

INTRODUCTORY REMARKS ON THE STATE SPACE MODELING  
OF WATER RESOURCE SYSTEMS

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## Preface

In the day-to-day management of river basins one of the crucial issues is the derivation of real-time operating policies, which are to be optimal in a certain sense, for the water resource systems. As telemetered systems are gradually coming into use this problem is becoming more and more important for decision makers operating such systems. In view of the inherent random nature of water resource systems there is a lot of room for methodological research as well, therefore the IIASA Research Plan for 1976 provides a task on the Methodology of Real-Time Forecasting and Control of Water Resource Systems; the aim of which is to contribute to the solution of the aforementioned problems.

As it turned out in the past few years the state space modeling techniques are particularly well suited for studying the problems of real-time forecasting/control in water resource systems. Quite a number of papers have been published on this subject, nevertheless water resource engineers were and still are reluctant to use in the course of their every day practice the techniques advocated. One probable reason for that is that the papers mentioned assume a certain amount of knowledge on the state space techniques, an assumption which sometimes does not really hold. Therefore, stimulated by needs of many practitioners, this paper aims to give a short introduction to state space modeling with particular reference to water resources systems. Through a number of examples the notion and structural properties of states of water resource systems are discussed, both for the deterministic and stochastic cases, since, as Yevjevich (1974) states, "only an integration of both deterministic and stochastic approaches promises the best mathematical-physical understanding and description of hydrologic processes and environment". It will be seen that the state space techniques are indeed capable of offering such an integrated approach.

As the purpose of this paper is merely to give an insight into the applicability of modern systems theory to water resource systems, the mathematics will be kept on a lower level, however, we will follow Einstein's dictum that "an explanation should be as simple as possible but no simpler". The recursive filtering and prediction algorithms are not discussed here; they are left for another paper where the final results of the aforementioned IIASA task will also be reported.

Finally, a technicality. The equations and examples in each of the four sections are numbered independently. If a reference is made from one section to an equation in another section, the number of the section stands first followed by the number of the particular equation referred to, e.g., equation (10) in section 1 is referred as (1-10) in any section except section 1, where it is referred simply as (1). The same holds true for the examples.

## Abstract

In section 1 the notion of state and state equations for water resource systems are discussed both for continuous and discrete dynamics. Section 2 presents the solution of state equation for linear systems including the derivation of state transition and impulse response matrices. In section 3 the structural properties such as observability, controllability, indentifiability and minimal realizations are discussed. Finally, in section 4 the state concept for stochastic systems is reexamined. The state and measurement disturbances are considered as being white Gaussian noise processes and it is showed how the case of of sequentially correlated uncertainties can be reduced to an augmented system model having white Gaussian state disturbance only. The paper concludes with the generalization of structural properties for stochastic systems. To illustrate the underlying concepts examples taken from a broad range of water resources problems, such as rainfall analysis, rainfall/runoff relation, reservoir and lake/aquifer problems, water quality control etc., are presented.



1. THE NOTION OF STATE AND STATE EQUATIONS FOR  
WATER RESOURCE SYSTEMS

The concept of state has its roots in the cause-effect relation of classical mechanics and in fact is not an entirely new concept but rather a unifying framework in which the relatively easy handling of complex systems with many interactions and/or input/output variables becomes possible. The state space approach is based upon the internal description of the systems as opposed to the classical internal description which considers the input/output relations only.

The concept of the state of the system  $\Sigma$  (which is actually the system model of the real system  $\mathcal{S}$  and the word "system" is unfortunately used for the sake of shortness even though it might be ambiguous) is a mathematical entity which mediates between the inputs and the outputs, i.e. the inputs act on the state which, in turn, generates the outputs. As Casti (1976) states, it is important to emphasize that the state, in general, has no intrinsic meaning and is introduced solely as a mathematical convenience in order to inject the notions of causality and internal structure into the description of  $\Sigma$ . The only quantities which have physical meaning are those which can generate or observe, namely the inputs and outputs. It should be stressed, however, that it is desirable for the model to reflect and use as much physical information as possible, i.e. the state variables, if possible, should have physical meaning. This is the principle of physicality.

Another, more intuitive, interpretation of the state is that it is the least amount of information which, together with the current input, uniquely determines the state at the next moment of time; in other words it is the minimal amount of information about the past history of the system which is required to predict its future behavior (Åström, 1970). Of course, this is a somewhat circular definition, but it does

convey the intuitive flavor of the state concept.

### Continuous Systems

Generally, systems are distributed over space and time and can be described by partial differential equations (Butkovsky, 1969). Our discussion here will be restricted to lumped systems described by ordinary differential or difference equations. First, consider the continuous case where the system dynamics is given by a set of differential equations. Thus, if

$$x_1(t), x_2(t), \dots, x_n(t)$$

are the state variables (or simply the states) of the process at time  $t$ , and

$$u_1(t), u_2(t), \dots, u_p(t)$$

are the input or control variables to the process at time  $t$ , then the system may be described by  $n$  first-order differential equations

$$\begin{aligned} \dot{x}_1(t) &= f_1[x_1(t), x_2(t), \dots, x_n(t), u_1(t), u_2(t), \dots, u_p(t), t] \\ \dot{x}_2(t) &= f_2[x_1(t), x_2(t), \dots, x_n(t), u_1(t), u_2(t), \dots, u_p(t), t] \\ &\vdots \\ \dot{x}_n(t) &= f_n[x_1(t), x_2(t), \dots, x_n(t), u_1(t), u_2(t), \dots, u_p(t), t] \end{aligned} \tag{1}$$

where  $\dot{x}_i(t)$  is in general a nonlinear time varying function  $f_i$  of the states, the inputs and time. The  $n$  state variables may be associated with separate axes in an  $n$ -dimensional (Euclidian) space called the state space and denoted by  $X$ . The path or motion of a system's states in their state space is called the state trajectory which, in other words, describes the history of state values in a given time interval. By defining



$$x(t) \hat{=} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = [x_1(t), x_2(t), \dots, x_n(t)]^T$$

as the state vector of the system, and

$$u(t) = [u_1(t), u_2(t), \dots, u_p(t)]^T$$

as the input vector, the state or system equation can be written

$$\dot{x}(t) = f_t[x(t), u(t)] \quad , \quad (2)$$

where the definition of  $f_t$  is apparent by comparison with (1). If the input vector is missing from (2) the system is said to be free; otherwise it is forced. As a matter of fact (2) gives the relation how the inputs acts on the states which in turn generate the outputs according to the algebraic relation

$$y(t) = h_t[x(t)] \quad , \quad (3)$$

where  $y(t)$  is an  $m$ -vector of the output variables and  $h_t$  is a nonlinear vector function. In the literature (3) is called output equation. Obviously  $m \leq n$ , indicating that sometimes not all the state variables are connected directly to the output.

As an illustration of these concepts in a hydrological context consider the following

**E x a m p l e 1** discussed in detail by Duong et al (1975).

As is well-known, direct runoff may be considered as the result of the transformation of rainfall excess by the basin. The physical process of this transformation is very complex, depending mainly upon the storage

effects in the basin. (The reader interested in the details and interconnections between the processes involved is referred to Dooge's (1973) comprehensive review.) To take into account these effects Kulandaiswamy (1964) derived the following general expression

$$S(t) = \sum_{n=0}^N a_n(q,u) \frac{d^n q}{dt^n} + \sum_{m=0}^M b_m(q,u) \frac{d^m u}{dt^m} ,$$

where  $S$  is the storage,  $t$  is time,  $N$  and  $M$  are integers, and  $a_n(q,u)$  and  $b_m(q,u)$  are parametric functions of the direct runoff  $q$  and the excess rainfall  $u$ . To apply the above storage relations to the study of the rainfall-runoff processes in a particular watershed, the values of  $N$  and  $M$ , and the form of  $a_n(\cdot)$  and  $b_m(\cdot)$ , respectively must be determined. Unfortunately, sometimes it is not feasible in practice. Therefore Prasad (1967) suggested the use of a simplified storage equation in the form of

$$S(t) = K_1 q^N(t) + K_2 \frac{dq(t)}{dt} ,$$

where  $K_1$ ,  $K_2$  and  $N$  are the unknown parameters to be estimated. In his study, Prasad assumed that these parameters are constant for a particular hydrograph. Using the continuity equation the following differential equation is obtained for the rainfall-runoff process

$$K_2 \frac{d^2 q}{dt^2} + K_1 N q^{N-1} \frac{dq}{dt} + q = u .$$

This can be written as

$$\frac{d^2 q}{dt^2} = -\left(\frac{1}{K_2}\right) K_1 N q^{N-1} \frac{dq}{dt} - \left(\frac{1}{K_2}\right) q + \left(\frac{1}{K_2}\right) u . \quad (E1-1)$$

By defining the following set of state variables

$$\begin{aligned}
 x_1(t) &= q(t) \\
 x_2(t) &= \dot{q}(t) \\
 x_3(t) &= K_1 \\
 x_4(t) &= \frac{1}{K_2} \\
 x_5(t) &= N \quad ,
 \end{aligned}$$

and assuming that the model coefficients are time invariant, the Prasad model (E1-1) becomes

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \\ \dot{x}_5(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -x_3(t)x_4(t)x_5(t)x_1^{x_5(t)-1}(t)x_2(t)+x_4(t)[u(t)-x_1(t)] \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(E1-2)

or, in abbreviated notation,

$$\dot{x}(t) = f_t[x(t), u(t)] \quad , \quad (E1-3)$$

which, like (2), is a nonlinear state equation of time invariant type. As for the output equation, one can immediately realize that by choosing the output process  $q(t)$  as being a state variable itself, it is in the form of

$$y(t) = [1,0,0,0,0] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix} \quad (E1-4)$$

or, like (3), in abbreviated notation as

$$y(t) = h_t[x(t)] . \quad (E1-5)$$

In fact, the output equation for the Prasad model is a linear one and the output process is scalar. The conclusions of this example are:

It is not at all necessary that nonlinear output equation be attached to a nonlinear state equation; Variables with no direct physical meaning can also be chosen as being state variables.

We mention that in a recent work by Maidment (1976) the linearized form of the Kulandaiswamy model is illustrated in a state space fashion.

### Discrete Systems

Until now we have been discussing systems which evolve on a continuous time set  $T_c = \{t: t_0 < t < t_f\}$ ,  $t_0$  and  $t_f$  being the starting and finishing times respectively of the processes. From now on we set  $t_0 = 0$  and  $t_f$  may be either finite,  $N$ , or infinite. We can define, similarly, a discrete time set  $T_d = \{t: t = 0, 1, 2, \dots\}$ , and we are interested in the state space modeling of processes evolving on such a discrete time set. By analogy with (2) and (3) the following nonlinear difference equations can be derived for discrete time systems:

$$x(t+1) = f_t[x(t), u(t)] \quad (4)$$

for the states and

$$y(t) = h_t[x(t)] \quad (5)$$

for the output process. For the sake of illustration consider the following example.

Example 2. In the hydrological literature many papers (Amorocho, 1963; Hino et al., 1971; Amorocho and Brandstatter, 1971; Bidwell, 1971; Diskin and Boneh, 1972; Diskin and Boneh, 1973; Zand and Harder, 1973; Quimpo, 1975) deal with the Volterra series representation of the nonlinear rainfall-runoff system. Such a representation has the form

$$\begin{aligned}
 y(t) = & \int_0^{\infty} h_1(\tau_1)u(t-\tau_1)d\tau_1 + \iint_{00}^{\infty\infty} h_2(\tau_1,\tau_2)u(t-\tau_1)u(t-\tau_2)d\tau_1d\tau_2 \\
 & + \dots + \iint_{00}^{\infty\infty} \dots \int_0^{\infty} h_{\mu}(\tau_1,\dots,\tau_{\mu})u(t-\tau_1)\dots u(t-\tau_{\mu})d\tau_1\dots d\tau_{\mu}
 \end{aligned}
 \tag{E2-1}$$

where  $\mu$  is the highest index in the truncated representation,  $u(t)$  is the input to the nonlinear system and  $h_1, h_2, \dots$  are the Volterra kernels. The problem is that of determining these kernel functions. To do that Amorocho and Brandstetter (1971) used Laguerre polynomials, Bidwell (1971) regression analysis, and Hino et al (1971) and Quimpo (1975) a technique developed by Lee and Schetzen which is based upon white noise input process. Here, it is assumed that the nonlinear system can be separated into cascaded blocks of linear dynamic system and a zero memory non-linear system as shown in Fig. 1. The linear subsystem has an impulse response  $g(t)$  and the nonlinear subsystem has a power representation given by

$$y(t) = b_1 y_{\ell}(t) + b_2 y_{\ell}^2(t) + \dots + b_{\mu} y_{\ell}^{\mu}(t) ,
 \tag{E2-2}$$

where  $y(t)$  is the output and  $y_{\ell}(t)$  is the input to the nonlinear subsystem; in fact the latter is the output of the linear subsystem at the same time. So, the problem is that neither the impulse response  $g(t)$  of the linear subsystem nor the coefficients  $b_1, b_2, \dots, b_{\mu}$  of the

nonlinear subsystem are known. As a matter of fact the Volterra series for  $y(t)$  of this particular system can be expressed in terms of  $g(t)$  and the coefficients of the nonlinear part. To show this, it is to be noted that the output  $y_\ell(t)$  of the linear subsystem is given by the convolution integral

$$y_\ell(t) = \int_0^{\infty} g(\tau) u(t-\tau) d\tau . \quad (\text{E2-3})$$

Substituting this into (E2-2) the output  $y(t)$  can be written as

$$\begin{aligned} y(t) = & b_1 \int_0^{\infty} g(\tau_1) u(t-\tau_1) d\tau_1 + b_2 \int_0^{\infty} g(\tau_1) u(t-\tau_1) d\tau_1 \times \\ & \times \int_0^{\infty} g(\tau_2) u(t-\tau_2) d\tau_2 + \dots + b_\mu \int_0^{\infty} g(\tau_1) u(t-\tau_1) d\tau_1 \times \dots \\ & \times \int_0^{\infty} g(\tau_\mu) u(t-\tau_\mu) d\tau_\mu . \end{aligned} \quad (\text{E2-4})$$

Comparing this expression with (E2-1) one concludes that the Volterra kernels for the system concerned are given by

$$h_i(\tau_1, \tau_2, \dots, \tau_i) = b_i g^i(\tau_i), \quad i = 1, 2, \dots, \mu. \quad (\text{E2-5})$$

Now, let us assume that the input is an impulse function. Then, by definition, the output  $y_\ell(t)$  of the linear subsystem is equal to the impulse response  $g(t)$ . Consider a truncated series approximation of  $g(t)$

$$g(t) \approx \sum_{i=1}^p a_i \phi_i(t) , \quad (\text{E2-6})$$

where  $a_i$  are some yet unknown constant and  $\phi_i(t)$  are a set of chosen orthogonal polynomials, for example Langerre polynomials. Substituting this series

approximation into (E2-2) the following expression is obtained:

$$y(t) = b_1 \sum_{i=1}^p a_i \phi_i(t) + b_2 \left( \sum_{i=1}^p a_i \phi_i(t) \right)^2 + \dots + b_\mu \left( \sum_{i=1}^p a_i \phi_i(t) \right)^\mu. \quad (\text{E2-7})$$

By defining a vector of unknown coefficients

$$x = [a_1, a_2, \dots, a_p, b_1, b_2, \dots, b_\mu]^T,$$

(E2-7) can be rewritten as

$$y(t) = h[x], \quad (\text{E2-8})$$

where  $h[\cdot]$  is a nonlinear function. Comparing this expression with (5) it becomes apparent that it is an output equation acting on discrete states which are, in fact, the parameters. As the coefficients  $a_1, a_2, \dots, a_p$  and  $b_1, b_2, \dots, b_\mu$  are all constant one can construct a linear state equation in the form

$$x(t+1) = x(t), \quad (\text{E2-9})$$

where  $t \in T_d$ . The conclusions of this example are: It is not at all necessary that a nonlinear state equation be attached to a nonlinear output equation, and reversely as it was shown in Example 1; Again, variables with no physical meaning can be chosen as being state variables.

We mention that Laguerre polynomials, due to the fact that they can be computed **recursively**, are particularly well suited for such an analysis. For details, see Dooge (1965) and Amorocho and Brandstatter (1971).

