FUZZY SET CONCEPTS FOR RISK ASSESSMENT

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I. Introduction

The risk assessment problems are essentially those of decision-making under uncertainty. If there is no uncertainty as to the result of activity considered, then there will be no sense of risk. Also if there is no possibility for us to make a choice, it would be a fate rather than a risk.

There are many types of uncertainties which can be divided into two major categories: the uncertainty due to randomness and that due to ambiguity. Conventional methods of treating the uncertainty are to apply statistical methods of estimation which are, in turn, based upon the concept of probability. Even in the case where the source of uncertainty is of non-statistical nature, formal application of statistical methods of analysis is often done to deal quantitatively with the uncertainty by tacitly accepting the premise that uncertainty - whatever its nature - can be equated with randomness (1). Most of the works on risk assessment have been done using such methods.

However in risk assessment problems, where societal or humanistic systems as well as technological or mechanistic systems are concerned, we encounter a variety of sources of

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uncertainty which are due to ambiguity in our cognition or perception of objects. For systematic treatment of this type of uncertainty, the concepts of fuzzy sets could be applied to construct system models, thus taking the ambiguity into consideration.

The aim of this note is to introduce basic concepts of fuzzy sets and to point out some possible application of this concept to the problems of risk assessment and decision-making.

II. Types of Uncertainty

It would be useful to consider some typical types of uncertainty which may be encountered in the problems of risk assessment before introducing the concepts of fuzzy sets. Only four types of uncertainty will be presented in this chapter. The simplest examples of picking up a ball from an urn will be used.

1. Uncertainty with precise and complete information about non-fuzzy outcome.

A simple example of this type of uncertainty is that of picking up a white ball from the urn which contains only a known number of white balls and a known number of black balls. Here we assume that all balls are made identically except for their colour, so that one cannot identify their colour unless he looks at them. The process of picking up a ball is, however, assumed to be completely random.

If there are m white balls and n black balls in the urn, the probability that one will pick up a white ball can be defined as $p = m/(m + n)$. This type of probability, which can be determined with objective information, is usually called the objective probability. This probability concept is consistent with that based on the relative frequency of occurrence of the event, and it can be verified experimentally with the aid of the law of large numbers if the ratio m/n is moderate. However, its experimental verification will become practically impossible if the ratio m/n becomes extremely small, 10^{-6} for instance. The probability of this type can, however, be determined exactly in the mathematical sense, even if it is extremely small. It can then be applied to develop probabilistic assessment of the problems in which a complete set of information necessary to determine such probabilities is available.

The uncertainty associated with fair lottery falls into this category.

2. Uncertainty with imprecise and/or incomplete information about non-fuzzy outcome.

Now let us consider a variation of the preceding example by assuming that the number m of white balls is not accurately known and that we are told only an approximate number $\tilde{m}$ of white balls. In this case, we cannot determine precise probability of picking up a white ball from the urn. It could also be said that the uncertainty is twofold, in that there are uncertainties due to randomness of the picking-up process and to imprecise information about the number of white balls. Usually, one would apply the same formula that was used in the previous example to define an "approximate" value of the probability by $\tilde{p} = \tilde{m}/(\tilde{m} + n)$. However, the meaning of the word "approximate" is usually more or less fuzzy and the true number of white balls may differ from the approximate number by 2% to 5% or more, depending upon the individual case. If we have the opportunity to make a sampling of balls, we can improve the value of probability by applying

Bayesian statistics. But if we have no opportunity at all to make such a sampling test, we must accept the fuzzyness which is involved in the given information. In conventional methods, it is often assumed that such fuzzyness can be captured by the terms of a probability distribution (2).

3. Uncertainty with precise and complete information about fuzzy outcome.

Let us suppose that the urn contains a total number of N balls but that the colour of the balls ranges almost continuously from sufficiently white to sufficiently black. We also assume that the degree of whiteness of each ball can be uniquely determined using some opto-electronic equipment for colour measurement and that a complete table showing the degree of whiteness of each ball contained in the urn has been given to us.

Under these conditions, what is the probability of picking up a "white" ball from this urn? One might claim that the question is not properly posed because the meaning of "white" is fuzzy. But we often encounter these types of problems in actual systems analysis.

The conventional practice in answering this question is as follows: We put a threshold on the degree of whiteness and define a ball as white if its degree of whiteness is higher than this threshold. By defining a white ball in this way, we can construct a non-fuzzy system model in which only "white" and "non-white" balls exist and thus we can determine the probability of picking up a "white" ball by applying the same formula that was used in the first example. But where should the threshold be placed? Subjectivity cannot be avoided in placing the threshold.

