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Optimal Control of R&D Investment in a Techno-Metabolic System

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Abstract

The nonlinear model of economic growth involving production, technology stock and their rates is considered. Two trends - growth and decline, in interaction between production and R&D investment are examined in the balance dynamics. The optimal control problem of R&D investment is studied for the balance dynamics and discounted utility function of consumption index. Pontryagin's optimality principle is applied for designing optimal nonlinear dynamics. The existence and uniqueness result is proved for the saddle type equilibrium and the convergence property of optimal trajectories is shown. Quasioptimal feedbacks of the rational type for balancing the dynamical system are proposed. Growth properties of production rate, R&D intensity and technology intensity are examined on generated trajectories. In the test example explicit formulas for the optimal feedback and the value function are obtained.

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Introduction

We consider a dynamical model which is connected with the problem of optimal R&D investment in a techno-metabolic system. The key idea of the model consists in the fact that there are two trends which describe interaction between manufacturing and R&D investment. On one hand growth of the firms output (deflated sales) is affected by the accumulated R&D investment. On the other hand the current R&D investment demands resources which are taken out from the manufacturing process. The first trend provides the stable effect of the sustainable growth. The second one introduces the risky factor of the R&D innovation. The model includes the integral utility function which correlates the amount of sales and production diversity. The amount of sales is determined by the production growth and production diversity depends on the accumulated and current R&D investment. The problem is to find the optimal R&D investment policy which maximizes the utility function in presence of two trends – "growth" and "decline" in dynamics of manufacturing and R&D investment.

In this research we deal with the classical problems of economic growth and optimal allocation of resources (see [Arrow, 1985], [Arrow, Kurz, 1970]). In our analysis we refer to the endogenous growth theory (see [Grossman, Helpman, 1991]) in which control models for optimal allocation of resources into manufacturing are studied and discounted utility functions with the consumption index of logarithmic type and equal elasticity of substitution of invented products are introduced. As a result they obtain dynamical systems which describe the optimal (equilibrium) growth of the knowledge stock – the accumulated R&D investment. The generalized model of the endogenous growth for countries with absorptive capacities and the asymptotic behavior of the ensuing non-linear dynamics were analyzed in [Hutschenreiter, Kaniovski, Kryazhimskii, 1995]. Unlike these models we are basing our analysis on dynamics which describes growth of sales with respect to R&D investment. Let us note that the origin of this dynamics can be found in the research [Watanabe, 1992] on substitution of the production factors to technology. We also use basic elements for constructing the model of R&D investment proposed in [Intriligator, 1971], [Griliches, 1984], [Arrow, 1985].

We compose the optimal control problem and solve it using the principle maximum of Pontryagin (see [Pontryagin, Boltyanskii, Gamkrelidze, Mishchenko, 1962]). For analysis of optimal solutions: value functions, optimal feedback and its approximations, we apply optimality principles of the theory of Hamilton-Jacobi equations [Crandall, Lions,

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1983], [Subbotin, 1995] and differential games [Krasovskii A.N., Krasovskii N.N., 1995]. Let us mention also approximate solutions for value functions and optimal feedbacks in control problems with discounted payoff integrals [Dolcetta, 1983] and differential games [Tarasyev, 1995], [Mel'nikova, Tarasyev, 1997].

We obtain the nonlinear system of differential equations which describes the dynamics of the optimal R&D investment policy, manufacturing and corresponding prices. We find the first integral for this system and reduce it to the system of the second order. Further we derive the following results for the reduced dynamics. The existence and uniqueness result for equilibrium is proved for the indicated range of parameters. Eigenvalues and eigenvectors of the Jacobi matrix are estimated and the saddle type of equilibrium is indicated. The existence of optimal trajectories leading to equilibrium is established. The growth properties of optimal trajectories are shown.

The optimal feedback is given implicitly and can't be expressed through explicit formulas. To approximate optimal feedback we propose several explicit formulas of the rational type – quasioptimal feedbacks. For these quasioptimal feedbacks we demonstrate the convergence of generated trajectories to equilibrium of the optimal control system. Qualitatively quasioptimal feedbacks are parametrized by tangent slopes of R&D intensities. Properties of quasioptimal regimes are analyzed and growth results for production rates, R&D intensities and technology intensity are proved.

For explicit analysis in one test example we reduce the dynamical model to the nonstationary balance equation. We obtain analytic solutions in this test model for optimal feedback, production rate, R&D intensity and technology intensity. Our explicit analysis shows that the optimal policy provides the proportional growth of manufacturing and R&D investment and explains the dependence of R&D intensity with respect to the parameter of substitution, the discount rate, the cost of R&D investment and the rate of return to R&D investment. For the value function of the test model we compose the Hamilton-Jacobi equation and evaluate its solution. We indicate the decomposition property of the value function: it consists of two terms, one of which presents the logarithmic dependence on initial level of production, another one introduces the aggregated influence of dynamics.

Part I. Control Design Problem for Dynamical Model

1 Dynamical Model of R&D Investment

For constructing the dynamical model of manufacturing and R&D investment we use the following variables:

$y = y(t)$ - manufacturing, production;

$T = T(t)$ - accumulated R&D investment, technology;

$\dot{T} = r = r(t)$ - change in technology T, the technology rate;

r_t - R&D investment;

$r_{(t-m)} = (1-\sigma)r + \sigma T$ - R&D investment in initial stage;

$\dot{y}/y$ - production rate;

r/y - R&D intensity;

$r_{(t-m)}/y$ - R&D intensity in initial stage;

T/y - technology intensity, y/T - technology productivity;

L - labor, K - capital, M - materials, E - energy, involved in manufacturing and R&D;

L_T - the labor input, K_T - the capital input, M_T - the materials input, E_T - the energy input, directed to R&D.

$\psi_1 = \psi_1(t)$ - the "price" of production;
$\psi_2 = \psi_2(t)$ - the "price" of accumulated R&D investment;
$\psi_1 y$ - the "cost" of production;
$\psi_2 T$ - the "cost" of technology;
$n = n(t)$ - measure of invented products;
$x = y/n$ - quantity of production of each brand.

For constructing dynamics we use the classical production function (see, for example, [Arrow, Kurz, 1970], [Intriligator, 1971], [Griliches, 1984], [Watanabe, 1992])

$$y = F(t, (L - L_T), (K - K_T), (M - M_T), (E - E_T), T) \tag{1.1}$$

For example, one can take the production function as the exponential function of Cobb-Douglas type

$$F = Ae^{\lambda t}(L - L_T)^{b_1}(K - K_T)^{b_2}(M - M_T)^{b_3}(E - E_T)^{b_4}T^{b_5} \tag{1.2}$$

We assume that the functional dependence between the labor L_T, capital K_T, materials M_T, energy E_T inputs and the accumulated R&D investment T is given by the function of the substitution type

$$T = T(L_T, K_T, M_T, E_T) = \min\{h_1(L_T), h_2(K_T), h_3(M_T), h_4(E_T)\} \tag{1.3}$$

and the inverse relations exist

$$\begin{aligned} L_T = L_T(T) = h_1^{-1}(T), \quad K_T = K_T(T) = h_2^{-1}(T) \\ M_T = M_T(T) = h_3^{-1}(T), \quad E_T = E_T(T) = h_4^{-1}(T) \end{aligned} \tag{1.4}$$

For example, one can accept the exponential structure of the R&D objectives function T (1.3)

$$T = \min\{\frac{L_T^{c_1}}{\gamma_L}, \frac{K_T^{c_2}}{\gamma_K}, \frac{M_T^{c_3}}{\gamma_M}, \frac{E_T^{c_4}}{\gamma_E}\} \tag{1.5}$$

and inverse maps also have the form of exponential functions

$$L_T = (\gamma_L T)^{\frac{1}{c_1}}, \quad K_T = (\gamma_K T)^{\frac{1}{c_2}}, \quad M_T = (\gamma_M T)^{\frac{1}{c_3}}, \quad E_T = (\gamma_E T)^{\frac{1}{c_4}} \tag{1.6}$$

Differentiating the production function (1.1) by time t and taking into account (1.4) we obtain the following equation

$$\begin{aligned} &\frac{\dot{y}}{y} = \frac{\partial F}{\partial t}\frac{1}{y} + \frac{\partial F}{\partial L}\frac{L}{y}\frac{\dot{L}}{L} + \frac{\partial F}{\partial K}\frac{K}{y}\frac{\dot{K}}{K} + \frac{\partial F}{\partial M}\frac{M}{y}\frac{\dot{M}}{M} + \frac{\partial F}{\partial E}\frac{E}{y}\frac{\dot{E}}{E} - \\ &\frac{\partial F}{\partial L}\frac{\partial L_T}{\partial T}\frac{\dot{T}}{y} - \frac{\partial F}{\partial K}\frac{\partial K_T}{\partial T}\frac{\dot{T}}{y} - \frac{\partial F}{\partial M}\frac{\partial M_T}{\partial T}\frac{\dot{T}}{y} - \frac{\partial F}{\partial E}\frac{\partial E_T}{\partial T}\frac{\dot{T}}{y} + \\ &\frac{\partial F}{\partial T}\frac{\dot{T}}{y} \end{aligned} \tag{1.7}$$

Let us rewrite equation (1.7) in the form

$$\frac{\dot{y}}{y} = f - p\frac{r}{y} + q\frac{r}{y} \tag{1.8}$$

where terms related to the production factors L, K, M, E, learning and scale effects λ are combined into function f

$$f = \lambda + \frac{\partial F}{\partial L}\frac{L}{y}\frac{\dot{L}}{L} + \frac{\partial F}{\partial K}\frac{K}{y}\frac{\dot{K}}{K} + \frac{\partial F}{\partial M}\frac{M}{y}\frac{\dot{M}}{M} + \frac{\partial F}{\partial E}\frac{E}{y}\frac{\dot{E}}{E} \tag{1.9}$$

decrease in manufacturing due to R&D spending L_T, K_T, M_T, E_T is collected into function p

$$p = p(t) = \frac{\partial F}{\partial L}\frac{\partial L_T}{\partial T} + \frac{\partial F}{\partial K}\frac{\partial K_T}{\partial T} + \frac{\partial F}{\partial M}\frac{\partial M_T}{\partial T} + \frac{\partial F}{\partial E}\frac{\partial E_T}{\partial T} \tag{1.10}$$

increase of R&D knowledge stock is described by function q which coincides with the marginal productivity of technology

$$q = q(t) = \frac{\partial F}{\partial T} \tag{1.11}$$

the control parameter r stands for the current change $\dot{T}$ in technology T

$$\dot{T} = r \tag{1.12}$$

Remark 1.1 *Change $\dot{T} = r$ in technology T is derived from R&D investment r_t, however, due to time lag m and obsolescence effect σ in technology, R&D investment r_t is not precisely equal to change in technology $\dot{T} = r$, but could be treated as its approximation*

$$r_t \approx r = \dot{T}$$

Such approximation could be supported by the following concept. Assume that contribution $r_{(t-m)}$ to the current change $\dot{T} = r$ in technology T is specified by the time lag m and the rate coefficient of obsolescence of technology σ according to the formula

$$\dot{T} = r = r(t) = \frac{1}{(1-\sigma)}(-\sigma T + r_{(t-m)}), \quad 0 \le \sigma < 1 \tag{1.13}$$

For small enough values of parameters m, σ we can derive from equation (1.13) the approximate relation

$$r \approx r_t \approx r_{(t-m)}$$

Relation (1.13) means that a part of contribution $r_{(t-m)}$ to R&D at time $(t-m)$ is spent for compensation of obsolescence σT of technology T and the rest $(r_{(t-m)} - \sigma T) \ge 0$ affects on the current change of technology $\dot{T} = r$ with the time lag m.

Differential equation (1.13) is a continuous analogue of the finite difference formula with the time step $\Delta = 1$ for dynamics of technology T_t depending on the knowledge stock $T_{(t-1)}$ in the previous year with the effect of obsolescence given by the rate coefficient σ and R&D investment in initial stage $r_{(t-m)}$ with the time lag m

$$T_t = r_{(t-m)} + (1-\sigma)T_{(t-1)}, \quad 0 \le \sigma < 1 \tag{1.14}$$

For the given knowledge stock $T = T(t)$ and known current change $r = r(t)$ the contribution $r_{(t-m)}$ is expressed by relation

$$r_{(t-m)} = (1-\sigma)r + \sigma T \tag{1.15}$$

The last relation shows that for the given knowledge stock T and its current change r contribution $r_{(t-m)}$ is their convex combination with coefficients $\sigma_1 = (1-\sigma)$, $\sigma_2 = \sigma$, $\sigma_1 + \sigma_2 = 1$, $\sigma_1 > 0$, $\sigma_2 \ge 0$. If the value of knowledge stock T is strictly larger than its velocity r, $T > r$, then contribution $r_{(t-m)}$ should be larger the greater is the rate coefficient of obsolescence σ.