## 2. STATE EQUATIONS FOR LINEAR SYSTEMS

### Continuous Case

The state equations for linear systems can be obtained as a special case of (1-2). The dynamic behaviour of such a system can be modeled by a set of first order linear differential equations

$$\dot{x}(t) = F(t)x(t) + G(t)u(t) , \quad (1)$$

where  $t \in T_c$ ,  $x(t)$  is an  $n$ -vector of states of the system,  $u(t)$  is a  $p$ -vector of input variables,  $F(t)$  is an  $n \times n$  matrix, and  $G(t)$  is an  $n \times p$  matrix. These latter matrices, commonly called the system matrices, are assumed to be continuous in  $t$ . The initial state is given by  $x(0)$ .

It is assumed that the output equation (1-3) is degenerated to the following linear relation:

$$y(t) = H(t)x(t) , \quad (2)$$

where  $y(t)$  is an  $m$ -vector of output variables, and  $H(t)$  is a continuous  $m \times n$  matrix which relates the states to the outputs.

The above model, from (1) and (2), is of time-varying type. Clearly when  $F$ ,  $G$  and  $H$  are constant we obtain a time-invariant description. The system is thus specified by the triplet  $(F,G,H)$ , which will be denoted as  $\Sigma = (F,G,H)$ .

Illustrating the above concepts through a series of examples, first a simple catchment model is considered.

**E x a m p l e 1.** Figure 2 shows a simple hydrological system in which  $u_1(t)$  and  $u_2(t)$  are the rainfall inputs



(say at different locations); the states are defined as the surface storages  $x_1(t)$ ,  $x_2(t)$  and  $x_3(t)$  and the groundwater storage as  $x_4(t)$  respectively. The constants in each case are:  $k$ 's for surface water flow,  $l_1$  and  $l_2$  for infiltration. The expression  $l_3[x_4(t) - x_3(t)]$  signifies the exchange between the groundwater and the stream. The outputs are  $y_1(t)$  and  $y_2(t)$ , the streamflow output and the contribution of groundwater to streamflow, respectively. The continuity equations for this problem are

$$\begin{aligned} \dot{x}_1(t) &= -(k_1 + l_1)x_1(t) + u_1(t) \\ \dot{x}_2(t) &= -(k_2 + l_2)x_2(t) + u_2(t) \\ \dot{x}_3(t) &= k_1x_1(t) + k_2x_2(t) + l_3[x_4(t) - x_3(t)] - k_3x_3(t) \\ \dot{x}_4(t) &= l_1x_1(t) + l_2x_2(t) - l_3[x_4(t) - x_3(t)] . \end{aligned} \tag{E1-1}$$

In vector-matrix form we have the following time invariant continuous state equation, with the initial condition  $x(0) = C$ .

$$\dot{x}(t) = F x(t) + G u(t) \tag{E1-2}$$

where

$$F = \begin{bmatrix} -(k_1 + l_1) & 0 & 0 & 0 \\ 0 & -(k_2 + l_2) & 0 & 0 \\ k_1 & k_2 & -(k_3 + l_3) & l_3 \\ l_1 & l_2 & l_3 & -l_3 \end{bmatrix} ,$$

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} .$$

The output equation becomes

$$y(t) = H x(t) , \quad (E1-3)$$

where

$$H = \begin{bmatrix} 0 & 0 & k_3 & 0 \\ 0 & 0 & -\ell_3 & \ell_3 \end{bmatrix} .$$

In the above example the states were being defined as storages, i.e. a discrete physical meaning can be attached to them. One might argue that, though this formulation is conceptually simple and elegant it is not applicable to practical problems, simply due to the fact that the parameters in the matrices F, G and H are very much uncertain, if not unknown completely. To surmount these difficulties the adaptive parameter estimation technique, discussed in detail in Szöllösi-Nagy (1976), can be used.

**E x a m p l e 2.** Duckstein and Kisiel (1972) investigated the role of linear control theory as an aid to the integral control of hydrologic systems for the case of a combined lake and aquifer storage system that supplies water demand. For illustrative purposes they demonstrated the case of Lake Kinneret in Israel. The system shown in Fig. 3 has a single output  $y(t)$  and two state variables  $x_1(t)$  and  $x_2(t)$  that define the lumped linear storage in the lake and aquifer respectively. By continuity, the system output is

$$y(t) = (1-c)bx_1(t) + (1-g)fx_2(t) + (1-a)u(t) , \quad (E2-1)$$

where  $u(t)$  is the flow, and  $a, b$  and  $f, g$  are constants that may be subjected to control also. Now they are

considered as being given fixed numbers. (E2-1) can be written in the familiar form of output equations as

$$y(t) = H x(t) + L u(t) \quad (E2-2)$$

where  $H = [(1-c)b, (1-g)f]$ ,  $L = (1-a)$  and  $x(t) = [x_1(t), x_2(t)]^T$ . Similarly, by continuity, the state equations for each of the lumped elements are, respectively,

$$\dot{x}_1(t) = -bx_1(t) + gfx_2(t) + au(t)$$

$$\dot{x}_2(t) = cbx_1(t) - fx_2(t) .$$

In vector-matrix form these coupled differential equations are

$$\dot{x}(t) = F x(t) + G u(t) , \quad (E2-3)$$

where

$$F = \begin{bmatrix} -b & gf \\ cb & -f \end{bmatrix} \quad G = \begin{bmatrix} a \\ 0 \end{bmatrix} .$$

To investigate stability form the following determinant equation in  $\lambda$

$$| F - \lambda I | = \begin{vmatrix} -b-\lambda & gf \\ cb & -f-\lambda \end{vmatrix} = 0 ,$$

where  $I$  is the identity matrix. Expanding the above determinant one obtains the characteristic equation

$$\lambda^2 + (b+f)\lambda + bf(1-cg) = 0 ,$$

whose discriminant

$$\Delta^2 = (b - f)^2 + 4bfcg$$

is always greater than zero. Hence, the eigenvalues  $\lambda_1$  and  $\lambda_2$  are always real

$$\left. \begin{array}{l} \lambda_1 \\ \lambda_2 \end{array} \right\} = \frac{-(b + f) \pm \Delta}{2} .$$

Also,  $(b + f) > \Delta$  because  $(b + f)^2 > \Delta^2 = (b + f)^2 - 4bf(1 - cg)$ . Hence  $\lambda_1$  and  $\lambda_2$  are always less than zero, and the system is highly damped. Both the lake and the aquifer act as filters or dampers so long as no energy (or water head) is added to the system from another source; otherwise the system may become oscillatory. The reader interested in stability problems is referred to Willems (1970) for further details.

Another illustration of the use of continuous state space modeling as applied to hydraulics can be found in Muzik (1974), where a model is developed describing the unsteady non-uniform flow in terms of a set of first order ordinary differential equations. Conceptually the model consists of a series of interacting reaches with unsteady uniform flow subjected to impulse input.

#### Discrete Case

As in the foregoing, relating to the continuous case, state space models can easily be defined for discrete processes. By analogy with (1) and (2) the state equation is defined as

$$x(t+1) = \phi(t+1,t) x(t) + \Gamma(t) u(t) , \quad (3)$$

where  $\phi(t+1,t)$  is called the state transition matrix, which in the case of free systems maps the state at time  $t$  to the new state at time  $t+1$ . The control transition matrix  $\Gamma(t)$  is similar to  $G(t)$  but it is denoted by  $\Gamma$  to emphasize that it stands for a discrete system. As in (2) the output equation is given by

$$y(t) = H(t) x(t) . \quad (4)$$

The matrix block diagram of this discrete linear system is shown in Fig. 4. To differentiate from scalar block diagrams the signal flow is depicted by fat arrows.

To illustrate the concept and solution of discrete state space equations in water resources systems, the simplified storage process of a reservoir is discussed below as

**E x a m p l e 3.** The volume  $x(t+1)$  of stored water at time  $t+1$  in a reservoir with capacity  $V$  can be calculated as follows:

$$x(t+1) = \begin{cases} 0 & \xi(t) + \phi x(t) \leq D(t) \\ \phi x(t) + \xi(t) - D(t), & \xi(t) + \phi x(t) > D(t) \\ V & \xi(t) + \phi x(t) \geq V + D(t) \end{cases} \quad (E3-1)$$

where  $\xi(t)$  means the inflow to the reservoir at time  $t$  and  $D(t)$  is the water demand at the same time. (For simplicity both are regarded as being deterministic variables.)  $\phi$  is a reducing factor to account for losses due to evaporation, seepage, etc. The volume of stored water is chosen as being the (scalar) state variable. Introducing a new variable  $u(t) = \xi(t) - D(t)$ , which might be called 'net inflow', the storage equation

(E3-1) can be written as

$$x(t+1) = \phi x(t) + u(t) \quad (\text{E3-2})$$

with the initial conditions  $x(0), u(0)$  given. It is clear that the state 'space'  $X$  is bounded by 0 and  $V$ . Equation (E3-2) can easily be solved by recursive substitutions:

$$\begin{aligned} x(1) &= \phi x(0) + u(0) \\ x(2) &= \phi x(1) + u(1) = \phi^2 x(0) + u(1) + \phi x(0) \\ &\vdots \\ x(t) &= \phi^t x(0) + \sum_{\tau=0}^{t-1} \phi^{t-\tau-1} u(\tau) . \end{aligned} \quad (\text{E3-3})$$

The last one is the solution itself and is composed of two parts, firstly the free or transient response, which depends only on the initial state and in practice contains all the information about the past of the system, and secondly the forced response, which depends upon the input (i.e. the net inflow). Using this example it might be interesting to investigate the stability of the system. Assume that the input is identically equal to one:  $u(t) \equiv 1$ . It might be thought of as an outflow from a regulated reservoir located on an upper reach of the river, with an outflow of  $2D(t)$ . Then (E3-3) becomes

$$x(t) = \phi^t x(0) + (1 + \phi + \phi^2 + \dots + \phi^{t-1}) ,$$

which has a solution

$$x(t) = \begin{cases} \phi^t x(0) + \frac{1 - \phi^t}{1 - \phi} , & \text{for } \phi \neq 1 \\ x(0) + t , & \text{for } \phi = 1 . \end{cases}$$

If we introduce  $x^* = 1/(1 - \phi)$ , then

$$x(t) = \begin{cases} \phi^t [x(0) - x^*] + x^* , & \text{for } \phi \neq 1 \\ x(0) + t & , \text{for } \phi = 1 . \end{cases}$$

The possible transient parts of this solution are depicted in Fig. 5, from which one concludes that the necessary condition for stability is that the absolute value of the reducing factor must be less than one,  $|\phi| < 1$ . Otherwise the system either 'blows up' or does not damp to an equilibrium state.

Solution of the time invariant discrete vector state equation can be carried out along the same lines as in the above example and is

$$x(t) = \phi^t x(0) + \sum_{\tau=0}^{t-1} \phi^{t-\tau-1} \Gamma u(\tau) . \quad (5)$$

When the  $\phi$  and  $\Gamma$  matrices are time dependent, which is the case in (3), the solution is

$$x(t) = \prod_{\tau=0}^{t-1} \phi(\tau+1, \tau) x(0) + \sum_{\nu=0}^{t-2} \sum_{\tau=\nu+1}^{t-1} \phi(\tau+1, \tau) \Gamma(\tau) u(\tau) + \Gamma(t-1) u(t-1) \quad (6)$$

Again, this is obtained by recursive substitutions.

To further amplify the applicability of the state space approach in hydrology, the free discrete state equation model of the rainfall process is given below as

**E x a m p l e 4.** Gabriel and Neumann (1962) found that a two-state Markov chain gives a good description of wet and dry days. If  $\phi_1$  denotes the probability that a dry day is followed by a wet day, then  $1 - \phi_1$  means the probability of the event that a dry day is followed by another dry day. Similarly, if  $\phi_2$  denotes the probability

that a wet day is followed by a dry day etc., then the following transition probability matrix can be constructed:

$$\begin{array}{rcc}
 & \text{Actual Day} & \\
 & \begin{array}{cc} \text{Dry} & \text{Wet} \\ \text{(Stage 0)} & \text{(Stage 1)} \end{array} & \\
 \text{Preceding Day} & \begin{array}{cc} \text{Dry (Stage 0)} & \left[ \begin{array}{cc} 1 - \phi_1 & \phi_1 \\ \phi_2 & 1 - \phi_2 \end{array} \right] & = \phi, \\
 & \text{Wet (Stage 1)} & \end{array} & \text{(E4-1)}
 \end{array}$$

which will here play the role of state transition matrix and is assumed to be time-invariant. Of course,  $0 < \phi_1 < 1$  and  $0 < \phi_2 < 1$ . Let the vector  $x(t+1) = [x_0(t+1), x_1(t+1)]^T$  denote the probability of finding the system in stage 0 (dry day) or in stage 1 (wet day) at time  $t+1$ . Let the initial condition,  $t=0$ , for this vector be  $x(0) = [x_0(0), x_1(0)]^T$ . First, consider the event of being in stage 0 at time  $t+1$ . This event can occur in two mutually exclusive ways:

- (1) stage 0 prevails at time  $t$  and no transition out of stage 0 occurs at time  $t+1$ . This has a probability of  $x_0(t)(1 - \phi_1)$ ;
- (2) alternatively, stage 1 prevails at time  $t$  and a transition from stage 1 to stage 0 occurs at time  $t+1$ . This has a probability of  $x_1(t)\phi_2$ .

The probability of being in stage 1 at time  $t+1$  could be obtained similarly. The probabilities at time  $t+1$  are given by the recurrence relations

$$\begin{aligned}
 x_0(t+1) &= x_0(t)(1 - \phi_1) + x_1(t)\phi_2 \\
 x_1(t+1) &= x_0(t)\phi_1 + x_1(t)(1 - \phi_2),
 \end{aligned}$$

or, in vector-matrix form,

$$x(t+1) = \phi x(t), \tag{E4-2}$$



which is, cf. (3), an unforced or free state equation with the solution

$$x(t) = \phi^t x(0) , \quad (E4-3)$$

according to (5),  $t \in T_d$ . In fact, the related output equation has the form

$$y(t) = H x(t) , \quad (E4-4)$$

where  $H=I$  is the identity matrix; i.e. the states themselves are the output variables. The power  $t$  of the state transition matrix in (E4-3) can easily be calculated. e.g. by the use of the Cayley-Hamilton theorem, and has the form

$$\phi^t = \frac{1}{\phi_1 + \phi_2} \begin{bmatrix} \phi_2 & \phi_2 \\ \phi_1 & \phi_1 \end{bmatrix} + \frac{(1 - \phi_1 - \phi_2)^t}{\phi_1 + \phi_2} \begin{bmatrix} \phi_1 & -\phi_2 \\ -\phi_1 & \phi_2 \end{bmatrix} , \quad (E4-5)$$

provided  $\phi_1 + \phi_2 \neq 0$ . Since  $\lambda_1 = 1$  and  $\lambda_2 = 1 - \phi_1 - \phi_2$  are eigenvalues of  $\phi$ , and taking into consideration the fact that  $x_0(0) = 1 - x_1(0)$ , the final results for the probabilities in (E4-3) are

$$\begin{aligned} x_0(t) &= \frac{\phi_2}{\phi_1 + \phi_2} + (1 - \phi_1 - \phi_2)^t \left[ x_0(0) - \frac{\phi_2}{\phi_1 + \phi_2} \right] \\ x_1(t) &= \frac{\phi_1}{\phi_1 + \phi_2} + (1 - \phi_1 - \phi_2)^t \left[ x_1(0) - \frac{\phi_1}{\phi_1 + \phi_2} \right] . \end{aligned} \quad (E4-6)$$

One question that arises is whether after a sufficiently long period of time the system settles down to a condition of statistical equilibrium in which the stage occupation probabilities are independent of the initial conditions.

