2C_A a_{22} x - 2C_A a_{22} x^2)/(2C_A x^2)\end{aligned}$$

In the last equation we took into account the following expressions for derivatives dx/ds, dy/ds, dp/ds and exponentials e^{-p}, e^{-2p}, $(1-e^{-p})$, $(1-e^{-2p})$ as functions of variables (x,y)

$$\begin{aligned}\frac{dx}{ds} &= 1-x \\ \frac{dy}{ds} &= -y \\ \frac{dp}{ds} &= \frac{1-x}{x} \\ e^{-p} &= \frac{\alpha_2}{C_A x} \\ e^{-2p} &= \frac{\alpha_2^2}{C_A^2 x^2} \\ 1-e^{-p} &= \frac{C_A x - \alpha_2}{C_A x} \\ 1-e^{-2p} &= \frac{C_A^2 x^2 - \alpha_2^2}{C_A^2 x^2}\end{aligned}$$

Summarizing derivatives dI_1/ds, dI_2/ds, carrying the expression to the common denominator and equating the sum to zero we obtain the following equation for the switching curve

$$\frac{C_A^2 x^2 y - 2\alpha_1 C_A x^2 - \alpha_2^2 y + 2C_A a_{22} x}{2C_A x^2} = 0$$

Taking into account that $v_A = 0$ (see (6.6)) we come to the final expression for the switching curve M_A^1

$$y = \frac{2\alpha_1 x}{C_A x + \alpha_2} \tag{6.19}$$

The curve M_A^1 is a hyperbola which passes through the points $(0,0)$, (x_A, y_A) and has the horizontal asymptote

$$y = \frac{2\alpha_1}{C_A}$$

Remark 6.1 *The derivative of the curve M_A^1 at the point (x_A, y_A) is determined by the formula*

$$\frac{dy}{dx} = \frac{2\alpha_1\alpha_2}{(C_A x + \alpha_2)^2}\bigg|_{x=x_A} = \frac{\alpha_1}{2\alpha_2}$$

It is less than the derivative $dy/dx = \alpha_1/\alpha_2$ of the switching line K_A^2, $y = (\alpha_1/\alpha_2)x$ (see (4.2)) in the game with the multiterminal functional. Hence, the curve M_A^1 is situated below the line K_A^2. And we can conclude that in the game with the integral functional coalitions have more opportunities for maneuver (optimal feedback controls are more flexible) since individuals can keep for a longer time strategies (not switch them) which coincide with their "local" interests determined by the gradient $(\partial g_A/\partial x, \partial g_A/\partial y)$ of the payoff function $g_A(x,y)$ and so stay longer in preferable domains.

In order to construct the complete switching curve M_A for the optimal strategy of the first coalition in the game with the integral payoff we should add to the curve M_A^1 the similar curve M_A^2 in the domain where $y \le y_A$

$$\begin{aligned}
M_A &= M_A^1 \cup M_A^2 \\
M_A^1 &= \{(x,y) \in [0,1]\times[0,1]: \quad y = \frac{2\alpha_1 x}{C_A x + \alpha_2}, \quad y \ge \frac{\alpha_1}{C_A}\} \\
M_A^2 &= \{(x,y) \in [0,1]\times[0,1]: \\
&\quad y = -\frac{2(C_A - \alpha_1)(1-x)}{C_A(1-x) + (C_A - \alpha_2)} + 1, \quad y \le \frac{\alpha_1}{C_A}\}
\end{aligned} \tag{6.20}$$

In the case when $C_A < 0$ the curves M_A^1 and M_A^2 are described by formulas

$$\begin{aligned}
M_A^1 &= \{(x,y) \in [0,1]\times[0,1]: \quad y = \frac{2\alpha_1(1-x)}{C_A(1-x) + (C_A - \alpha_2)}, \quad y \ge \frac{\alpha_1}{C_A}\} \\
M_A^2 &= \{(x,y) \in [0,1]\times[0,1]: \quad y = -\frac{2(C_A - \alpha_1)x}{C_A x + \alpha_2} + 1, \quad y \le \frac{\alpha_1}{C_A}\}
\end{aligned} \tag{6.21}$$

The curve M_A divides the unit square $[0,1]\times[0,1]$ into two parts - the upper part

$$D_A^u \supset \{(x,y): \quad x = x_A, \quad y > y_A\}$$

and the lower part

$$D_A^l \supset \{(x,y): \quad x = x_A, \quad y < y_A\}$$

The "positive" feedback u_A^{ds} has the following structure

$$u_A^{ds} = u_A^{ds}(x,y) = \begin{cases} \max\{0, -sgn(C_A)\} & \text{if } (x,y) \in D_A^u \\ \max\{0, sgn(C_A)\} & \text{if } (x,y) \in D_A^l \\ \in [0,1] & \text{if } (x,y) \in M_A \end{cases} \tag{6.22}$$

For the second coalition one can obtain the similar switching curves M_B for the optimal control problem with the integral functional. Precisely, in the case when $C_B > 0$ the switching curve M_B is given by relations

$$\begin{aligned} M_B &= M_B^1 \cup M_B^2 \\ M_B^1 &= \{(x,y) \in [0,1] \times [0,1] : \quad y = \frac{\beta_1 x}{2\beta_2 - C_B x}, \quad x \geq \frac{\beta_2}{C_B}\} \\ M_B^2 &= \{(x,y) \in [0,1] \times [0,1] : \\ & y = -\frac{(C_B - \beta_1)(1-x)}{2(C_B - \beta_2) - C_B(1-x)} + 1, \quad x \leq \frac{\beta_2}{C_B}\} \end{aligned} \tag{6.23}$$

