International Institute for Applied Systems Analysis
Schlossplatz 1 • A-2361 Laxenburg • Austria
Telephone: (+43 2236) 807 342 • Fax: (+43 2236) 71313
E-mail: publications@iiasa.ac.at • Internet: www.iiasa.ac.at

Interim Report **IR-00-022**

Dynamic Model of Market of Patents and Equilibria in Technology Stocks

Arkadii Kryazhimskii (kryazhim@mi.ras.ru)
Chihiro Watanabe (chihiro@me.titech.ac.jp)

Approved by

Gordon MacDonald (macdon@iiasa.ac.at)
Director, IIASA

April, 2000

Contents

Abstract

The paper presents a dynamic model of trading on market of patents. It is assumed that each firm participating in market produces its own technologies, whereas its manufacturing sector utilizes both originally produced technologies and those produced in other firms. The firms are therefore interdependent through the technology stocks used in manufacturing, which provides a basis for the emergence of market of patents. In our model a firm has three actions in market, *prior announcement*, *offering payoffs* and *making decisions*. Three-stage trading is repeated periodically and thus drives the evolution of the firms' technology stocks. We show that, under reasonable assumptions, the proposed pattern allows the firms to act so that, first, their individual decisions are subjectively best in every period of trading, and, second, current combinations of their technology stocks gradually approach a state which maximizes the total profit of the firms' community. An important feature of the model is that the described market operations imply the minimum exchange in individual information.

About the Authors

Arkadii Kryazhimskii
Mathematical Steklov Institute
Russian Academy of Sciences
Moscow, Russia
and
Dynamic Systems
International Institute for Applied Systems Analysis
A-2361 Laxenburg, Austria

Chihiro Watanabe
Department of Industrial Engineering & Management
Tokyo Institute of Technology
Tokyo, Japan

Acknowledgements

The authors would like to thank Charla Criffy-Brown and Gernot Hutschenreiter for their valuable comments, and the participants of the Technical Meeting between IIASA and Tokyo Institute of Technology on *The Comparative Analysis of the Endogenous Techno-Economic Process: Technology Spillovers in Japan, the USA, Europe and APEC Countries* (11-12 December 1999, Tokyo Institute of Technology, Tokyo) for a fruitful discussion.

Dynamic Model of Market of Patents and Equilibria in Technology Stocks

Arkadii Kryazhimskii
Chihiro Watanabe

Introduction

Globalization processes in economy have elevated the importance of knowledge as a critical techno-economic driver. Many economists describe knowledge as the engine of modern technological development. The key impacts of knowledge dissemination on long-run economic growth performance have been captured in the analytic framework of endogenous growth theory (see Grossman and Helpman, 1991). Within this analytic approach, the situation where a country has immediate, complete and costless access to the knowledge stocks of other countries, and also effects of assymmetric and symmetric flows of knowledge have been explored (see Hutschenreiter, et. al., 1995; Borisov, et. al., 1999). In these studies the country's ability to utilize externally produced knowledge is characterized by its absorbtive capacity, a variable index positively related to the country's accumulated knowledge stock. The absorptive capacity (or the assimilation capacity) consists of capacities of (i) distinguishing profitable knowledge from different knowledge, (ii) internalizing accepted knowledge, and (iii) embodying the internalized stock of knowledge to production process (Watanabe, et. al., 1998).

The absorbtive capacity represents the aggregate result of different knowledge exchange mechanisms without explaining them in detail. However, the importance of explaining and classifying such mechanisms on the international, national and corporate levels (which is obviously related to the issue of optimizing knowledge networks) is rapidly growing due to the explosion in information technology. Governments are increasingly challenged to build "knowledge-based" economies by creating "knowledge" infrastructure for "knowledge" intensive industries. In national systems of innovation there are a number of formal and informal mechanisms to acquire, create, exploit and accumulate new knowledge. For example, firms can create strategic alliances, undertake mergers and acquisitions, invest in R&D, enhance personnel movement, or simply use technology of selling and buying patents. For understanding the nature of all of these mechanisms and systematizing them a mathematical treatment would be critical. Such a systematic analysis could be a serious research program which is obviously far beyond the scope of the present work. Here we focus specifically on the exchange of patents.

Our starting conjecture is that repeated "fair" trading on market of patents is able to organize well-structurized flows of knowledge. "Fair" trading, in our understanding, implies that a firm which buys technologies produced by another firm offers, confidentially, a “fair” payoff to the producer. In our model a firm has an R&D department and a manufacturing deparment. The R&D department produces its own original technologies and the manufacturing deparment utilizes both originally produced technologies and technologies produced in other firms. The technological interdependence of the firms provides a basis for the emergence of market of patents.

In our model, each firm has three actions in market, *prior announcement*, *offering payoffs* and *making decisions*. At the first stage (prior announcement), each firm demonstrates a relatively small portion of its new project technologies without producing them. At the second stage (offering payoffs), each firm studies the announced technologies of other firms and offers its payoffs for these technologies. At the third stage (making decisions), each firm analyzes the total payoff offer of other firms and makes its decision on producing and selling or not producing (and not selling) its announced technologies. The three-stage procedure is repeated periodically and, thus, drives the evolution of the firms' technology stocks. Our goal is to show that the proposed pattern is flexible enough to let the firms produce and sell new portions of technologies in such a way that, first, the individual decisions are best for every firm in every period, and, second, the current combinations of firms' technology stocks eventually approach a state, which maximizes the total profit of the firms' community. An important feature of the discussed model is that the associated market operations imply the minimum exchange in information on firms' individual key characteristics such as the production functions, the costs for producing and maintaning technologies, and the structure of the technology stocks accumulated in manufacturing.