If this is so then there is an equilibrium probability distribution  $x^* = [x_0^*, x_1^*]^T$  and, on letting  $t \rightarrow \infty$  in (E4-2),  $x^*$  will clearly satisfy

$$x^* = \phi x^*$$

or

(E4-7)

$$x^*(I - \phi) = 0 ,$$

which will have non-zero solutions if the determinant  $|I - \phi|$  vanishes. With this and with the condition

$$x_0^* + x_1^* = 1$$

in mind one obtains the equilibrium probabilities

$$x_0^* = \frac{\phi_2}{\phi_1 + \phi_2} , \quad x_1^* = \frac{\phi_1}{\phi_1 + \phi_2} , \quad (E4-8)$$

which are indeed independent of the initial condition  $x(0)$ . To gain further interesting insight substitute (E4-8) into (E4-6). The conclusions are left to the reader. The equilibrium probabilities might in fact be obtained by taking limits,  $t \rightarrow \infty$ , in (E4-5) or (E4-6), since  $|\lambda_2| < 1$ . Finally, for the sake of completeness, consider the degenerate cases. It means that if  $\phi_1 = \phi_2 = 0$  then

$$x(t+1) = x(t) = x(0) ,$$

i.e. the system remains forever in its initial state. This follows from (E4-1), (E4-2) and (E4-3). On the other hand, if  $\phi_1 = \phi_2 = 1$  then

$$\begin{aligned} x_0(t+1) &= x_1(t) = x_0(t-1) = \dots \\ x_1(t+1) &= x_0(t) = x_1(t-1) = \dots , \end{aligned}$$

i.e. the system oscillates deterministically between two

stages, and if the initial state is given, the behavior of the system is non-random. See the remarks about equilibrium states in Example 3.

### State Transition Matrix

Now let us turn to the continuous case and consider the solution of the continuous state equation, given by (1). First we deal with the unforced case,  $u(t) = 0$ . For such a case the state equation is the homogeneous vector differential equation

$$\dot{x}(t) = F(t) x(t) . \quad (7)$$

Assume that the solution of (7) is known in the form

$$x(t) = \phi(t,0)x(0) , \quad (8)$$

where again  $x(0)$  is the vector of initial states at  $t_0 = 0$  and  $\phi(t,0)$  is a state transition matrix since it maps the initial state into a state at any later time  $t > 0$ . Obviously,

$$\phi(0,0) = I , \quad (9)$$

i.e. at the initial time the state transition matrix is the identity matrix itself. Taking derivatives of the assumed solution (8), one obtains

$$\dot{x}(t) = \frac{d}{dt} \phi(t,0) x(0) = \dot{\phi}(t,0) x(0) . \quad (10)$$

On the other hand, premultiplying (8) by  $F(t)$  shows that (7) becomes

$$\dot{x}(t) = F(t) \phi(t,0) x(0) ,$$

which combined with (10) gives

$$\dot{\Phi}(t,0) x(0) = F(t) \Phi(t,0) x(0) . \quad (11)$$

Since (11) must hold for all possible initial states it yields

$$\dot{\Phi}(t,0) = F(t) \Phi(t,0) . \quad (12)$$

That is, one concludes that the state transition matrix satisfies a matrix differential equation, given by (12), and the solution is unique.

For time invariant systems  $F(t) = F$ . The state transition matrix depends only on the lag  $t - t_0$ , i.e. in our case on  $t$  only,  $\Phi(t,0) = \Phi(t)$ . For this case the solution of (12) is readily obtained as

$$\Phi(t) = \exp (Ft) , \quad (13)$$

which is known as the matrix exponential.

The state transition matrix has some interesting properties. The first, namely (9), has already been mentioned. For realizing the second property consider the expressions

$$x(t_2) = \Phi(t_2, t_1) x(t_1)$$

$$x(t_1) = \Phi(t_1, t_0) x(t_0)$$

since (8) holds for any initial state. Here, to give a more general flavour we used  $t_0$  to indicate the initial time. That is, by substitution

$$x(t_2) = \Phi(t_2, t_1) \Phi(t_1, t_0) x(t_0) .$$

By definition of the state transition matrix, on the other

hand,

$$x(t_2) = \Phi(t_2, t_0) x(t_0) .$$

Taking into account that  $x(t_0)$  is arbitrary, by combining the above two expressions we have

$$\Phi(t_2, t_0) = \Phi(t_2, t_1) \Phi(t_1, t_0) \quad (14)$$

for any  $t_0$ ,  $t_1$  and  $t_2$ , independently of the order of them. It is clear from (14) that the state transition matrix is never singular

$$|\Phi(t, t_0)| \neq 0$$

and does have an inverse. To examine the third property consider the expressions

$$x(t_2) = \Phi(t_2, t_1) x(t_1)$$

$$x(t_1) = \Phi(t_1, t_2) x(t_2) .$$

Premultiplying the latter by the inverse state transition matrix

$$\Phi^{-1}(t_1, t_2) x(t_1) = x(t_2)$$

and comparing this with the former, we have the relationship

$$\Phi^{-1}(t_1, t_2) = \Phi(t_2, t_1) \quad (15)$$

for any  $t_1$  and  $t_2$  in any order. That is to say to change subscripts the state transition matrix must be inverted. A diagrammatic representation of these properties is shown in Fig. 6. In summary, the state transition matrix has the following properties in continuous case:

	Time varying systems	Time invariant systems
(I)	$\Phi(t_0, t_0) = I$	$\exp [F(t_0 - t_0)] = I$
(II)	$\Phi(t_2, t_0)$ $= \Phi(t_2, t_1)\Phi(t_1, t_0)$	$\exp [F(t_2 - t_0)]$ $= \exp [F(t_2 - t_1)]\exp[F(t_1 - t_0)]$
(III)	$\Phi^{-1}(t_1, t_2) = \Phi(t_2, t_1)$	$\exp [-F(t_1 - t_2)] = \exp[F(t_2 - t_1)]$

Other relationships involving the determinant of the state transition matrix are

$$\frac{d}{dt} |\Phi(t, t_0)| = [\text{tr } F(t)] |\Phi(t, t_0)|$$

and

$$|\Phi(t, t_0)| = \exp \left| \int_{t_0}^t \text{tr } F(\tau) d\tau \right|,$$

where  $\text{tr}$  denotes the trace of the transition matrix, i.e. the sum of its elements along the main diagonal. The proof is simple; for details consult Meditch (1969)

### Solution for Continuous Systems

Now, we can turn to the solution of the state equation of forced linear dynamic systems given by (1). Here the Lagrangean method of variation of constants will be used. In this method a function is to be constructed which, upon multiplication with the homogeneous solution, satisfies the given state equations. It is assumed that the initial condition, say  $c$ , in the solution

$$x(t) = \Phi(t, t_0) c(t) \tag{16}$$

of the homogeneous equation is also a function of time which is to be determined. Derivation of (16) gives

$$\dot{x}(t) = \Phi(t, t_0) \dot{c}(t) + \dot{\Phi}(t, t_0) c(t) .$$

Substitution of (12) into the above expression gives

$$\dot{x}(t) = \Phi(t, t_0) \dot{c}(t) + F(t) \Phi(t, t_0) c(t) \quad (17)$$

On the other hand, by substituting the assumed solution (16) into the original differential equation (1) one obtains

$$\dot{x}(t) = F(t) \Phi(t, t_0) c(t) + G(t) u(t)$$

which, combined with (17), yields

$$\Phi(t, t_0) \dot{c}(t) = G(t) u(t) .$$

Premultiplying this expression by the inverse of the state transition matrix the following differential equation is obtained:

$$\dot{c}(t) = \Phi^{-1}(t, t_0) G(t) u(t) .$$

Integrating over  $[t_0, t]$  and considering that by (8) and (16)  $c(t_0) = x(t_0)$ , the function  $c(t)$  is

$$c(t) = x(t_0) + \int_{t_0}^t \Phi^{-1}(\tau, t_0) G(\tau) u(\tau) d\tau ,$$

which, if substituted into (16), gives the solution of the state equation (1) as

$$x(t) = \Phi(t, t_0) x(t_0) + \int_{t_0}^t \Phi(t, \tau) G(\tau) u(\tau) d\tau , \quad (18)$$

where the relations (14) and (15) respectively have been applied.

According to the output equation (2) the output is then given by

$$y(t) = H(t) \Phi(t, t_0) x(t_0) + \int_{t_0}^t H(t) \Phi(t, \tau) G(\tau) u(\tau) d\tau \quad (19)$$

For time invariant systems, by considering (13) one can immediately obtain the solution for the states as

$$x(t) = e^{F(t-t_0)} x(t_0) + \int_{t_0}^t e^{F(t-\tau)} G u(\tau) d\tau, \quad (20)$$

and for the output as

$$y(t) = H e^{F(t-t_0)} x(t_0) + \int_{t_0}^t H e^{F(t-\tau)} G u(\tau) d\tau. \quad (21)$$

In fact the solutions in both cases can be split into two parts

$$x(t) = \mathcal{F}_1[x(t_0)] + \mathcal{F}_2[\mathcal{U}(t_0, t)] \quad (22)$$

where the first part is the free response which depends only on the initial state and in practice contains all the information about the past of the system, while the second part is the forced response which depends upon the input segment

$\mathcal{U}(t_0, t) = \{u(\tau) : \tau = t_0, t_1, \dots, t\}$ ; and of course both of them depend upon the structure of the system represented by the matrices  $\Phi(\cdot)$  and  $G(\cdot)$ . To emphasize the similarities between the continuous (20) and discrete (5) solutions, a table is presented below for the time invariant system. A similar one can of course be set up for time varying systems. To make the similarities even more apparent, (13) is used.



	Continuous Time	Discrete Time
Free Component	$\Phi(t) x(0)$	$\Phi^t x(0)$
Forced Component	$\int_{t_0}^t \Phi(t-\tau) G u(\tau) d\tau$	$\sum_{\tau=0}^{t-1} \Phi^{t-\tau-1} \Gamma u(\tau)$

Example 5. As an example the Nash model of the rainfall-runoff process is discussed here. Nash (1960) modeled the surface runoff by a series of  $n$  reservoirs each of which has the same storage coefficient  $K$  which is a dimensionless constant. It is assumed that the outflow from one reservoir is proportional by  $k$  to the content of the reservoir in question (Fig. 7). Let the content of the  $i$ th reservoir at time  $t$ ,  $x_i(t)$ , be the  $i$ th state variable. Then by continuity, the state equation is

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} -k & & & 0 \\ k & -k & & \\ & k & -k & \\ & & & 0 \\ & & & k & -k \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(t) \quad (E5-1)$$

or, in vector-matrix form,

$$\dot{x}(t) = F x(t) + G u(t) . \quad (E5-2)$$

Since the outflow from the last reservoir is the output of the system, the output equation becomes

$$y(t) = H x(t) , \quad (E5-3)$$

where  $H = [0, 0, \dots, k]$ . If the system is initially relaxed and the input is a Dirac function,  $u(t) = \delta(t)$ , then the output is the impulse response of the system,  $y(t) = h(t)$ , which is in fact the instantaneous unit hydrograph. Considering that the input is an impulse function, (E5-1) can be solved successively instead of by the general solution (20). Thus

$$\begin{aligned} \dot{x}_1(t) + kx_1(t) &= \delta(t) & \rightarrow & \quad x_1(t) = e^{-kt} \\ \dot{x}_2(t) + kx_2(t) &= k e^{-kt} & \rightarrow & \quad x_2(t) = kt e^{-kt} \\ & & & \quad \vdots \\ & & & \quad \vdots \\ & & & \quad x_i(t) = \frac{(kt)^{i-1}}{(i-1)!} e^{-kt} \\ & & & \quad \vdots \\ & & & \quad \vdots \end{aligned}$$

That is, according to (E5-3), the impulse response is

$$h(t) = kx_n(t) = k \frac{(kt)^{n-1}}{(n-1)!} e^{-kt}, \quad (\text{E5-4})$$

which, by letting  $k = 1/K$ , gives the well-known Nash IUH

$$h(t) = \frac{1}{K} \left( \frac{t}{K} \right)^{n-1} \frac{1}{(n-1)!} e^{-\frac{t}{K}}.$$

It is interesting to note that the same expression for the use of successive routing through a characteristic reach for channel routing has been derived by Kalinin and Milyukov (1957). Their procedure is based upon the linearization of the unsteady flow equation. The similarities with Muzik's approach (1974) are apparent.

### Impulse Response Matrix

When the system is initially relaxed,  $x(t_0) = 0$ , then the impulse response matrix of the system is given by

$$\mathcal{H}(t, \tau) = H(t) \Phi(t, \tau) G(\tau) \quad (23)$$

since (18) becomes

$$y(t) = \int_{t_0}^t \mathcal{H}(t, \tau) u(\tau) d\tau . \quad (24)$$

The name of the impulse response matrix derives from the fact that each element  $h_{ij}(t, \tau)$  of  $\mathcal{H}(t, \tau)$  is the response of the  $i$ th component of  $y(t)$  for a unit impulse input in the  $j$ th component of  $u(\cdot)$  applied at time  $\tau$ . The use of the system impulse matrix is convenient when one wishes only an input-output relation and is not concerned with the system's state variables.

For time invariant systems  $\mathcal{H}(t, \tau) = \mathcal{H}(t - \tau)$ , i.e.

$$y(t) = \int_{t_0}^t \mathcal{H}(t - \tau) u(\tau) d\tau , \quad (25)$$

which is the well-known convolution. It is clear from (21) that  $\mathcal{H}(t - \tau) = H \exp F(t - \tau) G$ . Equations (24) and (25) give the external description of a linear dynamic system. This means that the input-output behavior is described by a Volterra integral equation.

### Discrete Formulation

In practice the aforementioned procedures are applied mainly for digital computers that work in a discrete environment. Care must be exercised, however, when one wishes to set up a

discrete linear model for a system which has continuous linear dynamics in reality.

Let a discrete time set  $T_d = \{t: t=t_0, t_0+1, \dots, t_k, t_{k+1}, \dots\}$  be given and consider the time interval  $t_k \leq t \leq t_{k+1}$  for some  $k=0, 1, \dots$ . We assume that  $x(t_k)$  is given and  $u(t) = u(t_k)$  is constant for  $t_k \leq t \leq t_{k+1}$ . Then it follows from (18) that

$$x(t_{k+1}) = \phi(t_{k+1}, t_k)x(t_k) + \left[ \int_{t_k}^{t_{k+1}} \phi(t_{k+1}, \tau) G(\tau) d\tau \right] u(t_k) \quad (26)$$

Defining

$$x(t+1) = x(t_{k+1})$$

$$x(t) = x(t_k)$$

$$\phi(t+1, t) = \phi(t_{k+1}, t_k)$$

and

$$\Gamma(t) = \int_{t_k}^{t_{k+1}} \phi(t_{k+1}, \tau) G(\tau) d\tau, \quad (27)$$

we can write (26) as

$$x(t+1) = \phi(t+1, t) x(t) + \Gamma(t) u(t), \quad (28)$$

for  $t = 0, 1, \dots$ , which is identical to (3). It is important to emphasize that the above discrete system is described from a continuous system, therefore the invertibility of the state transition matrix is always assured and computational difficulties do not arise. The output equation is the same as in the pure discrete case.

**Example 6.** Here the discrete state space formulation of the continuous Streeter-Phelps model is discussed.

Engineers used this model for many years to describe the changes in water quality of a river. Although there are much more sophisticated techniques that have been developed in the past few years for water quality modeling, the Streeter-Phelps model is still in use, mainly because of its simplicity. In many cases it still gives meaningful answers and elegant results. To prove this the reader is referred to the extremely rich literature on this subject (Koivo and Phillips, 1971; Young and Beck, 1974; Singh, 1975; Szöllösi-Nagy, 1975; Gourishankar and Lawson, 1975).