Collecting the terms $(r/y)p$, $(r/y)q$ which depend on the control parameter r into the net contribution by R&D intensity $(r/y)g$ we obtain the first equation for the dynamical control process

$$\frac{\dot{y}}{y} = f - g\frac{r}{y} \tag{1.16}$$

Let us assume in (1.16) that

$$g = g(t) = p(t) - q(t) > 0 \tag{1.17}$$

In the general case function f depends on the accumulated R&D investment T. Let us assume that this dependence is given by the formula

$$f = f_1 + f_2\left(\frac{T}{y}\right)^{\gamma}, \quad f_1 = f_1(t), \quad f_2 = f_2(t) \tag{1.18}$$

The exponential structure of the growth function f (1.18) is rather reasonable since the production function F (1.1) and the R&D objectives function T (1.5) are of the exponential type.

Combining formulas (1.12), (1.16), (1.18) we obtain the dynamic process described by the system of differential equations

$$\begin{aligned} &\frac{\dot{y}}{y} = f_1 + f_2\left(\frac{T}{y}\right)^{\gamma} - g\frac{r}{y} \\ &\dot{T} = r \end{aligned} \tag{1.19}$$

One can consider dynamic process (1.19) as balance equations of spending resources between the productivity rate $\dot{y}/y$ and R&D intensity r/y. The term f_2T^{γ} shows the growth effect of the accumulated R&D investment T on production y. Function f_1 presents the non-R&D contribution. The negative sign $(-g(t))$ of the net contribution by R&D means that in the short-run spending $p(t)$ into R&D prevales on the rate of return $q(t)$ to R&D and provides the decline and risky factor of R&D investment.

The production y and the accumulated R&D investment T stand for the phase parameters in dynamics (1.19). The current change r in technology T is the control parameter. The control parameter $r = r(t)$ is not fixed and can be chosen for obtaining "good" properties of trajectories of dynamics (1.19).

2 Utility Function for R&D Investment Process

Now we need to formalize the goal for designing the control parameter $r = r(t)$ and indicate the profit of R&D investment in the long-run. For this purpose we consider the utility function represented by the integral with the discount coefficient ρ (see, for example, [Arrow, 1985], [Arrow, Kurz, 1970], [Grossman, Helpman, 1991])

$$U_t = \int_t^{\infty} e^{-\rho(s-t)} \ln D(s)ds \tag{2.1}$$

Here consumption index $D(s)$ represents an utility of products (technologies) at time s, ρ is the discount rate, s is the running time, t is the fixed (initial time).

For D we choose a specification that imposes a constant and equal elasticity of substitution between every pair of products

$$D = D(s) = (\int_0^n x^{\alpha}(j)dj)^{1/\alpha}, \quad n = n(s) \tag{2.2}$$

Here j is the current index of invented products, $x(j)$ is the quantity of production of the brand with index j, n is the quantity of available (invented) products, α, $0 < \alpha < 1$, is the parameter of elasticity, ε – elasticity of substitution between any two products,

$$\varepsilon = 1/(1-\alpha) > 1 \tag{2.3}$$

Let us make the following assumptions. Assume that quantities $x(j)$ are equal for each index j

$$x(j) = \frac{y}{n}, \quad y = y(s), \quad n = n(s) \tag{2.4}$$

and the quantity of invented products n depends on the accumulated R&D investment T and change in technology r according to the exponential rule

$$n = n(s) = be^{\kappa s}T^{\beta_1}r^{\beta_2}, \quad T = T(s), \quad r = r(s) \tag{2.5}$$

Formulas (2.4), (2.5) means that innovation n depends upon the forefront R&D activities demonstrated by the technology rate r. At the same time it owed accumulation of past R&D activity given by technology stock T. In addition innovation n has such general tendency to decaying nature which can be expressed by term $e^{\kappa s}$. All three effects lead to decrease in the respective brand production x and imply diversification.

Combining equations (2.1)-(2.5) we obtain the following expression for the utility function

$$\begin{aligned} W_t \;&=\; \int_t^\infty e^{-\rho(s-t)}(\ln y(s) + a_1 \ln T(s) + a_2 \ln r(s))ds + \\ &\int_t^\infty e^{-\rho(s-t)}(\kappa s + \ln b)ds \end{aligned}$$

Here

$$a_1 = A\beta_1, \quad a_2 = A\beta_2, \quad A = \frac{(1-\alpha)}{\alpha}$$

Let us note that the second term $e^{-\rho(s-t)}(\kappa s + \ln b)$ in the utility function U_t does not depend on main variables y, T, r and, hence, one can consider the utility function U_t

$$U_t = \int_t^\infty e^{-\rho(s-t)}(\ln y(s) + a_1 \ln T(s) + a_2 \ln r(s))ds \tag{2.6}$$

instead of the utility function W_t taking in mind relation

$$W_t = U_t + \int_t^\infty e^{-\rho(s-t)}(\kappa s + \ln b)ds = U_t + \frac{\kappa}{\rho}(t + \frac{1}{\rho}) + \frac{1}{\rho}\ln b$$

The structure of the utility function U_t (2.6) means that investors (governments, financial groups) are interested in growth of production y as well as in growth of the accumulated R&D investment T and the current change of technology r (new goods, technologies, etc.).

3 Optimality Principles for Investment Dynamics

The problem is to find such level of the technology rate $r^0 = r^0(t)$ (optimal investment or optimal control) in the class of piecewise-constant functions $r(t)$, the corresponding optimal production $y^0 = y^0(t)$ and the optimal accumulated R&D investment $T^0 = T^0(t)$ subject to dynamics (1.19) which maximize the utility function (2.6).

Let us note that problem (1.19), (2.6) is a classical problem of the optimal control theory. For its solution one can use the maximum principle of L.S. Pontryagin (see [Pontryagin, Boltyanskii, Gamkrelidze, Mishchenko, 1962]). Applications of this optimality principle to problems of economic growth were developed in [Arrow, 1985], [Arrow, Kurz, 1970].

According to this principle it is necessary to compose the system of the following equations. The first two equations are given by dynamic process (1.19). We rewrite them as follows

$$\begin{aligned} \dot{y} &= f_1 y + f_2 T^{\gamma} y^{(1-\gamma)} - gr \\ \dot{T} &= r \end{aligned} \tag{3.1}$$

Let us compose the Hamiltonian of the problem (1.19), (2.6)

$$\begin{aligned} H(y, T, r, \psi_1, \psi_2) &= \ln y + a_1 \ln T + a_2 \ln r + \\ &\quad \psi_1 (f_1 y + f_2 T^{\gamma} y^{(1-\gamma)} - gr) + \psi_2 r \end{aligned} \tag{3.2}$$

The Hamiltonian H (3.2) is the current flow of utility from all sources. The current control $r = r(t)$ is chosen to maximize this flow. Calculating maximum of the Hamiltonian (3.2) by parameter r we obtain the following relations

$$\frac{\partial H}{\partial r} = a_2 \frac{1}{r} - g\psi_1 + \psi_2 = 0 \tag{3.3}$$

So the maximum value is attained at the optimal technology rate r^0

$$r^0 = a_2 \frac{1}{(g\psi_1 - \psi_2)} \tag{3.4}$$

For dynamics of the conjugate (adjoint) variables ψ_1, ψ_2 which can be interpreted as "price" of production y and "price" of the accumulated R&D investment T one can compose the adjoint equations

$$\dot{\psi}_1 = \rho\psi_1 - \frac{\partial H}{\partial y} = \rho\psi_1 - \frac{1}{y} - (1-\gamma)\psi_1 f_2 T^{\gamma} \frac{1}{y^{\gamma}} - \psi_1 f_1 \tag{3.5}$$

$$\dot{\psi}_2 = \rho\psi_2 - \frac{\partial H}{\partial T} = \rho\psi_2 - a_1 \frac{1}{T} - \gamma\psi_1 f_2 y^{(1-\gamma)} \frac{1}{T^{(1-\gamma)}} \tag{3.6}$$

Prices ψ_1, ψ_2 measure the marginal contribution of variables y, T to the utility function. Differential equations (3.5), (3.6) for prices ψ_1, ψ_2 can be interpreted as an equilibrium condition: the increment in flow plus the change in price should be zero.

Combining equations (3.1)-(3.6) we obtain the following closed system of differential equations (3.7)-(3.10)

$$\frac{\dot{y}}{y} = f_1 + f_2 \left(\frac{T}{y}\right)^{\gamma} - g a_2 \frac{1}{(g\psi_1 - \psi_2) y} \tag{3.7}$$

$$\dot{T} = a_2 \frac{1}{(g\psi_1 - \psi_2)} \tag{3.8}$$

$$\frac{\dot{\psi}_1}{\psi_1} = \rho - \frac{1}{\psi_1 y} - (1-\gamma) f_2 \left(\frac{T}{y}\right)^{\gamma} - f_1 \tag{3.9}$$

$$\frac{\dot{\psi}_2}{\psi_2} = \rho - a_1 \frac{1}{\psi_2 T} - \gamma f_2 \frac{\psi_1 y}{\psi_2 T} \left(\frac{T}{y}\right)^{\gamma} \tag{3.10}$$

We need to find a solution of the system (3.7)-(3.10) which meet the transversality condition of the maximum principle

$$\lim_{t \to \infty} e^{-\rho t} z(t) = 0 \tag{3.11}$$

Here function z is the cost of production y and the accumulated R&D investment T

$$z = \psi_1 y + \psi_2 T \tag{3.12}$$

Transversality condition (3.11) means that the total cost $z = z(t)$ (3.12) should not grow rapidly than exponent $e^{\rho t}$. In fact we will show below that the total cost $z(t)$ should be constant in the optimal regime.

Let us introduce the value function $(t, y, T) \to w(t, y, T)$ which assigns the optimal result w of the utility function (2.6) along the optimal process (y^0, T^0, r^0) to an initial position (t, y, T)

$$\begin{aligned} w(t, y, T) &= \max_{r(\cdot)} \int_t^{\infty} e^{-\rho(s-t)} (\ln y(s) + a_1 \ln T(s) + a_2 \ln r(s)) ds = \\ &\int_t^{\infty} e^{-\rho(s-t)} (\ln y^0(s) + a_1 \ln T^0(s) + a_2 \ln r^0(s)) ds, \\ & y^0(t) = y, \quad T^0(t) = T \end{aligned} \tag{3.13}$$

Assuming that the value function w is a differentiable one we can compose for it the Hamilton-Jacobi equation

$$\begin{aligned} &\frac{\partial w}{\partial t} + \frac{\partial w}{\partial y} f_1 y + \frac{\partial w}{\partial y} f_2 T^{\gamma} y^{(1-\gamma)} + e^{-\rho t} (\ln y + a_1 \ln T) + \\ &\max_r \{(-\frac{\partial w}{\partial y} g + \frac{\partial w}{\partial T}) r + e^{-\rho t} a_2 \ln r\} = 0 \end{aligned} \tag{3.14}$$

For the differentiable value function w adjoint variables ψ_i, $i = 1, 2$ are defined as its impulses

$$\psi_1 = e^{\rho t} \frac{\partial w}{\partial y}, \quad \psi_2 = e^{\rho t} \frac{\partial w}{\partial T} \tag{3.15}$$

Let us note that the theory of nondifferentiable solutions of Hamilton-Jacobi equations was developed in [Crandall, Lions, 1983], [Subbotin, 1995]. Special results for Hamilton-Jacobi equations in control problems with discount were obtained in [Dolcetta, 1983]. The method of stochastic programming maximin (see [Krasovskii, A.N., Krasovskii, N.N., 1995]) provides the instrument for estimating adjoint variables ψ_i, at points of nondifferentiability of the value function w (3.13). The grid schemes for constructing nondifferentiable solutions of Hamilton-Jacobi equations and optimal feedbacks were proposed in [Tarasyev, 1995], [Mel'nikova, Tarasyev, 1997].

We will be looking for the value function w in the following form

$$w(t, y, T) = e^{-\rho t} (u(y, T) + v(t)) \tag{3.16}$$

Taking into account expression (3.4) for the optimal control r^0 and the fact that prices ψ_1, ψ_2 are the impulse variables

$$\psi_1 = e^{\rho t} \frac{\partial w}{\partial y} = \frac{\partial u}{\partial y}, \quad \psi_2 = e^{\rho t} \frac{\partial w}{\partial T} = \frac{\partial u}{\partial T} \tag{3.17}$$

we obtain from (3.14) the Hamilton-Jacobi equation for components $u(y,T)$, $v(t)$

$$-\rho(u(y,T)+v(t))+\dot v(t)+\frac{\partial u}{\partial y}f_1 y+\frac{\partial u}{\partial y}f_2T^{\gamma}y^{(1-\gamma)}+$$
$$(\ln y+a_1\ln T)+a_2(-1+\ln a_2-\ln(g\frac{\partial u}{\partial y}-\frac{\partial u}{\partial T}))=0 \qquad (3.18)$$

Dividing equation (3.18) into two parts with respect to functions $v=v(t)$, $u=u(y,T)$ we obtain two differential equations

$$\dot v(t)=\rho v(t)+a_2-a_2\ln a_2 \qquad (3.19)$$

$$-\rho u(y,T)+\frac{\partial u}{\partial y}f_1 y+\frac{\partial u}{\partial y}f_2T^{\gamma}y^{(1-\gamma)}+$$
$$(\ln y+a_1\ln T)+a_2\ln(g\frac{\partial u}{\partial y}-\frac{\partial u}{\partial T})=0 \qquad (3.20)$$

Our task is to analyze the system (3.7)-(3.10) for the optimal dynamics of production y, the accumulated R&D investment T, the current technology rate r, prices ψ_1, ψ_2 of production and the accumulated R&D investment together with the transversality conditions (3.11), (3.12) for cost z. In parallel in Section 4 we calculate analytically for the test control problem the value function w as the solution of the Hamilton-Jacobi equation (3.14).