The paper is organized as follows. In section 1 we introduce a model of a community of firms. A model of “fair” round-by-round trading on market of patents is presented in section 2. Here we describe an intuitively clear rule for making individual decisions in one round (we call them *boundedly rational decisions*) and justify their (intuitively clear) optimality for every firm in every round. Section 3 is devoted to the analysis of the global evolution of the technology stocks driven by boundedly rational market decisionmakers. In particular, we state that the area of low technology stocks (which is covered in an initial period of the evolution) is very favourable for the technological development and operations on market of patents. In this area the boundedly rational firms never interrupt production of new technologies and all patents are sold on market. Our main result, which closes the section, states that the boundedly rational decisionmakers drive their technology stocks to a state which maximizes the total profit of the firms' community. In other words, the firms behaving boundedly rational in every round eventually find a combination of their technologies, which is best for their community as a whole. Section 4 presents conclusions. Section 5, the Appendix, contains the proves.

We conclude the Introduction with a short characterization of our methodology. In our setting, the firms operate in the situation where their profits depend on the technologies developed in other firms. This falls entirely into the scope of theory of noncooperative games (see Germeyer, 1976; Basar and Olsder, 1982; Vorobyov, 1985). To characterize the combinations of firms' technologies, which may be acceptable for their community, we refer to the notion of a Pareto equilibrium. Among the Pareto equilibria we select the one which maximizes the total profit of the firms' community and, thus, represents the best combination of firms' technologies. This combination can obviously be viewed as a target of the firms' community as a whole. However, the immediate switch to the target of all firms simultaneously is not possible thanks to the informational barriers. To make a cooperative decision on the radical switch to the target the firms must communicate to each other their production functions, thier costs for producing and maintaining technologies, and the structure of thier technology stocks. This total exchange in privat information is hardly imaginable; moreover, the firms may not be able to reconstruct instantaneously the global shape of their own production and cost functions. Instead of the radical switch, slow evolutionary paths towards the target should be studied. This view is in good agreement with theory of repeated games which assumes interacting agents to learn in an infinite sequence of rounds (see Smale, 1980; Axelrod, 1984; Fudenberg and Krebs, 1993; Nowak and Sigmund, 1992; Kaniovski and Young, 1995). In this context, we state that boundedly rational firms which gradually approach the target through market of patents learn successfully.

1 Firms: static relations

1.1 Production functions

Let us assume that n firms, numbered $1, \ldots, n$, operate on market. Following the classical approach (Arrow and Kurz, 1970; Intriligator, 1971; see also Watanabe, 1992), we assume that production of each firm i, y_i, is a function of labor, L_i, capital, K_i, materials, M_i, energy use, E_i, and the technology stock, ξ_i, accumulated in the manufacturing sector,

$$y_i = F_i(L_i, K_i, M_i, E_i, \xi_i); \tag{1.1}$$

as usual, we call this function the *production function*. We assume that, given a size of the firm's accumulated technology stock, ξ_i, particular amounts of labor, capital, materials, and energy are required. Usually, these amounts are found as minimum ones supporting the technology stock ξ_i which is, in turn, represented as

$$\xi_i = \min\{h_{1i}(L_i), h_{2i}(K_i), h_{3i}(M_i), h_{4i}(E_i)\};$$

here h_{ji} $(j = 1, \ldots, 4)$ are strictly increasing functions of the quantities of labor, capital, materials, and energy use, respectively. The requireness of the minimum values for these quantities leads to

$$L_i = h_{1i}^{-1}(\xi_i), \quad K_i = h_{2i}^{-1}(\xi_i), \quad M_i = h_{3i}^{-1}(\xi_i), \quad E_i = h_{4i}^{-1}(\xi_i).$$

Substituting into (1.1), we represent production, y_i, as a function of ξ_i only:

$$y_i = f_i(\xi_i). \tag{1.2}$$

The function f_i will also be called the production function of firm i. We assume f_i to be monotonically increasing. This assumption agrees with the observation that firm's production, $y_i(t)$, grows in time, t, as the R&D investment, $r_i(t-m)$ (where m is a commercialization time lag), exceeds the rate of technology obsolescence, $\rho_i\xi_i(t)$, in the firm. In this situation the rate of the firm's technology stock, which is represented as $\dot{\xi}_i(t) = r_i(t-m) - \rho_i\xi_i(t)$, is positive; in other words, the firm's technology stock, $\xi_i(t)$, grows. The fact that the growth in the technology stock, $\xi_i(t)$, is accompanied by growth in production, $y_i(t)$, implies that the production function f_i in (1.2) is increasing.

1.2 Exchange in technologies

We assume that each firm i works on new technologies, whereas the technology stock used in its maunfacturing sector, ξ_i, consists not only of technologies produced in firm i but also comprises some of those developed in other firms. Thus, $\xi_i = a_{i1}x_1 + \ldots + a_{in}x_n$, where x_j is the stock of the technologies developed in firm j. The coefficient a_{ij} located between 0 and 1 represents the fraction of the technology stock developed in firm j, which is used in firm i. Generally, a_{ij} may depend on the size and structure of the firms' technology stocks. For example, Jaffe (1986) defined this coefficient as a "technological distance" between firms i and j:

$$a_{ij} = \left[\sum_{k=1}^{n} \frac{x_{ik}}{x_i} \sum_{k=1}^{n} \frac{x_{jk}}{x_j}\right] \left[\sum_{k=1}^{n} \left(\frac{x_{ik}}{x_i}\right)^2 \sum_{k=1}^{n} \left(\frac{x_{jk}}{x_j}\right)^2\right]^{-1/2};$$

here x_{ik} is the fraction of the technology stock produced in firm i, which is devoted to area k ($x_i = x_{i1} + \ldots + x_{in}$). In the present study we, for simplicity, assume that a_{ij} are constant. Substituting $\xi_i = a_{i1}x_1 + \ldots + a_{in}x_n$ into (1.2), we represent production, y_i, of firm i as a function of the technology stocks developed in firms $1, \ldots, n$:

$$y_i = f_i(a_{i1}x_1 + \ldots + a_{in}x_n). \tag{1.3}$$

1.3 Profits

Let $c_i = c_i(x_i)$ be the cost for producing and maintaining the technology stock x_i in firm i; the function c_i is monotonically increasing. We assume for simplicity that the whole output, y_i, of firm i is sold on market with a unit price. Then the profit of firm i is given by $u_i = y_i - c_i$. Using (1.3), we represent the profit as a function of the firms' technology stocks:

$$u_i = u_i(x_1, \ldots, x_n) = f_i(a_{i1}x_1 + \ldots + a_{in}x_n) - c_i(x_i). \tag{1.4}$$