In the case when parameter C_B is negative $C_B < 0$ the curves M_B^1 and M_B^2 are determined by formulas

$$\begin{aligned} M_B^1 &= \{(x,y) \in [0,1] \times [0,1] : \quad y = -\frac{(C_B - \beta_1)x}{2\beta_2 - C_B x} + 1, \quad x \geq \frac{\beta_2}{C_B}\} \\ M_B^2 &= \{(x,y) \in [0,1] \times [0,1] : \quad y = \frac{\beta_1(1-x)}{2(C_B - \beta_2) - C_B(1-x)}, \quad x \leq \frac{\beta_2}{C_B}\} \end{aligned} \tag{6.24}$$

The curve M_B divides the unit square $[0,1] \times [0,1]$ into two parts - the left part

$$D_B^l \supset \{(x,y) : \quad x < x_B, \quad y = y_B\}$$

and the right part

$$D_B^r \supset \{(x,y) : \quad x > x_B, \quad y = y_B\}$$

The "positive" feedback v_B^{ds} has the following structure

$$v_B^{ds} = v_B^{ds}(x,y) = \begin{cases} \max\{0, -sgn(C_B)\} & \text{if } (x,y) \in D_B^l \\ \max\{0, sgn(C_B)\} & \text{if } (x,y) \in D_B^r \\ \in [0,1] & \text{if } (x,y) \in M_B \end{cases} \tag{6.25}$$

On Fig.6 switching curves M_A, M_B are shown for matrixes A (4.13), B (4.14). Directions of velocities $\dot{x}$ are depicted by horizontal (left and right) arrows and directions of velocities $\dot{y}$ - by vertical (up and down) arrows.

6.4 Values Guaranteed by Optimal Feedbacks in the Problem with the Integral Payoffs

Let us formulate the statement which affirms that the "positive" optimal feedback control $u_A^{ds}(x,y)$ (6.22) with the switching curve M_A determined by formulas (6.20),(6.21) guarantees that the averaged value of the integral functional is more or equal than the value $v_A = D_A/C_A$ of the static matrix game.

Proposition 6.2 *For any initial position $(x_0, y_0) \in [0,1] \times [0,1]$ and for any trajectory*

$$(x^{ds}(\cdot), y^{ds}(\cdot)) \in X(x_0, y_0, u_A^{ds}), \quad x^{ds}(t_0) = x_0, \quad y^{ds}(t_0) = y_0, \quad t_0 = 0$$

generated by the optimal feedback control $u_A^{ds} = u_A^{ds}(x,y)$ there exists a finite moment $t_* \in [0, T_A]$ such that at this moment the trajectory $(x^{ds}(\cdot), y^{ds}(\cdot))$ comes to the line where $x = x_A$, i.e. $x^{ds}(t_*) = x_A$. Then according to construction of optimal feedback control u_A^{ds} (it maximizes the integral (6.16)) the following estimate holds

$$\int_{t_*}^{T} g_A(x^{ds}(t), y^{ds}(t))dt \geq v_A(T - t_*), \quad \forall T \geq t_* \tag{6.26}$$

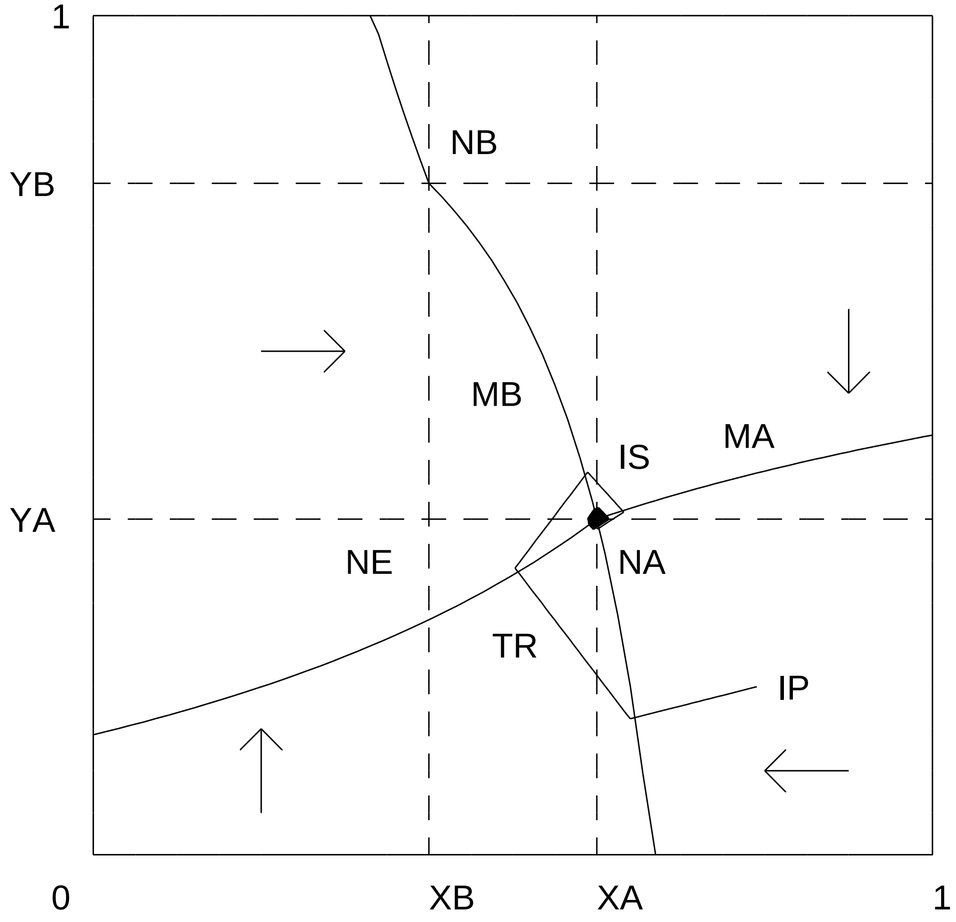

Figure 6: Guaranteeing dual-step synthesis u_A^{ds}, v_B^{ds} for integral payoffs and the Nash equilibrium trajectory $TR = (x^{ds}(\cdot), y^{ds}(\cdot))$.