4. Uncertainty with imprecise and/or incomplete information about fuzzy outcome.

Let us make a little modification in the preceding example: Instead of a complete table which lists the grade of whiteness of each ball contained in the urn, suppose we are given an incomplete table in which the information about the grade of whiteness is missing for a certain number of balls. The question is the same: What is the probability that one will pick a white ball from the urn?

A conventional method which is often used in systems analysis is as follows: First, the "white" ball is defined rather subjectively as in the previous example, Then, using the data available in the table, the distribution of the number of balls as a function of the grade is determined. By assuming the same distribution for the balls for which the information about the grade is missing, an appropriate grade is assigned to each ball of missing information. Then the probability of picking up a white ball is determined using the same formula that was applied in the previous example. In this example, subjectivity is twofold, that is , subjectivity is involved in making the definition of a white ball, and is assigning grades to the balls of missing information.

The above examples represent a static system. In dynamic systems, variation of system parameters will cause other types of uncertainty. Many types of uncertainty are mixed in the actual complex system.

III. Basic Concepts of Fuzzy Sets

The theory of fuzzy sets, which was first proposed by Zadeh (3) in 1965, provides a novel method of treating fuzziness in a systematic manner. In fact, much of our real world is more or less fuzzy. The traditional scientific method of investigation, however, is to try to eliminate fuzziness in the model to be built rather than to accept the fuzzziness as it is perceived. The development of the fuzzy set theory has opened a new possibility of handling very complex problems which are very difficult to treat by conventional methods of mathematical analysis. In this chapter, some basic concepts of fuzzy sets will be presented in order to introduce such concepts to those who are not familiar with the fuzzy set theory. Detailed description of the theory can be found in the references (3) to (6).

A fuzzy set is a set of elements whose limit is not sharply defined but fuzzy. That is, the transition between membership and non-membership of the elements to a given attribute is gradual and an element can "more or less" belong to the set, in contrast to the classical set, for which membership of the elements can be characterized by a binary form such as "yes" or "no, "true" or "false", "1" or "0", etc.

The set of the adults which is defined by the age only will serve as a very simple example of a classical set. We may define the adults as those people whose age has attained twenty, for instance, regardless of any other states of those people such as mental development. Such a definition of the adult is traditionally adopted and is widely applied in the society. It is, however, clearly unreasonable in many cases to make such a clear-cut classification for reasons other than those of tradition or convenience, especially when one considers the larger differences. It should be noted, therefore, that such a method of classification like this is often used, consciously or unconsciously, in the systems analysis without adequate forethought.

If we apply the concept of characteristic functions to characterize membership of the elements to the set of the adults as a function of age, then we obtain a binary function described in Fig. 3.1. We asssociate the value 1 with the age greater than and including twenty, and the value 0 with the age less than twenty. In mathematical expression we can write it as:

$$f_A(X) \begin{cases} = 1 \text{ if } X \geq 20 & (\text{or } X \in A) \\ = 0 \text{ if } X < 20 & (\text{or } X \in A) \end{cases}$$

where $f_A(X)$ is the characteristic function, A the set of the adults and X the age. Thus, for example:

$$f_A(15) \quad = \quad 0; \qquad\qquad f_A(20.5) \quad = \quad 1 \quad .$$

Now let us consider the set of "old" men which will serve as a simple example of fuzzy set. Here we will think of the word "old" as used in daily life. In this case, we have no actual clear-cut definition of the word "old", that is, there is no definite threshold of age above which it should be called to be old. In fact, a man becomes old, although this may not necessarily be in linear relationship with his age. A man of age 80 is usually considered to be quite old and a man of age 50 is said to be less old than a man of age 60. This implies that, even if we think of the meaning of the word "old" in connection with age only, the delineation is very fuzzy. However, our daily conversation is usually not based on words which have clear-cut definitions as is done in mathematics, but on very fuzzy words; yet it usually goes smoothly.