1.4 Technology game

Each firm, i, desires to maximize its profit, u_i. However, the firm's profit depends on the technology stocks developed in other firms (see (1.4)). Therfore, an actual combination of the firms' technology stocks, $(x_1, \ldots, x_n)$, may not be satisfactory for all firms, and some firms may wish to change it. This situation falls into the scope of game theory (see, e.g., [Basar and Olsder, 1982]). Following the theory, we consider the set of all hypothetically admissible combinations of firms' technology stocks,

$(x_1, \ldots, x_n)$, and define combinations which are acceptable to the firms' community as a whole. The acceptable combinations represent the so-called *Pareto equilibria*; we call them Pareto equilibrium combinations of thechnology stocks.

The definition is as follows. A combination of the thechnology stocks, $(\hat{x}_1, \ldots, \hat{x}_n)$, is said to be a *Pareto equilibrium* if there is no other combination of those, $(x_1, \ldots, x_n)$, which is more preferrable to the firms' community in the following sense: if all the firms pass (virtually) from $(\hat{x}_1, \ldots, \hat{x}_n)$ to $(x_1, \ldots, x_n)$, at least one of them wins in profit, that is, $u_i(x_1, \ldots, x_n) > u_i(\hat{x}_1, \ldots, \hat{x}_n)$ for some i, whereas all the others do not lose, that is, $u_j(x_1, \ldots, x_n) \geq u_j(\hat{x}_1, \ldots, \hat{x}_n)$ for all $j \neq i$. So, every combination $(x_1, \ldots, x_n)$ differing from the Pareto equilibrium $(\hat{x}_1, \ldots, \hat{x}_n)$ is either not better than $(\hat{x}_1, \ldots, \hat{x}_n)$ for all firms, that is, $u_j(x_1, \ldots, x_n) \leq u_j(\hat{x}_1, \ldots, \hat{x}_n)$ for all j, or it is strictly worse than $(\hat{x}_1, \ldots, \hat{x}_n)$ for at least one firm, that is, $u_i(x_1, \ldots, x_n) < u_i(\hat{x}_1, \ldots, \hat{x}_n)$ for some i.

It is remarkable that every maximizer of the *weighted sum* of the firms' profits,

$$u(x_1, \ldots, x_n) = \mu_1 u_1(x_1, \ldots, x_n) + \ldots + \mu_n u_n(x_1, \ldots, x_n),$$

where $\mu_1, \ldots \mu_n$ are arbitrary positive weight coefficients, is a Pareto equilibrium. The case where $\mu_1 = \ldots = \mu_n = 1$ is of special interest. In this case the weighted sum represents the *total profit* of the firm's community,

$$u(x_1, \ldots, x_n) = u_1(x_1, \ldots, x_n) + \ldots + u_n(x_1, \ldots, x_n). \tag{1.5}$$

For the firms' community as a whole, Pareto equilibria which maximize the total profit (1.5) are obviously the best combinations of technology stocks.

2 Dynamic market of patents

2.1 Dynamics of technology stocks

Let us introduce dynamics in our model. For simplicity, we discretize time. Namely, we fix an infinite sequence of instants, t_k, $k = 0, 1, 2, \ldots$, and study changes between them. We set $t_0 = 0$ and assume that instants t_k appear with a fixed (small) positive step δ: $t_{k+1} = t_k + \delta$. The technology stock produced in firm i and actually used in manufacturing at time t_k will be denoted x_i^k. We assume that in period k located between t_k and t_{k+1} each firm i introduces $r_i^k \delta$ new technologies and its $r_i^{k-}\delta$ obsolesced technologies are washed off from manufacturing. Therefore,

$$\Delta^k x_i = x_i^{k+1} - x_i^k = (r_i^k - r_i^{k-})\delta. \tag{2.6}$$

Note that the size of the new portion of technologies, $r_i^k \delta$, is controlled by firm i, whereas the size of old technologies washed off from manufacturing, $r_i^{k-}\delta$, is determined by the production process. In period k production of firm i changes from $y_i^k = f_i(a_{i1}x_1^k + \ldots + a_{in}x_n^k)$ to $y_i^{k+1} = f_i(a_{i1}x_1^{k+1} + \ldots + a_{in}x_n^{k+1})$ and, therefore, grows for

$$\begin{aligned}
\Delta^k y_i &= f_i(a_{i1}x_1^{k+1} + \ldots + a_{in}x_n^{k+1}) - f_i(a_{i1}x_1^k + \ldots + a_{in}x_n^k) \\
&= f'(a_{i1}x_1^k + \ldots + a_{in}x_n^k)(a_{i1}\Delta^k x_1 + \ldots + a_{in}\Delta^k x_n) + o(\delta) \\
&= f'(a_{i1}x_1^k + \ldots + a_{in}x_n^k)[a_{i1}(r_1^k - r_1^{k-}) + \ldots + a_{in}(r_n^k - r_n^{k-})]\delta + o(\delta);
\end{aligned}$$

here and in what follows $o(\delta)$ stands for a small value which tends to 0 faster than δ ($o(\delta)$ is second order in δ), and f_i' is the derivative of f_i (we assume that f_i is differentiable); The cost for maintaning technologies in firm i changes from $c_i(x_i^k)$ to $c_i(x_i^{k+1})$ and grows for

$$\Delta^k c_i = c_i(x_i^{k+1}) - c_i(x_i^k) = c_i'(x_i^k)(r_i^k - r_i^{k-})\delta + o(\delta)$$

where c_i' is the derivative of c_i (we assume that c_i is differentiable). Therefore, in period k the profit of firm i grows for

$$\begin{aligned}\Delta^k u_i &= \Delta^k y_i - \Delta^k c_i \\ &= f'(a_{i1}x_1^k + \ldots + a_{in}x_n^k)[a_{i1}(r_1^k - r_1^{k-}) + \ldots + a_{in}(r_n^k - r_n^{k-})]\delta - \\ &\quad c_i'(x_i^k)(r_i^k - r_i^{k-})\delta + o(\delta).\end{aligned}$$

The above formulas describe the firms' dynamics. The dynamics is controlled by $r_1^k, \ldots, r_n^k$, the rates of new technology inputs in firms $1, \ldots, n$.