In particular, this inequality remains valid when time T tends to infinity

$$\liminf_{T\to+\infty} \frac{1}{(T-t_*)} \int_{t_*}^{T} g_A(x^{ds}(t), y^{ds}(t))dt \geq v_A \tag{6.27}$$

Inequalities (6.26),(6.27) mean that the averaged value of the integral functional is not worse than the value v_A of the static matrix game.

The analogous result is valid for trajectories generated by optimal control v_B^{ds} (6.25) corresponding to the switching curve M_B (6.23),(6.24).

Remark 6.2 *Consider the acceptable trajectory $(x_{AB}^{ds}(\cdot), y_{AB}^{ds}(\cdot))$ generated by "positive" feedbacks u_A^{ds} (6.22), v_B^{ds} (6.25). Then according to Proposition 6.2 the following inequalities take place*

$$\liminf_{T\to+\infty} \frac{1}{(T-t_*)} \int_{t_*}^{T} g_A(x_{AB}^{ds}(t), y_{AB}^{ds}(t))dt \geq v_A \tag{6.28}$$

$$\liminf_{T\to+\infty} \frac{1}{(T-t_*)} \int_{t_*}^{T} g_A(x_{AB}^{ds}(t), y_{AB}^{ds}(t))dt \geq v_B \tag{6.29}$$

and hence the acceptable trajectory $(x_{AB}^{ds}(\cdot), y_{AB}^{ds}(\cdot))$ provides better results for both coalitions than trajectories leading to points of static Nash equilibrium at which corresponding payoffs are equal to values v_A and v_B.

On Fig.6 the acceptable trajectory $TR = (x_{AB}^{ds}(\cdot), y_{AB}^{ds}(\cdot))$ is shown for the game with the payoff matrixes A (4.13) and B (4.14). It starts from the initial position $IP = (0.8, 0.2)$ and moves at first along the straight line to the corner $(0, 0)$ of the unit square $[0, 1] \times [0, 1]$ with control signals $u = 0$, $v = 0$. Then it crosses the switching line M_B and the second coalition switches its control v from 0 to 1. Further the trajectory TR moves to the corner $(0, 1)$ till the meeting with the switching line M_A. Here the first coalition changes its signal u from 0 to 1. After that the trajectory directs to the corner $(1, 1)$. Next it meets the switching line M_B and turns to the corner $(1, 0)$. Then it intersects the switching line M_A and again as at the beginning moves to the corner $(0, 0)$. In the process of this cycling the trajectory TR converges to the point $IS = M_A \bigcap M_B$ of intersection of the switching curves M_A, M_B.

7 Quasioptimal Feedback Controls in Games with the Coordinated "Long-Term" and "Short-Term" Interests of Populations and Individuals

7.1 Three-Step Optimal Control Problems

In this section we examine the three-step optimal control problem for the first coalition when control parameter u is coordinated with "short-term" interests of individuals. The coordination means that components $\dot{x}$, $\dot{y}$ of velocity vector of the system (1.1) should have the same sign as components of gradients $\partial g_A/\partial x$ and $\partial g_B/\partial y$.

In Sections 7.1, 7.2 we find solutions for auxiliary three-step optimal control problems. Then we construct switching curves N_A (7.22),(7.23), L_A (7.24), N_B (7.26),(7.27), L_B (7.28) for "positive" triple-step feedbacks $u_A^0 = u_A^{ts}(x, y)$ (7.25), $v_B^0 = v_B^{ts}(x, y)$ (7.29) maximizing integral payoff functionals (6.1),(6.2) in presense of additional restrictions on control parameters u and v. Substituting "positive" feedbacks u_A^{ts} (7.25), v_B^{ts} (7.29) into dynamics (1.1) we obtain the acceptable trajectory $(x_{AB}^{ts}(\cdot), y_{AB}^{ts}(\cdot))$ which forms the basis of a dynamical Nash equilibrium.

Let us turn our attention to the three-step optimal control problem. We assume that there are restrictions $P(x,y)$ on control parameter u which depend on the sign of the partial derivative $\partial g_A/\partial x$ of the payoff function g_A expressing "short-term" interests of individuals

$$P(x,y) = \{u: \quad x \leq u \leq 1 \quad \text{if } \frac{\partial g_A}{\partial x} = C_A y - \alpha_1 \geq 0, \\ 0 \leq u \leq x \quad \text{if } \frac{\partial g_A}{\partial x} = C_A y - \alpha_1 < 0\} \tag{7.1}$$

In other words the following "noncontradictory" relations are valid for restrictions $P(x,y)$ and "short-term" interests $g_A(x,y)$

$$\max_{u \in P(x,y)} \frac{\partial g_A}{\partial x} u \geq 0, \quad (x,y) \in [0,1] \times [0,1]$$

Actions $v = v(t)$ of the second coalition satisfy inequalities

$$0 \leq v \leq 1 \tag{7.2}$$

and can be the most unfavorable.

Thus, we analyze the optimal control problem with the payoff functional

$$J_A^f = \int_0^{T_f} g_A(x(t), y(t)) dt \tag{7.3}$$

on trajectories $(x(t), y(t))$ of the dynamical system (1.1) starting from initial position $x(0) = x_0$, $y(0) = y_0$ and generated by controls $u(\cdot)$, $v(\cdot)$ satisfying restrictions (7.1),(7.2). The terminal instant $T_f = T_f(x_0, y_0)$ in formula (7.3) will be specified below.