We prove the following qualitative results for the system (3.7)-(3.10). We determine its first integral expressed in terms of the cost function

$$z=\psi_1 y+\psi_2 T=p^0=\frac{a_1+a_2+1}{\rho} \qquad (3.21)$$

We introduce new variables

$$x_1=\frac{y}{T},\quad x_2=\psi_1 y,\quad x_3=\frac{1}{T},\quad x_4=\psi_2 T \qquad (3.22)$$

where x_1 – technology productivity, x_2 – the cost of production, x_3 – the inverse technology, x_4 – the cost of technology.

Taking into account the first integral (3.21) we obtain in Section 5 the equivalent reduced system with separable variables x_1, x_2 in one block and x_3 in another. Further we establish the existence and uniqueness result for the equilibrium x^0 of the reduced system. Then in Section 6 we estimate eigenvalues and eigenvectors of the linearized system and show that equilibrium is the saddle point. This fact means that there exist trajectories of the optimal dynamics (3.7)-(3.10) which lead to the equilibrium.

In the general case optimal control r^0 which provides convergence to equilibrium x^0 has very complicated structure. To substitute optimal control r^0 we propose in Section 7 a series of quasioptimal feedbacks depending on parameter ω with the rational structure r^*

$$r^*=r^*(y,T)=\frac{a_2 y}{(d+(k_1\omega+k_2)((y/T)-x_1^0)+\omega((y/T)-x_1^0)^2)} \qquad (3.23)$$

We prove the convergence result for trajectories of the controlled process (3.7), (3.8) generated by feedbacks r^* (3.23). We indicate parameter w_0 connected with the slope of eigenvector of the linearized system with the negative eigenvalue. The rational feedback $r^*=r^*(\omega_0)$ with the slope ω_0 can be interpreted as the linear approximation of the optimal control r^0.

In Section 8 we study the behavior of R&D intensities r/y, $r_{(t-m)}/y$ and show that there exist intervals for parameter ω which give different combinations of growing and declining properties. Especially we analyze these properties for feedback $r^*=r^*(\omega_0)$ with the "optimal" slope ω_0.

4 Analytic Solution of the Test Optimal Control Problem

Let us note that the nonlinear system (3.7)-(3.10) for the optimal process (y^0, T^0, r^0) is rather complicated and at the first glance does not have the analytic solution expressed in the explicit functions. In the further sections we will give analysis of the system behavior based on implicit formulas. In order to obtain explicit solutions we consider now the reduced version – the test optimal control problem, as the first approximation of the nonlinear system (3.7)-(3.10).

For obtaining the simplified dynamics assume that $\gamma = 0$ in (1.16) and, hence, function f does not depend on the technology parameter T

$$f = f_1 + f_2$$

So we deal with the following dynamics

$$\frac{\dot{y}}{y} = f(t) - g(t)\frac{r}{y} \tag{4.1}$$

Let us assume in (4.1) that

$$g(t) = p(t) - q(t) > 0 \tag{4.2}$$

Unlike the implicit analysis given in the next sections we consider here a nonstationary model with the time dependent functions $f = f(t)$, $g = g(t)$.

We consider equation (4.1) as a balance equation of spending resources between the productivity rate $\dot{y}/y$ and R&D intensity r/y. The negative sign $(-g(t))$ of the net contribution by R&D means that in the short-run technology consumption $p(t)$ exceeds the rate $q(t)$ of return to R&D.

Let us assume that the utility function U_t (2.6) does not depend on the accumulated R&D investment T and so $\beta_1 = 0$, $\beta_2 = 1$

$$U_t = \int_t^{\infty} e^{-\rho(s-t)}(\ln y(s) + A\ln r(s))ds, \quad A = \frac{(1-\alpha)}{\alpha} \tag{4.3}$$

We consider the optimal control problem for dynamics (4.1) and the utility function (4.3) as a reduction of the nonlinear system (3.7)-(3.10). The structure of the utility function (4.3) means that investors are interested as in growth of production y as in growth of new products which is provided by the technology rate r (or by R&D investment r_t). The balance equation (4.1) describes the dynamical relation between production y and the technology rate r, and gives restrictions on the growth of the technology rate r.

The problem is to find such technology rate $r^0 = r^0(t)$ in the class of piecewise-constant functions $r(t)$ and the corresponding optimal production $y^0 = y^0(t)$ subject to dynamics (4.1) which maximize the utility function (4.3).

Applying the Pontryagin principle of maximum to the reduced control problem (4.1), (4.3) we obtain the following system of equations. The first equation is the balance dynamics (4.1)

$$\dot{y} = fy - gr \tag{4.4}$$

Let us compose the Hamiltonian of the problem (4.1), (4.3)

$$H(y, r, \psi) = \ln y + \frac{(1-\alpha)}{\alpha}\ln r + \psi(fy - gr) \tag{4.5}$$

Its maximum by parameter r is determined by the formula

$$\frac{\partial H}{\partial r} = \frac{(1-\alpha)}{\alpha}\frac{1}{r} - g\psi = 0 \tag{4.6}$$

So its maximum value is attained at the optimal R&D investment r^0

$$r^0 = \frac{(1-\alpha)}{\alpha}\frac{1}{g\psi} \tag{4.7}$$

For dynamics of the conjugate variable ψ which can be interpreted as "price" of production y one can compose the adjoint equation

$$\dot{\psi} = \rho\psi - \frac{\partial H}{\partial y} = \rho\psi - \frac{1}{y} - f\psi \tag{4.8}$$

Combining equations (4.6)-(4.8) we obtain the following closed system of differential equations

$$\frac{\dot{y}}{y} = f - \frac{(1-\alpha)}{\alpha}\frac{1}{y\psi} \tag{4.9}$$

$$\frac{\dot{\psi}}{\psi} = \rho - f - \frac{1}{y\psi} \tag{4.10}$$

Introducing notation $z = y\psi$ for the production cost and summarizing equations (4.9), (4.10) we obtain the differential equation

$$\dot{z} = \rho z - \frac{1}{\alpha} \tag{4.11}$$

The general solution of equation (4.11) has the following form

$$z(t) = Ce^{\rho t} + \frac{1}{\rho\alpha} \tag{4.12}$$

The unique solution which meets the transversality condition of the maximum principle is the steady state solution

$$\lim_{t\to\infty} e^{-\rho t} z(t) = 0 \tag{4.13}$$

For the steady state solution we obtain the following formula

$$z = z(t) = \frac{1}{\rho\alpha} \tag{4.14}$$

Substituting solution (4.14) into equations (4.9), (4.10) we obtain dynamics of the optimal process

$$\frac{\dot{y}}{y} = f - (1-\alpha)\rho \tag{4.15}$$

$$\frac{\dot{\psi}}{\psi} = (1-\alpha)\rho - f \tag{4.16}$$

Let us formulate properties of solution.

Assuming that function $f = f(t)$ is a nondecreasing one $f'(t) \geq 0$ with positive value of the difference $(f - (1-\alpha)\rho) > 0$ and introducing notations

$$Q(t) = \int_{t_0}^{t} (f(\tau) - (1-\alpha)\rho)d\tau > (f(t_0) - (1-\alpha)\rho)(t - t_0) \tag{4.17}$$

we get the optimal model with the exponentially growing production y

$$y = y(t) = y_0 e^{Q(t)}, \quad y(t_0) = y_0 \tag{4.18}$$

and the exponentially decreasing price ψ

$$\psi = \psi(t) = \psi_0 e^{-Q(t)}, \quad \psi(t_0) = \psi_0 \tag{4.19}$$

The production rate $\dot{y}/y$ is determined by difference $(f-(1-\alpha)\rho)$ (4.15) and is growing with the growth of function $f = f(t)$.

Substituting (4.14) into the optimal control (4.7) we obtain relations between the optimal investment r and the optimal production y

$$r = \frac{(1-\alpha)\rho}{g} y \tag{4.20}$$

Equation (4.20) means that the optimal R&D investment r increases proportionally to the growth of the optimal production y with coefficient $((1-\alpha\rho)/g)$.

For R&D intensity r/y we have the following formula

$$\frac{r}{y} = \frac{(1-\alpha)\rho}{g} = \frac{(1-\alpha)\rho}{(p-q)} \tag{4.21}$$

which describes the dependence of the optimal R&D intensity on the substitution parameter α, the subjective discount rate ρ and the discounted marginal productivity of technology $(-g)$. When the cost p for sustaining the accumulated R&D investment T is high, then the research intensity r/y is low. Vice versa, increase of the rate of return to R&D q leads to the growth of the research intensity r/y. Assuming that the positive function $g = g(t)$ is nonincreasing over time t, $g'(t) \leq 0$ we get the growth property of the research intensity r/y. Let us note that the similar properties of R&D intensity were obtained for the empirical data of Japanese manufacturing industry (see [Watanabe, 1997, 1998]).

The scaled R&D intensity $(r/y)g$ is a constant value and depends only on parameters α, ρ which are universal in the model

$$\frac{r}{y} g = (1-\alpha)\rho \tag{4.22}$$

According to formula (4.20) the optimal R&D investment $r(t)$ has more complicated dependence than R&D intensity. Besides the dependence on parameters α, ρ it directly depends on parameter g and indirectly - on parameter f via the optimal production y. Let us remind that parameters $f = f(t)$ and $g = g(t)$ are determined by the specific economical conditions of the given country.

Taking into account relation (4.20) for optimal R&D investment r with growing coefficient $1/g$ and solution (4.18) for optimal growth of production y we can derive the growth process for technology

$$T = T_0 + (1-\alpha)\rho y_0 \int_{t_0}^{t} \frac{e^{Q(\tau)}}{g(\tau)} d\tau, \quad T(t_0) = T_0 \tag{4.23}$$

For technology intensity $P = T/y$ one can derive the following differential equation

$$\dot{P} = \frac{(\dot{T}y - \dot{y}T)}{y^2} = -\frac{\dot{y}}{y}P + \frac{r}{y} = \\ -(f(t) - (1-\alpha)\rho)P + \frac{(1-\alpha)\rho}{g(t)} \tag{4.24}$$

and its solution according to the Cauchy formula

$$P(t) = P_0 e^{-Q(t)} + (1-\alpha)\rho \int_{t_0}^{t} \frac{e^{(-Q(t)+Q(\tau))}}{g(\tau)} d\tau, \quad P(t_0) = P_0 \tag{4.25}$$

Technology intensity P has the zero velocity on the curve

$$P^0(t) = \frac{(1-\alpha)\rho}{g(t)(f(t)-(1-\alpha)\rho)} \tag{4.26}$$

If the initial position (t_0, P_0) is situated below the curve P^0 (4.26), $P_0 < P^0(t_0)$, then technology intensity $P(t)$ is growing. If the initial position (t_0, P_0) is situated beyond the curve P^0 (4.26), $P_0 > P^0(t_0)$, then technology intensity $P(t)$ is declining over time t.

Finally we consider the value function $(t, y) \to \varphi(t, y)$ which assigns the optimal result φ of the utility function (4.3) along the optimal process (y^0, r^0) to an initial position (t, y). The value function φ is the solution of the Hamilton-Jacobi equation for the reduced control problem

$$\begin{aligned} &\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial y} f y + e^{-\rho t} \ln y + \\ &\max_r \{-\frac{\partial \varphi}{\partial y} g r + e^{-\rho t} A \ln r\} = 0 \end{aligned} \tag{4.27}$$

Let us find the value function φ in the class of the following structure

$$\varphi(t, y) = e^{-\rho t}(\mu(y) + \nu(t)) \tag{4.28}$$

Substituting the optimal control r^0 (4.7) into the Hamilton-Jacobi equation (4.27) and considering price ψ as the the impulse variable

$$\psi = \frac{\partial \mu}{\partial y} \tag{4.29}$$

we derive from (4.27) the Hamilton-Jacobi equation for components $\mu(y)$, $\nu(t)$

$$\begin{aligned} &-\rho(\mu + \nu) + \dot{\nu} + \frac{\partial \mu}{\partial y} f y + \ln y \\ &-A + A(\ln A - \ln g - \ln \frac{\partial \mu}{\partial y}) = 0 \end{aligned} \tag{4.30}$$

Using an indeterminate coefficient in the expression for function μ

$$\mu(y) = B \ln y \tag{4.31}$$

we obtain after substituting (4.31) into the Hamilton-Jacobi equation (4.30) the explicit expression for parameter B

$$B = \frac{(A+1)}{\rho} = \frac{1}{\rho\alpha} \tag{4.32}$$

and the linear differential equation for function ν

$$\dot{\nu}(t) = \rho\nu(t) + h(t), \quad h(t) = A \ln g - Bf - A(\ln(1-\alpha) - \ln \rho - 1) \tag{4.33}$$

The general solution of equation (4.33) has the following form

$$\nu(t) = Ce^{\rho t} + F(t), \quad F(t) = \int_0^t e^{-\rho(s-t)} h(s) ds \tag{4.34}$$

The transversality condition for component ν

$$\lim_{t\to\infty} e^{-\rho t}\nu(t) = 0 \tag{4.35}$$

provides the explicit expression for parameter C

$$C = -\int_0^{+\infty} e^{-\rho s} h(s) ds \tag{4.36}$$

Finally we obtain the following explicit expressions for functions μ and ν

$$\mu(y) = \frac{1}{\rho\alpha} \ln y \tag{4.37}$$

$$\nu(t) = -\int_t^{+\infty} e^{-\rho(s-t)} h(s) ds \tag{4.38}$$

In particular, if h is a constant, then ν is also a constant determined by the formula

$$\nu = -\frac{h}{\rho} \tag{4.39}$$

According to the explicit expressions for the value function φ (4.28), (4.37), (4.38) we can conclude that in the considered model the optimal result has the decomposition property. The first term μ depends only on the discount parameter ρ, the elasticity of substitution α and in the logarithmic way (not very intensively) on the initial production y and does not depend on the specific characters – functions f and g, of the dynamical system (4.1). On the contrary the second term ν is determined mainly by dynamics (4.1) aggregated in function h (4.33) and does not depend on the initial production y.