2.2 Market of patents. Boundedly rational decisions

Our goal is to show that market of patents can make the firms choose new portions of technologies, $r_i^k\delta$, in such a way that, first, the choices are profitable in each period k and, second, the current combination of technology stocks, $(x_1^k, \ldots, x_n^k)$, eventually approaches the Pareto equilibrium $(\hat{x}_1, \ldots, \hat{x}_n)$, which maximizes the total profit of the firms' community (see (1.5)).

In our model, each firm has three actions in period k: *prior announcement*, *offering payoffs*, and *making decisions*. At the first stage (prior announcement), each firm demonstrates a relatively small portion of its new technologies without producing them. At the second stage (offering payoffs), each firm studies the announced technologies of other firms and offers its payoffs for these technologies. At the third stage, (making decisions) each firm analyzes the total payoff offer of other firms and makes its decision on producing and selling or not producing (and not selling) its announced technologies. The three-stage procedure is repeated in each period k and thus drives the evolution of the firms' technology stocks.

Let us describe the firms' behavior in period k in detail. Each firm i starts period k with the announcement of its new technologies of a relatively small size $r_i^{k+}\delta$, which is, however, greater than $r_i^{k-}\delta$. Every other firm, j, examines the announced technologies of firm i and offers to firm i its payoff, q_{ji}^k, for those technologies (among the announced ones) which will be used in the manufacturing sector of firm j. Now we argue for firm j. By assumption fraction a_{ji} of the technologies produced in firm i is used in firm j. Therefore, firm j is interested in $a_{ji}r_i^{k+}\delta$ announced technologies of firm i. The incorporation of these technologies in the manufacturing sector of firm j yields production growth of size

$$\begin{aligned}\Delta^{k+} y_{ji} &= f_j(a_{j1}x_1^k + \ldots + a_{ji}[x_i^k + (r_i^{k+} - r_i^{k-})\delta] + \ldots + a_{jn}x_n^k) - \\ &\quad f_j(a_{j1}x_1^k + \ldots + a_{ji}x_i^k + \ldots + a_{jn}x_n^k) \\ &= f_j'(a_{j1}x_1^k + \ldots + a_{jn}x_n^k)a_{ji}(r_i^{k+} - r_i^{k-})\delta + o(\delta).\end{aligned}$$

Firm j decides how much to pay to firm i for its $a_{ji}r_i^{k+}\delta$ new technologies. Firm j has all reasons to view $\Delta^{k+}y_{ji}$ as an upper bound for its payoff, q_{ji}^k, to firm i. Generally, firm j would offer to firm i some q_{ji}^k smaller than $\Delta^{k+}y_{ji}$. However, the difference between q_{ji}^k and $\Delta^{k+}y_{ji}$ can be small compared to their absolute values. The more "fairly" firm j operates, the smaller is the difference. Here we resrict our analysis to extremely fair behaviors. Thus, we assume that the payoff offered by firm j to firm i is

$$q_{ji}^k = f_j'(a_{j1}x_1^k + \ldots + a_{jn}x_n^k)a_{ji}(r_i^{k+} - r_i^{k-})\delta \tag{2.7}$$

(we neglect $o(\delta)$).

Let us come back to firm i. Its own production growth due to its new technologies is given by

$$q_{ii}^k = f_j'(a_{j1}x_1^k + \ldots + a_{jn}x_n^k)a_{ii}(r_i^{k+} - r_i^{k-})\delta$$

(again, $o(\delta)$ is neglected). Firm i finds the total payoff offer, q_i^k, for the announced new technologies as the sum of the payoffs offered by all other firms and its own income due to production growth, q_{ii}^k:

$$q_i^k = q_{1i}^k + \ldots + q_{ii}^k + \ldots + q_{ni}^k. \tag{2.8}$$

Next, firm i computes its expenditure for developing and maintaining the announced new technologies as the cost increment

$$\Delta^{k+}c_i = c_i(x_i^k + (r_i^{k+} - r_i^{k-})\delta) - c_i(x_i^k).$$

Neglecting $o(\delta)$, we represent this as

$$p_i^k = c_i'(x_i^k)(r_i^{k+} - r_i^{k-})\delta. \tag{2.9}$$

Finally, firm i compares the payoff offer, q_i^k, and the expenditure, p_i^k. If the payoff offer is not smaller than the expenditure, $q_i^k \geq p_i^k$, firm i produces the announced technologies of size $r_i^{k+}\delta$, that is, sets $r_i^k = r_i^{k+}$ (see (2.6)), and sells its patents to all other firms; in this case every firm j pays q_{ji}^k to firm i for patents for $a_{ji}r_i^{k+}\delta$ technologies. If the payoff offer to firm i is smaller than the expenditure, $q_i^k < p_i^k$, firm i does not produce the announced technologies in period k; it sets $r_i^k = 0$ (see (2.6)). This closes period k. We call the above decisions of firm i *boundedly rational.* The decisionmaking rule for finding boundedly rational decisions is, therefore,

$$r_i^k = \begin{cases} r_i^{k+} & \text{if} \quad q_i^k \geq p_i^k, \\ 0 & \text{if} \quad q_i^k < p_i^k. \end{cases} \tag{2.10}$$

2.3 Optimality of boundedly rational decisions

Now we will show that the boundedly rational decisions are in fact *best* in every period k (in our argument we neglect the second order terms $o(\delta)$).

Let us decide for firm i, which of the two options is better in period k:

(i) to produce and sell the announced $r_i^{k+}\delta$ technologies, or

(ii) not to produce (and not to sell) them.

In terms of the transition formula (2.6), options (i) and (ii) prescribe $r_i^k = r_i^{k+}$ and $r_i^k = 0$, respectively.