We consider the case when initial positions (x_0, y_0) satisfy relations

$$x_0 = x_A, \quad y_0 > y_A \tag{7.4}$$

Let us suppose that actions of the second coalition are the most unfavorable. For trajectories of the system (1.1) starting from initial positions (x_0, y_0) (7.4) these actions $v_A^0 = v_A^{cl}(x,y)$ are determined by formula

$$v_A^{ts}(x,y) \equiv 0 \tag{7.5}$$

The optimal actions $u_A^{ts} = u_A^{ts}(x,y)$ of the first coalition which maximize the payoff functional J_A^f are constructed as the three-step bang-bang control

$$u_A^{ts}(t) = u_A^{ts}(x(t), y(t)) = \begin{cases} 1 & \text{if } t_0 \leq t < s \\ x(s) & \text{if } s \leq t < T_{y_A} \\ 0 & \text{if } T_{y_A} \leq t < T_f \end{cases} \tag{7.6}$$

The parameter s here is the parameter of optimization. The switching instant T_{y_A} is the moment when the trajectory $(x(\cdot), y(\cdot))$ crosses the line $y = y_A$, i.e. $y(T_{y_A}) = y_A$, and the final instant T_f is determined by the condition that the trajectory $(x(\cdot), y(\cdot))$ comes back to the line $x = x_A$, i.e. $x(T_f) = x_A$.

So we consider three collections of characteristics. Characteristics of the first collection satisfy the system of differential equations

$$\begin{aligned} \dot{x} &= -x + 1 \\ \dot{y} &= -y \end{aligned} \tag{7.7}$$

and can be represented by the Cauchy formula

$$\begin{aligned} x(t) &= (x_0 - 1)e^{-t} + 1 \\ y(t) &= y_0 e^{-t}, \qquad 0 \le t \le s \end{aligned} \tag{7.8}$$

where initial positions (x_0, y_0) satisfy conditions (7.4) and the first switching instant s is the parameter of optimization.

The second aggregate of characteristics is determined by the system of differential equations

$$\begin{aligned} \dot{x} &= 0 \\ \dot{y} &= -y \end{aligned} \tag{7.9}$$

solutions of which are determined by the Cauchy formula

$$\begin{aligned} x(t) &= x \\ y(t) &= ye^{-t}, \quad 0 \le t \le p(s) \end{aligned} \tag{7.10}$$

Initial positions $(x, y) = (x(s), y(s))$ are determined by relations

$$\begin{aligned} x = x(s) &= (x_0 - 1)e^{-s} + 1 \\ y = y(s) &= y_0 e^{-s} \end{aligned} \tag{7.11}$$

The second switching instant $p = p(s)$ and the position $(x(p), y(p))$ of the characteristic (7.10) are given by formulas

$$\begin{aligned} ye^{-p} &= y_A = \frac{\alpha_1}{C_A}, \quad p = p(s) = \ln \frac{C_A y(s)}{\alpha_1} \\ x(p) &= x \\ y(p) &= y_A \end{aligned} \tag{7.12}$$

The third collection of characteristics is given by the system of differential equations

$$\begin{aligned} \dot{x} &= -x \\ \dot{y} &= -y \end{aligned} \tag{7.13}$$

and are determined by the Cauchy formula

$$\begin{aligned} x(t) &= xe^{-t} \\ y(t) &= y_A e^{-t}, \quad 0 \le t \le r(s) \end{aligned} \tag{7.14}$$

Here the final instant $r = r(s)$ and the final position $(x(r), y(r))$ of the characteristic trajectory (7.14) are given by formulas

$$\begin{aligned} xe^{-r} &= x_A = \frac{\alpha_2}{C_A}, \quad r = r(s) = \ln \frac{C_A x(s)}{\alpha_2} \\ x(r) &= x_A \\ y(r) &= y_A e^{-r} \end{aligned} \tag{7.15}$$

The optimal control problem consists in finding such time s and corresponding switching point $(x, y) = (x(s), y(s))$ on the trajectory $(x(\cdot), y(\cdot))$ that the integral $I = I(s)$

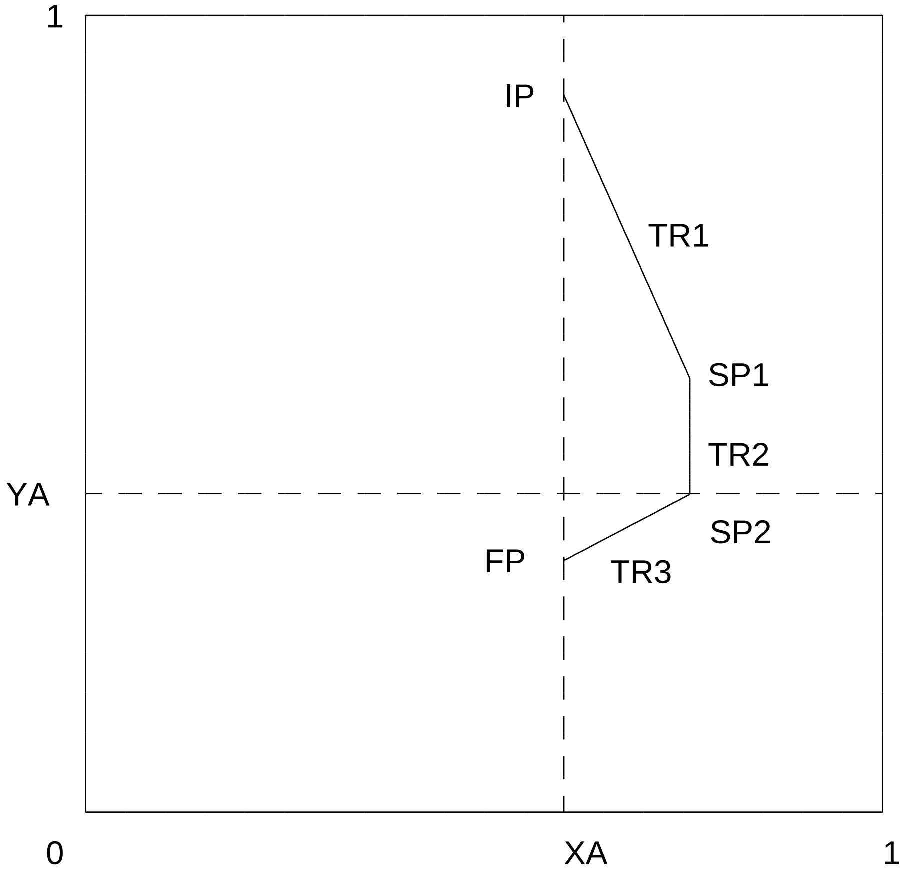

Figure 7: Three families of characteristics $TR1$, $TR2$, $TR3$ in the triple-step problem.