If we apply the concept of characteristic functions to characterize membership of the elements to the set of "old" men, a rather continuous function that represents gradual transition between membership and non-membership and not a binary function must be used. The characteristic function, or the membership function as it is called in the fuzzy set theory, might have a form as described in Fig. 3.2., for example. Thus, the membership function which characterizes the fuzzy set can be considered as a generalization of the characteristic function applied to the classical set. In terms of a more general mathematical expression, a fuzzy set A in X, where X is a space of points with a generic element of X denoted by X, is characterized by a membership function $f_A(X)$ which associates a real number in the interval {0,1} with the value of $f_A(X)$ at (X) representing the grade of membership of X in A with each point in X.

$$X \xrightarrow[f_A]{} \{0,\ 1\}$$

One of the difficulties in applying the fuzzy set theory to actual complex problems is how to determine the membership functions. This is a problem which is fortunately somewhat similar to that of determining the utility function in utility theory.

To show that the concepts of fuzzy sets are an extension of those of classical sets, some basic definitions will be listed below:

- Empty Set

  The definition of empty set is the same as for the classical set. A fuzzy set is said to be empty if and only if its membership function is identically zero on the space X.

- Equality

Two fuzzy sets A and B are equal if and only if the membership functions $f_A(X)$ and $f_B(X)$ are equal for all X in X. Written more simply, A = B if and only if $f_A = f_B$.

- Complement (Fig. 3.3)

The complement of a fuzzy set A (denoted by A') is the fuzzy set which is characterized by the membership function

$$f_{A'} = 1 - f_A \quad .$$

- Containment

A fuzzy set A is said to be contained in a fuzzy set B′ if and only if $f_A \leq f_B$ for all X in X.
This is equivalent to saying that A is a sub-set of B, or that A is smaller than or equal to B.
Symbolically, this relation is written as

$$A \subset B \Longleftrightarrow f_A \leq f_B \quad .$$

- Union (Fig. 3.4.)

The union of two fuzzy sets A and B with respective membership functions $f_A(X)$ and $f_B(X)$ is defined as a fuzzy set C whose membership function $f_C(X)$ is related to those of A and B by

$$f_C = \mathrm{Max}\ \{f_A(X),\ f_B(X)\} \quad , \quad X \in X$$

or in abbreviated form

$$f_C = f_A \vee f_B \quad .$$

The union of A and B is symbolically written

$$C = A \cup B \quad .$$

- Intersection (Fig. 3.5)

  The intersection of two fuzzy sets A and B with respective membership functions $f_A(X)$ and $f_B(X)$ is a fuzzy set C whose membership function $f_C(X)$ is related to those of A and B by

$$f_C(X) = \text{Min}\ \{f_A(X),\ f_B(X)\} \quad , \qquad X \in X \quad .$$

  In abbreviated form, the above expression is written as:

$$f_C = f_A \wedge f_B \quad .$$

  The intersection of A and B is symbolically written as:

$$C = A \cap B \quad .$$

- Probability of Fuzzy Event

  As an extension of the expression of the probability for the non-fuzzy event characterized by an ordinary set, the probability P(A) of a fuzzy event characterized by a fuzzy set A in n-dimensional Euclidian space R^n can be expressed by the following integral

$$P(A) = \int_{R^n} f_A(X)\, dP = E(f_A) \quad ,$$

  where f_A is the membership function associated with the fuzzy set A, P is the probability measure and the operation E denotes the mathematical expectation.

Using this basic definition, the probability theory for the fuzzy events can be developed in a manner similar to that for the non-fuzzy events.

We will not go any further into the mathematical aspects of fuzzy set and conclude this chapter by pointing out that a quite broad area of the theory has been developed in the past ten years. This includes fuzzy topology, fuzzy matrices, fuzzy convex sets, fuzzy logic, fuzzy systems, fuzzy language, L-fuzzy sets, and others. Fuzzy theory is being applied to a variety of problems in the field of social sciences, i.e., economics, psychology and sociology, as well as in the field of engineering.

IV. Possible Application to Risk Assessment

The term risk may generally refer to the undesirable effect associated with a specific activity considered in connection with their respective uncertainties (7). Since the notion of risk here must be conceived in terms of the interaction between the object (environment, activity, technological products, etc.) and the subject (individual, group, society, etc.) a concrete definition of risk must be provided with the specifications of the type "of what" and "to whom" (Fig. 4.1).

Presence of the subject in the risk concept is an essential source of fuzziness in the risk assessment,because the same object may be differently risky to different subjects who are in different positions and who have different amounts of information, different types of perception and preference with respect to the object. In this sense, risk is a notion which is not of absolute or objective nature but rather of relative or subjective nature.