Part II. Equilibrium Solution of Optimal Growth

5 Existence of Equilibrium and Optimal Solution

We begin with finding the first integral for the optimal dynamics (3.7)-(3.10).

Proposition 5.1 *The optimal dynamics (3.7)-(3.10) has the first integral (3.21).*

Proof. Differentiating the cost function $z = \psi_1 y + \psi_2 T$ along system (3.7)-(3.10) we obtain differential equation

$$\dot{z} = \rho z - (a_1 + a_2 + 1) \tag{5.1}$$

Its general solution is given by formula

$$z(t) = Ce^{\rho t} + \frac{a_1 + a_2 + 1}{\rho} \tag{5.2}$$

The unique solution of the type (5.2) which meets the transversality condition (3.11) is the constant function $z = (a_1 + a_2 + 1)/\rho$ when the exponential part is canceled $C = 0$. $\square$

Introducing new variables (3.22) we obtain the following statement.

Proposition 5.2 *Change of variables (3.22) leads to the system with the separable structure*

$$\begin{aligned}
\dot{x}_1 &= f_1 x_1 + f_2 x_1^{(1-\gamma)} - \frac{a_2(x_1+g)x_1}{(gx_2 - x_1x_4)} \\
\dot{x}_2 &= \rho x_2 + \gamma f_2 x_2 \frac{1}{x_1^{\gamma}} - 1 - \frac{a_2 g x_2}{(gx_2 - x_1x_4)} \\
\dot{x}_3 &= -\frac{a_2 x_1 x_3}{(gx_2 - x_1x_4)} \\
\dot{x}_4 &= \rho x_4 - \gamma f_2 x_2 \frac{1}{x_1^{\gamma}} - a_1 + \frac{a_2 x_1 x_4}{(gx_2 - x_1x_4)}
\end{aligned} \tag{5.3}$$

Proof. Let us derive for example the first equation in the system (5.3). We have

$$\dot{x}_1 = \frac{\dot{y}T - y\dot{T}}{T^2} = \left(f_1 y + f_2 T^{\gamma} y^{(1-\gamma)} - \frac{g a_2}{(g\psi_1 - \psi 2)}\right)\frac{1}{T} -$$
$$\frac{y}{T}\frac{1}{T}\frac{a_2}{(g\psi_1 - \psi 2)} = f_1 x_1 + f_2 x_1^{(1-\gamma)} - \frac{a_2(x_1+g)x_1}{(gx_2 - x_1x_4)}$$

Other equations in system (5.3) are obtained analogously. □

Taking into account the first integral (3.21) we reduce system with four variables (5.3) to the three dimensional system with the block structure

$$\begin{aligned}
\dot{x}_1 &= f_1 x_1 + f_2 x_1^{(1-\gamma)} - \frac{a_2(x_1+g)x_1}{((x_1+g)x_2 - p^0 x_1)} = F_1(x_1, x_2, x_3) \\
\dot{x}_2 &= \rho x_2 + \gamma f_2 x_2 \frac{1}{x_1^{\gamma}} - 1 - \frac{a_2 g x_2}{((x_1+g)x_2 - p^0 x_1)} = F_2(x_1, x_2, x_3) \\
\dot{x}_3 &= -\frac{a_2 x_1 x_3}{((x_1+g)x_2 - p^0 x_1)} = F_3(x_1, x_2, x_3)
\end{aligned} \tag{5.4}$$

In our further analysis in this section we assume that functions f_1, f_2, g have constant values and the following inequalities hold

$$0 \leq \gamma \leq 1 \tag{5.5}$$

$$f_1 - \rho = \nu > 0 \tag{5.6}$$

Condition (5.5) indicates the moderate influence of growth of the technology stock T on the production rate $\dot{y}$. Condition (5.6) means that the main rate f_1 of production growth is strictly greater than the discount rate ρ. Only under these conditions we may find further stationary points of system (5.4) and indicate the corresponding growth properties in the original dynamics (3.1).

One can prove that system (5.4) and, hence, system (5.3) has stationary points $x^0 = (x_1^0, x_2^0, x_3^0, x_4^0)$

$$\begin{aligned}
f_1 x_1^0 + f_2 (x_1^0)^{(1-\gamma)} - \frac{a_2(x_1^0+g)x_1^0}{(gx_2^0 - x_1^0 x_4^0)} &= 0 \\
\rho x_2^0 + \gamma f_2 x_2^0 \frac{1}{(x_1^0)^{\gamma}} - 1 - \frac{a_2 g x_2^0}{(gx_2^0 - x_1^0 x_4^0)} &= 0 \\
-\frac{a_2 x_1^0 x_3^0}{(gx_2^0 - x_1^0 x_4^0)} &= 0 \\
\rho x_4^0 - \gamma f_2 x_2^0 \frac{1}{(x_1^0)^{\gamma}} - a_1 + \frac{a_2 x_1^0 x_4^0}{(gx_2^0 - x_1^0 x_4^0)} &= 0
\end{aligned} \tag{5.7}$$

More precisely the following statement is valid.

Proposition 5.3 *Assume that the growth conditions (5.5), (5.6) hold. Then systems (5.3), (5.4) have stationary points x^0 with the following properties*

$$0 < r_1 < x_1^0 \leq z_1 \tag{5.8}$$

$$0 \leq r_2 < x_2^0 \leq z_2 \tag{5.9}$$

$$(x_1^0 + g)x_2^0 - p^0 x_1^0 > 0 \tag{5.10}$$

$$x_3^0 = 0 \tag{5.11}$$

$$0 \leq x_4^0 < p^0, \quad x_2^0 + x_4^0 = p^0 \tag{5.12}$$

Here parameters r_1, z_1 are unique positive solutions of the following equations

$$\frac{g}{(r_1 + g)} = \frac{(\rho r_1^\gamma + \gamma f_2)}{(f_1 r_1^\gamma + f_2)} \tag{5.13}$$

$$\frac{p^0 g}{(z_1 + g)} = \frac{a_2 z_1^\gamma}{(f_1 z_1^\gamma + f_2)} \tag{5.14}$$

respectively.

Parameters r_2 and z_2 are defined by relations

$$r_2 = p^0 \min\{1 - \gamma,\ 1 - \frac{a_1}{(f_1 p^0 + 1)}\} \tag{5.15}$$

$$z_2 = p^0 \tag{5.16}$$

If the growth rate γ and the corresponding transition coefficient f_2 are sufficiently small

$$f_2 \gamma^2 \leq \frac{a_2}{p^0} \min\{1, \frac{g(a_1 + 1)}{a_2}\} \tag{5.17}$$

then point x^0 is unique.

Proof. Let us consider the system of nonlinear algebraic equations for stationary points of system (5.4)

$$\begin{aligned}
f_1 x_1 + f_2 x_1^{(1-\gamma)} - \frac{a_2 (x_1 + g) x_1}{((x_1 + g)x_2 - p^0 x_1)} &= 0 \\
\rho x_2 + \gamma f_2 x_2 \frac{1}{x_1^\gamma} - 1 - \frac{a_2 g x_2}{((x_1 + g)x_2 - p^0 x_1)} &= 0 \\
-\frac{a_2 x_1 x_3}{((x_1 + g)x_2 - p^0 x_1)} &= 0
\end{aligned} \tag{5.18}$$

Resolving the first equation with respect to variable x_2 we obtain the formula for the monotonically growing hyperbola

$$x_2 = \frac{p^0 x_1}{(x_1 + g)} + \frac{a_2 x_1^\gamma}{(f_1 x_1^\gamma + f_2)} \tag{5.19}$$

since its derivative is strictly positive

$$x_2' = \frac{p^0 g}{(x_1 + g)^2} + \frac{\gamma a_2 f_2}{x_1^{(1-\gamma)}} (f_1 x_1^\gamma + f_2)^2 > 0 \tag{5.20}$$

Hyperbola (5.19) has the horizontal asymptote

$$x_2 = p^0 + \frac{a_2}{f_1} \tag{5.21}$$

Expressing the ratio $1/((x_1+g)x_2 - p^0 x_1)$ from the first equation, substituting it to the second equation and resolving the obtained relation with respect to variable x_2 we obtain the formula for the hyperbola

$$x_2 = \frac{(x_1+g)x_1^\gamma}{((\rho x_1^\gamma + \gamma f_2)(x_1+g) - (f_1 x_1^\gamma + f_2)g)} \tag{5.22}$$

On the interval

$$x_1 > r_1, \quad (\rho r_1^\gamma + \gamma f_2)(r_1+g) - (f_1 r_1^\gamma + f_2)g = 0 \tag{5.23}$$

hyperbola (5.22) is strictly positive, has the vertical asymptote

$$x_1 = r_1 \tag{5.24}$$

and the horizontal asymptote

$$x_2 = \frac{1}{\rho} \tag{5.25}$$

It is clear that hyperbolas (5.21), (5.25) have points of intersection (x_1^0, x_2^0) on the interval (5.23). Really, hyperbola (5.19) grows to infinity when $x_1 \downarrow r_1$ while hyperbola (5.22) is finite at r_1, then hyperbola (5.19) tends to the upper asymptote (5.21) and hyperbola (5.22) tends to the lower asymptote (5.25) when $x_1 \to +\infty$. Obviously the second coordinate of a stationary point satisfies inequalities

$$0 < x_2^0 < p^0 + \frac{a_2}{f_1} \tag{5.26}$$

Let us prove that indeed relation (5.9) takes place. Expressing the ratio $g/(x_1+g)$ from the first hyperbola (5.19) and substituting it to the second one (5.22) we obtain the following relation

$$x_2 = \frac{p^0}{(\rho^* p^0 - a_2 - f^*(p^0 - x_2))}, \quad \rho^* = \rho + \gamma f_2 x_1^{-\gamma}, \quad f^* = f_1 + f_2 x_1^{-\gamma} \tag{5.27}$$

which is equivalent to the quadratic equation

$$f^* x_2^2 - (f^* p^0 - \rho^* p^0 - a_2)x_2 - p^0 = (x_2 - p^0)(f^* x_2 + 1) + (\rho^* p^0 - (a_2+1))x_2 = 0 \tag{5.28}$$

Since $\rho^* p^0 - (a_2+1) \geq 0$ and $x_2 > 0$ then relation (5.28) evidently implies the second part of inequalities (5.9)

$$x_2^0 \leq p^0$$

Let us derive the first part of inequalities (5.9). To this end we rewrite relation (5.28) as follows

$$x_2 = p^0 - \frac{(\rho^* p^0 - (a_2+1))x_2}{(f^* x_2 + 1)} > \tag{5.29}$$

$$p^0 - \frac{(\rho^* p^0 - (a_2+1))p^0}{(f^* p^0 + 1)} = p^0 \left(1 - \frac{((\rho^* - \rho)p^0 + a_1)}{(f^* p^0 + 1)}\right) =$$

$$p^0 \left(1 - \frac{(\gamma f_2 p^0 + a_1 x_1^\gamma)}{((f_1 x_1^\gamma + f_2)p^0 + x_1^\gamma)}\right) \geq p^0 \min\{1-\gamma,\ 1 - \frac{a_1}{(f_1 p^0 + 1)}\} = r_2$$

Inequalities (5.9) are thus proved. In order to obtain relations (5.8) it is necessary to mention only that hyperbola (5.19) is a monotonically increasing function and therefore

$$x_2^0 \leq p^0 \quad \Longrightarrow \quad x_1^0 \leq z_1$$

Combining all inequalities we obtain that coordinates of points x^0 satisfy the necessary relations (5.8), (5.9).