Recall that as the firms make their decisions in period k, the firms' community is spilt in the two groups, the group of those that produce and sell the announced technologies (*sell firms*) and the group of those that do not produce (and do not sell) the announced technologies (*not-sell firms*). Every sell firm j sells patents for its $a_{ij}r_j^+(t_k)\delta$ technologies to firm i. The implementation of these technologies brings firm i the income

$$q_{ij}^k = f'(a_{i1}x_1^k + \ldots + a_{in}x_n^k)a_{ij}(r_j^{k+} - r_j^{k-})\delta$$

due to production growth (see (2.7) where i and j change their places). This income equals the payoff of firm i to firm j. Hence, every operation with a sell firm brings firm i the income 0.

Every not-sell firm j does not sell its new technologies to firm i, which implies the production loss of size

$$l_{ij}^k = f'(a_{i1}x_1^k + \ldots + a_{in}x_n^k)a_{ij}r_j^{k-}\delta$$

in firm i. The total loss of firm i due to the lack of new technologies of the not-sell firms is given by

$$l_i^k(J_i^k) = \sum_{j \in J_i^k} l_{ij}^k; \tag{2.11}$$

here J_i^k is the set of all not-sell firms in period k with the exception of firm i, and $\sum_{j \in J_i^k}$ denotes summation over all j from J_i^k.

Now consider i as a potential seller. Recall that every firm j offers firm i payoff q_{ji}^k for its announced technologies of size $a_{ij}r_i^{k+}\delta$. Hence,

$$\bar{q}_{ji}^k = \frac{q_{ji}^k}{a_{ji}r_i^{k+}\delta}$$

is the price set by firm j for a new technology unit of firm i. Therefore,

$$\bar{q}_{ji}^k a_{ji} r_i^k \delta = q_{ji}^k \frac{r_i^k}{r_i^{k+}}$$

is the payoff of firm j for $a_{ji}r_i^k\delta$ technologies of firm i. Similarly we find that $q_{ii}^k r_i^k / r_i^{k+}$ is the income of firm i through the implementation of its own $a_{ii}r_i^k\delta$ technologies, and $p_i^k r_i^k / r_i^{k+}$ (see (2.9)) is its cost for producing and maintainig $r_i^k\delta$ new technologies. The income gained by firm i through producing, selling and implementing $r_i^k\delta$ technologies is, therefore,

$$q_{1i}^k \frac{r_i^k}{r_i^{k+}} + \ldots + q_{ni}^k \frac{r_i^k}{r_i^{k+}} - p_i^k \frac{r_i^k}{r_i^{k+}} = (q_i^k - p_i^k)\frac{r_i^k}{r_i^{k+}}$$

(see (2.8)). Distracting the loss $l_i^k(J_i^k)$ (2.11), we find the total income, $Q_i^k(r_i^k, J_i^k)$, of firm i in period k:

$$Q_i^k(r_i^k, J_i^k) = \frac{q_i^k - p_i^k}{r_i^{k+}} r_i^k - l_i^k(J_i^k). \tag{2.12}$$

Recall that firm i must choose between $r_i^k = r_i^{k+}$ (option (i)) and $r_i^k = 0$ (option (ii)). The best choice for firm i is, obviously, the one which provides a higher value to $Q_i^k(r_i^k, J_i^k)$. Substituting $r_i^k = r_i^{k+}$ and $r_i^k = 0$ in (2.12), we get

$$Q_i^k(r_i^{k+}, J_i^k) = (q_i^k - p_i^k) - l_i^k(J_i^k),$$

$$Q_i^k(0, J_i^k) = -l_i^k(J_i^k).$$

Consequently, if $q_i^k - p_i^k > 0$, the best choice for firm i is $r_i^k = r_i^{k+}$ (option (i)), and if $q_i^k - p_i^k < 0$, the best choice for firm i is $r_i^k = 0$ (option (ii)). If $q_i^k - p_i^k = 0$, both choices yield $Q_i^k(r_i^k, J_i^k)) = -l_i^k(J_i^k)$, in this case they are equivalent for firm i. Thus, the bounded rationality decisionmaking rule (2.10), which prescribes firm i to choose (i) if $q_i^k - p_i^k \geq 0$ and (ii) otherwise, is best for firm i in period k.

2.4 Robustness and multioptimality of boundedly rational decisions

The boundedly rational (and best) decisions of firm i are *robust* with respect to the decisions of other firms. Namely, the boundedly rational decisions of firm i do not depend on the decisions of other firms on selling or not selling their announced technologies in period k, although the income of firm i in this period, $Q_i^k(r_i^k, J_i^k)$ (2.12), depends on the decisions of other firms through J_i^k (the set of all not-sell firms in period k with the exception of firm i).

The above rubustness property can also be interpreted as *mulitioptimality*. To make this interpretation clear, let us replace J_i^k in (2.12) (and in (2.11)) by an arbitrary subgroup, J, of firms $1, \dots, i-1, i+1, \dots, n$. "Free" Js have a very clear meaning. When firm i does not know the actual decisions of other firms, it has to view all Js as equally admissible candidates for being the actual not-sell group in period k. Thus, we make firm i deal with the family of *virtual incomes*

$$Q_i^k(r_i^k, J) = \frac{q_i^k - p_i^k}{r_i^{k+}} r_i^k - l_i^k(J) = \frac{q_i^k - p_i^k}{r_i^{k+}} r_i^k - \sum_{j \in J} l_{ij}^k \tag{2.13}$$

depending on an uncertain set J. For any J, firm i can find its *best response* to J, that is, its best decision under the hypothesis that J is the actual not-sell group. The best response of firm i is given by r_i^k which solves the next maximization problem:

$$\text{maximize } Q_i^k(r_i^k, J) \text{ over } r_i^k \in \{r_i^{k+}, 0\}; \tag{2.14}$$

here $r_i^k \in \{r_i^{k+}, 0\}$ indicates that r_i^k is restricted to the two-element set with elements r_i^{k+} and 0. We see that the boundedly rational decision which is made by firm i irrespective of J, responds best to *any* J. This decision is therefore multioptimal in the sense that it solves *all* maximization problems (2.14) parametrized by J simultaneously.