In this chapter some posssible applications of the fuzzy set concepts to the risk assessment problems will be pointed out by using very basic conceptual illustrations.

1. Event-Consequence Model

In Reference (7), risk estimation refers to the identification of the second (and higher) order consequences of a decision and the subsequent estimation of the magnitude of the associated risks. For this purpose, it is necessary to construct a model based on which the analysis is to be performed. However, for a complex real system such as the energy system, it is usually very difficult to construct a detailed model which can simulate exactly the behaviour of the real system. In such a case, there exists a certain gap between the real system and its model (Fig. 4.2).

It is, however, difficult to predict how large this gap will be because it is usually very difficult to know the range of the real system behaviour and also to predict exactly how the model itself will behave. Usually, it is possible to know about the gap only very vaguely. If we can know about the gap even in a vague way the model may be improved by taking this vagueness into consideration.

Let us take an example of the core melting model for a nuclear reactor. In Reference (8), it is assumed that "if conditions are such that some core melting results, then essentially complete core melting results". This is an assumption made for the sake of conservatism, and is due to lack of sufficient data. The categorization of the set is similar to the example of the set of adults described in the previous chapter (Fig. 4.3a). However, as the past accident records show, partial core melting is more likely to occur than complete core melting which has never been experienced up to now in actual reactors. Therefore, if we want to make a more "realistic" assessment rather than a "conservative" one, it would be better to take partial core melting into consideration in the model. In terms of fuzzy set, a macroscopic model of core melting can be represented by a fuzzy set of "core melting" which includes 50% to 100% core melting. The associated membership function here is possibly linear with the percentage of molton core (Fig. 4.3b).

2. Inadequate Statistical Data

When the data available for analysis are very poor, there will be a large amount of uncertainty in the results of analysis. If the measurement technique for the event of interest is not well developed, or if the measured data are much distorted by various types of disturbances, available data will be very inaccurate. Also, if the number of samples is very small, the data will be statistically inadequate. Let us suppose, for example, that there is a big urn which contains a large number of small balls which are all identical except for their colour. Also suppose that somebody gave us, as a sample, only ten balls picked up from the urn, of which one is white and the remaining nine are black. Now we are asked to estimate the ratio m of the number of white balls to the total number of balls contained in the urn. What can we say with only the data given above? Is the ratio m nearly equal to 0.1? Such a sample may be obtained even if the true value of m is 0.01 in this case. We do not know whether or not the sampling was made under sufficiently randomized condition. We do not know if there are balls of other colour in the urn or not. In this situation, it is almost impossible to make a statistically meaningful estimation of the ratio m. If we are informed that the sampling was made under randomized conditions and that there are only white balls and black balls in the urn, then the situation will become much better. However, the uncertainty in the estimate of m is still large; and we cannot eliminate such uncertainty.

One of the methods which is often employed in such situations is to use the value 0.1 as a base estimate of m and then to make a sort of sensitivity analysis by varying the value of m around the base point 0.1. A more complicated method is to apply the statistical theory of estimation, although formal application of the theory will not produce good results when available data are poor. A less complicated method would be to apply fuzzy set concept. Since we must start our analysis using available data, it would be justifiable to think that m is nearly equal to 0.1. Even if somebody else knows that the true value of m is 0.01, it does not effect our analysis, since we do not know it.

Now, in fuzzy set terms, the statement "nearly equal to 0.1" is replaced by a fuzzy set of numbers with an associated membership function which takes its largest value at 0.1. This membership function may be determined subjectively, depending on how much we can rely upon the available data. If one does not like this type of subjectivity, the membership function may be determined, for example, as the likelihood function, which assumes randomized conditions for the sampled data (Fig. 4.4). However, subjectivity cannot be avoided in assuming the randomized condition also in this case.

In the assessment of risks associated with low frequency-large consequence events for which adequate statistical data are usually not available, a large amount of uncertainty and ambiguity may exist in the data as well as in the model.

3. Decision-Making Involving Risks

Risk-benefit problems essentially involve multiple object functions,of which at least two functions are related to the risk and the benefit. The decision-making problem with multiple object functions is generally very difficult to treat "objectively", because simultaneous satisfaction of multiple object functions is not always possible. Therefore, there will usually arise the problem of weighting or trade-off among different object functions, for which subjective judgement is more or less inevitable. Even in the problem of mechanistic system related decision-making, such as the optimization of control system of a power plant having multiple object functions, determination of the weight for each object function is usually made only very vaguely through designer's experience and rather subjectively. For more complex decision problems related to humanistic systems, subjectivity will become more inevitable than for the case of mechanistic systems.