Inequality (5.10) follows from the fact that all points of the first hyperbola (5.19) including point x^0 satisfy relations

$$x_2 = \frac{p^0 x_1}{(x_1+g)} + \frac{a_2 x_1^\gamma}{(f_1 x_1^\gamma + f_2)} > \frac{p^0 x_1}{(x_1+g)}, \quad x_1 > 0 \tag{5.30}$$

To complete verification of inequalities (5.8) - (5.12) let us note that conditions (5.7), (5.8), (5.10) imply relation (5.11) and conditions (3.21), (5.9) imply relation (5.12).

Let us pass now to the question of uniqueness of solution x^0. In this connection we examine the first derivative of hyperbola (5.22)

$$x_2' = \frac{(f_2(x_1+g)^2\gamma^2 - f_2 g(x_1+g)\gamma - g(f_1 x_1^\gamma + f_2)x_1)}{x_1^{(1-\gamma)}((\rho x_1^\gamma + \gamma f_2)(x_1+g) - (f_1 x_1^\gamma + f_2)g)} \tag{5.31}$$

For $\gamma = 0$ this derivative is strictly negative $x_2' < 0$. When $0 < \gamma < 1$ derivative x_2' (5.31) changes sign from $-$ to $+$ while x_1 grows from r_1 to $+\infty$. Hence hyperbola (5.22) does not have monotone properties and in the general case several points of intersection of hyperbola (5.19) and hyperbola (5.22) may exist. If derivative x_2' (5.31) is nonpositive at point z_1 (5.14) then hyperbola (5.22) is a monotonically decreasing function on the interval $(r_1, z_1]$ and stationary solution x^0 is unique. Let us estimate the numerator of derivative x_2' (5.31) at point z_1 (5.14)

$$\begin{aligned}&\gamma^2 f_2(z_1+g)^2 - \gamma f_2 g(z_1+g) - (f_1 z_1^\gamma + f_2)g z_1 = \\ &(z_1+g)(f_2 g\gamma(\gamma-1) + z_1(f_2\gamma^2 - \frac{a_2 z_1^\gamma}{p^0}))\end{aligned} \tag{5.32}$$

It is clear that inequality

$$f_2\gamma^2 \leq \frac{a_2 z_1^\gamma}{p^0} \tag{5.33}$$

together with relations (5.32) imply the desired condition

$$x_2'(z_1) \leq 0 \tag{5.34}$$

Let us estimate coordinate z_1 from below

$$a_2(z_1+g) = p^0 g(f_1 + f_2 z_1^{-\gamma}) > p^0 g f_1 \tag{5.35}$$

Inequality (5.35) implies the estimate

$$z_1 > \frac{g(f_1 p^0 - a_2)}{a_2} > \frac{g(a_1+1)}{a_2} \tag{5.36}$$

Combining inequalities (5.33), (5.36) we obtain that condition (5.17) implies the desired relation (5.34) and the uniqueness result consequently. □

Remark 5.1 *For parameter r_1 (5.13) the following estimate takes place*

$$r_1 \geq g \min\{\frac{(f_1-\rho)}{\rho},\ \frac{(1-\gamma)}{\gamma}\} \tag{5.37}$$

Proof. Rewriting relation (5.13) as a ratio

$$r_1 + g = \frac{g(f_1 r_1^{\gamma} + f_2)}{(\rho r_1^{\gamma} + \gamma f_2)} \tag{5.38}$$

and taking into account that hyperbolic function in the right hand side of equation (5.38) is a monotone one we obtain the following estimate

$$r_1 + g \geq g \min\{\frac{f_1}{\rho},\ \frac{1}{\gamma}\} \tag{5.39}$$

The last inequality implies the necessary estimate (5.37). □

6 Qualitative Properties of Optimal Investment

In order to describe properties of the optimal control r^0 (3.4) we analyze stability of the stationary point x^0 (5.7). More precisely, we indicate the saddle character of this equilibrium and show the existence of optimal trajectories which converge to it. To this end we calculate the Jacobi matrix of the right hand side of system (5.4)

$$DF = \begin{pmatrix} \partial F_1/\partial x_1 & \partial F_1/\partial x_2 & \partial F_1/\partial x_3 \\ \partial F_2/\partial x_1 & \partial F_2/\partial x_2 & \partial F_2/\partial x_3 \\ \partial F_3/\partial x_1 & \partial F_3/\partial x_2 & \partial F_3/\partial x_3 \end{pmatrix} \tag{6.1}$$

For partial derivatives $\partial F_i/\partial x_j$, $i, j = 1, 2, 3$ we have the following relations

$$\begin{aligned}
\partial F_1/\partial x_1 &= f_1 + (1-\gamma) f_2 x_1^{-\gamma} - \frac{a_2(x_1((x_1+g) - p^0 x_1) + g x_2 (x_1+g))}{((x_1+g)x_2 - p^0 x_1)^2} \\
\partial F_1/\partial x_2 &= \frac{a_2 (x_1+g)^2 x_1}{((x_1+g)x_2 - p^0 x_1)^2} \\
\partial F_1/\partial x_3 &= 0 \\
\partial F_2/\partial x_1 &= -\gamma^2 f_2 x_2 x_1^{-(1+\gamma)} - \frac{a_2 g x_2 (p^0 - x_2)}{((x_1+g)x_2 - p^0 x_1)^2} \\
\partial F_2/\partial x_2 &= \rho + \gamma f_2 x_1^{-\gamma} + \frac{a_2 g p^0 x_1}{((x_1+g)x_2 - p^0 x_1)^2} \\
\partial F_2/\partial x_3 &= 0 \\
\partial F_3/\partial x_1 &= -\frac{a_2 g x_2 x_3}{((x_1+g)x_2 - p^0 x_1)^2} \\
\partial F_3/\partial x_2 &= \frac{a_2 (x_1+g) x_1 x_3}{((x_1+g)x_2 - p^0 x_1)^2} \\
\partial F_3/\partial x_3 &= -\frac{a_2 x_1}{((x_1+g)x_2 - p^0 x_1)}
\end{aligned} \tag{6.2}$$

We indicate signs of coefficients $\partial F_i/\partial x_j$ in the following statement.

Proposition 6.1 *Coefficients $\partial F_i/\partial x_j$ of the Jacobi matrix DF at the stationary point x^0 are determined as follows*

$$\begin{aligned}
\partial F_1(x^0)/\partial x_1 &= -\gamma f_2 x_1^{-\gamma} - \frac{g p^0 x_1 (f_1 x_1^{\gamma} + f_2)^2}{a_2 x_1^{2\gamma}(x_1+g)^2} < 0 \\
\partial F_1(x^0)/\partial x_2 &= \frac{x_1(x_1+g)^2(f_1 x_1^{\gamma} + f_2)^2}{a_2 x_1^{2\gamma}(x_1+g)^2} > 0 \\
\partial F_1(x^0)/\partial x_3 &= 0 \\
\partial F_2(x^0)/\partial x_1 &= -\gamma^2 f_2 x_2 x_1^{-(1+\gamma)} - \frac{g x_2 (p^0 - x_2)(f_1 x_1^{\gamma} + f_2)^2}{a_2 x_1^{2\gamma}(x_1+g)^2} < 0 \\
\partial F_2(x^0)/\partial x_2 &= \rho + \gamma f_2 x_1^{-\gamma} + \frac{g p^0 x_1 (f_1 x_1^{\gamma} + f_2)^2}{a_2 x_1^{2\gamma}(x_1+g)^2} > 0 \\
\partial F_2(x^0)/\partial x_3 &= 0 \\
\partial F_3(x^0)/\partial x_1 &= 0 \\
\partial F_3(x^0)/\partial x_2 &= 0 \\
\partial F_3(x^0)/\partial x_3 &= -\frac{x_1^{(1-\gamma)}(f_1 x_1^{\gamma} + f_2)}{(x_1+g)} < 0
\end{aligned} \tag{6.3}$$

Here $x_i = x_i^0$, $i = 1, 2, 3$.

Proof. Expressing the ratio $1/((x_1+g)x_2 - p^0 x_1)$ from hyperbolic equation (5.19), substituting it to formulas for partial derivatives (6.2) and taking into account Proposition (5.3) for the stationary point x^0 we obtain relations (6.3) with definite signs. □

Taking into account Proposition (6.1) one can formulate the following statements with respect to eigenvalues of the Jacobi matrix DF.

Proposition 6.2 *The Jacobi matrix DF has at least one eigenvalue with positive real part and hence the stationary point x^0 is unstable.*

Proof. Let us consider the matrix of the second order

$$D = \begin{pmatrix} \partial F_1(x^0)/\partial x_1 & \partial F_1(x^0)/\partial x_2 \\ \partial F_2(x^0)/\partial x_1 & \partial F_2(x^0)/\partial x_2 \end{pmatrix} \tag{6.4}$$

The block structure of matrix DF (6.3) at the stationary point x^0 imply that eigenvalues of matrix D are eigenvalues of matrix DF.

According to relations (6.3) the trace of matrix D (6.4) is positive

$$TR = \partial F_1(x^0)/\partial x_1 + \partial F_2(x^0)/\partial x_2 = \rho > 0 \tag{6.5}$$

The positiveness of the trace TR (6.5) means that at least one eigenvalue of matrix D and hence of matrix DF has positive real part. □

We will prove now that for the small enough rates of growth γ the Jacobi matrix DF has real eigenvalues one of which is positive and two others are negative. We introduce the following assumption for parameter a_1

$$a_1 \leq 1 \tag{6.6}$$

and restrictions on growth rates γ

$$\gamma \le \gamma^0, \quad \gamma^0 = \min\{\frac{1}{2}, \frac{\rho}{f_1}, \min_k\{e(p_k)\}\} \tag{6.7}$$
$$e = e(p) = 2p(1-p)$$
$$p_1 = \frac{\rho a_2}{(a_1+a_2+1)(f_1+f_2)}$$
$$p_2 = \frac{\rho a_2 g(a_1+1)}{(a_1+a_2+1)(f_1 g(a_1+1)+f_2 a_2)}$$
$$p_3 = \frac{\rho}{f_1}$$

Proposition 6.3 *Assume that conditions (5.5), (5.6), (5.17), (6.6), (6.7) hold. Then the Jacobi matrix DF has real eigenvalues: one - positive, and two - negative. Hence the stationary point x^0 is a saddle point.*

Proof. The block structure of the Jacobi matrix DF and negative sign of the diagonal element $\partial F_3(x^0)/\partial x_3$ imply that at least one eigenvalue μ_3 is real and negative

$$\mu_3 = -\frac{x_1^{(1-\gamma)}(f_1 x_1^\gamma + f_2)}{(x_1+g)} \tag{6.8}$$

and the corresponding eigenvector h_3 is the unit vector

$$h_3 = (0,0,1) \tag{6.9}$$

Let us turn our attention to matrix D of the second order. It has real eigenvalues: one - positive, and one - negative, if and only if its discriminant DI

$$DI = \frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial x_2} - \frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial x_1} \tag{6.10}$$

is negative.

Let us show that under conditions (5.5), (5.6), (5.17), (6.6), (6.7) discriminant DI is negative and this completes the proof.

In the expression for discriminant DI we combine terms with common denominators

$$\begin{aligned}
DI = &-\left(\gamma f_2 x_1^\gamma + \frac{gp^0 x_1(f_1 x_1^\gamma + f_2)^2}{a_2 x_1^{2\gamma}(x_1+g)^2}\right)\left(\rho + \gamma f_2 x_1^\gamma + \frac{gp^0 x_1(f_1 x_1^\gamma + f_2)^2}{a_2 x_1^{2\gamma}(x_1+g)^2}\right) + \\
&\frac{x_1(x_1+g)^2(f_1 x_1^\gamma + f_2)^2}{a_2 x_1^{2\gamma}(x_1+g)^2})\left(\gamma^2 f_2 x_2 x_1^{-(1+\gamma)} + \frac{g x_2(p^0 - x_2)(f_1 x_1^\gamma + f_2)^2}{a_2 x_1^{2\gamma}(x_1+g)^2}\right) = \\
&-\left(x_2 - \frac{p^0 x_1}{(x_1+g)}\right)\left(x_2 - \frac{p^0 g}{(x_1+g)}\right)\frac{gp^0(f_1 x_1^\gamma + f_2)^4}{a_2^2 x_1^{4\gamma}(x_1+g)^2} - \\
&-\frac{\rho g p^0 x_1(f_1 x_1^\gamma + f_2)^2}{a_2 x_1^{2\gamma}(x_1+g)^2} - \gamma\rho f_2 x_1^{-\gamma} - \gamma^2 f_2^2 x_1^{-2\gamma} - \\
&-2\gamma\frac{gp^0 f_2 x_1(f_1 x_1^\gamma + f_2)^2}{a_2 x_1^{3\gamma}(x_1+g)^2} + \gamma^2\frac{f_2 x_2(f_1 x_1^\gamma + f_2)^2}{a_2 x_1^{3\gamma}}
\end{aligned} \tag{6.11}$$

Let us consider the first term in relation (6.11). According to property (5.10) the first multiplier is positive

$$x_2 - \frac{p^0 x_1}{(x_1+g)} > 0 \tag{6.12}$$

Let us prove that under conditions (6.6), (6.7) the second multiplier is also positive. Taking into account equation (5.19) and estimate (5.29) we obtain the following chain of inequalities

$$
\begin{aligned}
& x_2 - \frac{p^0 g}{(x_1+g)} = 2x_2 - p^0 - \frac{a_2 x_1^\gamma}{(f_1 x_1^\gamma + f_2)} \geq \\
& p^0 \left(1 - \frac{2(\gamma f_2 p^0 + a_1 x_1^\gamma)}{((f_1 x_1^\gamma + f_2) + x_1^\gamma)} - \frac{a_2 x_1^\gamma}{(f_1 x_1^\gamma + f_2) p^0}\right) > \\
& p^0 \left(1 - \frac{2\gamma f_2 p^0 + (2a_1 + a_2) x_1^\gamma}{(f_1 x_1^\gamma + f_2) p^0}\right) \geq \\
& p^0 \min\{1 - 2\gamma,\ 1 - \frac{(2a_1 + a_2)}{f_1 p^0}\}
\end{aligned}
\tag{6.13}
$$

It is clear that under conditions (6.6), (6.7), namely,

$$
\gamma \leq \frac{1}{2}, \quad a_1 \leq 1
$$

expression (6.13) is positive

$$
x_2 - \frac{p^0 g}{(x_1+g)} > 0 \tag{6.14}
$$

Combining inequalities (6.12), (6.14) we obtain that the first term in relation for discriminant (6.11) is negative.