2.5 Local game. Nash equilibricity of boundedly rational decisions

Let us represent the virtual income $Q_i^k(r_i^k, J)$ as an explicit function of the firms' choices $r_1^k, \dots, r_1^k$. Note that fixing J is the same as assuming that all j from J

choose $r_j^k = 0$ and all j not belonging to J choose $r_j^k = r_j^{k+}$. Then, introducing

$$\lambda_{ij}^k(r_{ij}^k) = \begin{cases} 1 & \text{if} \quad r_j^k = 0, \\ 0 & \text{if} \quad r_j^k = r_j^{k+}, \end{cases}$$

we represent $Q_i^k(r_i^k, J)$ (2.13) as

$$\begin{aligned} Q_i^k(r_1^k, \ldots, r_n^k) &= \frac{q_i^k - p_i^k}{r_i^{k+}} r_i^k - \lambda_{i1}^k(r_{i1}^k) l_{i1}^k - \ldots - \lambda_{i\ i-1}^k(r_{i\ i-1}^k) l_{i\ i-1}^k - \\ &\quad \lambda_{i\ i+1}^k(r_{i\ i+1}^k) l_{i\ i+1}^k - \ldots - \lambda_{in}^k(r_{in}^k) l_{in}^k. \end{aligned}$$

This representation shows that the firms – each maximizing its income – act as players in an n-person game. In this *local* game $r_i^k \in \{r_i^{k+}, 0\}$ is the admissible action of player (firm) i and $Q_i^k(r_1^k, \ldots, r_n^k)$ is the payoff to this player.

The multioptimality of the boundedly rational decisions implies that in the local game the boundedly rational decision (2.10) of any firm i responds best to arbitrary combination, $(r_1^k, \ldots, r_{i-1}^k, r_{i+1}^k, \ldots, r_n^k)$, of admissible actions of other firms. In other words, no matter how r_j^k for $j \neq i$ are chosen, the boundedly rational decision of firm i maximizes $Q_i^k(r_1^k, \ldots, r_n^k)$ over r_i^k.

In particular, the boundedly rational decision of each firm responds best to the boundedly rational decisions of all other firms. This property characterizes the entire combination of the firm's boundedly rational decisions as a *Nash equilibrium* in the local game in period k (see, e.g., Vorobyov, 1985).

2.6 Informational aspect

Every firm is, obviously, interested in not spreading information about its key characteristics such as the production function, the costs for producing and maintaning technologies, and the structure of the technology stock accumulated in manufacturing. In the context of our model, each firm i views functions f_i and c_i, the technology stock ξ_i, and the coefficients a_{ij} characterizing the structure of ξ_i as its privat information. The technological evolution and exchange in technologies via market of patents should imply the minimum exchange in privat information.

Market of patents with boundedly rational decisionmakers meets this condition entirely. Indeed, in period k every two boundedly rational firms, i and j, communicate three times. First time, they announce their new technologies. Second time, they exchange with their payoff offers. Third time, they sell each other patents for new technologies (at this stage, one of the firms, or both of them, may decide not to sell the patents). At the first and third stages there is no information exchange between the firms. The most informative stage is offering payoffs. At this stage firm j indicates to firm i the required part, $a_{ji} r_i^{k+} \delta$, of the announced $r_i^{k+} \delta$ technologies. Thus, implicitly, the structural coefficient a_{ji} is communicated to firm i. Recall that the payoff offer of firm j to firm i, q_{ji}^k, is given by (2.7). Using this formula and knowing q_{ji}^k and a_{ji}, firm i is able to identify $f_j'(\xi_j^k)$, the derivative of the production function of firm j (marginal productivity of firm j) at the currently accumulated technology stock $\xi_j^k = a_{j1} x_1^k + \ldots + a_{jn} x_n^k$. This, obviously, does not give to firm i any information on the global shape of function f_j, the size of the currently accumulated technology stock ξ_j^k, and the structural coefficients a_{js} for $s \neq i$. Similar signals go from firm i to firm j. The exchange in information is, evidently, minimal.

3 Analysis of market trajectories

3.1 Assumptions and definitions

In the previous section we showed that the firms' boundedly rational decisions, $r_1^k, \ldots, r_n^k$, are best with respect to the firms' current interests in any period k. Our main gaol in this section is to show that the boundedly rational decisions drive the collection of the firms' technology stocks, $(x_1^k, \ldots, x_n^k)$, to the state $(\hat{x}_1, \ldots, \hat{x}_n)$, which is best for the firms' community as a whole; namely, $(\hat{x}_1, \ldots, \hat{x}_n)$ is the Pareto equilibrium, which maximizes the total profit of the firms' community (see (1.5)). A strict formulation of this key property will be given in Proposition 3.4 which will close our analysis. The existence and uniqueness of $(\hat{x}_1, \ldots, \hat{x}_n)$ maximizing the total profit will be stated in Proposition 3.3. The domain of attraction of boundedly rational trajectories and their behavior in the area of low technology stocks (covered in an initial interval of the evolution) will be characterized in Propositions 3.1 and 3.2.

In our analysis, we use several assumptions.

We assume that production of firm i, $f_i(\xi_i)$, grows with the technology stock ξ_i, and its growth rate, $f_i'(\xi_i)$, declines as ξ_i grows. So, the higher is the level of the accumulated technology stock, the smaller is the production increment gained through the implementation of a new technology unit.

We also assume that the cost of firm i for producing and maintaining x_i technologies, $c_i(x_i)$, grows with x_i and its growth rate, $c_i'(x_i)$, infinitely grows as ξ_i grows. Thus, the higher is the level of the active firm's technology stock, the higher is its cost for producing and maintaining a new technology unit, moreover, the latter cost approaches infinity at extremely high levels of the technology stock and vanishes at the origin.