The model assumes that the water quality of the river can be characterized by the dynamic interrelationship between the biochemical oxygen demand (BOD) and the dissolved oxygen (DO). Further, it assumes a first order reaction kinetic for the BOD

$$\frac{d B(t)}{dt} = - K_R B(t) , \quad (E6-1)$$

where  $B(t)$  is the BOD concentration in  $\text{mg}/\ell$  and  $K_R$  is the BOD removal or decay coefficient in  $\text{day}^{-1}$ . By continuity

$$\frac{d D(t)}{dt} = - K_a D(t) - K_R B(t) + K_a D_S , \quad (E6-2)$$

where  $D(t)$  is the DO concentration in  $[\text{mg}/\ell]$ ,  $K_a$  is the re-aeration coefficient in  $[\text{day}^{-1}]$ , and  $D_S$  is the saturation level of the dissolved oxygen. Defining the state variables as  $x_1(t) = B(t)$  and  $x_2(t) = D(t) - D_S$  respectively, the latter being known as oxygen deficit and having direct physical meaning, the state equation for the Streeter-Phelps model is

$$\dot{x}(t) = F x(t) , \quad (E6-3)$$

where

$$F = \begin{bmatrix} -K_r & 0 \\ -K_r & -K_a \end{bmatrix}$$

is considered to be constant. As a matter of fact one of the objectives of setting up a water quality model is to control the water quality itself in order to achieve a desirable level of quality. The water quality of a river might for example be controlled by, amongst other things, treatment plants and artificial aeration facilities located along the river. We define the control vector as  $u(t) = [u_1(t), u_2(t)]^T$ , where  $u_1(t)$  is for control of effluent dumping from the sewage treatment plant and  $u_2(t)$  is for artificial aeration carried out. The first control might mean, say, the operation rule for a retention reservoir situated right after the treatment plant; the second control is the timing schedule for the aeration facilities. So, considering (E6-3), the process model becomes

$$\dot{x}(t) = F x(t) + G u(t) , \quad (E6-4)$$

where

$$G = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

The minus sign refers to the fact that the more the artificial aeration the less the oxygen deficit, and vice versa. And now we are ready to derive a discrete model for the continuous process given by (E6-4). According to (13) the state transition between two discrete time epochs  $t$  and  $t+1$ , respectively, is characterized by the

$$\phi(t+1, t) = \exp F \quad (E6-5)$$

matrix exponential. Since the eigenvalues of  $F$  are

negatives,  $\lambda_1 = -K_r$  and  $\lambda_2 = -K_a$ , the system (E6-4) is stable. Using the well-known Sylvester expansion theorem, the one-step state transition matrix is obtained as

$$\Phi(t+1,t) = \begin{bmatrix} \exp(-K_r) & \vdots & 0 \\ \dots & \vdots & \dots \\ \frac{-K_r}{K_a - K_r} [\exp(-K_r) - \exp(-K_a)] & \vdots & \exp(-K_a) \end{bmatrix}, \quad (E6-6)$$

provided that  $K_a \neq K_r$ . As for the determination of the control transition matrix  $\Gamma(t)$ , (27) is evaluated and, due to the special structure of  $G$ , gives the same form as (E6-6) except that the matrix element in the lower right hand corner is negative. In fact both the state and control transition matrices are time invariant and hence the discrete state equation of the continuous process is

$$x(t+1) = \Phi x(t) + \Gamma u(t) . \quad (E6-7)$$

As far as the output of the system is concerned, the situation is that the evaluation of BOD concentration usually needs several days in a laboratory and to determine real-time control policies DO measurements are available only. That is

$$y(t) = H x(t) , \quad (E6-8)$$

where  $H = [0,1]$ . The system thus is specified by the triplet  $(\Phi, \Gamma, H)$ , which will be denoted as  $\Sigma = (\Phi, \Gamma, H)$ . The dynamics of this water quality control system is shown in Fig. 8.

Example 7. Here we determine the impulse responses of the water quality system discussed above. Let us assume that the system at  $t_0 = 0$  is initially relaxed, i.e.  $x(0) = 0$  (or it is transformed into an initially relaxed system by the transformation  $x'(0) = x(0) - x^*$ , where  $x^*$  is some equilibrium state). Then the state transition matrix is

$$\Phi(\tau, 0) = \begin{bmatrix} e^{-K_r \tau} & 0 \\ -\frac{K_r}{K_a - K_r} \left( e^{-K_r \tau} - e^{-K_a \tau} \right) & e^{-K_a \tau} \end{bmatrix} .$$

According to (23) the impulse response vector for the water quality system is

$$\mathcal{H}(\tau) = H \Phi(\tau, 0) G = \begin{bmatrix} -\frac{K_r}{K_a - K_r} \left( e^{-K_r \tau} - e^{-K_a \tau} \right), -e^{-K_a \tau} \end{bmatrix} ,$$

and the output is given by (25) as

$$y(t) = \int_0^t \begin{bmatrix} -\frac{K_r}{K_a - K_r} \left( e^{-K_r(t-\tau)} - e^{-K_a(t-\tau)} \right), \\ -e^{-K_a(t-\tau)} \end{bmatrix} \begin{bmatrix} u_1(\tau) \\ u_2(\tau) \end{bmatrix} d\tau .$$

Subsequently we deal with systems which are either discrete by nature or have been transformed from the continuous description into a discrete one.



### 3. STRUCTURAL PROPERTIES FOR LINEAR DISCRETE SYSTEMS

In this section we consider two fundamental concepts of linear system theory which are intimately related to the basic ideas of estimation and control. These notions, termed observability and controllability are due to Kalman (1961). If these conditions are not met, optimal control, estimation, and identification cannot be obtained.

Previously we have seen that many systems have only one output variable even though many state variables might be necessary to describe the internal behavior of the systems. Similarly, there may be only one or a few inputs to the system and there might be states which are not affected by the input(s). The pertinent questions are:

1. Under what conditions is it possible to establish, in a finite interval of time, the time history of the state of a dynamic system given the time history of the output over the same interval?
2. Under what conditions is it possible to transfer the state of a dynamic system from a given state to a desired state in a finite interval of time using a control  $u$ ?

To illustrate the problems related to these questions consider the dynamic system  $\Sigma$  shown in Fig. 9 (Meditch, 1969), where  $d$  is a vector whose components consists of some or all of the elements  $x_1, \dots, x_k$ . Because of the system's structure there is no way that the values  $x_{k+1}, \dots, x_n$  can be determined from the output  $y$ , since these variables do not affect  $x_1, \dots, x_k$  nor do they appear in  $y$ . Such a system is said to be unobservable. On the other hand, since  $u$  affects all of the elements of  $x$  the system is controllable. By changing the direction of

the vector  $d$ , as shown in Fig. 10, an observable-uncontrollable system is obtained since  $u$  affects only the variables  $x_1, \dots, x_k$  but all of the state variables are observable. As Kalman (1962) showed in his canonical structure theorem any linear dynamic system can be decomposed into four subsystems: (1) a controllable and observable subsystem; (2) a subsystem which is controllable but not observable; (3) a subsystem which is not controllable but observable; and finally, (4) a subsystem which is neither controllable nor observable. This decomposition is shown in Fig. 11. Note, that in Figs. 9, 10, and 11 the subscripts  $c, \bar{c}, o, \bar{o}$  stand, respectively, for controllable, uncontrollable, observable and unobservable.

### Observability

Consider the discrete linear dynamic systems given either by (2-3) and (2-4) or by (2-28)

$$x(t+1) = \Phi(t+1, t)x(t) + \Gamma(t)u(t) \quad (1)$$

$$y(t) = H(t)x(t) \quad , \quad (2)$$

where  $t=0, 1, \dots$ . We assume that the input sequence  $\{u(0), u(1), \dots\}$  is given but  $x(0)$  is unknown. Now, the problem is, as posed by question 1, that we wish to determine  $x(t)$  from an examination of  $y(t)$  over some finite interval of time. Obviously if  $H(t)$  is  $n \times n$  and nonsingular for all  $t \geq 0$ , then

$$x(t) = H^{-1}(t)y(t)$$

and the question of observability is resolved trivially. The same can be inferred if  $H(t)$  is  $n \times n$  but nonsingular for only one value of  $t \geq 0$ , say  $t_i$ .

The real problems arise when either  $H(t)$  is  $n \times n$  but singular for all  $t \geq 0$  or  $H(t)$  is  $m \times n$ ,  $m \neq n$ . For these cases it is not at all clear how  $x(t)$  can be determined from  $y(\tau)$ ,  $0 \leq \tau \leq N$ , for some finite  $N$ .

With this in mind we define observability in the following way.

The discrete linear dynamic system given by (1) and (2) is observable if  $x(0)$  can be determined from the set of outputs  $y(1), \dots, \{y(1), \dots, y(N)\}$  for some finite  $N$ . If this is true for any initial time, the system is said to be completely observable.

Since  $u(t)$  is assumed known for all  $t \geq 0$ , its contribution to  $x(t)$ , which is given by the second terms of either (2-6) or (2-26), is easily determined. Hence, it is sufficient to consider only the unforced system

$$x(t+1) = \Phi(t+1, t)x(t) \tag{3}$$

$$y(t) = H(t)x(t) \tag{4}$$

Consider now the sequence of outputs  $\{y(t), \dots, y(N)\}$  beginning with  $t=1$ . From (3) and (4) we have, by recursive substitutions

$$y(1) = H(1)x(1) = H(1)\Phi(1, 0)x(0)$$

$$y(2) = H(2)x(2) = H(2)\Phi(2, 1)x(1) = H(2)\Phi(2, 1)\Phi(1, 0)x(0)$$

$\vdots$

$$y(N) = H(N)x(N) = H(N)\Phi(N, N-1)x(N-1) = H(N)\Phi(N, N-1)\dots\Phi(1, 0)x(0) \tag{5}$$

Defining

$$y_N = \begin{bmatrix} y(1) \\ \vdots \\ y(N) \end{bmatrix}$$

and

$$\Phi(i,0) = \Phi(i,i-1)\dots\Phi(1,0)$$

for  $i=1,\dots,N$  , it is clear that  $y_N$  is an  $mN$  vector. Letting

$$H_N = \begin{bmatrix} H(1)\Phi(1,0) \\ \vdots \\ H(N)\Phi(N,0) \end{bmatrix} , \quad (6)$$

which is an  $mN \times n$  matrix, we obtain for (5) that

$$y_N = H_N x(0) . \quad (7)$$

Premultiplying this relation by  $H_N^T$  we get

$$H_N^T H_N x(0) = H_N^T y_N . \quad (8)$$

From the definition (6) of  $H_N$  we can realize that

$$H_N^T H_N = \sum_{i=1}^N \Phi^T(i,0) H^T(i) H(i) \Phi(i,0) , \quad (9)$$

which is an  $n \times n$  symmetric matrix and is denoted by  $\mathcal{O}(0,N)$  and called as observability matrix. Then it follows from (8) that

$$x(0) = \mathcal{O}^{-1}(0,N) H_N^T y_N , \quad (10)$$

which shows that the system is completely observable if  $\mathcal{O}(0,N)$  is positive definite for some  $N > 0$ . This is the sufficient condition for observability. It can be easily shown (Desoer, 1970) that it is a necessary condition at the same time.

Criterion for observability for time invariant systems can be established along the same lines. That is, consider the unforced system

$$x(t+1) = \Phi x(t) \quad (11)$$

$$y(t) = Hx(t) \quad (12)$$

and a sequence of outputs  $\{y(0), \dots, y(n-1)\}$ . Again, we have

$$\begin{aligned} y(0) &= Hx(0) \\ y(1) &= Hx(1) = H\phi x(0) \\ &\vdots \\ y(n-1) &= Hx(n-1) = H\phi^{n-1}x(0) \end{aligned} .$$

By similar definitions as before

$$Y_n = \begin{bmatrix} H \\ H\phi \\ \vdots \\ H\phi^{n-1} \end{bmatrix} x(0) = \mathcal{O}^T x(0) \quad . \quad (13)$$

If  $x(0)$  is to be determined uniquely the matrix  $\mathcal{O}^T$  (or equivalently  $\mathcal{O}$ ) must have an inverse, i.e., be nonsingular. This statement is the same as to require that the  $n \times mn$  observability matrix  $\mathcal{O}$

$$\mathcal{O} = [H^T \quad \vdots \quad \phi^T H^T \quad \vdots \quad \dots \quad \vdots \quad (\phi^T)^{n-1} H^T] \quad (14)$$

be of rank  $n$ ,  $\rho(\mathcal{O})=n$ , since the rank of any matrix is the order of the largest square array in the matrix, formed by deleting rows and/or columns, that is nonsingular.

**E x a m p l e 1.** Here we examine whether the rainfall system in Example 2.4 is observable. Since  $H$  in (2-E 4-4) is equal to the identity matrix  $H^T=I$  and consequently  $\phi^T H^T = \phi^T$ . That is the observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 1-\phi_1 & \phi_2 \\ 0 & 1 & \phi_1 & 1-\phi_2 \end{bmatrix}$$

which has a rank of 2 so the system is observable.

Example 2. Can we say the same about the water quality system discussed in Example 2.6? For notational simplicity let (2-E6-6) be

$$\Phi = \begin{bmatrix} \phi_{11} & 0 \\ \phi_{21} & \phi_{22} \end{bmatrix} .$$

Since

$$H^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix} ,$$

the observability matrix (14) for this case,  $n=2$ , becomes

$$\mathcal{O} = \begin{bmatrix} 0 & \phi_{21} \\ 1 & \phi_{22} \end{bmatrix} ,$$

which has a rank of 2, or is invertible, only if  $\phi_{21} \neq 0$ , i.e. if

$$\frac{-K_r}{K_a - K_r} (e^{-K_r} - e^{-K_a}) \neq 0 .$$

First, consider the case when  $K_r \neq K_a$ . Obviously

$$0 < \left| \frac{1}{K_a - K_r} \right| < C_1 < \infty$$

and

$$0 < \left| e^{-K_r} - e^{-K_a} \right| < C_2 < \infty .$$

Consequently

$$0 < \left| \frac{1}{K_a - K_r} \right| \left| e^{-K_r} - e^{-K_a} \right| < C_1 C_2 < \infty ,$$

therefore if  $K_a \neq K_r$ , then  $\phi_{21} \neq 0$ . Now consider the possibility that  $K_a = K_r$ . Then

$$\frac{-K_r e^{-K_r} \left[ e^{-(K_a - K_r)} - 1 \right]}{K_a - K_r} = \frac{0}{0}$$

which is an indeterminate form. Thus let  $K_a - K_r = K$  and consider

$$\lim_{K \rightarrow 0} K_r e^{-K_r} \left[ \frac{1 - e^K}{K} \right]$$

for which the L'Hospital' rule is applied giving

$$\lim_{K \rightarrow 0} K_r e^{-K_r} \left[ \frac{-Ke^K}{1} \right] = -K_r e^{-K_r} \neq 0 .$$

Thus if  $K_r$  and  $K_a$  are non-zero and bounded the observability matrix is nonsingular. Consequently, the system is completely observable. To gain more insight to the notion of observability let us make a change in the water quality system, namely assume that only BOD data are available for control. Then for this new system with structure  $\Sigma_*$  the output matrix is  $H_* = [1, 0]$  and the observability matrix becomes

$$O_* = \begin{bmatrix} 1 & \phi_{11} \\ 0 & 0 \end{bmatrix}$$

which is of rank one, i.e. the system  $\Sigma_*$  is unobservable. In fact by such a structural change Fig. 8 becomes similar to Fig. 9.

Remark. The relation between observability and estimation should be clear at this point. In fact (10) is an algorithm for determining  $x(0)$  from available output data. Along with (1) the determination of the states' history becomes possible, in other words, we have solved the estimation problem under ideal conditions, that is, for deterministic systems.

### Controllability

To establish the criterion of controllability we proceed analogously to that of observability. We consider the discrete linear system

$$x(t+1) = \Phi(t+1,t)x(t) + \Gamma(t)u(t) \quad (15)$$

for  $t=0,1,\dots$ , where  $x(0)$  is known but  $\{u(0),u(1),\dots\}$  is not specified. We concern ourselves here with the problem of transferring the state of the system (15) from  $x(0)$  to some desired terminal state  $x(N)$  where  $N$  is finite.

We define controllability in the following way:

The discrete linear dynamic system of (15) is controllable at time  $t=0$  if there exists a control sequence  $\{u(0),u(1),\dots,u(N-1)\}$  such that the state  $x(0)$  can be driven to any arbitrary state  $x(N)$  where  $N$  is finite. If this is true for any  $x(0)$  and initial times the system is said to be completely controllable.