The next three terms in relation (6.11) are also negative.

Finally let us prove that under condition (6.7) the sum of the last two terms in relation (6.11) is negative. We have the following inequality

$$
\begin{aligned}
& -2\gamma \frac{g p^0 f_2 x_1 (f_1 x_1^\gamma + f_2)^2}{a_2 x_1^{3\gamma} (x_1+g)^2} + \gamma^2 \frac{f_2 x_2 (f_1 x_1^\gamma + f_2)^2}{a_2 x_1^{3\gamma}} \leq \\
& \gamma \frac{f_2 p^0 (f_1 x_1^\gamma + f_2)^2}{a_2 x_1^{3\gamma}} \left(\gamma - \frac{2 g x_1}{(x_1+g)^2}\right)
\end{aligned}
\tag{6.15}
$$

Introducing notations

$$
p = p(x_1) = \frac{g}{(x_1+g)}, \quad e(p) = 2p(1-p) \tag{6.16}
$$

we estimate the second multiplier in relation (6.15)

$$
\begin{aligned}
& \gamma - \frac{2 g x_1}{(x_1+g)^2} = \gamma - 2\frac{g}{(x_1+g)}\left(1 - \frac{g}{(x_1+g)}\right) = \\
& \gamma - 2p(x_1)(1 - p(x_1)) = \gamma - e(p(x_1)) \leq \\
& \gamma - \min\{e(p(r_1)),\ e(p(z_1))\}
\end{aligned}
\tag{6.17}
$$

Here the following relations take place

$$
0 < r_1 < x_1 \leq z_1 < +\infty, \quad 0 < p(z_1) \leq p(x_1) < p(r_1) < 1 \tag{6.18}
$$

Using definition of the lower bound r_1 (5.13) we estimate value $p(r_1)$ from above

$$
p(r_1) = \frac{(\rho r_1^\gamma + \gamma f_2)}{(f_1 r_1^\gamma + f_2)} \leq \frac{\rho}{f_1}, \quad \gamma \leq \frac{\rho}{f_1} \tag{6.19}
$$

Taking into account definition of the upper bound z_1 (5.14) and its estimate (5.36) we evaluate quantity $p(z_1)$ from below

$$p(z_1) = \frac{a_2 z_1^{\gamma}}{p^0(f_1 z_1^{\gamma} + f_2)} \geq \min\{\frac{a_2}{p^0(f_1+f_2)}, \frac{a_2 g(a_1+1)}{p^0(f_1 g(a_1+1) + f_2 a_2)}\} \tag{6.20}$$

Combining condition (6.7) and estimates (6.15) - (6.20) we derive that the sum of the last two terms in relation (6.11) is negative. Taking into account negative signs of all previous terms in relation for discriminant DI (6.11) we conclude that discriminant DI is negative. The last property implies the existence of one positive and one negative eigenvalues of the Jacobi matrix D. □

Let us indicate properties of eigenvalues and eigenvectors of the Jacobi matrix D.

Remark 6.1 *If discriminant DI (6.11) of the Jacobi matrix is negative then the positive eigenvalue μ_1 provides the greater growth rate for trajectories of system (5.4) than the growth rate ρ*

$$\mu_1 > \rho > 0 \tag{6.21}$$

and the negative eigenvalue μ_2 can be presented through the positive one

$$\mu_2 = -(\mu_1 - \rho) < 0 \tag{6.22}$$

Proof. Since trace TR (6.5) is equal to the growth rate ρ and discriminant DI is negative then the positive eigenvalue is determined by formula

$$\mu_1 = \frac{\rho}{2} + \left(\frac{\rho^2}{4} + |DI|\right)^{1/2} > \rho$$

and the negative eigenvalue satisfies equality

$$\mu_1 + \mu_2 = \rho$$

which imply conditions (6.21), (6.22). □

Remark 6.2 *Eigenvectors h_1, h_2 corresponding to eigenvalues μ_1, μ_2 have positive coordinates*

$$h_1 = \frac{1}{n_1}(b, a+\mu_1, 0), \quad n_1 = (b^2 + (a+\mu_1)^2)^{1/2} \tag{6.23}$$

$$h_2 = \frac{1}{n_2}(a+\mu_1, c, 0), \quad n_2 = (c^2 + (a+\mu_1)^2)^{1/2} \tag{6.24}$$

Here

$$a = |\partial F_1/\partial x_1|, \quad b = \partial F_1/\partial x_2, \quad c = |\partial F_2/\partial x_1| \tag{6.25}$$

If discriminant DI is negative then arguments

$$\varphi_i = \arctan\frac{h_i^2}{h_i^1}, \quad i = 1, 2 \tag{6.26}$$

of eigenvectors h_i, $i = 1, 2$ are connected by inequality

$$0 \leq \varphi_2 < \varphi_1 < \frac{\pi}{2} \tag{6.27}$$

Proof. Substituting the positive eigenvalue μ_1 (6.21) into equation for eigenvectors

$$\begin{pmatrix} \mu + a & -b \\ c & \mu - (\rho + a) \end{pmatrix} \begin{pmatrix} h^1 \\ h^2 \end{pmatrix} = 0 \tag{6.28}$$

and considering its first line with components of eigenvector h_1

$$(\mu_1 + a)h_1^1 - bh_1^2 = 0 \tag{6.29}$$

we obtain expression (6.23).

Analogously substituting the negative eigenvalue μ_2 (6.22) into equation for eigenvectors (6.28), and considering its second line with components of eigenvector h_2

$$ch_2^1 + (\mu_2 - (\rho + a))h_2^2 = ch_2^1 - (\mu_1 + a)h_2^2 = 0 \tag{6.30}$$

we derive relation (6.24).

Taking into account that discriminant DI is negative

$$DI = -a(\rho + a) + bc < 0$$

we obtain the following chain of inequalities

$$0 > -a(\rho + a) + bc > -(\rho + a)^2 + bc > -(\mu_1 + a)^2 + bc$$

The last inequality provides relation

$$\tan \varphi_1 = \frac{(\mu_1 + a)}{b} > \frac{c}{(\mu_1 + a)} = \tan \varphi_2 \tag{6.31}$$

which in turn implies the necessary condition for arguments φ_i, $i = 1, 2$ (6.27). □

Let us consider the linearization for nonlinear system (5.4) at stationary point $x^0 = (x_1^0, x_2^0, 0)$

$$\begin{aligned} \dot{x_1} &= \frac{\partial F_1(x^0)}{\partial x_1}(x_1 - x_1^0) + \frac{\partial F_1(x^0)}{\partial x_2}(x_2 - x_2^0) \\ \dot{x_2} &= \frac{\partial F_2(x^0)}{\partial x_1}(x_1 - x_1^0) + \frac{\partial F_2(x^0)}{\partial x_2}(x_2 - x_2^0) \\ \dot{x_3} &= \frac{\partial F_3(x^0)}{\partial x_3}x_3 \end{aligned} \tag{6.32}$$

Summarizing properties of the Jacobi matrix indicated in Propositions 6.1 - 6.3 and Remarks 6.1 - 6.2 one can formulate properties of linear system (6.32).

Proposition 6.4 *Under conditions (5.5), (5.6), (5.17), (6.6), (6.7) linear system (6.32) has the following properties.*

1. Equilibrium x^0 is the unique saddle point.

2. For any pair x_1^*, x_3^* there exists the unique component x_2^* such that initial position $x^* = (x_1^*, x_2^*, x_3^*)$ is situated on the plane generated by eigenvectors h_2, h_3 corresponding to negative eigenvalues μ_2, μ_3. Trajectory $x^*(\cdot)$ of linear system (6.32) starting at initial position x^* tends to equilibrium x^0.

3. If initial value x_2^* is a proper one

$$0 \leq x_2^* \leq p^0 \tag{6.33}$$

then trajectory $x^*(\cdot)$ meets the necessary condition

$$0 \leq x_2^*(t) \leq p^0, \quad \forall \quad t \tag{6.34}$$

4. The second component $x_2(\cdot)$ of other trajectories $x(\cdot)$ starting at points $x = (x_1^*, x_2, x_3^*)$, $x_2 \neq x_2^*$ tends to infinity with the exponential growth rate $\mu_1 > \rho$

$$x_2(t) \to \infty, \quad t \to \infty \tag{6.35}$$

Proof. Property 1 follows immediately from Propositions 5.3, 6.3.

Property 2 follows from the fact that the first coordinate h_2^1 of eigenvector h_2 and the third coordinate h_3^3 of eigenvector h_3 are strictly positive and hence the coordinate plain generated by the first and third components x_1, x_3 can be orthogonally projected on the plane generated by eigenvectors h_2, h_3.

Property 3 is deduced from the fact that eigenvalues μ_2, μ_3 are negative and initial position x^* is situated on the plane generated by eigenvectors h_2, h_3.

Property 4 is derived from the nonstable character of the saddle point x^0 and the property of the positive eigenvalue $\mu_1 > \rho$ (6.21). □

Let us indicate the sense of properties 1-4 of linear system (6.32) for optimality of trajectories of nonlinear system (5.4).

According to the Grobman-Hartman theorem (see [Hartman, 1964]) nonlinear system (5.4) admits a trajectory as well as linear system (6.32) which converges to equilibrium x^0.

Proposition 6.5 *Nonlinear system (5.4) inherits the convergence property of linear system (6.32):*
there exists a trajectory $x^0(\cdot)$ which leads nonlinear system (5.4) from initial position x^* to equilibrium x^0

$$\lim_{t \to \infty} x_i^0(t) = x_i^0, \quad x_i^0(t_0) = x_i^*, \quad i = 1, 2, 3 \tag{6.36}$$

Let us make more strong assumption about the uniqueness of a convergent trajectory.

Hypothesis 6.1 *Assume that nonlinear system (5.4) inherits the uniqueness properties 2,4 of linear system (6.32) for a convergent trajectory. It means that a trajectory $x^0(\cdot)$ of nonlinear system (5.4) which tends to equilibrium x^0 and hence satisfies the necessary conditions (3.7)-(3.10) of the Pontryagin's maximum principle as well as the transversality condition (3.11) is unique. Due to concavity of the integrand in the utility function (2.6) trajectory $x^0(\cdot)$ is optimal since the maximum principle is sufficient in this case.*

We indicate now the growth properties of the optimal trajectory $x^0(\cdot)$ which tends to equilibrium x^0 of nonlinear system (5.4).

Remark 6.3 *The third component $x_3^0(\cdot) = 1/T^0$ converges to zero $x_3^0 = 0$ (5.11) with negative velocity (5.4). It means that optimal technology stock $T^0 = T^0(t)$ monotonically grows to infinity.*

The first component $x_1^0(\cdot) = y^0/T^0$ converges to the positive equilibrium value $x_1^0 > 0$ (5.8). It shows that optimal production $y^0 = y^0(t)$ also grows to infinity with the same growth rate as technology T^0. In particular, this growth property of production y^0 means that its derivative in dynamics (3.7) is strictly positive $\dot{y}^0(t) > 0$, $t \geq t_0$.

If the initial ratio $x_1^* = y^0(t_0)/T^0(t_0)$ is greater than at equilibrium x_1^0, $x_1^0 \leq x_1^*$, then the optimal ratio $x_1^0(t) = y^0(t)/T^0(t)$ is decreasing from the initial state x_1^* to equilibrium x_1^0. It indicates that optimal technology stock T^0 is growing rapidly than production y^0.