Let us give more accurate formulations of the assumptions. We assume that for every $i = 1, \ldots, n$ function f_i is defined and twice continuously differentiable on the nonnegative half-interval $[0, \infty)$, strictly increasing, that is, $f_i'(\xi_i) > 0$ for all $\xi_i \geq 0$, and strictly concave, that is, $f_i''(\xi_i) < 0$ for all $\xi_i \geq 0$. Here and in what follows the right derivative is considered at the origin.

We assume that for every $i = 1, \ldots, n$ function c_i is defined and twice continuously differentiable on the nonnegative half-interval $[0, \infty)$, strictly increasing, that is, $c_i'(x_i) > 0$ for all $\xi_i \geq 0$, strictly convex, that is, $c_i''(\xi_i) \geq 0$ for all $\xi_i \geq 0$, and, finally, has the zero growth rate at the origin and infinite growth rate at infinity, that is,

$$c_i'(0) = 0, \tag{3.15}$$

$$\lim_{x_i \to \infty} c_i'(x_i) = \infty. \tag{3.16}$$

We assume $0 \leq a_{ij} \leq 1$ for all $i, j = 1, \ldots, n$ and $a_{ii} > 0$ for all $i = 1, \ldots, n$ (the latter assumption says that each firm utilizes a nonzero fraction of self-produced technologies).

Now we introduce constraints on the maximum rates of the firms' technology inputs, r_i^{k+}, and the rates of technology outflows, r_i^{k-} (see section 1). Namely, we assume that for every collection of technology stocks in period k, $(x_1^k, \ldots, x_n^k)$, where

$x_1^k, \dots, x_n^k > 0$, and every $i = 1, \dots, n$ we have

$$\rho_i^{++}(x_i^k) \geq r_i^{k+} \geq \rho_i^{+-}(x_i^k), \tag{3.17}$$

$$\rho_i^{-+}(x_i^k) \geq r_i^{k-} \geq \rho_i^{--}(x_i^k) > 0, \tag{3.18}$$

$$\rho_i^{+-}(x_i^k) > \rho_i^{-+}(x_i^k), \tag{3.19}$$

$$x_i^k > \rho_i^{+-}(x_i^k), \tag{3.20}$$

where ρ_i^{++}, ρ_i^{+-}, ρ_i^{-+}, ρ_i^{--} are given nonnegative continuous functions defined on the half-axis $[0, \infty)$.

In what follows, $(x_1^0, \dots, x_n^0)$ is an arbitrary initial combination of the firms' technology stocks, $x_1^0, \dots, x_n^0 > 0$.

Note that assumption (3.20) implies that every sequence $(x_1^k, \dots, x_n^k)$, $k = 0, 1, \dots$, of firms' technology stocks, which develops under the general transition formula (2.6) with arbitrary $r_i^k \geq 0$ $(i = 1, \dots n)$, satisfies the natural constraints $x_1^k, \dots, x_n^k > 0$, or, in other words, never abandons the positive orthant O^+; the latter is by definition the collection of all $(x_1, \dots, x_n)$ such that $x_1, \dots, x_n > 0$.

Let a sequence $(x_1^k, \dots, x_n^k)$, $k = 0, 1, \dots$, of firms' technology stocks be driven by the boundedly rational decisions, that is, for each $i = 1, \dots, n$ and each $k = 0, 1, \dots$ the next conditions hold:

(i) x_i^k and x_i^{k+1}, the technology stocks of firm i in periods k and $k+1$, satisfy the transition formula (2.6),

(ii) r_i^k, the decision of firm i in period k, is determined by the bounded rationality decisionmaking rule (2.10),

(iii) r_i^{k+} and r_i^{k-}, the maximum rate of the technology input and the rate of the technology outflow of firm i in period k, satisfy the constraints (3.17) and (3.18),

(iv) q_i^k, the total payoff offer to firm i in period k, is given by (2.8), (2.7), and

(v) p_i^k, the expenditure of firm i in period k, is given by (2.9).

We call the above sequence $(x_1^k, \dots, x_n^k)$, $k = 0, 1, 2, \dots$, a *boundedly rational trajectory* of the technology stocks on market of patents.

Note that a boundedly rational trajectory is, generally, not unique, although the bonded rationality decisionmaking rule (2.10) is well determined. The reason is that the maximum inflow rate r_i^{k+} and the outflow rate r_i^{k-} are determined not uniquely; they may take arbitrary values between $\rho_i^{+-}(x_i^k)$ and $\rho_i^{++}(x_i^k)$, and between $\rho_i^{--}(x_i^k)$ and $\rho_i^{-+}(x_i^k)$, respectively (see (3.17) and (3.18)).

3.2 Domain of attraction

All boundedly rational trajectories have a common *domain of attraction*, A, which is bounded and strictly separated from the origin. The domain A can be defined as the collection of all $(x_1, \dots, x_n)$ such that $\kappa \leq x_i \leq K$ $(i = 1, \dots, n)$ for some $\kappa > 0$ and $K > \kappa$. In standard mathematical notations,

$$A = \{(x_1, \dots, x_n) : \kappa \leq x_i \leq K \ (i = 1, \dots, n)\}. \tag{3.21}$$

The attraction property of A is interpreted as follows: every boundedly rational trajectory enters A in some finite period and then circulates in A. Thus, if market of patents is boundedly rational, then the technology stock of each firm can neither stay below κ forever, nor become lower than κ after visiting the region above κ; similarly, it can neither stay above K forever, nor become higher than K after visiting the region below K.

Let us give the accurate formulation.

Proposition 3.1 *There exist $\kappa > 0$ and $K > \kappa$ such that the set A given by (3.21) is the domain of attraction for the boundedly rational trajectories of the technology stocks in the following sense: if the time step δ is sufficiently small, then for every boundedly rational trajectory of the technology stocks, $(x_1^k, \ldots, x_n^k)$, $k = 0, 1, \ldots$, there is period k_* such that $(x_1^k, \ldots, x_n^k)$ lies in A for all $k \geq k_*$.*

The proposition is proved in Appendix.