attains the maximum value

$$I(s) = I_1(s) + I_2(s) + I_3(s) \tag{7.16}$$

$$\begin{aligned} I_1(s) &= \int_0^s (C_A((x_0-1)e^{-t}+1)y_0e^{-t} - \\ &\quad \alpha_1((x_0-1)e^{-t}+1) - \alpha_2 y_0 e^{-t} + a_{22})dt \end{aligned} \tag{7.17}$$

$$\begin{aligned} I_2(s) &= \int_0^{p(s)} (C_A x(s)y(s)e^{-t} - \\ &\quad \alpha_1 x(s) - \alpha_2 y(s)e^{-t} + a_{22})dt \end{aligned} \tag{7.18}$$

$$\begin{aligned} I_3(s) &= \int_0^{r(s)} (C_A x(s) y_A e^{-2t} - \\ &\quad \alpha_1 x(s)e^{-t} - \alpha_2 y_A e^{-t} + a_{22})dt \end{aligned} \tag{7.19}$$

On Fig.7 initial position $IP = (0.6, .09)$, characteristics of three different types $TR1$ (7.8), $TR2$ (7.10), $TR3$ (7.14), switching points $SP1 = (x(s), y(s))$, $SP2 = (x(p), y(p))$ and final point $FP = (x(r), y(r))$ are shown.

7.2 Solution of the Three-step Optimal Control Problem

Let us indicate the scheme for solution of the three-step optimal control problem (7.7)-(7.19). We express integrals I_k, $k = 1, 2, 3$ as functions $I_k(x, x_0, y_0)$ depending on the first coordinate x of optimal switching points $(x, y) = (x(s), y(s))$ and initial positions (x_0, y_0), calculate the derivative by the variable x of the integral $I(x, x_0, y_0)$

$$I(x, x_0, y_0) = I_1(x, x_0, y_0) + I_2(x, x_0, y_0) + I_3(x, x_0, y_0)$$

equate this derivative to zero $dI/dx = 0$ and find the equation $F(x, x_0, y_0) = 0$ for optimal switching coordinates x. Then we connect initial positions (x_0, y_0) with the switching positions (x, y) by corresponding relations and obtain the equation $F(x, x_0(x, y), y_0(x, y)) = 0$. for the optimal switching curve.

So, we calculate at first the integrals I_1, I_2, I_3 as functions of variables x, x_0, y_0

$$\begin{aligned} I_1 = I_1(x, x_0, y_0) \quad &= \quad \frac{C_A y_0((x_0-1)^2 - (x-1)^2)}{2(x_0-1)} + \frac{C_A y_0(x_0 - x)}{(x_0-1)} - \\ & \alpha_1((x_0 - x) + \ln(1 - x_0) - \ln(1 - x)) - \frac{\alpha_2 y_0(x_0 - x)}{(x_0-1)} + \\ & a_{22}(\ln(1 - x_0) - \ln(1 - x)) \end{aligned}$$

$$\begin{aligned} I_2 = I_2(x, x_0, y_0) \quad &= \quad \frac{(C_A x - \alpha_2)(C_A y_0(x-1) - \alpha_1(x_0-1))}{C_A(x_0-1)} + \\ & (a_{22} - \alpha_1 x)(\ln\frac{C_A y_0}{\alpha_1(1-x_0)} + \ln(1-x)) \end{aligned}$$

$$I_3 = I_3(x, x_0, y_0) = \frac{\alpha_1(\alpha_2^2 - C_A^2 x^2)}{2C_A^2 x} + a_{22}(\ln x + \ln\frac{C_A}{\alpha_2})$$

Now we calculate derivatives dI_1/dx, dI_2/dx, dI_3/dx

$$\begin{aligned} \frac{dI_1}{dx} &= -\frac{y_0(C_A x - \alpha_2)}{(x_0-1)} + \frac{(\alpha_1 x - a_{22})}{(x-1)} \\ \frac{dI_2}{dx} &= \frac{(C_A y_0(x-1) - \alpha_1(x_0-1) + C_A y_0 x - \alpha_2 y_0)}{(x_0-1)} - \\ & \alpha_1(\ln\frac{C_A y_0}{\alpha_1(1-x_0)} + \ln(1-x)) + \frac{(a_{22} - \alpha_1 x)}{(x-1)} \\ \frac{dI_3}{dx} &= -\frac{\alpha_1(C_A^2 x^2 + \alpha_2^2)}{2C_A^2 x^2} + \frac{a_{22}}{x} \end{aligned}$$

Summarizing derivatives dI_1/dx, dI_2/dx, dI_3/dx and equating the sum to zero we obtain the following equation

$$\frac{C_A y_0(1-x)}{(1-x_0)} - \alpha_1 - \alpha_1 \ln\frac{C_A y_0(1-x)}{\alpha_1(1-x_0)} - \frac{(\alpha_1 C_A^2 x^2 - 2C_A^2 a_{22} x + \alpha_1 \alpha_2^2)}{2C_A^2 x^2} = 0$$

Taking into account that $v_A = a_{22}C_A - \alpha_1\alpha_2 = 0$ in (6.6), and parameters x, y, x_0, y_0 are connected by formula

$$y = \frac{y_0(1-x)}{(1-x_0)}$$

we obtain the expression for the switching curve N_A^1

$$\frac{(C_A y - \alpha_1)}{\alpha_1} - \ln(1 + \frac{(C_A y - \alpha_1)}{\alpha_1}) - \frac{(C_A x - \alpha_2)^2}{2C_A^2 x^2} = 0 \qquad (7.20)$$