7 Quasioptimal Feedback of R&D Investment

It should be noted that the problem of searching the optimal trajectory $x^0(\cdot)$ which leads system (5.4) to the saddle point x^0 is very complicated due to the unstable properties of this equilibrium. Let us consider several constructive procedures for finding a quasioptimal feedback which leads coordinates $x_1(\cdot)$, $x_3(\cdot)$ of the system to equilibrium x_1^0, x_3^0. To this end we consider the linear regime for the second coordinate $x_2(\cdot)$

$$x_2 = x_2^0 + \omega(x_1 - x_1^0), \quad \omega \geq 0 \tag{7.1}$$

$$\begin{aligned} \dot{x}_1 &= f_1 x_1 + f_2 x_1^{(1-\gamma)} - \frac{a_2(x_1+g)x_1}{(d + k(\omega)(x_1 - x_1^0) + \omega(x_1 - x_1^0)^2)}, \quad x_1(t_0) = x_1^* \\ \dot{x}_3 &= -\frac{a_2 x_1 x_3}{(d + k(\omega)(x_1 - x_1^0) + \omega(x_1 - x_1^0)^2)}, \quad x_3(t_0) = x_3^* \end{aligned} \tag{7.2}$$

Here parameters d, k are determined by relations

$$d = gx_2^0 - (p^0 - x_2^0)x_1^0, \quad k = k(\omega) = k_1\omega + k_2, \quad k_1 = x_1^0 + g, \quad k_2 = -(p^0 - x_2^0) \tag{7.3}$$

and initial conditions x_1^*, x_3^* should satisfy conditions

$$x_1^0 \leq x_1^* < x_1^0 + \overline{x}_1(\omega), \quad x_3^* > 0 \tag{7.4}$$

Here parameter $\overline{x}_1$ is defined by relations

$$\overline{x}_1(\omega) = \begin{cases} 2d/(|k(\omega)| + (k^2(\omega) - 4\omega d)^{1/2}), & k^2(\omega) - 4\omega d \geq 0 \\ +\infty, & \text{otherwise} \end{cases} \tag{7.5}$$

Bearing in mind formulas for new variables (3.22) we extract the expression for feedback $r = r(y, T)$ from system (7.2)

$$\begin{aligned} r^* = \dot{T} = -\frac{\dot{x}_3}{x_3^2} = \frac{a_2 x_1}{x_3(d + k(\omega)(x_1 - x_1^0) + \omega(x_1 - x_1^0)^2)} = \\ \frac{a_2 y}{(d + k(\omega)((y/T) - x_1^0) + \omega((y/T) - x_1^0)^2)} \end{aligned} \tag{7.6}$$

Let us formulate the convergence result for dynamics (7.2).

Proposition 7.1 *Assume that the slope coefficient ω of the second coordinate x_2 (7.1) satisfies conditions*

$$0 \leq \omega \leq \frac{gp^0}{(x_1^0 + g)^2} = \omega_1 \tag{7.7}$$

Then the quasioptimal rational feedback r^* (7.6) leads trajectories $x^*(\cdot)$ of system (7.2) from initial conditions x_1^*, x_3^* (7.4) to equilibrium x_1^0, x_3^0.

Proof. Since the first equation in system (7.2) is independent of the second one then convergence of component $x_1^*(\cdot)$ to the equilibrium value x_1^0 follows from the property of asymptotic stability - the corresponding partial derivative should have the negative sign

$$\frac{dF_1(x^0)}{dx_1} < 0 \tag{7.8}$$

Taking into account relations (6.3) for partial derivatives $\partial F_1/\partial x_i$ and linear dependence (7.1) of coordinates x_i, $i = 1, 2$ we obtain the following formula for the full derivative dF_1/dx_1

$$\frac{dF_1(x^0)}{dx_1} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial x_2}\omega = -\gamma f_2 x_1^{-\gamma} - \frac{x_1(f_1 x_1^{\gamma} + f_2)^2}{a_2 x_1^{2\gamma}(x_1 + g)^2}(gp^0 - (x_1 + g)^2\omega) \tag{7.9}$$

Definitely under conditions (7.7) the derivative dF_1/dx_1 is negative and this provides the necessary asymptotic stability.

Relations

$$\frac{\partial F_3(x^0)}{\partial x_1} = 0, \quad \frac{\partial F_3(x^0)}{\partial x_3} < 0 \tag{7.10}$$

imply properties of asymptotic stability for the third equation of system (7.2) and provide the convergence of the third component $x_3^*(\cdot)$ to zero. □

Finally let us indicate properties of quasioptimal trajectories $x^*(\cdot)$ which converge to equilibrium x^0.

Remark 7.1 *The third component $x_3^*(\cdot) = 1/T^*$ converges to zero in the quasioptimal regime (7.2) with the negative velocity (7.10). It means that technology stock T^* monotonically grows to infinity with the asymptotic growth rate $|\mu_3| > (f_1 - \rho) > 0$ (6.8).*

The first component $x_1^*(\cdot) = y^*/T^*$ converges to the positive equilibrium value $x_1^0 > 0$. It shows that quasioptimal production y^* also grows to infinity with the same growth rate as technology T^*.

If the initial ratio x_1^* of production y^* to technology T^* is greater than the corresponding value at equilibrium x_1^0

$$x_1^0 \leq x_1^* < \frac{gx_2^0}{(p^0 - x_2^0)} \tag{7.11}$$

then the quasioptimal ratio $x_1^*(\cdot) = y^*/T^*$ is decreasing from the initial state x_1^* to equilibrium x_1^0. It indicates that in this case technology stock T^* is growing rapidly than production y^*.

Let us consider properties of quasioptimal feedback r^* (7.6) which generates trajectories $x^*(\cdot)$.

Remark 7.2 *In the expression for quasioptimal control r^* (7.6) denominator tends to the positive constant value*

$$(gx_2^0 - (p^0 - x_2^0)\frac{y^*}{T^*}) \to (gx_2^0 - (p^0 - x_2^0)x_1^0) > 0, \quad t \to \infty \tag{7.12}$$

and numerator $a_2 y^*$ is linear with respect to production y^*. It demonstrates that the value of quasioptimal control r^* (7.6) is also growing to infinity with the same asymptotic growth rate as production y^* and technology T^*.

8 Behavior of R&D Intensities

In this section we examine the question about the evolutionary behavior of R&D intensities which is expressed by ratios r/y, $r_{(t-m)}/y$ on quasioptimal trajectories.

Proposition 8.1 *There exists the interval of slopes*

$$\omega_2 \leq \omega \leq \omega_1, \quad \omega_1 = \frac{gp^0}{(x_1^0 + g)^2}, \quad \omega_2 = \frac{(p^0 - x_2^0)}{(x_1^0 + g)} \tag{8.1}$$

such that quasioptimal feedback $r^* = r^*(\omega)$ (7.6) leads trajectories $x^*(\cdot)$ from initial position x_1^*, x_3^* (7.4) to equilibrium x_1^0, x_3^0 with evolutionary decline of ratio $x_1 = y/T$ and growth of ratio r/y.

Proof.
Let us prove first the inequality

$$0 \leq \omega_2 < \omega_1 \tag{8.2}$$

Really taking into account relation (5.19) we have the chain of inequalities

$$\omega_1 = \frac{p^0 g}{(x_1 + g)^2} > \frac{p^0 g}{(x_1 + g)^2} - \frac{a_2 x_1^\gamma}{(x_1 + g)(f_1 x_1^\gamma + f_2)} = \frac{(p^0 - x_2)}{(x_1 + g)} = \omega_2 \geq 0$$

According to Proposition 7.1 for slopes $0 \leq \omega \leq \omega_2$ trajectories $x^*(\cdot)$ generated by feedbacks $r^*(\omega)$ from initial position x_1^*, x_3^* (7.4) converge to equilibrium x_1^0, x_3^0. From Remark 7.1 it follows that the ratio $x_1 = y/T$ is declining to x_1^0. Let us consider the ratio r/y in the quasioptimal regime (7.6)

$$\frac{r}{y} = \frac{a_2}{(d + k(\omega)(x_1 - x_1^0) + \omega(x_1 - x_1^0)^2)} \tag{8.3}$$

The derivative of feedback (8.3) with respect to x_1 at point x_1^0 is determined by formula

$$\frac{d(r/y)}{dx_1}\Big|_{x_1 = x_1^0} = -\frac{a_2 k(\omega)}{d^2} \tag{8.4}$$

If parameter $k = k(\omega)$ is nonnegative and hence we have relation

$$\omega \geq \frac{(p^0 - x_2^0)}{(x_1^0 + g)} = \omega_2 \tag{8.5}$$

then derivative (8.5) is nonpositive and ratio r/y is growing while ratio $x_1 = y/T$ is declining.

In the opposite case when $0 \leq \omega < \omega_2$ ratio r/y is declining when ratio $x_1 = y/T$ is declining. □

Let us consider the natural candidate for the slope of the quasioptimal feedback (7.6) – the slope ω_0 of eigenvector h_2 (6.24) of the Jacobi matrix D (6.4) which corresponds to the negative eigenvalue μ_2 (6.22)

$$\omega_0 = \frac{(a + \mu_2)}{b} = \frac{(a + \rho - \mu_1)}{b} = \frac{c}{(a + \mu_1)} \tag{8.6}$$

Proposition 8.2 *The slope ω_0 (8.6) of eigenvector h_2 (6.24) corresponding to the negative eigenvalue μ_2 (6.22) satisfies relations*

$$0 \leq \omega_0 < \omega_1 \tag{8.7}$$

and, hence, the quasioptimal feedback $r^* = r^*(\omega_0)$ (7.6) with slope ω_0 leads trajectories $x^*(\cdot)$ from initial position x_1^*, x_3^* to equilibrium x_1^0, x_3^0.

Proof. Taking into account relation (6.21) we have the following chain of inequalities

$$\omega_1 = \frac{a}{b} > \frac{a}{b} - \frac{(\mu_1 - \rho)}{b} = \frac{(a + \rho - \mu_1)}{b} = \omega_0 = \frac{c}{(a + \mu_1)} \geq 0$$

□

Let us make comparison of the quasioptimal slope ω_0 (8.6) with the slope ω_2 (8.5) which is responsible for the growing property of ratio r/y.

Proposition 8.3 *There exists a threshold $\gamma^* > 0$ such that for parameters $0 \leq \gamma < \gamma^*$, $a_1 \geq 0$ the quasioptimal slope ω_0 satisfies inequalities*

$$0 \leq \omega_0 \leq \omega_2 \tag{8.8}$$

If the strict inequalities $0 < \gamma < \gamma^*$ or $a_1 > 0$ take place then the strict relations

$$0 < \omega_0 < \omega_2 \tag{8.9}$$

are valid.

Inequality (8.9) means that the quasioptimal control $r^* = r^*(\omega_0)$ (7.6) with slope ω_0 provides the declining property of ratio r/y.

Proof. Let us estimate the difference

$$\omega_2 - \omega_0 = \frac{(p^0 - x_2)}{(x_1 + g)} - \frac{(a + \rho - \mu_1)}{b} =$$
$$\frac{1}{b}(\mu_1 - \rho - \gamma f_2 x_1^{-\gamma} - \frac{a_2 b}{(x_1 + g)F}) =$$
$$\frac{1}{b}((\frac{\rho^2}{4} + a(a + \rho) - bc)^{1/2} - (\frac{\rho}{2} + \gamma f_2 x_1^{-\gamma} + \frac{x_1 F}{(x_1 + g)})) =$$
$$\frac{(a - e)(a + e + \rho) - bc}{bS}$$

Here parameters a, b, c are the absolute values of elements of the Jacobian matrix (6.25) and parameters F, e, S are determined by relations

$$F = f_1 + f_2 x_1^{-\gamma}, \quad e = \gamma f_2 x_1^{-\gamma} + \frac{x_1 F}{(x_1 + g)}$$
$$S = (\frac{\rho^2}{4} + a(a + \rho) - bc)^{1/2} + (\frac{\rho}{2} + e)$$

Taking into account expression for the difference

$$a - e = \frac{x_1 F^2}{a_2 (x_1 + g)}(\frac{g p^0}{(x_1 + g)} - \frac{a_2}{F}) = \frac{x_1 F^2}{a_2 (x_1 + g)}(p^0 - x_2)$$

we obtain the following relation

$$\omega_2 - \omega_0 = \frac{(p^0 - x_2)(2\gamma f_2 x_1^{-\gamma} + \rho + F(x_1 - g)/(x_1 + g))}{(x_1 + g)S} -$$
$$\frac{\gamma^2 f_2 x_2 (x_1 + g) x_1^{-(1+\gamma)}}{(x_1 + g)S}$$

Finally substituting instead of ratio $F(x_1-g)/(x_1+g)$ its expression through parameter $(p^0 - x_2)$ we get formula

$$\omega_2 - \omega_0 = \frac{(p^0 - x_2)(2\gamma f_2 x_1^{-\gamma} + \rho + F - 2(a_2 + F(p^0 - x_2))/p^0)}{(x_1 + g)S} - \frac{\gamma^2 f_2 x_2 (x_1 + g) x_1^{-(1+\gamma)}}{(x_1 + g)S} \tag{8.10}$$

We need to estimate difference $(\omega_2 - \omega_0)$ (8.10) in the neighborhood of point $(\xi, \eta) = (x_1(\gamma), x_2(\gamma))$, $\gamma = 0$. In order to calculate the Taylor expansion of the first order for difference $(p^0 - x_2(\gamma))$ we need to estimate the value x_2 and its derivative $dx_2/d\gamma$ at the origin $\gamma = 0$.