One of the classical approaches to such decision problems is to apply the utility theory. Interesting discussions on decision problems from the psychological standpoint can be found in Reference (9). A mathematical theory of decision-making based on the ordinary set theory can be found in Reference (10) also. In the classical theories, the problem statement must be clearly made so that in the system model, the goal and the constraints could also be clearly defined. The formulation of the utility theory, for example, is based on the concepts of ordinary sets and postulates a set of axioms with regard to preference ordering which must be respected by a "rational" subject, although such an abstract and mathematical notion of rationality cannot be achieved so easily in a real world situation. In order to apply the theory, it is usual practice to construct a clear-cut model in which fuzziness is absent by abstracting, from the vast complexity of the real situation, manageable features relevant to the problems under consideration. In practice, there is always more or less subjective judgement as to what should be taken as the model which is appropriate for our study. However, the fact that all modelling process involves in part certain subjective judgement is usually obscured in the use of such a clear-cut model by the absence of any explicit representation of the subjective elements within the model itself. In this respect, the fuzzy-set-theory approach might improve such situations by accepting the fuzziness or the ambiguity in the model.

When the problem is stated very vaguely, the appropriateness of the fuzzy set concepts becomes more pronounced. Just to explain conceptually the fuzzy set theoretic approach to decision problems, let us consider a very simple example. Suppose that the problem is to find a value or a set of values which satisfies the following goal and constraint:

Goal:	x is required to be sufficiently larger than 1.
Constraint:	x must be roughly equal to 5.

Using conventional mathematical notation, it can be expressed by:

Goal:	$x \gg 1$
Constraint:	$x \simeq 5$.

How can we handle this problem using conventional methods? A conventional method would be to reformulate the problem as follows:

Taking $y = x$ as the object function, and assuming that the meaning of "roughly equal to 5" can be represented by a number which lies between $5 - d$ and $5 + d$, we maximize the object function under the constraint which is subjectively interpreted. In mathematical expression, it can be written by:

Goal: $\max_{x} y = x$

Constraint: $5 - d \le x \le 5 + d$.

The solution is simple:

$$x = 5 + d \ .$$

This is shown graphically in Figure 4.5a.

In fuzzy-set-theoretic approach, the vagueness in the problem statement is expressed more or less subjectively in the form of a membership function. Assuming the membership functions for the goal $f_g(x)$ and for the constraint $f_c(x)$, which are respectively associated with the fuzzy goal set G and the fuzzy constraint set C, the solution is expressed by a fuzzy set which is the intersection of the fuzzy goal set G and the fuzzy constraint set C.

Using mathematical notation, the solution set is:

$$D = G \cap C$$

with the associated membership function:

$$f_d = f_g \wedge f_c \ .$$

If it is required to find a particular number x instead of a set, one may choose, for example, the value of x which maximizes the membership function f_d. This is called the maximizing decision. A graphical illustration is given in Fig. 4.5b. Determination of $f_g(x)$ and $f_c(x)$ will depend upon the nature of the problem. Thus the expression "$x \gg 1$", for instance, may be represented by different membership functions in different problems, depending upon the judgement as to what is "sufficiently" larger than 1 in a particular problem.

V. Concluding Remarks

There will be raised a question about the merits which can be gained by the application of fuzzy set theory to risk assessment problems. It will be difficult to give a clear-cut answer to such a question because no sophisticated methods can be very powerful if we do not take into consideration the real world situation. Successful application of the theory will therefore depend as much upon the skill of the analyst as upon the nature of the problem itself. Determination of the membership function and interpretation of various fuzzy relations are the major key points for successful application of the theory. Although the fuzzy set concept can be applied to construct a microscopic model as well as a macroscopic model, its application on the level of microscopic model may not be interesting since it will only complicate the model without adding any substantial gain to the result of the analysis. Application of the fuzzy set theory for constructing a macroscopic model for risk assessment would be more appropriate since certain fine structures in the microscopic model can be retained to some extent, while using the fuzzy set concepts in the macroscopic model. The possibility of constructing better macroscopic models based on the fuzzy set concept will be advantageous, from the computational viewpoint, for its application to the development of reactor diagnostics which must be performed on an on-line, real-time base.

A mathematical formulation of risk assessment problems based on the fuzzy set theory will be presented in a forthcoming paper (11). The present paper is intended as an introduction to this.