The curve N_A^1 with the accuracy of the second order with respect to variable y is a hyperbola which passes through the point (x_A, y_A)

$$\frac{(C_A y - \alpha_1)}{\alpha_1} - \frac{(C_A x - \alpha_2)}{C_A x} = 0 \tag{7.21}$$

It has the horizontal asymptote

$$y = \frac{2\alpha_1}{C_A}$$

and the vertical asymptote

$$x = 0$$

Remark 7.1 *The derivative of the curve N_A^1 at the point (x_A, y_A) is determined by relation*

$$\frac{dy}{dx} = \left.\frac{\alpha_1 \alpha_2}{C_A^2 x^2}\right|_{x=x_A} = \frac{\alpha_1}{\alpha_2}$$

It is equal to the derivative $dy/dx = \alpha_1/\alpha_2$ of the switching line K_A^2, $y = (\alpha_1/\alpha_2)x$ (see (4.2)) in the game with the multiterminal functional. Hence, the curve N_A^1 is situated below the line K_A^2.

To construct the complete switching curve N_A for the optimal strategy of the first coalition we need to supplement the curve N_A^1 by the analogous curve N_A^2 in the domain where $y \leq y_A$

$$\begin{aligned}
N_A &= N_A^1 \cup N_A^2 \\
N_A^1 &= \{(x,y) \in [0,1] \times [0,1] : \\
&\quad \frac{(C_A y - \alpha_1)}{\alpha_1} - \ln\left(1 + \frac{(C_A y - \alpha_1)}{\alpha_1}\right) - \\
&\quad \frac{(C_A x - \alpha_2)^2}{2C_A^2 x^2} = 0, \quad y \geq \frac{\alpha_1}{C_A}\} \\
N_A^2 &= \{(x,y) \in [0,1] \times [0,1] : \\
&\quad \frac{(C_A(1-y) - (C_A - \alpha_1))}{(C_A - \alpha_1)} - \ln\left(1 + \frac{(C_A(1-y) - (C_A - \alpha_1))}{(C_A - \alpha_1)}\right) - \\
&\quad \frac{(C_A(1-x) - (C_A - \alpha_2))^2}{2C_A^2 (1-x)^2} = 0, \quad y \leq \frac{\alpha_1}{C_A}\}
\end{aligned} \tag{7.22}$$

In the case when $C_A < 0$ the curves N_A^1 and N_A^2 are described by formulas

$$\begin{aligned}
N_A^1 &= \{(x,y) \in [0,1] \times [0,1] : \\
&\quad \frac{(C_A y - \alpha_1)}{\alpha_1} - \ln\left(1 + \frac{(C_A y - \alpha_1)}{\alpha_1}\right) - \\
&\quad \frac{(C_A(1-x) - (C_A - \alpha_2))^2}{2C_A^2 (1-x)^2} = 0, \quad y \geq \frac{\alpha_1}{C_A}\} \\
N_A^2 &= \{(x,y) \in [0,1] \times [0,1] : \\
&\quad \frac{(C_A(1-y) - (C_A - \alpha_1))}{(C_A - \alpha_1)} - \ln\left(1 + \frac{(C_A(1-y) - (C_A - \alpha_1))}{(C_A - \alpha_1)}\right) - \\
&\quad \frac{(C_A x - \alpha_2)^2}{2C_A^2 x^2} = 0, \quad y \leq \frac{\alpha_1}{C_A}\}
\end{aligned} \tag{7.23}$$

Let us remind that the line

$$L_A = \{(x,y) \in [0,1] \times [0,1] : \quad 0 \le x \le 1, \quad y = y_A\} \tag{7.24}$$

is also the switching one for the first coalition according to (7.6). The curves N_A, L_A divide the unit square $[0,1] \times [0,1]$ into three parts - the upper part

$$S_A^u \supset \{(x,y) : \quad x = x_A, \quad y > y_A\}$$

the lower part

$$S_A^l \supset \{(x,y) : \quad x = x_A, \quad y < y_A\}$$

and the middle part

$$S_A^m = S_A^{m1} \bigcup S_A^{m2}$$

$$S_A^{m1} \supset \{(x,y) : \quad x < x_A, \quad y = y_A\}$$

$$S_A^{m2} \supset \{(x,y) : \quad x > x_A, \quad y = y_A\}$$

The "positive" feedback u_A^{ts} has the following structure

$$u_A^{ts} \;=\; u_A^{ts}(x,y) = \begin{cases} \max\{0, -sgn(C_A)\} & \text{if } (x,y) \in S_A^u \\ \max\{0, sgn(C_A)\} & \text{if } (x,y) \in S_A^l \\ x & \text{if } (x,y) \in S_A^m \\ \in [0,x] \text{ or } \in [x,1] & \text{if } (x,y) \in N_A \bigcup L_A \end{cases} \tag{7.25}$$

For the second coalition one can obtain the similar switching curves N_B. Namely, in the case when $C_B > 0$ the switching curve N_B is given by relations

$$\begin{aligned} N_B &= N_B^1 \cup N_B^2 \\ N_B^1 &= \{(x,y) \in [0,1] \times [0,1] : \\ &\quad \frac{(C_B x - \beta_2)}{\beta_2} - \ln(1 + \frac{(C_B x - \beta_2)}{\beta_2}) - \\ &\quad \frac{(C_B y - \beta_1)^2}{2C_B^2 y^2} = 0, \quad x \ge \frac{\beta_2}{C_B}\} \\ N_B^2 &= \{(x,y) \in [0,1] \times [0,1] : \\ &\quad \frac{(C_B(1-x) - (C_B - \beta_2))}{(C_B - \beta_2)} - \ln(1 + \frac{(C_B(1-x) - (C_B - \beta_2))}{(C_B - \beta_2)}) - \\ &\quad \frac{(C_B(1-y) - (C_B - \beta_1))^2}{2C_B^2(1-y)^2} = 0, \quad x \le \frac{\beta_2}{C_B}\} \end{aligned} \tag{7.26}$$