Assuming $\gamma = 0$ in the system (5.18) which defines the equilibrium x^0 and resolving the first two equations with respect to $\eta = x_2(0)$ we obtain the following relation

$$f^0\eta^2 - (f^0p^0 + a_2 - \rho p^0)\eta - p^0 = f^0\eta(\eta - p^0) + (\rho p^0 - a_2)\eta - p^0 = 0 \tag{8.11}$$

Here

$$f^0 = f_1 + f_2$$

Using relation (8.11) and estimating difference $(p^0 - \eta)$ from above we obtain the following chain

$$0 \le (p^0 - \eta) = \frac{(\rho p^0 - a_2)}{f^0} - \frac{p^0}{f^0\eta} = \frac{(a_1 + 1)}{f^0} - \frac{p^0}{f^0\eta} \le \frac{a_1}{f^0} \tag{8.12}$$

Let us estimate difference $(p^0 - \eta)$ from below. To this end we consider the tangent line to parabola (8.11) at point (p^0, a_1p^0)

$$y = (f^0p^0 + (a_1 + 1))(x_2 - p^0) + a_1p^0 \tag{8.13}$$

Due to convexity of parabola (8.11) we obtain the low bound for $(p^0 - \eta)$ by the zero point $(p^0 - \overline{\eta})$ of the tangent line (8.13)

$$(p^0 - \eta) \ge (p^0 - \overline{\eta}) = \frac{a_1p^0}{(f^0p^0 + (a_1 + 1))} \tag{8.14}$$

Let us calculate derivative $dx_2/d\gamma$ of function $x_2 = x_2(\gamma)$ with respect to parameter γ at point $\gamma = 0$. To this end let us differentiate two first equations of system (5.18) with respect to γ

$$\begin{aligned} \frac{\partial F_1}{\partial x_1}\frac{dx_1}{d\gamma} + \frac{\partial F_1}{\partial x_2}\frac{dx_2}{d\gamma} + \frac{\partial F_1}{\partial \gamma} = 0 \\ \frac{\partial F_2}{\partial x_1}\frac{dx_1}{d\gamma} + \frac{\partial F_2}{\partial x_2}\frac{dx_2}{d\gamma} + \frac{\partial F_2}{\partial \gamma} = 0 \end{aligned} \tag{8.15}$$

To resolve system (8.15) with respect to derivatives $dx_1/d\gamma$, $dx_2/d\gamma$ we need to find the inverse matrix D^{-1} to the Jacobian matrix D (6.4)

$$D^{-1} = \begin{pmatrix} (a+\rho)/DI & -b/DI \\ c/DI & -a/DI \end{pmatrix} \tag{8.16}$$

and calculate partial derivatives

$$\frac{\partial F_1}{\partial \gamma} = -f_2 \xi^{(1-\gamma)} \ln \xi|_{\gamma=0} = -f_2 \xi \ln \xi$$
$$\frac{\partial F_2}{\partial \gamma} = (f_2 \eta \xi^{-\gamma} - \gamma f_2 \eta \xi^{-\gamma} \ln \xi)|_{\gamma=0} = f_2 \eta \tag{8.17}$$

Taking into account relations (8.16), (8.17) we find derivatives $dx_1/d\gamma$, $dx_2/d\gamma$

$$\frac{dx_1}{d\gamma} = \frac{((a+\rho)\xi \ln \xi + b\eta) f_2}{DI}$$
$$\frac{dx_2}{d\gamma} = \frac{(c\xi \ln \xi + a\eta) f_2}{DI} \tag{8.18}$$

Substituting relations (6.3), (6.25) for derivatives a, b, c to formula (8.18) we obtain the expression for derivative $dx_2/d\gamma$

$$\frac{dx_2}{d\gamma} = \frac{f_2 f^0 g \xi \eta}{a_2 DI (\xi + g)^2}((p^0 - \eta) \ln \xi + p^0) \tag{8.19}$$

Expanding expressions $(p^0 - x_2(\gamma))$, $F(\gamma)$ into Taylor series in the neighborhood of point $\gamma = 0$ we get formulas

$$(p^0 - x_2(\gamma)) = (p^0 - \eta) + \frac{dx_2}{d\gamma}\gamma + o(\gamma) \tag{8.20}$$

$$F(\gamma) = f_1 + (x_1(\gamma))^{-\gamma} f_2 = f^0 + \frac{dF}{d\gamma}\gamma + o(\gamma) \tag{8.21}$$

$$\frac{dF}{d\gamma} = -(x_1(\gamma))^{-\gamma} f_2(\ln(x_1(\gamma)) + \gamma \frac{1}{x_1(\gamma)} \frac{dx_1}{d\gamma})|_{\gamma=0} = -f_2 \ln \xi$$

Here $o(\gamma)$ is the infinitesimal value of high order

$$o(\gamma) = \gamma \varphi(\gamma), \quad \lim_{\gamma \to 0} \varphi(\gamma) = 0 \tag{8.22}$$

Let us estimate numerator NU in the right hand side of relation (8.10) for difference $(\omega_2 - \omega_0)$

$$NU = (p^0 - \eta)(\rho + f^0 - \frac{2(a_2 + f^0(p^0 - \eta))}{p^0}) +$$
$$\gamma(p^0 - \eta)(2f_2 + 2\frac{f^0}{p^0}\frac{dx_2}{d\gamma} + 2\frac{(p^0 - \eta)}{p^0} f_2 \ln \xi - f_2 \ln \xi) -$$
$$\gamma \frac{dx_2}{d\gamma}(\rho + f^0 - \frac{2(a_2 + f^0(p^0 - \eta))}{p^0}) + o(\gamma) \tag{8.23}$$

Let us consider two cases. In the first case assume $a_1 > 0$. Taking into account inequalities (8.12), (8.14) for difference $(p^0 - \eta)$ we obtain the following estimate for the first term in relation (8.23)

$$(p^0 - \eta)(\rho + f^0 - \frac{2(a_2 + f^0(p^0 - \eta))}{p^0}) \geq$$
$$\frac{(p^0 - \eta)}{p^0}(2 + f^0 - \rho) \geq \frac{a_1}{(f^0 p^0 + (a_1 + 1))}(2 + f^0 - \rho) > 0 \tag{8.24}$$

Since according to inequality (8.24) the first term in relation (8.23) is strictly positive then there exists threshold $\gamma^* > 0$ such that for parameters γ, $0 \leq \gamma < \gamma^*$ numerator NU and, hence, difference $(\omega_2 - \omega_0)$ is strictly positive.

In the second case, when $a_1 = 0$, estimates (8.12), (8.14) imply equality $(p^0 - \eta) = 0$. For numerator NU we have the following relations

$$NU = -\gamma \frac{dx_2}{d\gamma}(\rho + f^0 - \frac{2a_2}{p^0}) + o(\gamma) =$$
$$\gamma(\frac{\xi \eta f_2 g(f^0)^2}{a_2(\xi + g)^2 |DI|}(2 + f^0 - \rho) + \varphi(\gamma)) \tag{8.25}$$

Since the first term in relation (8.25) is strictly positive then there exists threshold $\gamma^* > 0$ such that for parameters γ, $0 < \gamma < \gamma^*$ numerator NU and, hence, difference $(\omega_2 - \omega_0)$ is strictly positive. □

Let us consider two important modifications of quasioptimal strategy R^* (7.6) which are characterized by preserving constant values of coordinates x_1, x_2.

Remark 8.1 *Assuming $\omega = 0$ in formula (7.6) for the quasioptimal control r^* one can obtain the quasioptimal process with the constant value for the cost of production $x_2 = x_2^0$*

$$r = \frac{a_2 y}{(d - (p^0 - x_2^0)((y/T) - x_1^0))} \tag{8.26}$$

In the quasioptimal process (8.26) ratio y/r is growing while ratio y/T is declining.

Setting the constant value for coordinate $x_1 = y/T = x_1^0$ in formula (8.26) one can derive the quasioptimal process

$$r = \frac{a_2 y}{d} \tag{8.27}$$

with the fixed second coordinate $x_2 = x_2^0$ and the constant ratio

$$\frac{r}{y} = \frac{a_2}{d} \tag{8.28}$$

Both quasioptimal feedbacks (8.26), (8.27) lead trajectories $x^*(\cdot)$ of system (7.2) from initial conditions x_1^*, x_3^* (7.4) to equilibrium x_1^0, x_3^0.

Let us examine the behavior of ratio $r_{(t-m)}/y$ of R&D investment in initial stage $r_{(t-m)}$ (1.15) to production y – R&D intensity in initial stage, in the quasioptimal regime (7.6). According to (1.15), (7.6) we have relations

$$\frac{r_{(t-m)}}{y} = (1-\sigma)\frac{r}{y} + \sigma\frac{T}{y} = \frac{(1-\sigma)a_2}{(d + k(\omega)(x_1 - x_1^0) + \omega(x_1 - x_1^0)^2)} + \frac{\sigma}{x_1} \tag{8.29}$$

For nonpositive derivative

$$\frac{d(r_{(t-m)}/y)}{dx_1}|_{x_1 = x_1^0} = -\frac{(1-\sigma)a_2 k(\omega)}{d^2} - \frac{\sigma}{(x_1^0)^2} \leq 0 \tag{8.30}$$

ratio $r_{(t-m)}/y$ is growing while ratio $x_1 = y/T$ is declining. Resolving inequality (8.30) with respect to parameter ω we find conditions for parameters ω and σ which provide the growth property of ratio $r_{(t-m)}/y$

$$\omega \geq \frac{1}{(x_1^0 + g)}((p^0 - x_2^0) - \frac{\sigma}{(1-\sigma)}\frac{1}{a_2}\frac{d^2}{(x_1^0)^2}) = \omega_3 \tag{8.31}$$

Remark 8.2 *Summarizing previous results one can derive the following properties of the quasioptimal control $r^* = r^*(\omega)$ (7.6):*

1. *if* $0 \leq \omega < \max\{0, \omega_3\}$ *then both ratios* $r_{(t-m)}/y$ *and* r/y *are declining;*
2. *if* $\max\{0, \omega_3\} \leq \omega < \omega_2$ *then ratio* $r_{(t-m)}/y$ *is growing and ratio* r/y *is declining.*
3. *if* $\omega_2 \leq \omega \leq \omega_1$ *then both ratios* $r_{(t-m)}/y$ *and* r/y *are growing;*

while ratio y/T *is declining.*

Finally let us analyze properties of production rate $\dot{y}/y$ (3.7) in the quasioptimal regime $r^* = r^*(\omega)$ (7.6). Differentiating production rate $\dot{y}/y$ with respect to parameter $x_1 = y/T$ at equilibrium x_1^0 we obtain the following formula

$$\frac{d(\dot{y}/y)}{dx_1}|_{x_1 = x_1^0} = -\gamma f_2(x_1^0)^{-(1+\gamma)} + \frac{g a_2((x_1^0 + g)\omega - (p^0 - x_2^0))}{d^2} \tag{8.32}$$

Remark 8.3 *If slope* ω *satisfies inequality*

$$\omega < \frac{(p^0 - x_2^0)}{(x_1^0 + g)} + \frac{\gamma f_2(x_1^0)^{-(1+\gamma)} d^2}{a_2 g} = \omega_4 \tag{8.33}$$

then production rate $\dot{y}/y$ *is growing while parameter* $x_1 = y/T$ *characterizing technology intensity is decreasing to equilibrium* x_1^0.

Conclusion

In this paper we examine the nonlinear model of optimal allocation of resources - R&D investment in a techno-metabolic system, which describes behavior of production and technology rates with respect to R&D investment. The growth and decline trends in interaction between production and R&D investment are described in the balance dynamics. The growth property is expressed by the exponential term of technology intensity. Investment to R&D leads to the redistribution of resources between production and technology stock and provides the risky factor of innovation. The discounted utility function correlates the amount of sales and production diversity. Qualitatively it expresses preferences of investors in the simultaneous growth of production, technology stock and technology rate. We apply the Pontryagin's optimality principle to the problem of optimal control design. Optimality principles are expressed in the nonlinear system of differential equations of the fourth order. Its equilibrium is connected with optimal solutions. We prove the existence and uniqueness of the saddle type equilibrium and show that optimal trajectories should converge to it. Since the optimal feedback which generates optimal trajectories is given implicitly we provide several explicit approximations of the rational type – quasioptimal feedbacks. We examine properties of quasioptimal feedbacks for different tangent slopes generated by possible R&D intensities. We study the particular case when the tangent slope is determined by the optimal solution of the linearized system – the slope of eigenvector corresponding to the negative eigenvalue of the Jacobi matrix. Growth properties of production rate, R&D intensity and technology intensity for quasioptimal feedbacks as functions of slopes are indicated on generated trajectories. In the test example we make an accent on the nonstationary character of the reduced dynamics. We obtain explicit solutions for optimal feedback, production rate, R&D intensity and technology intensity. The value function of the reduced control problem is calculated analytically by the method of indeterminate coefficients for the Hamilton-Jacobi equation. Algebraic expressions for the value function have the decomposition property: the first term depends in the logarithmic way on the initial production, the second one depends in the aggregated form on functions of nonstationary dynamics.

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