Following similar steps as in the observability analysis, the criterion of complete controllability for time varying systems that is that the  $n \times n$  controllability matrix

$$\mathcal{C}(0,N) = \sum_{i=1}^N \Phi(0,i)\Gamma(i-1)\Gamma^T(i-1)\Phi^T(0,i) \quad (16)$$

is positive definite for some finite  $N > 0$ , where

$$\Phi(0,i) = \Phi(0,1) \cdots \Phi(i-1,i) \quad , \quad i=1,1,\dots,N.$$



For time invariant systems consider

$$x(t+1) = \phi x(t) + \Gamma u(t) \quad , \quad (17)$$

with known initial state  $x(0)$ . Again, the question is that under what conditions can we determine the control necessary to drive the system to  $x(n)$ , where  $x(n)$  is arbitrary in the state space? As in the observability study we may apply recursive substitutions, yielding (c.f.: 2-E3-3)

$$\begin{aligned} x(1) &= \phi x(0) + \Gamma u(0) \\ x(2) &= \phi x(1) + \Gamma u(1) = \phi^2 x(0) + \phi \Gamma x(0) + \Gamma u(1) \\ &\vdots \\ x(n) &= \phi^n x(0) + \phi^{n-1} \Gamma u(0) + \dots + \Gamma u(n-1) \quad . \end{aligned}$$

Therefore

$$\begin{aligned} x(n) - \phi^n x(0) &= \phi^{n-1} \Gamma u(0) + \phi^{n-2} \Gamma u(1) + \dots + \Gamma u(n-1) \\ &= [\Gamma \vdots \phi \Gamma \vdots \dots \vdots \phi^{n-1} \Gamma] \begin{bmatrix} u(n-1) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix} \quad . \end{aligned}$$

Since  $x(n)$  and  $x(0)$  are given the condition for a unique solution for the control sequence to exist is that the  $n \times n$  controllability matrix  $\mathcal{C}$

$$\mathcal{C} = [\Gamma \vdots \phi \Gamma \vdots \dots \vdots \phi^{n-1} \Gamma] \quad (18)$$

has rank  $n$ ,  $\rho(\mathcal{C}) = n$ .

**Example 3.** Here we analyze whether the water quality system discussed in Example 2.6 is controllable or not.

The controllability matrix (18) for the system is

$$\mathcal{C} = \begin{bmatrix} \dots & e^{-K_r} & \dots & \vdots & 0 & \dots & e^{-2K_r} & \dots & \vdots & 0 \\ \dots & \dots & \dots & \vdots & \dots & \dots & \dots & \dots & \vdots & \dots \\ \frac{-K_r}{K_a - K_r} \left( e^{-K_r} - e^{-K_a} \right) & \dots & \dots & \vdots & -e^{-K_a} & \dots & \frac{-K_r}{K_a - K_r} \left( e^{-2K_r} - e^{-2K_a} \right) & \dots & \vdots & -e^{-2K_a} \end{bmatrix}$$

Following similar steps as in Example 2 it can be seen that the above matrix is of rank 2, consequently the system is controllable. Again, let us make a change in the water quality system of Fig. 9 and assume that only the dissolved oxygen is controlled. Then for this new system with structure  $\Sigma_{**}$  the control transition matrix becomes

$$\Gamma_{**} = \begin{bmatrix} 0 & 0 \\ 0 & -\phi_{22} \end{bmatrix},$$

i.e., the controllability matrix is

$$\mathcal{C}_{**} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\phi_{22} & 0 & -\phi_{22}^2 \end{bmatrix},$$

which is of rank one, i.e., the system  $\Sigma_{**}$  is uncontrollable. By such a structural change Fig. 8 becomes similar to Fig. 10. To make the analysis complete consider the situation when only the biochemical oxygen demand is controlled. Then for this system with structure  $\Sigma_{***}$  the control transition matrix becomes

$$\Gamma_{***} = \begin{bmatrix} \phi_{11} & 0 \\ \phi_{21} & 0 \end{bmatrix}$$

and the controllability matrix is

$$\mathcal{C}_{***} = \begin{bmatrix} \phi_{11} & 0 & \phi_{11}^2 & 0 \\ \phi_{21} & 0 & \phi_{21}(\phi_{11} + \phi_{22}) & 0 \end{bmatrix},$$

which is, again taking similar steps as in Example 2, of rank 2, i.e., this system is controllable. The conclusions of this example are: controlling the dissolved oxygen only the water quality system becomes uncontrollable while controlling the biochemical oxygen demand, alone or together with the DO, the system is controllable.

Remark. Similarities between observability and controllability are extremely interesting. For example, to derive the condition (16) for controllability it is enough to make the following changes in the observability condition (9):

Observable  $\longrightarrow$  Controllable

$$\mathcal{O}(0,N) \longrightarrow \mathcal{C}(0,N)$$

$$\Phi(i,0) \longrightarrow \Phi^T(0,i)$$

$$H(i) \longrightarrow \Gamma^T(i-1) \quad .$$

This property was first observed by Kalman (1961), who termed it duality. Thus observability and controllability are dual properties of linear dynamic systems.

### Identifiability

Consider a time invariant free system given by

$$x(t+1) = \Phi x(t) \quad , \quad x(0) \text{ is known.} \quad (19)$$

As Lee (1964) defines it, a system is said to be identifiable if it is possible to determine  $\Phi$  from the time history of the state variables. Again, by recursive substitutions

$$x(1) = \Phi x(0)$$

$$x(2) = \Phi x(1) = \Phi^2 x(0)$$

$$\vdots$$

$$x(n) = \Phi x(n-1) = \Phi^{n-1} x(0) \quad .$$

Since all the state variables are available, we can set up a matrix

$$\begin{aligned} [x(1) \vdots x(2) \vdots \cdots \vdots x(n)] &= [\Phi x(0) \vdots \Phi x(1) \vdots \cdots \vdots \Phi x(n-1)] \\ &= \Phi [x(0) \vdots x(1) \vdots \cdots \vdots x(n-1)] \end{aligned}$$

If  $\Phi$  is to be determined uniquely the matrix

$$[x(0) \vdots x(1) \vdots \cdots \vdots x(n-1)]$$

must be nonsingular. This statement is the same as to require that the identifiability matrix  $\mathcal{I}$

$$\mathcal{I} = [x(0) \vdots \Phi x(0) \vdots \cdots \vdots \Phi^{n-1} x(0)] \quad (20)$$

be of rank  $n$ ,  $\rho(\mathcal{I}) = n$ . Physically, it means that  $x(0)$  must excite all modes of the system.

**Example 4.** Consider again the second order water quality system discussed in Example 2.6. The identifiability matrix is

$$\mathcal{I} = [x(0) \vdots \Phi x(0)] = \begin{bmatrix} x_1(0) & \phi_{11} x_1(0) \\ x_2(0) & \phi_{21} x_1(0) + \phi_{22} x_2(0) \end{bmatrix}$$

The system is identifiable if  $\rho(\mathcal{I}) = 2$ ; unidentifiable, if the determinant of this matrix equals zero. This is the case if both columns of the matrix are linearly dependent. We can distinguish (1) a trivial case:  $x_1(0) = x_2(0) = 0$ , that is the relaxed system cannot be identified; and (2) a non-trivial case

$$\lambda \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} \phi_{11} & 0 \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

or

$$[\Phi - \lambda I] \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = 0 \quad ,$$

from which the eigenvalues  $\lambda_1$  and  $\lambda_2$  and the corresponding eigenvectors  $e_1$  and  $e_2$  can be obtained. If  $x(0) = \lambda_1 e_1$  then only one mode  $\exp(\lambda_1 t)$  of the process is excited by  $x(0)$  and the mode  $\exp(\lambda_2 t)$  is not identifiable. If  $x(0) = \lambda_2 e_2$ , then only the mode  $\exp(\lambda_2 t)$  can be identified. Consequently, the process is identifiable only if all modes of the process are excited by  $x(0)$ . For further details see Lee (1964).

### Minimal Realizations

We have seen in Section 2 that the input-output description of a linear dynamic system, (2-24) and (2-25), can be derived from its state-space representation. One might ask whether it is possible to construct a dynamic system in state-space form such that it generates the same input-output pairs as the dynamic system in input-output form. This is the problem realization. It can be shown (Willems and Mitter, 1971) that every input-output dynamic system has a state-space realization.

It is clear that the input-output system (2-24) is realized by a state space model if and only if its impulse response matrix has the form of (c.f.: (2-23))

$$\mathcal{H}(t, \tau) = H(t)\Phi(t, \tau)G(\tau) \quad ,$$

for all  $t > \tau$ . Furthermore, the input-output system has a finite dimensional linear realization if and only if the impulse response matrix has the separable form

$$\mathcal{H}(t, \tau) = \psi_1(t)\psi_2(\tau) \quad .$$

For time invariant systems this is obviously true since (2-25) can be written as

$$y(t) = \int_0^t H e^{F(t-\tau)} G u(\tau) d\tau ,$$

where the impulse response matrix

$$\mathcal{H}(t, \tau) = H e^{F(t-\tau)} G = (H e^{Ft}) (e^{-F\tau} G) = \psi_1(t) \psi_2(\tau)$$

is separable. The sufficiency can also be proven easily. For time varying systems we refer to Casti (1976).

Given a record of input-output pairs (an external description of the system) the realization  $(F, G, H)$  that can produce this record is not unique in the sense that many different sets of  $(F(\text{or } \Phi), G(\text{or } \Gamma), H)$  can give the same input-output behavior. The choice of a particular  $(F(\text{or } \Phi), G(\text{or } \Gamma), H)$  corresponds to the choice of a coordinate system. This choice can have considerable impact in numerical analyses as well as affecting system observability and controllability.

Assume that a state space model  $\Sigma = (\Phi, \Gamma, H)$  is a realization of the input-output system. Then it is said to be a minimal realization of the input-output system if every other realization of finite dimensional linear type has a state space of greater or equal dimension.

We cannot hope to identify states that are unobservable and for all practical purposes there is no point in specifying more states than can be controlled. Therefore the minimal realizations have the following properties (Kalman et al, 1969):

1. All minimal realizations of  $\Sigma$  are equivalent.
2. Any minimal realization of  $\Sigma$  is completely controllable and completely observable.
3. If a minimal realization of  $\Sigma$  is completely controllable and completely observable, it is a minimal realization.

A minimal realization of a system  $\Sigma$  is thus a subsystem of  $\Sigma$  having the same dimension as  $\Sigma$  only when  $\Sigma$  itself is minimal. The above properties of minimal realization can be stated formally (see Desoer, 1970) as: A realization  $(\Phi, \Gamma, H)$  is minimal if and only if the system

$$x(t+1) = \Phi x(t) + \Gamma u(t) \quad (21)$$

$$y(t) = H x(t) \quad (22)$$

is completely observable and completely controllable. Moreover, a minimal realization always exists and any two minimal realizations  $(\Phi_1, \Gamma_1, H_1)$  and  $(\Phi_2, \Gamma_2, H_2)$  are related via the similarity transformation

$$\Phi_2 = T \Phi_1 T^{-1} \quad ,$$

$$\Gamma_2 = T \Gamma_1 \quad ,$$

$$H_2 = H_1 T^{-1}$$

for some nonsingular matrix  $T$ . So, it turns out that the minimality of realizations is intimately related to the concept of controllability and observability, which is somewhat surprising since there is no a priori reason why this should be the case.

**E x a m p l e 5.** The water quality control system  $\Sigma$  discussed in Example 2.6 is a minimal realization of the processes involved. Obviously, the realizations  $\Sigma_*$  and  $\Sigma_{**}$  discussed in Examples 2 and 3, respectively, are not minimal.

Equations (21) and (22) can be written as

$$x(t+1) = [\Phi, \Gamma] \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$

$$y(t) = H x(t) \quad ,$$

and assuming that all the state variables are also output variables, i.e.,  $H = I$ , we have

$$y(t+1) = [\Phi, \Gamma] \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}, \quad (23)$$

which is a direct relation between the input and output, it does not have the state vector  $x(t)$  appearing explicitly. Using (23),  $[\Phi, \Gamma]$  can be obtained from the relation

$$\sum_{t=1}^N [y(t+1) \cdots y(t+n+p)] = [\Phi, \Gamma] \sum_{t=1}^N \begin{bmatrix} y(t) \cdots y(t+n+p-1) \\ u(t) \cdots u(t+n+p-1) \end{bmatrix}, \quad (24)$$

provided that the matrix multiplying  $[\Phi, \Gamma]$  is nonsingular. In this case these are a unique solution for  $[\Phi, \Gamma]$ .

For observable and controllable systems with output matrix  $H$  not equal to identity matrix  $H_0$  and Kalman (1966) constructed algorithms to obtain minimal realization using Hankel matrices formed from the impulse response matrices. This question is far beyond the scope of this paper, so the reader is referred to the literature (Kalman et al, 1969; Desoer, 1970; Budin, 1971; Casti, 1976). An effective minimal realization algorithm can be found in Silverman (1971).



#### 4. DISCRETE LINEAR STOCHASTIC DYNAMIC SYSTEMS

##### The Notion of State and the Derivation of State Equations

Up until now we dealt with strict deterministic systems where there are no uncertainties of any kind. Unfortunately, in practical water resources application this is certainly not the case, since, as Yevjevich (1974) states, it is extremely difficult to find a pure deterministic hydrologic process in nature. In this section we redefine the previous concepts of dynamic systems and give some insight into the behavior of systems in a random environment. The discussion will be restricted to discrete systems. For continuous stochastic systems the reader is referred to Fleming and Rishel (1975).

As Åström (1970) indicates for stochastic systems we naturally cannot require, as we did for deterministic systems in Section 1, that the future behavior be uniquely determined by the actual state  $x$ . A natural extension of the notion of state to stochastic systems would be to require that the probability distribution of the state  $x$  at future time should be uniquely determined by the actual value of the state. This means that, we require that the system be described as a Markov process. In other words, we assume that  $x(t+1)$  is not uniquely given by  $x(t)$  and  $u(t)$  as expressed by (1-3), but that  $x(t+1)$  is a random vector which also depends on a random variable  $w(t)$ , i.e.,

$$x(t+1) = \underset{\sim}{f}_t[x(t), u(t), w(t)] \quad , \quad t \in T_d \quad , \quad (1)$$

where  $\underset{\sim}{f}_t$  now is the conditional expectation of  $x(t+1)$  given  $x(t), u(t)$ , and  $w(t)$ . It is assumed here that  $w(t)$  has zero mean. The above equation is called a stochastic difference equation.

For stochastic linear systems, by analogy with (2-3), the state equation becomes

$$x(t+1) = \Phi(t+1,t)x(t) + \Gamma(t)u(t) + w(t), \quad (2)$$

where, beyond the already known notations,  $w(t)$  is a vector of white gaussian noise (WGN) sequences, with zero mean

$$\mathcal{E}\{w(t)\} = 0 \quad (3)$$

and covariance matrix

$$\mathcal{E}\{w(\tau)w^T(t)\} = Q(t)\delta_{\tau t}, \quad (4)$$

where  $\delta_{\tau t}$  is the Kronecker delta and  $Q(t)$  is a positive semi-definite matrix. Since the state itself is a random variable the initial state is given by its mean

$$\mathcal{E}\{x(0)\} = \hat{x}(0) \quad (5)$$

and covariance matrix

$$\mathcal{E}\{[x(0) - \hat{x}(0)][x(0) - \hat{x}(0)]^T\} = P(0), \quad (6)$$

where  $P(0)$  is a positive semidefinite  $n \times n$  matrix. The noise process is called process disturbance, or sometimes model uncertainty, and is assumed to be independent of  $x(0)$ , so that

$$\mathcal{E}\{[x(0) - \hat{x}(0)]w^T(t)\} = 0 \quad (7)$$

for all  $t \in T_d$ . If the control variable  $u(t)$  is missing from (2), then the system generates a Gauss-Markov sequence.

By similar arguments as before it is assumed that the output of the system is contaminated with some noise, i.e., (2-4) becomes the following measurement equation

$$z(t) = H(t) x(t) + v(t) \quad , \quad (8)$$

where  $x(t)$  is the  $m$ -vector of measurements, and  $v(t)$  is a vector of WGN sequences, called measurement error or measurement uncertainty with zero mean

$$\mathcal{E}\{v(t)\} = 0 \quad (9)$$

and covariance matrix

$$\mathcal{E}\{v(\tau)v^T(t)\} = R(t)\delta_{\tau t} \quad , \quad (10)$$

where  $R(t)$  is a positive semidefinite matrix. We assume that the measurement uncertainty is independent of  $x(0)$ , so that

$$\mathcal{E}\{[x(0) - \hat{x}(0)]v^T(t)\} = 0 \quad , \quad (11)$$

for all  $t \in T_d$ . Moreover, it is also assumed that the uncertainties are independent of each other

$$\mathcal{E}\{w(\tau) v^T(t)\} = 0 \quad , \quad (12)$$

for all  $t, \tau \in T_d$ . Clearly, the measurements  $z(t)$  generate an increasing measurement sequence

$$\mathcal{Z}_t = [z(1), z(2), \dots, z(t)]^T \quad (13)$$

with the obvious chain property of

$$\mathcal{Z}_t = [\mathcal{Z}_{t-1}, z(t)]^T \quad . \quad (14)$$

The matrix block diagram of the discrete stochastic linear system is shown in Fig. 12. Upon inspection with Fig. 4, the differences from and similarities with the deterministic case become immediately apparent.