In the case when $C_B < 0$ the curves N_B^1 and N_B^2 are determined by formulas

$$\begin{aligned} N_B^1 &= \{(x,y) \in [0,1] \times [0,1] : \\ &\quad \frac{(C_B x - \beta_2)}{\beta_2} - \ln(1 + \frac{(C_B x - \beta_2)}{\beta_2}) - \\ &\quad \frac{(C_B(1-y) - (C_B - \beta_1))^2}{2C_B^2(1-y)^2} = 0, \quad x \ge \frac{\beta_2}{C_B}\} \\ N_B^2 &= \{(x,y) \in [0,1] \times [0,1] : \\ &\quad \frac{(C_B(1-x) - (C_B - \beta_2))}{(C_B - \beta_2)} - \ln(1 + \frac{(C_B(1-x) - (C_B - \beta_2))}{(C_B - \beta_2)}) - \\ &\quad \frac{(C_B y - \beta_1)^2}{2C_B^2 y^2} = 0, \quad x \le \frac{\beta_2}{C_B}\} \end{aligned} \tag{7.27}$$

Remind that the line

$$L_B = \{(x,y) \in [0,1] \times [0,1] : \quad x = x_B, \quad 0 \le y \le 1\} \tag{7.28}$$

is also the switching one for the second coalition. The curves N_B, L_B divide the unit square $[0,1] \times [0,1]$ into three parts - the left part

$$S_B^l \supset \{(x,y) : \quad x < x_B, \quad y = y_B\}$$

the right part

$$S_B^r \supset \{(x,y) : \quad x > x_B, \quad y = y_B\}$$

and the middle part

$$S_B^m = S_B^{m1} \bigcup S_B^{m2}$$

$$S_B^{m1} \supset \{(x,y) : \quad x = x_B, \quad y < y_B\}$$

$$S_B^{m2} \supset \{(x,y) : \quad x = x_B, \quad y > y_B\}$$

The "positive" feedback v_B^{ts} has the following structure

$$v_B^{ts} \quad = \quad v_B^{ts}(x,y) = \begin{cases} \max\{0, -sgn(C_B)\} & \text{if } (x,y) \in S_B^l \\ \max\{0, sgn(C_B)\} & \text{if } (x,y) \in S_B^r \\ y & \text{if } (x,y) \in S_B^m \\ \in [0,y] \text{ or } \in [y,1] & \text{if } (x,y) \in N_B \bigcup L_B \end{cases} \tag{7.29}$$

On Fig.8 switching curves N_A^1, N_A^2, L_A and N_B^1, N_B^2, L_B are shown for matrixes A (4.13), B (4.14). Directions of velocities $\dot{x}$ are depicted by horizontal arrows: left arrows - in the upper domain S_A^u, right arrows - in the lower domain S_A^l, left-right arrows - in the middle domain S_A^m. Directions of velocities $\dot{y}$ are indicated by vertical arrows: up arrows - in the left domain S_B^l, down arrows - in the right domain S_B^r, up-down arrows - in the middle domain S_B^m.

7.3 The Guaranteed Value in the Three-Step Optimal Control Problem

Generalizing the considered three-step optimal control problem we can formulate the result which arises from the optimization nature of this problem and provides better index values than values of trajectories tending to static Nash equilibria.

Proposition 7.1 *Consider three-step optimal control $u_A^{ts}(x,y)$ (7.25) with the switching curves N_A (7.22),(7.23), L_A (7.24). Then for any initial position $(x_0, y_0) \in [0,1] \times [0,1]$ and for any trajectory*

$$(x^{ts}(\cdot), y^{ts}(\cdot)) \in X(x_0, y_0, u_A^{ts}), \quad x^{ts}(t_0) = x_0, \quad y^{ts}(t_0) = y_0, \quad t_0 = 0$$

generated by the optimal feedback control $u_A^{ts} = u_A^{ts}(x,y)$ there exists a finite moment $t_* \in [0, T_A]$ such that at this moment the trajectory $(x^{ts}(\cdot), y^{ts}(\cdot))$ comes to the line where $x = x_A$. Then according to construction of three-step optimal feedback control u_A^{ts} (it maximizes the integral (7.16) and conforms to "short-term" interests of individuals) the following estimate holds

$$\int_{t_*}^{T} g_A(x^{ts}(t), y^{ts}(t))dt \ge v_A(T - t_*), \quad \forall T \ge t_* \tag{7.30}$$