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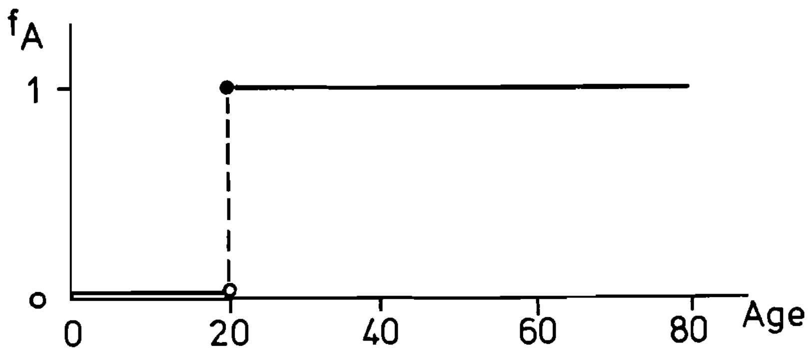

Fig. 3.1 Characteristic function of "Adult"

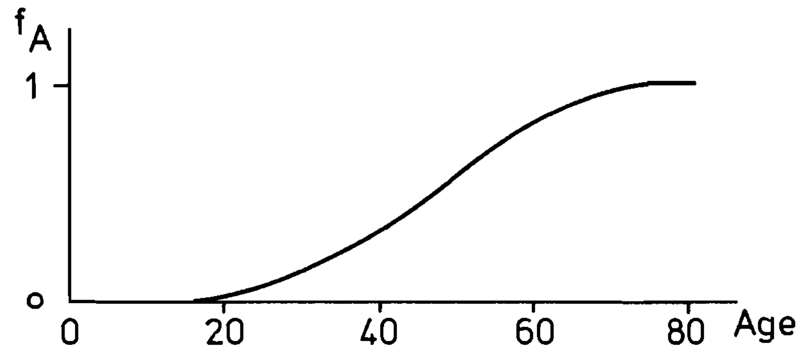

Fig. 3.2 A membership function of "Old" men

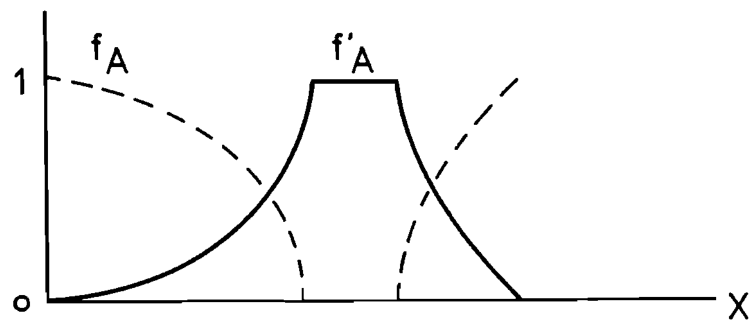

Fig. 3.3 The membership functions f_A and $f_{A'}$, associated with the fuzzy set A and its complement A', respectively