For illustrative purposes some examples on the state space formulation of time series models are presented below.

**E x a m p l e 1.** In water resources literature there is a long history of using autoregressive (AR) time series models wither to predict or to generate sequences. For a detailed account consult Clarke (1973). A discrete time AR model of order n has the form of

$$y(t+1) = \phi_1 y(t) + \phi_2 y(t-1) + \dots + \phi_n y(t-n+1) + w(t) \quad , \quad (E1-1)$$

where the  $\phi$ s are the autoregressive coefficients, and  $w(t)$  is a WGN sequence with the usual properties. Defining the state variables as  $x_1(t)=y(t-n+1)$ ,  $x_2(t)=y(t-n+2)$ , ...,  $x_n(t)=y(t)$ , (E1-1) can be written in the state space model

$$x(t+1) = \Phi x(t) + \Gamma w(t) \quad , \quad (E1-2)$$

where

$$\Phi = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ \vdots & & & 0 & 1 \\ \phi_n & \phi_{n-1} & \dots & \phi_2 & \phi_1 \end{bmatrix} \quad , \quad \Gamma = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

The measurement equation, which in this case is the output equation as well, is

$$z(t) = y(t) = H x(t) \quad , \quad (E1-3)$$

where

$$H = [0, 0, \dots, 1] \quad .$$

In fact in the above model the noise sequence plays the role of the input and the measurement uncertainty is not present.

Example 2. To model rainfall sequences a moving average (MA) model of order  $n$  is

$$y(t) = \theta_1 w(t-1) + \theta_2 w(t-2) + \dots + \theta_n w(t-n) \quad (\text{E2-1})$$

is frequently used in hydrology, where  $\theta$ s are the moving average parameters and  $w(\cdot)$  is a WGN sequence. For example Matalas (1963) has applied the above model to relate effective annual precipitation to annual runoff. Defining the state variables as  $x_1(t) = w(t-n)$ ,  $x_2(t) = w(t-n+1)$ , ...,  $x_n(t) = w(t-1)$ , (E2-1) can be written

$$x(t+1) = \Phi x(t) + \Gamma w(t) \quad , \quad (\text{E2-2})$$

where

$$\Phi = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix} \quad , \quad \Gamma = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

and

$$z(t) = y(t) = H x(t) \quad , \quad (\text{E2-3})$$

where

$$H = [\theta_n, \theta_{n-1}, \dots, \theta_1] \quad .$$

Example 3. Here the state space model of the combination of the previous time series models is derived. These time series models are called mixed autoregressive-moving average models (ARMA) and are extensively treated in Box and Jenkins (1970). For hydrological interpretation consult Spolia and Chander (1974) and Dooge (1972), where the intimate relation

between ARMA models and linear reservoirs is pinpointed. Consider an ARMA (n,n) model, which has n autoregressive and m moving average terms, respectively, i.e.,

$$y(t+1) + \phi_1 y(t) + \dots + \phi_n y(t-n+1) = \theta_1 w(t) + \theta_2 w(t-1) + \dots + \theta_n w(t-n+1) \quad , \quad (E3-1)$$

which can be transformed into a state space model

$$x(t+1) = \Phi x(t) + \Gamma w(t) \tag{E3-2}$$

$$y(t) = H x(t) \quad ,$$

where  $\Phi, \Gamma, H$  are given by (Lee, 1964)

$$\Phi = \begin{bmatrix} 0 & 1 & 1 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 1 \\ \hline -\phi_n & \dots & -\phi_{n-1} & \dots & \dots & -\phi_1 \end{bmatrix} \quad ,$$

$$\Gamma = \begin{bmatrix} 1 & \dots & \dots & 0 \\ \phi_1 & 1 & & \vdots \\ \phi_2 & \phi_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n-1} & \phi_n & \dots & \phi_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix} \quad ,$$

$$H = [ 1, 0, \dots, 0 ] \quad .$$

Other examples for stochastic state space modeling as applied to hydrologic systems can be found in Kontur (1975), McLaughlin (1975), and Yakowitz (1975).

State Model for Systems with Correlated Noise Sequences

As it turns out from (4) and (10) the uncertainties were assumed to be independent between sampling intervals, i.e., they were assumed to be WGN sequences. One might say that this is a somewhat serious restriction since in real-world water resources systems the uncertainties are often sequentially correlated. Below, we demonstrate, following Sorrenson (1966), how a state space model can be constructed when the uncertainties are serially correlated. For simplicity we omit the control vector from (2) and consider the system

$$x(t+1) = \Phi(t+1,t) x(t) + \mathcal{W}(t) \tag{15}$$

$$z(t) = H(t) x(t) + \mathcal{V}(t) \quad , \tag{16}$$

where the noise processes  $\mathcal{W}(t)$  and  $\mathcal{V}(t)$  are not necessarily independent between sampling times. It is assumed however, that they are zero mean processes and are still independent of each other, i.e.,

$$\mathcal{E}\{\mathcal{W}(\tau) \mathcal{V}^T(t)\} = 0 \quad , \tag{17}$$

for all  $\tau, t \in T_d$ . The covariance matrices of the noise processes are given by

$$\mathcal{E}\{\mathcal{W}(\tau) \mathcal{W}^T(t)\} = W(\tau,t) \tag{18}$$

$$\mathcal{E}\{\mathcal{V}(\tau) \mathcal{V}^T(t)\} = V(\tau,t) \quad . \tag{19}$$

It is known (see e.g. Box and Jenkins, 1970), however, that a correlated sequence can be looked upon as the output of a linear system whose input was a WGN sequence. Such a linear system is called shaping filter. This means that the correlated noise processes are generated by

$$\mathscr{W}(t+1) = \Phi_W(t+1,t) \mathscr{W}(t) + w(t) \quad (20)$$

$$\mathscr{V}(t+1) = \Phi_V(t+1,t) \mathscr{V}(t) + v(t) \quad , \quad (21)$$

where  $\Phi_W(\cdot)$  and  $\Phi_V(\cdot)$  are the transition matrices of the appropriate shaping filters, and  $w(\cdot)$  and  $v(\cdot)$  are WGN sequences acting as inputs to the shaping filters (Fig. 13). By augmenting the state vector with the correlated noise processes and combining (15), (16), (20), and (21) we have

$$\begin{bmatrix} x(t+1) \\ \mathscr{W}(t+1) \\ \mathscr{V}(t+1) \end{bmatrix} = \begin{bmatrix} \Phi(t+1,t) & I & 0 \\ 0 & \Phi_W(t+1,t) & 0 \\ 0 & 0 & \Phi_V(t+1,t) \end{bmatrix} \begin{bmatrix} x(t) \\ \mathscr{W}(t) \\ \mathscr{V}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ w(t) \\ v(t) \end{bmatrix} \quad (22)$$

$$z(t) = [H(t) \ 0 \ I] \begin{bmatrix} x(t) \\ \mathscr{W}(t) \\ \mathscr{V}(t) \end{bmatrix} \quad , \quad (23)$$

which is apparently a state space model for

$$x'(t) = [x(t), \mathscr{W}(t), \mathscr{V}(t)]^T \quad ,$$

in the form of (2) and (8) as

$$x'(t+1) = \Phi'(t+1,t) x'(t) + w'(t) \quad (24)$$

$$z(t) = H'(t) x'(t) \quad , \quad (25)$$

or even simpler since there is no WGN measurement uncertainty present since the correlated measurement uncertainty is embedded in the state equation of the augmented system. Thus, whenever the mathematical model includes correlated processes which are of such a nature as to permit the derivation of the appropriate shaping filter, the system can be reduced to the form of (24) and (25). In this case all the techniques developed for handling (2) and (8) are valid provided that the state transition matrices



$\phi_W(\cdot)$  and  $\phi_V(\cdot)$  are available. We turn now over attention to the determination of these matrices. Here we derive  $\phi_W(\cdot)$  only, noting that the very same procedure holds for  $\phi_V(\cdot)$ .

As it is known, a random sequence  $\mathcal{W}(t)$  with zero mean is said to be wide-sense Markov, or equivalently, serially correlated sequence if its covariance matrix  $W(\tau, t)$  satisfies the relation

$$W(\tau, u) W^{-1}(u, u) W(u, t) = W(\tau, t) \quad , \quad (26)$$

where

$$\tau \geq u \geq t \quad .$$

Assume that the correlated sequence  $\mathcal{W}(\cdot)$  is generated by (20), where the matrix  $\phi_W(\cdot)$  must obey the properties of the state transition matrices discussed in Section 2. Then, in accordance with (18) and (20) we have

$$\begin{aligned} \mathcal{E}\{\mathcal{W}(t+1)\mathcal{W}^T(t)\} &= W(t+1, t) \\ &= \mathcal{E}\{[\phi_W(t+1, t)\mathcal{W}(t) + w(t)]\mathcal{W}^T(t)\} \\ &= \phi_W(t+1, t) \mathcal{E}\{\mathcal{W}(t)\mathcal{W}^T(t)\} + \mathcal{E}\{w(t)\mathcal{W}^T(t)\} \quad . \end{aligned}$$

Since  $w(t)$  is independent of  $\mathcal{W}(t)$

$$\mathcal{E}\{w(t)\mathcal{W}^T(t)\} = 0$$

we have

$$W(t+1, t) = \phi_W(t+1, t) W(t, t) \quad .$$

Assuming that the covariance matrix  $W(t, t)$  is positive definite,

$$\phi_W(t+1, t) = W(t+1, t) W^{-1}(t, t) \quad (27)$$

Now, we verify that  $\Phi_W(\cdot)$  is really a state transition matrix. Obviously

$$\Phi_W(t, t) = I \quad ,$$

which corresponds to (2-9), and

$$\Phi_W(t+1, t) \Phi_W(t, t-1) = W(t+1, t) W^{-1}(t, t) W(t, t-1) W^{-1}(t-1, t-1) \quad ,$$

which, due to (26) and (27), reduces to

$$\begin{aligned} \Phi_W(t+1, t) \Phi_W(t, t-1) &= W(t+1, t-1) W^{-1}(t-1, t-1) \\ &= \Phi_W(t+1, t-1) \quad , \end{aligned}$$

thus  $\Phi_W(\cdot)$  satisfies the requirement for a state transition matrix stated in Section 2.

To complete the discussion the covariance matrix of the WGN process  $w(\cdot)$  is still to be derived. It follows from (20) that

$$\begin{aligned} \mathcal{E}\{w(t)w^T(t)\} &= \mathcal{E}\{[\mathcal{W}(t+1) - \Phi_W(t+1, t)\mathcal{W}(t)] [\cdot]^T\} \\ &= W(t+1, t+1) - W(t+1, t) W^{-1}(t, t) W(t, t+1) \quad , \quad (28) \end{aligned}$$

which is nonnegative-definite.

**E x a m p l e 4.** Consider a scalar correlated sequence  $\mathcal{W}(t)$  with an exponential covariance function

$$W(\tau, t) = e^{-|\tau-t|} \quad , \quad \tau \geq t \quad .$$

Then, according to (26),

$$\begin{aligned} W(\tau, t) &= W(\tau, u) W^{-1}(u, u) W(u, t) \\ &= e^{-|\tau-u|} e^{-|u-t|} = e^{-|\tau-t|} \end{aligned}$$

since  $\tau \geq u \geq t$ . Using (27) the state transition factor becomes

$$\Phi_W(t+1,t) = W(t+1,t)W^{-1}(t,t) = e^{-|t+1-t|} = 0.3679$$

and the covariance of the corresponding WGN sequence is

$$\begin{aligned} Q(t) &= \mathcal{E}\{w^2(t)\} = 1 - e^{-|t+1-t|} e^{-|t-t-1|} \\ &= 1 - e^{-2} = 0.8647 \end{aligned}$$

### Structural Properties

Here we shortly reexamine the structural properties such as observability, controllability, minimal realization, developed for deterministic systems in Section 3, for discrete stochastic linear dynamic systems. The relevant questions are the same as in Section 3 but here the system and measurement uncertainties make life much more interesting. The names of these structural properties will be the same modified only by the adjective stochastic.

For stochastic observability, by similar arguments as in Section 3, it is sufficient to consider only the unforced system if it is assumed that there is no system uncertainty, i.e.,

$$x(t+1) = \Phi(t+1,t)x(t) \tag{29}$$

$$z(t) = H(t)x(t) + v(t) \quad , \tag{30}$$

with noisy measurements, having the usual statistics, on it. Here again, if we can determine the state vector at any one time in  $T_d$ , such as  $x(0)$ , then from (29) we can determine all other state vectors. Because of noisy measurements, however, it is no longer possible to determine the state vector from a finite number of observations. Instead, we consider the problem

of obtaining the maximum likelihood estimate of  $x(0)$  by maximizing the likelihood function

$$p[\mathcal{Z}_t | x(0)] \quad (31)$$

with respect to the choice of  $x(0)$ . Again, determining, or rather estimating,  $x(0)$  in this fashion is equivalent to estimating  $x(\tau)$ , for  $0 \leq \tau \leq t$ , since  $x(\tau)$  evolves from  $x(0)$  according to (29).  $\mathcal{Z}_t$  has the same meaning as in (13). We have, for the first two conditional moments

$$\mathcal{E}\{x(i) | x(0)\} = H(i)x(i) = H(\tau)\Phi(\tau, 0)x(0)$$

$$\text{var}\{z(i) | x(0)\} = R(i) \quad ,$$

where

$$\Phi(i, 0) = \Phi(i, i-1) \cdots \Phi(1, 0) \quad .$$

The likelihood function, or probability density of  $\mathcal{Z}_t$  conditioned upon  $x(0)$ , is Gaussian and has the form of

$$p[\mathcal{Z}_t | x(0)] = \prod_{i=1}^t [(2\pi)^{-t/2} |R(i)|^{-1/2} \exp(-\frac{1}{2} ||z(i) - H(i)x(i)||_{R^{-1}(i)}^2)] \quad , \quad (32)$$

where, for the sake of brevity, the notation

$$||z(i) - H(i)x(i)||_{R^{-1}(i)}^2 = [z(i) - H(i)x(i)]^T R^{-1}(i) [z(i) - H(i)x(i)]$$

was used for the quadratic forms. As a matter of fact the maximization of the likelihood function is equivalent to the minimization of the quadratic forms, i.e.,

$$J = \frac{1}{2} \sum_{i=1}^t ||z(i) - H(i)x(i)||_{R^{-1}(i)}^2 \quad (33)$$

is to be minimized. This minimization is of least squares type which must be accomplished with respect to  $x(0)$ , where

$$x(i) = \Phi(i,0)x(0) \quad . \quad (34)$$

Combining the two foregoing equations, differentiating with respect to  $x(0)$ , and setting the result equal to zero the estimated initial state can be obtained as

$$\hat{x}(0) = \mathcal{O}_s^{-1}(0,t) \sum_{i=1}^t \Phi^T(i,0)H^T(i)R^{-1}(i)z(i) \quad (35)$$

where

$$\mathcal{O}_s(0,t) = \sum_{i=1}^t \Phi^T(i,0)H^T(i)R^{-1}(i)H(i)\Phi(i,0) \quad (36)$$

is an  $n \times n$  symmetric matrix called stochastic observability matrix. For the solution (35) to exist  $\mathcal{O}_s(0,t)$  must have an inverse. If such an inverse exists the system (29) and (30) is said to be stochastically observable. By comparing (36) with (3-9) it appears that the only difference between the deterministic and stochastic observability matrices is that the later, through the measurement noise covariance matrix, considers the uncertainties as well. For the case when the system is not free similar criterion can be established; for details see Aoki (1967).