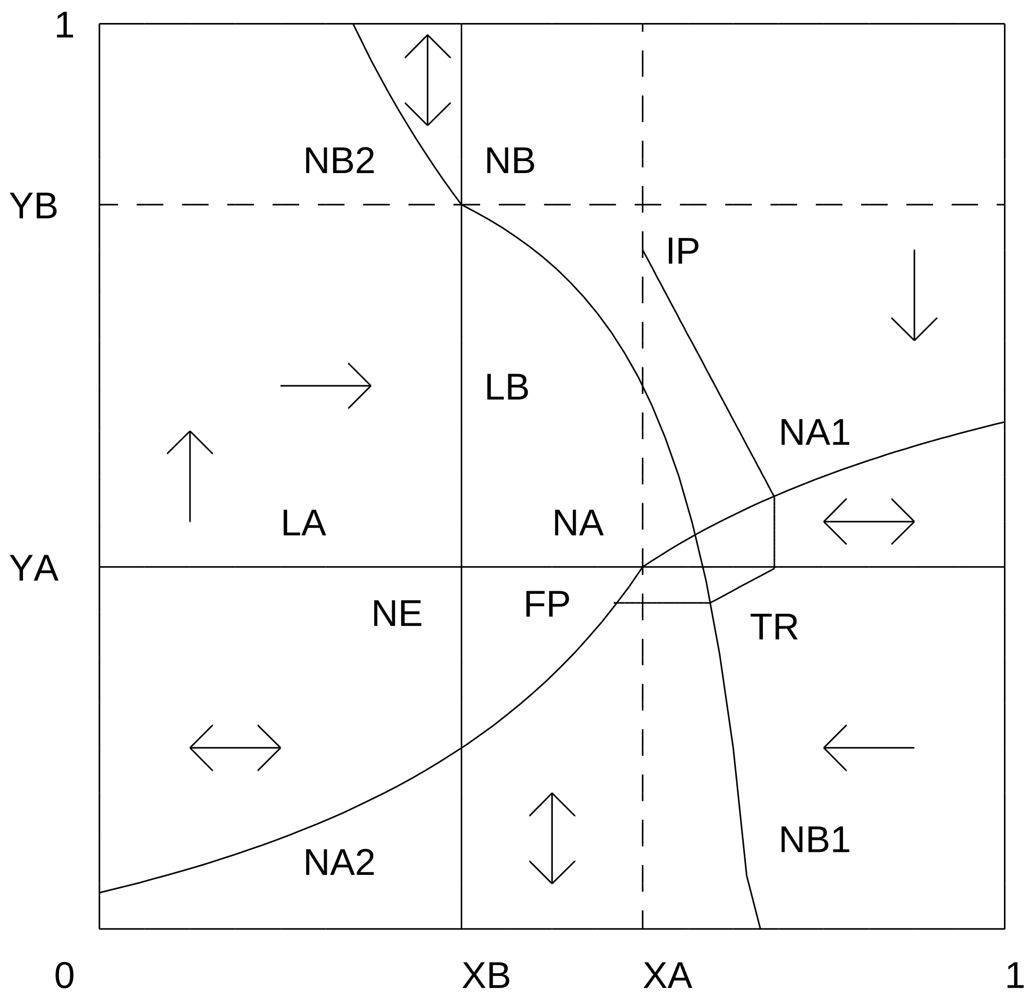

Figure 8: Guaranteeing triple-step design u_A^{ts}, v_B^{ts} for integral payoffs and the Nash equilibrium trajectory $TR = (x^{ts}(\cdot), y^{ts}(\cdot))$.

In particular, this inequality remains valid when time T tends to infinity

$$\liminf_{T\to+\infty} \frac{1}{(T-t_*)} \int_{t_*}^{T} g_A(x^{ts}(t), y^{ts}(t))dt \geq v_A \qquad (7.31)$$

Inequalities (7.30),(7.31) mean that the averaged value of the integral functional is not worse than the value v_A of the static matrix game.

The analogous result is valid for trajectories generated by three-step optimal control v_B^{ts} (7.29) corresponding to the switching curves N_B (7.26),(7.27), L_B (7.28).

Remark 7.2 *Consider the acceptable trajectory $(x_{AB}^{ts}(\cdot), y_{AB}^{ts}(\cdot))$ generated by "positive" three-step feedbacks u_A^{ts} (7.25), v_B^{ts} (7.29). Then according to Proposition 7.1 the following inequalities take place*

$$\liminf_{T\to+\infty} \frac{1}{(T-t_*)} \int_{t_*}^{T} g_A(x_{AB}^{ts}(t), y_{AB}^{ts}(t))dt \geq v_A \qquad (7.32)$$

$$\liminf_{T\to+\infty} \frac{1}{(T-t_*)} \int_{t_*}^{T} g_A(x_{AB}^{ts}(t), y_{AB}^{ts}(t))dt \geq v_B \qquad (7.33)$$

and hence the acceptable trajectory $(x_{AB}^{ts}(\cdot), y_{AB}^{ts}(\cdot))$ provides better results for both coalitions than trajectories leading to points of static Nash equilibrium at which corresponding payoffs are equal to values v_A and v_B.

On Fig.8 the acceptable trajectory $TR = (x_{AB}^{ts}(\cdot), y_{AB}^{ts}(\cdot))$ is shown for the game with the payoff matrixes A (4.13) and B (4.14). It starts from the initial position $IP = (0.6, 0.75)$ and moves at first along the straight line to the corner $(1, 0)$ of the unit square $[0,1] \times [0,1]$ with control signals $u = 1$, $v = 0$. Then it crosses the switching line N_A^1 and the first coalition switches its control u from 1 to x. Further the trajectory TR moves down along the straight line parallel to the axis y till the meeting with the switching line L_A. Here the first coalition changes its signal u from x to 0. After that the trajectory directs to the corner $(0, 0)$. Next it meets the switching line N_B^1 and the second coalition changes its control signal from 0 to y. Then the trajectory TR moves to the left along the straight line parallel to the axis x and intersects the switching curve N_A^2. On this curve the first coalition changes its control signal u from 0 to x. The trajectory TR stops here at point FP because velocities are equal to zero $\dot{x} = 0$, $\dot{y} = 0$.

Conclusion

In this paper we introduce the control generalization of evolutionary dynamics described by Kolmogorov's differential equations choosing coefficients of flows on the feedback principle. It is shown that in arising evolutionary (dynamical) nonzero sum games "positive" maximizing feedbacks (guaranteeing feedbacks) provide better results than trajectories which tend to (or circulate near) static Nash equilibria. We construct these "positive" feedbacks analytically for different (terminal, integral) types of functionals determined on infinite horizon. Feedbacks are calculated via methods of the theory of generalized solutions (value functions for corresponding differential games) of Hamilton-Jacobi equations and have bang-bang properties on switching curves. We use multiterminal payoffs which depend on future states and so take into account the foreseeing principle. It is proved that equilibrium trajectories generated by guaranteeing feedbacks are located in domains with better index values for both coalitions than at static Nash equilibria. We indicate the "evolutionary-revolutionary" properties of equilibrium trajectories and show examples when these trajectories converge to points of intersection of switching curves with the strictly better index values. The risk barrier surrounding the equilibrium trajectory is intended for considering the venturous factor. We give an appropriate economical model for justification of dynamics, payoffs, optimal feedbacks, dynamical Nash equilibria and equilibrium trajectories.

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