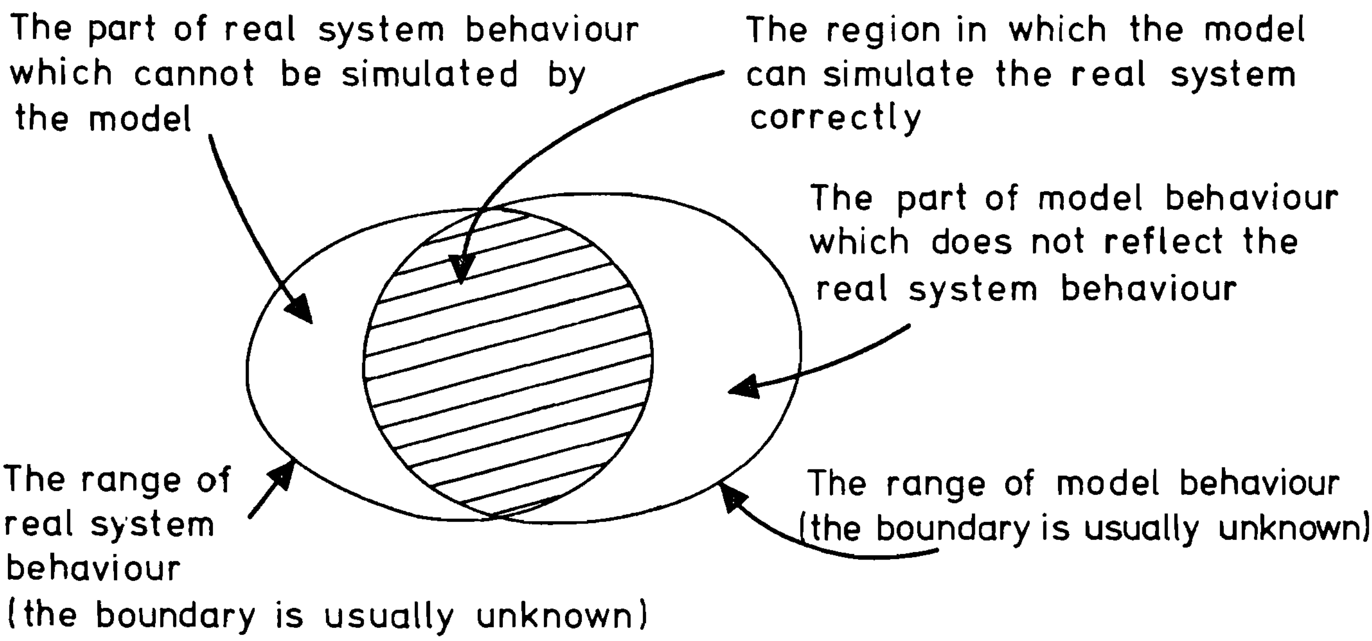

Fig. 4.2 The gap between the real system and the model

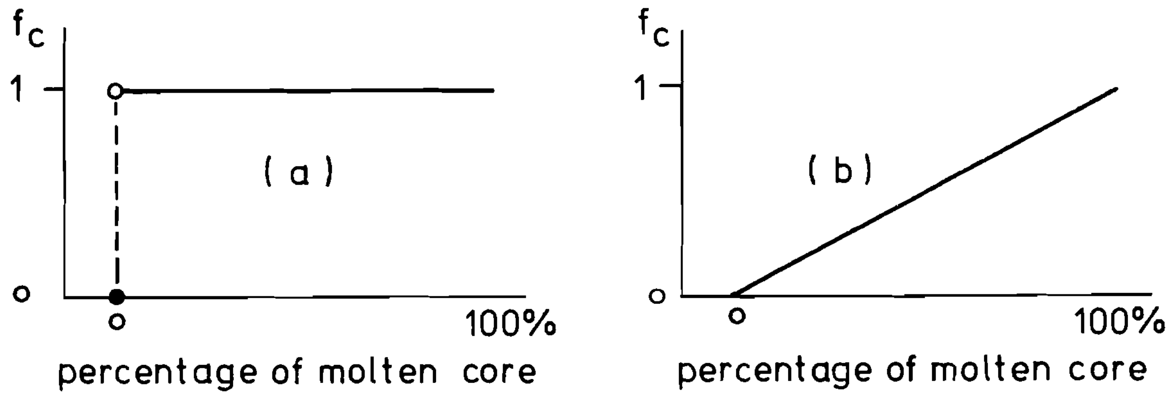

Fig. 4.2 (a): Characteristic function for the model in Reference (8)

(b): A possible membership function for the fuzzy set "core melting"