Here the same remark can be made as for the deterministic observability, namely that the observability and estimation of stochastic systems state are intimately related concepts. Indeed, using (29) along with (35) the estimation of the states' history becomes possible. We note that in practical computer applications recursive techniques are applied. These procedures are discussed in detail later.

Criterion for controllability of stochastic systems can be obtained along the same lines. The final result is that the

stochastic controllability matrix

$$\mathcal{C}_s(0,t) = \sum_{i=1}^t \Phi(0,i) \Gamma(i-1) Q(i-1) \Gamma^T(i-1) \Phi^T(0,i) \quad (37)$$

must be positive definite, i.e., the process noise must excite all the states in the system (c.f.: (3-16)). Again see Aoki (1967) where the identifiability conditions for stochastic systems are also discussed.

As for minimal realizations of stochastic systems Akaike (1974) showed recently that the natural representation of a state space for stochastic systems is given by the predictor space, the linear space spanned by the predictors when a system is driven by a WGN input sequence, and a minimal realization corresponds to a selection of a basis of this predictor space. As in the deterministic case, any two minimal realizations are connected by a nonsingular linear transformation between the two corresponding bases of the predictor space. Akaike's minimal realization algorithm is again based upon the analysis of the infinite dimensional Hankel matrix composed from the impulse response matrices.

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Figures

- Figure 1. Nonlinear System as Cascaded Blocks of Linear Dynamic and Zero Memory Nonlinear Subsystems.
- Figure 2. Simplified Catchment Model.
- Figure 3. Flow Chart of Lake-Aquifer System.
- Figure 4. Matrix Block Diagram of a Discrete Linear Dynamic System.
- Figure 5. State Trajectories for Transient Responses.
- Figure 6. Three Basic Properties of the State Transition Matrix
- Figure 7. State Representation of the Nash-Model.
- Figure 8. The Dynamics of the Discrete Time Water Quality Control System.
- Figure 9. An Unobservable-Controllable System.
- Figure 10. An Observable-Uncontrollable System.
- Figure 11. Canonical Decomposition of a Linear Dynamic System into Four Subsystems.
- Figure 12. Matrix Block Diagram of a Stochastic Discrete Linear Dynamic System.
- Figure 13. The Notion of Shaping Filter.

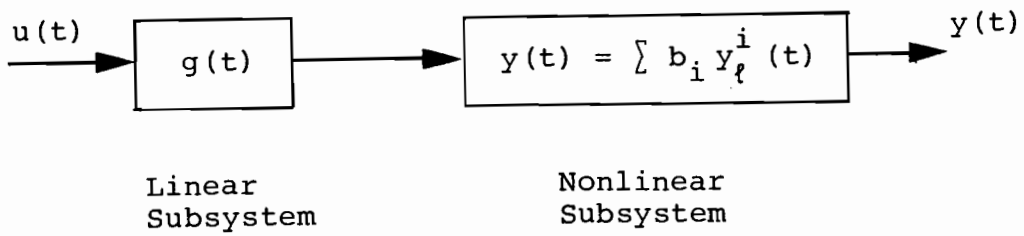


Figure 1. Nonlinear System as Cascaded Blocks of Linear Dynamic and Zero Memory Nonlinear Subsystems.

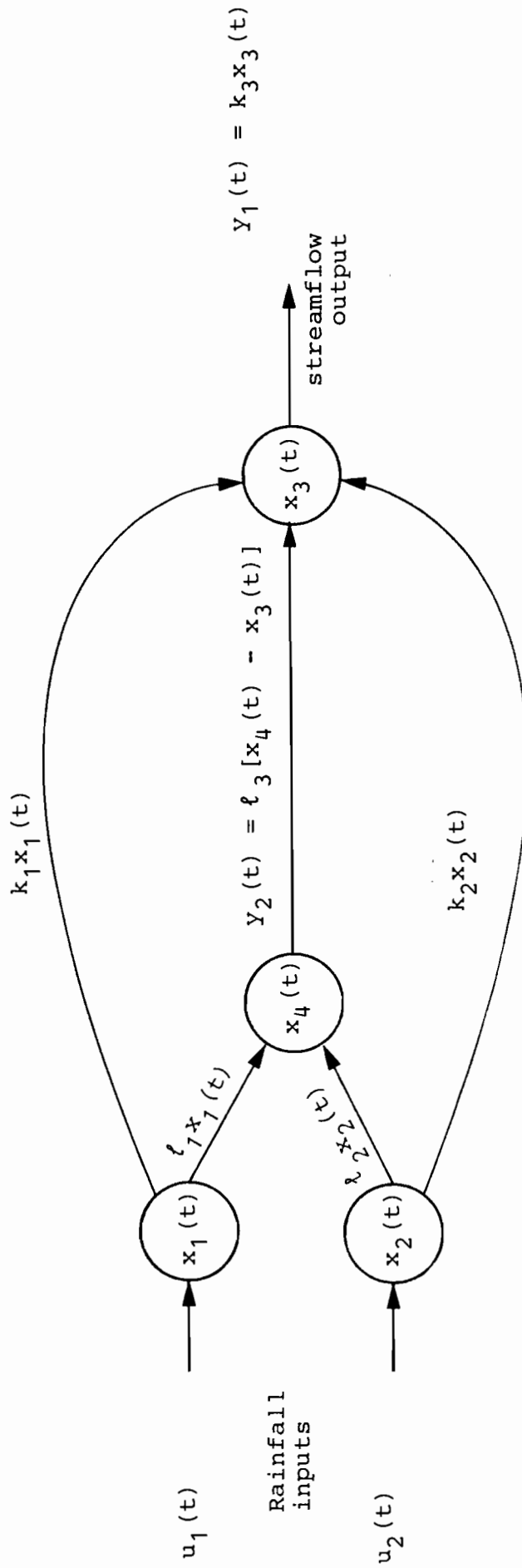


Figure 2. Simplified Catchment Model.

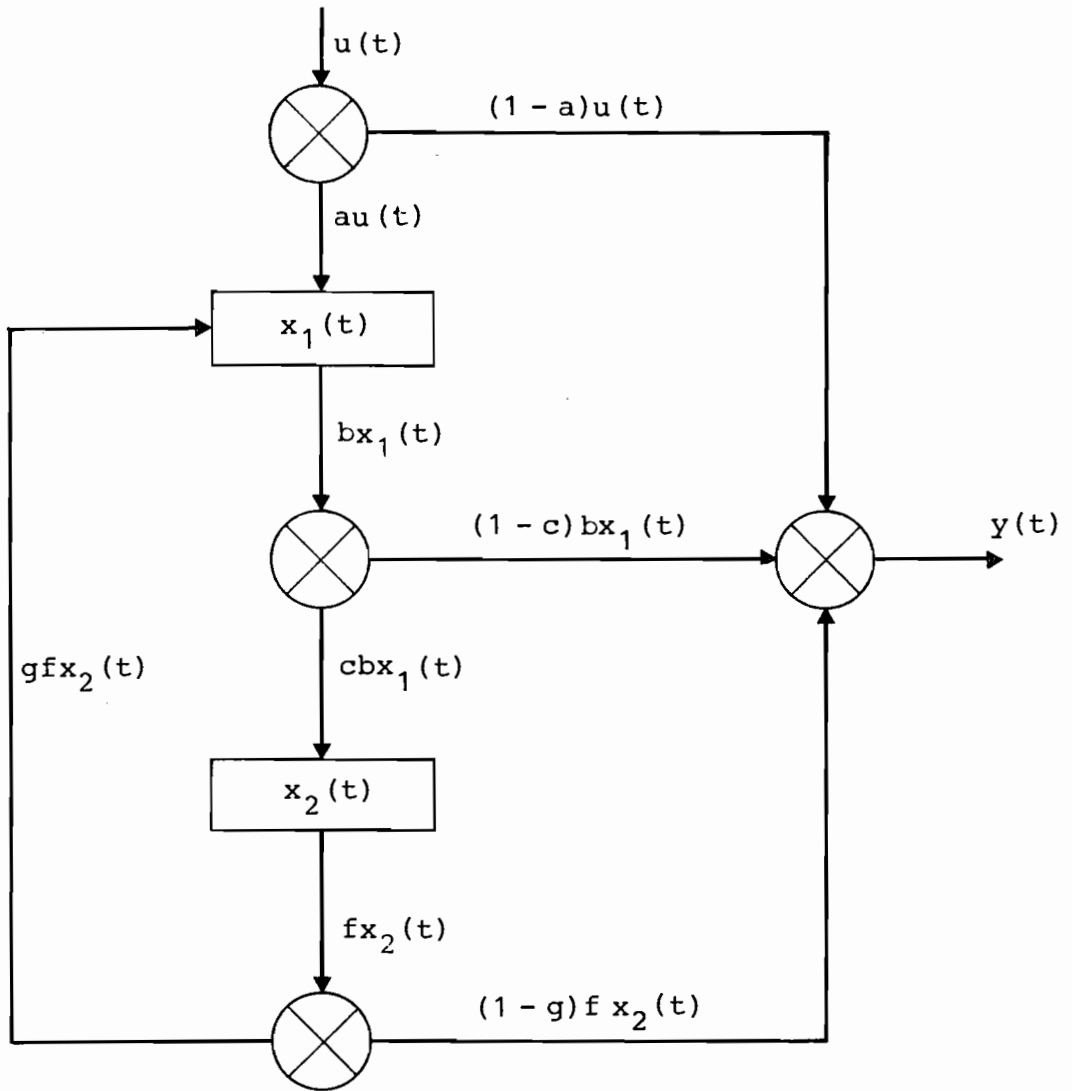


Figure 3. Flow Chart of Lake-Aquifer System.

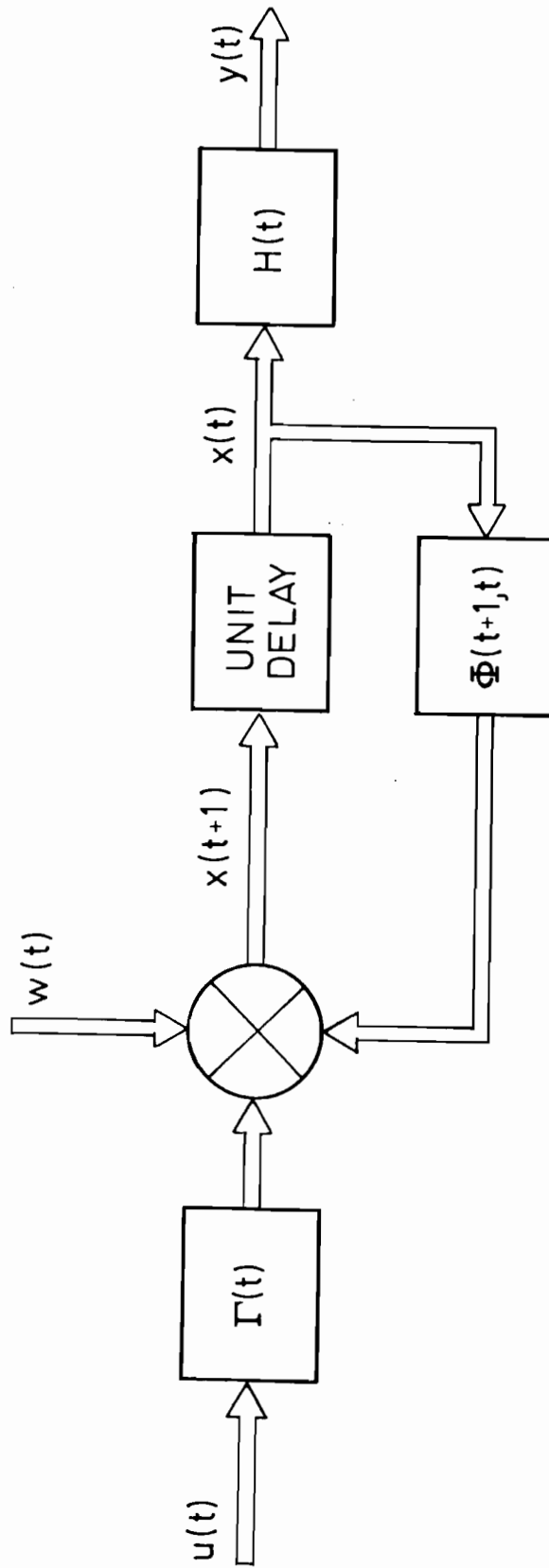


Figure 4. Matrix Block Diagram of a Discrete Linear Dynamic System.

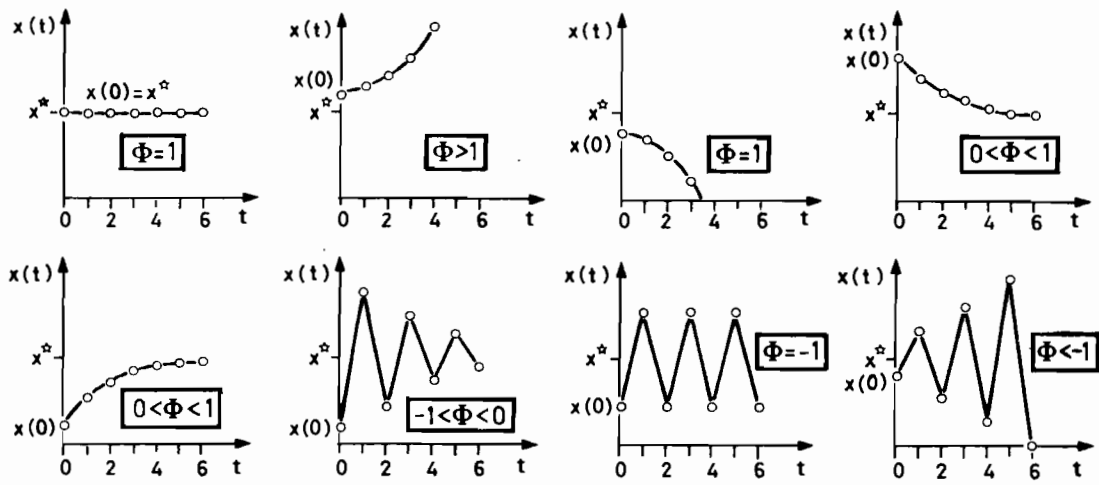


Figure 5. State Trajectories for Transient Responses.



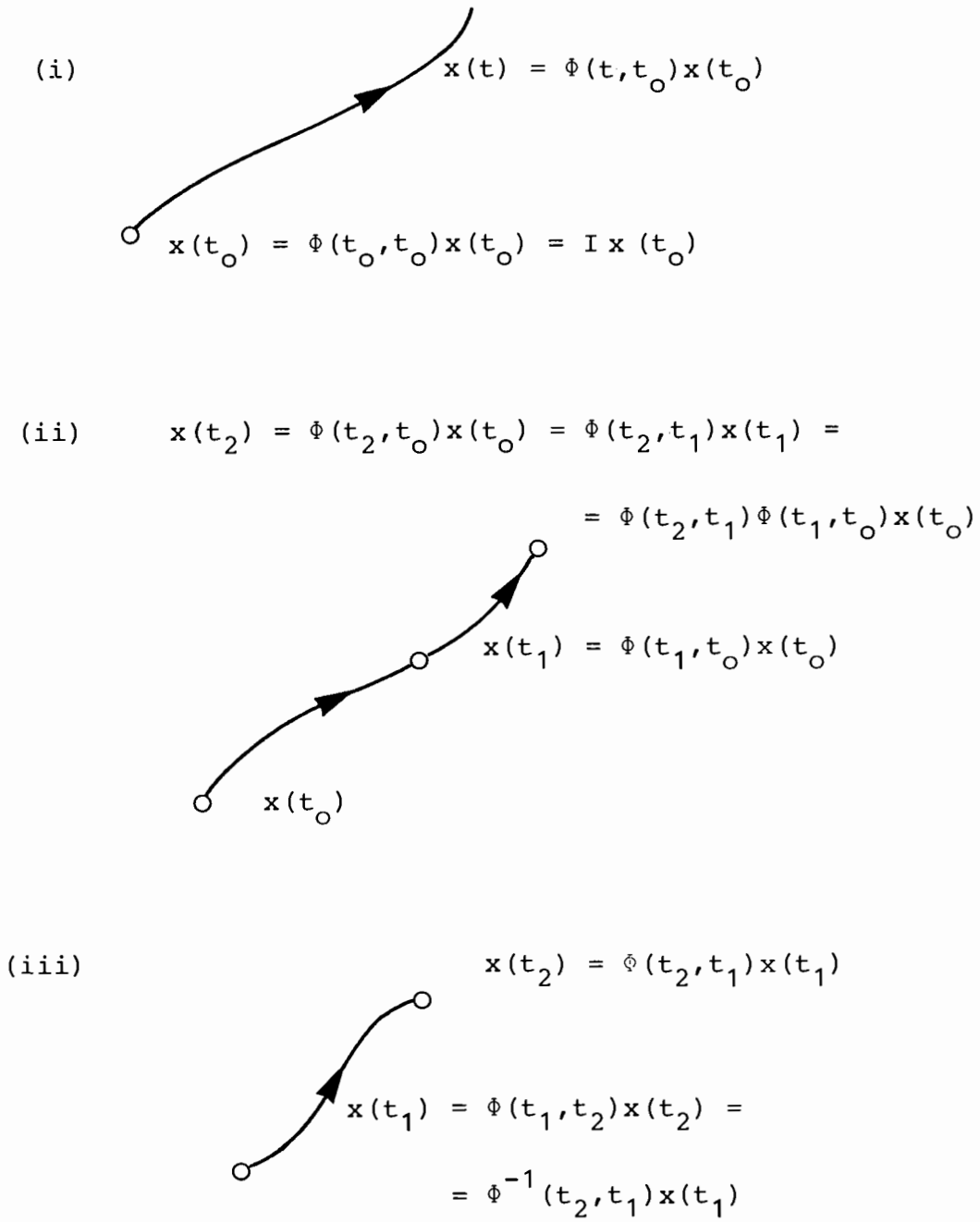


Figure 6. Three Basic Properties of the State Transition Matrix.