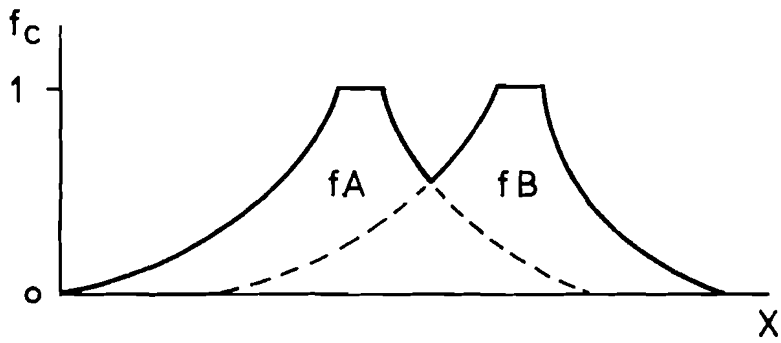

Fig. 3.4 The membership function associated with the union C of A and B

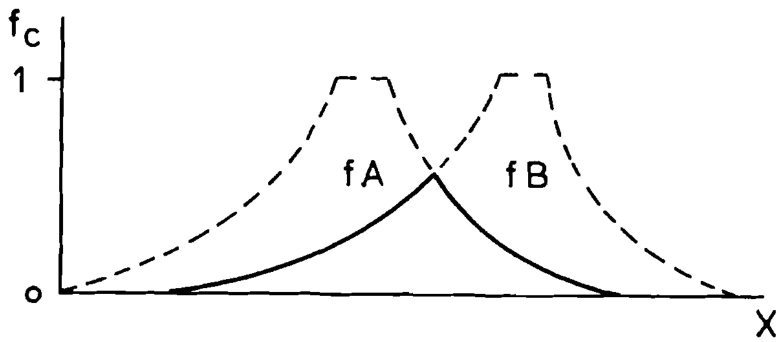

Fig. 3.5 The membership function associated with the intersection C of A and B

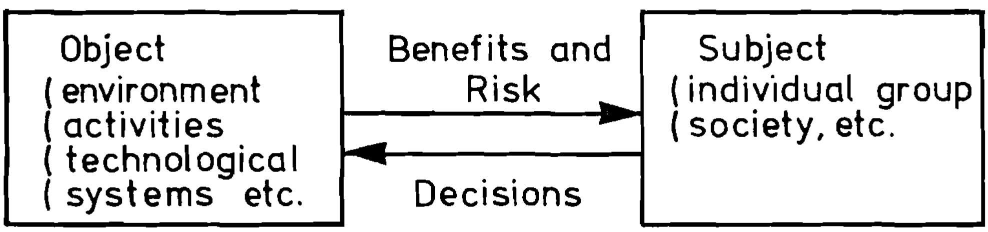

Fig. 4.1 Interaction between object and subject

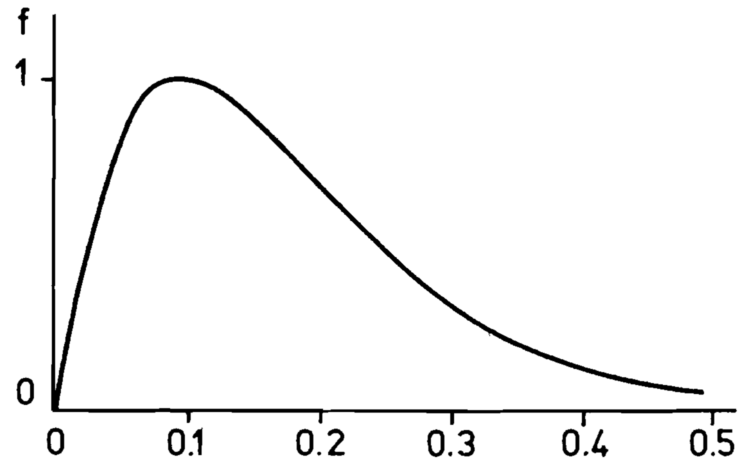

Fig. 4.4 Likelihood function used as a membership function

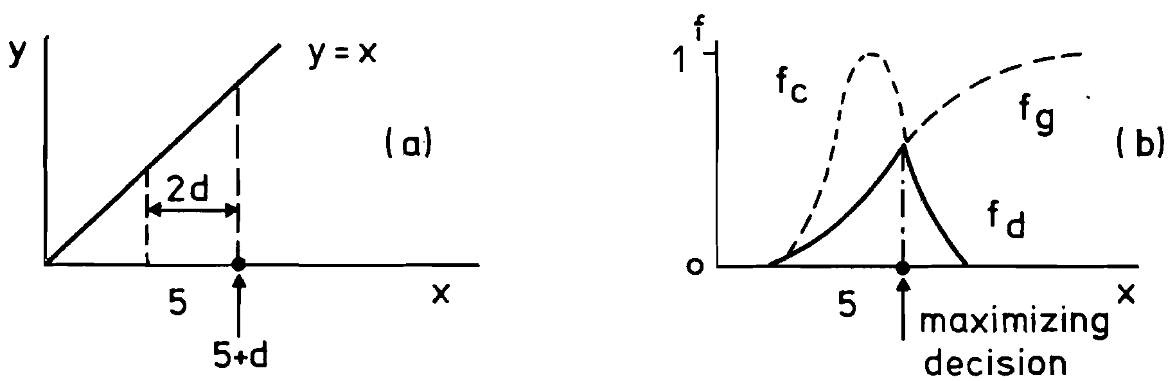

Fig. 4.5 (a): A conventional method
(b): A fuzzy set theoretic approach