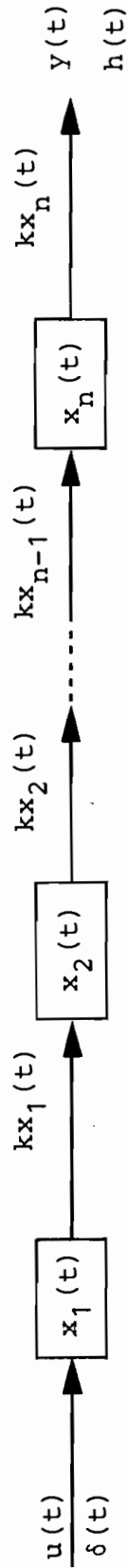


Figure 7. State Representation of the Nash-Model.

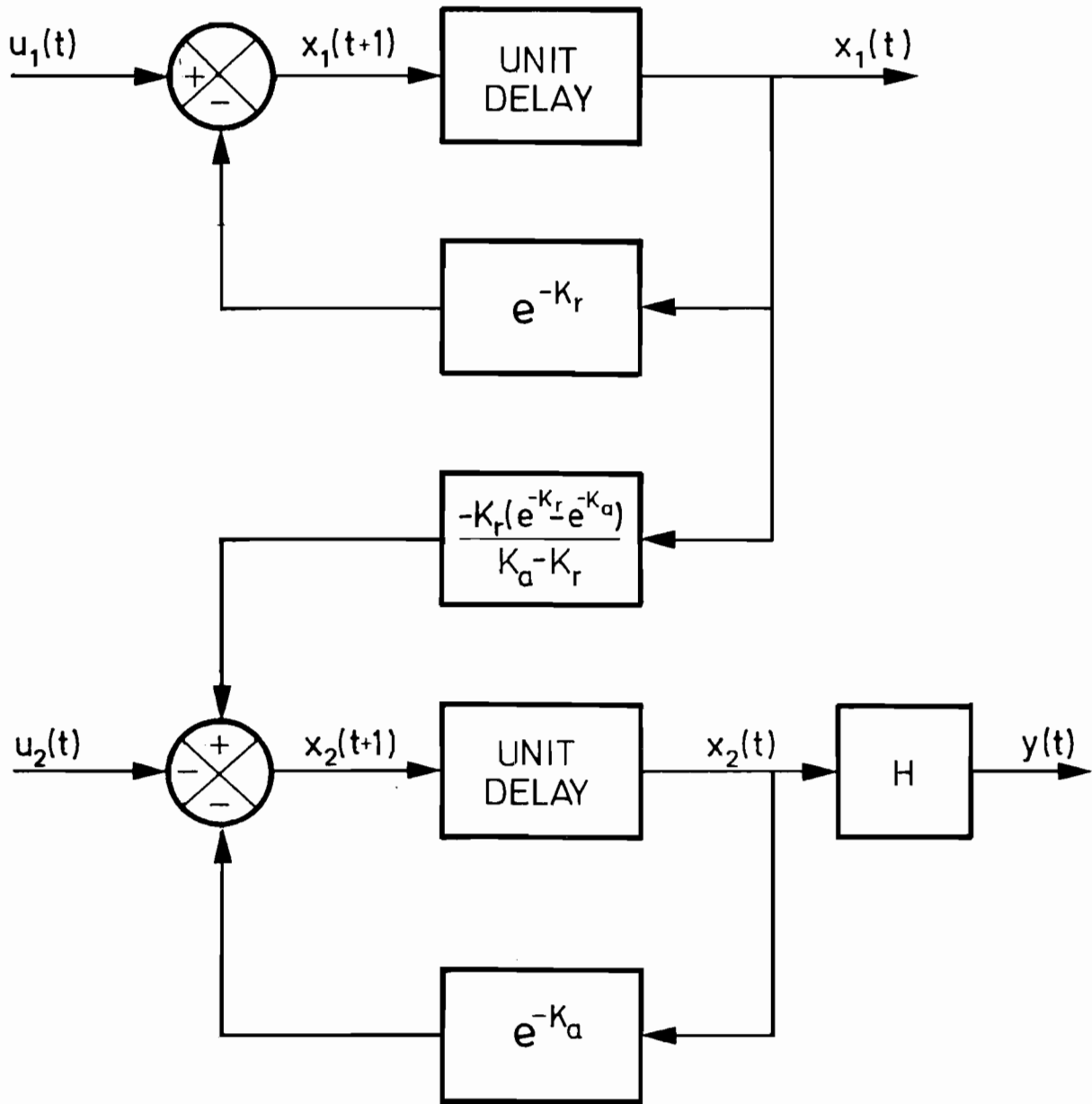


Figure 8. The Dynamics of the Discrete Time Water Quality Control System.

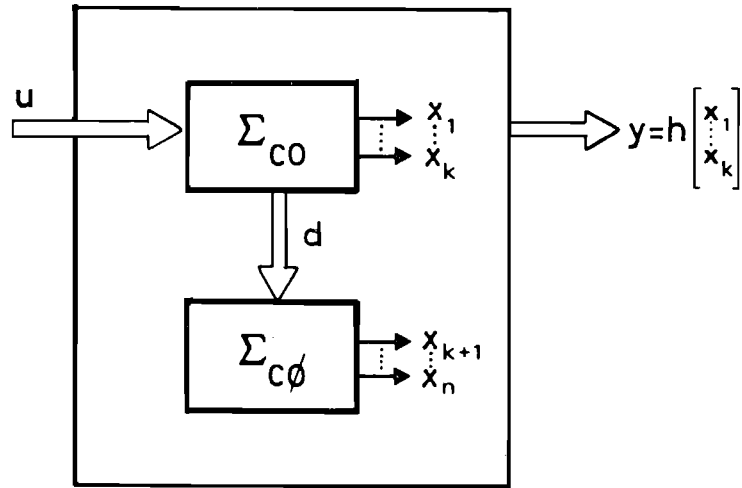


Figure 9. An Unobservable-Controllable System.

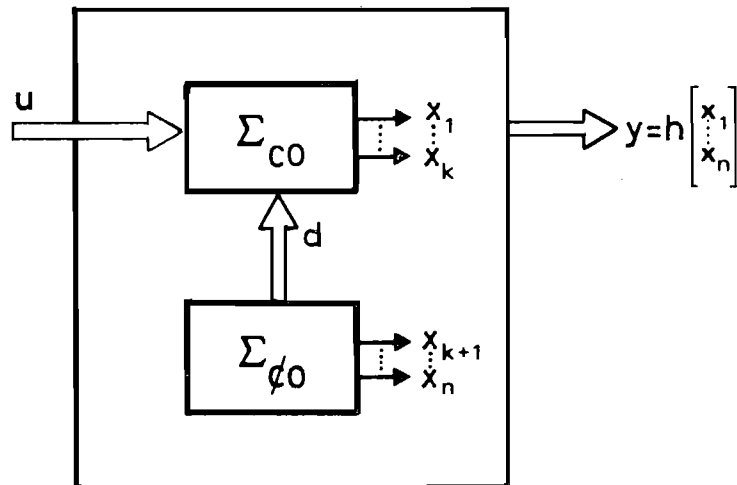


Figure 10. An Observable-Uncontrollable System.

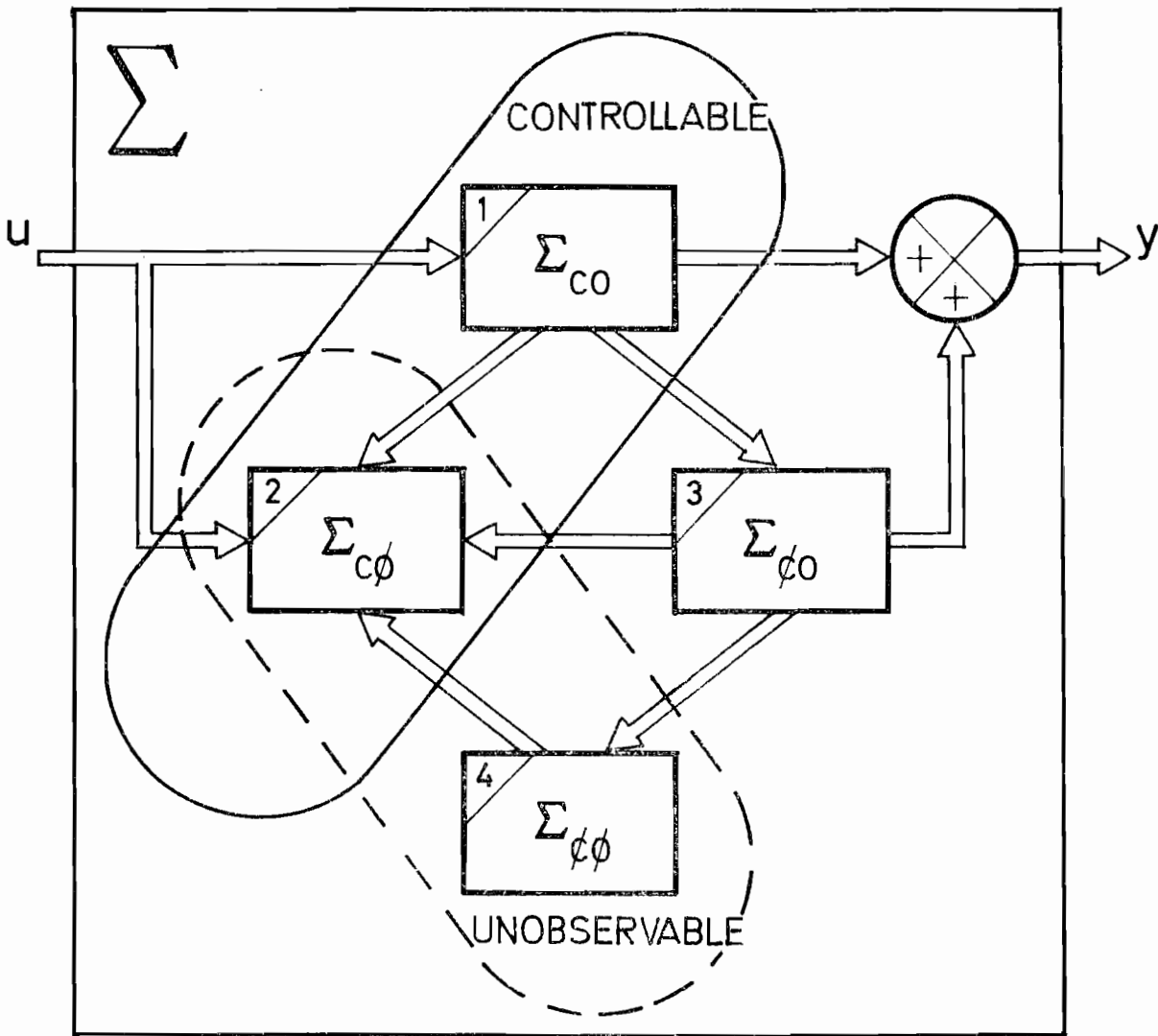


Figure 11. Canonical Decomposition of a Linear Dynamic System into Four Subsystems.

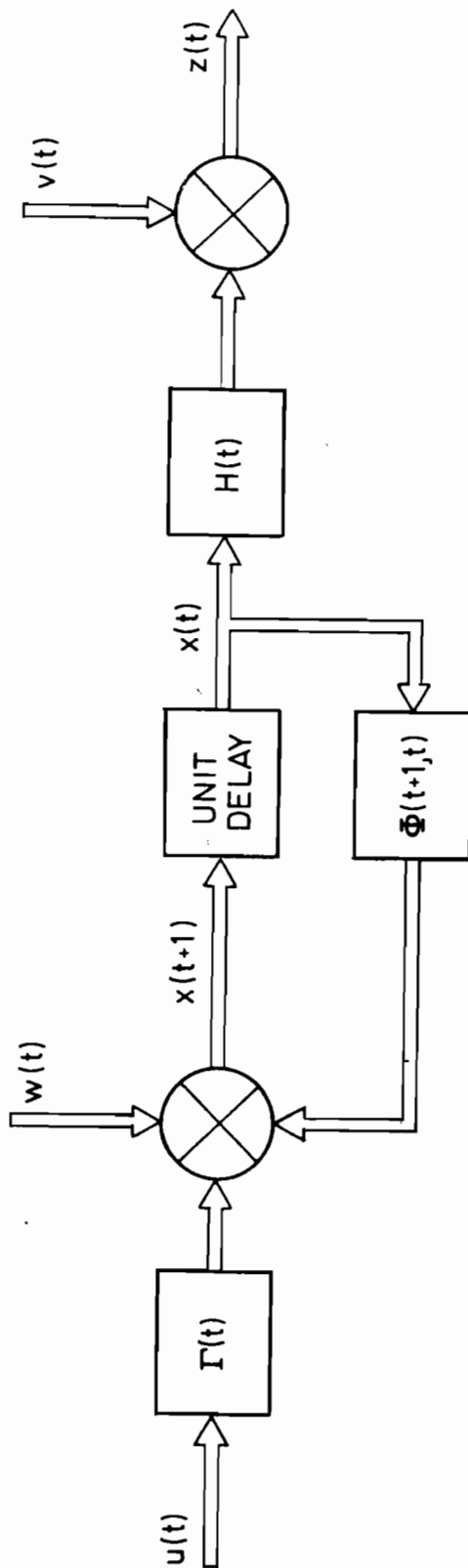


Figure 12. Matrix Block Diagram of a Stochastic Discrete Linear Dynamic System.

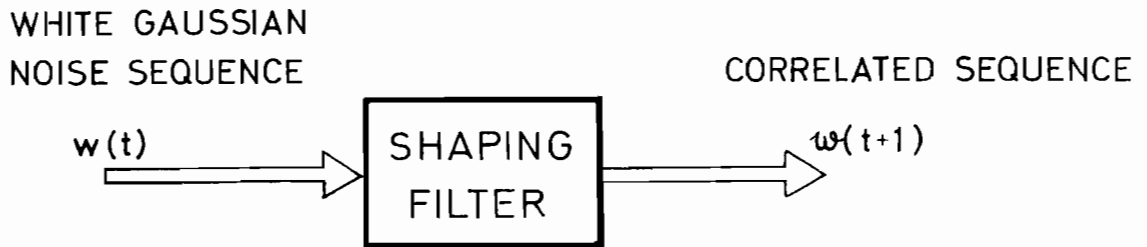


Figure 13. The Notion of Shaping Filter.