3.3 Evolution in the area of low technology stocks

Our model shows that the area of low technology stocks (which is covered in an initial period of the evolution) is very favourable for the technological development and operations on market of patents. In this area all boundedly rational firms never interrupt the production of new technologies, and all patents are sold on market.

Here is the exact formulation.

Proposition 3.2 *There exists $\sigma > 0$ such that for every boundedly rational trajectory of the technology stocks, $(x_1^k, \ldots, x_n^k)$, $k = 0, 1, \ldots$, and every period k, for which $x_1^k, \ldots, x_n^k \leq \sigma$, the boundedly rational decision r_i^k (2.10) of any firm i is r_i^{k+} (firm i develops and sells patents for $r_i^{k+}\delta$ new technologies).*

A proof is given in Appendix.

3.4 Convergence to the best Pareto equilibrium

Now we consider the firms' total profit function, u, given by (1.5).

Proposition 3.3 *The total profit function u (1.5) has the unique maximizer, $(\hat{x}_1, \ldots, \hat{x}_n)$, in the positive orthant O^+.*

As noted in section 1 $(\hat{x}_1, \ldots, \hat{x}_n)$ is a Pareto equilibrium.

Our main, and final, result says that if the time step δ is sufficiently small, then every boundedly rational trajectory enters an arbitrarily small neighborhood of $(\hat{x}_1, \ldots, \hat{x}_n)$ in a finite period and stays there forever. Thus, the boundedly rational firms find the combination of technologies which is best for the firms' community as a whole and keep their technology stocks close to it.

Here is the accurate formulation.

Proposition 3.4 *For every $\varepsilon > 0$ there is $\delta_0 > 0$ having the following property: if the time step δ is smaller than δ_0, then for every boundedly rational trajectory of the technology stocks, $(x_1^k, \ldots, x_n^k)$, $k = 0, 1, \ldots$, there exists period k_0 such that for all periods $k \geq k_0$ and all firms $i = 1, \ldots, n$ it holds that $|x_i^k - \hat{x}_i| < \varepsilon$.*

Propositions 3.3 and 3.4 are proved in Appendix.

4 Conclusions

We presented a dynamical model of a community of firms whose technology stocks overlap. It was shown that market of patents allows the firms to organize flows of knowledge in a globally optimal and locally rational manner. Namely, market-driven combinations of firms' technology stocks may in the long run approach a point favourable for the firms' community as a whole, whereas in every local market operation the individual decision of every firm agrees with its current interest.

The presented model is to a considerable extend stylized. The basic simplifying assumptions are the following:

(i) the accumulated technology stock of a firm stays in a strict correspondence with labor, capital, materials, and energy use (subsection 1.1);

(ii) the structural coefficients, a_{ji}, are constant (subsection 1.2);

(iii) the whole production is sold and prices are constant (subsection 1.3);

(iv) the prior descriptions of the announced new technologies are complete and payoff offers ideally "fair" (subsection 2.2);

(v) long-term R&D projects are neglected.

In this context one may think about extensions of the presented analysis under reasonably weakened assumptions (i) – (v) as natural further steps in the analytical treatment of market of patents.

5 Appendix

Here we prove Propositions 3.1 – 3.4.

We start with a technical lemma following from the assumptions on functions f_i and c_i.

Lemma 5.1 *There exist a positive K_0 and a positive $\kappa_0 < K$ such that*

$$c_i'(x_i) > \sum_{j=1}^{n} f_j'(a_{j1}x_1 + \ldots + a_{jn}x_n)a_{ji} \tag{5.22}$$

for all $i = 1, \ldots, n$, all $x_i \geq K_0$, and all $x_j > 0$, $j \neq i$, and

$$c_i'(x_i) < \sum_{j=1}^{n} f_j'(a_{j1}x_1 + \ldots + a_{jn}x_n)a_{ji} \tag{5.23}$$

for all $i = 1, \ldots, n$ and all positive $x_i \leq \kappa_0$ and $x_j \leq K_0$, $j \neq i$.

Proof. Take i between 1 and n. Fix a positive η. By assumption f_j' is decreasing. Then for all j, all $x_i \geq \eta$, and all $x_s > 0$, $s \neq i$, we have

$$f_j'(a_{j1}x_1 + \ldots + a_{ji}x_i + \ldots + a_{jn}x_n)a_{ji} \leq f_j'(a_{ji}x_i)a_{ji} \leq f_j'(a_{ji}\eta)a_{ji}.$$

Since c_i' is infinitely increasing (see (3.16)), there is $K_0 \geq \eta$ such that (5.22) holds for all $x_i \geq K_0$ and all $x_j > 0$, $j \neq i$. Without loss of generality we set K_0 to be the same for all i.

By assumption f_i' is positive, $c_i'(0) = 0$ (see (3.15)) and $a_{ii} > 0$. Then there is a positive $\kappa_0 < K_0$ such that

$$f_i'(a_{11}K_0 + \ldots + a_{ii}\kappa_0 + \ldots + a_{nn}K_0)a_{ii} > c_i'(x_i)$$

for all positive $x_i \le \kappa_0$ and $x_j \le K$, $j \ne i$. Taking into account that f_i' is decreasing, we get

$$\begin{aligned}\sum_{j=1}^n f_j'(a_{j1}x_1 + \ldots + a_{jn}x_n)a_{ji} &\ge f_i'(a_{11}x_1 + \ldots + a_{nn}x_n)a_{ii} \\ &\ge f_i'(a_{11}K_0 + \ldots + a_{ii}\kappa_0 + \ldots + a_{nn}K_0)a_{ii} \\ &> c_i'(x_i)\end{aligned}$$

for all positive $x_i \le \kappa_0$ and $x_j \le K$, $j \ne i$. Hence, for all these $x_1, \ldots, x_n$ we have (5.23). Without loss of generality we set κ_0 to be the same for all i. The lemma is proved.

Below, we fix constants κ_0 and K_0 introduced in Lemma 5.1. In what follows $(x_1^k, \ldots, x_n^k)$, $k = 0, 1, \ldots$, is an arbitrary boundedly rational trajectory of the technology stocks. Recall that by definition for all $i = 1, \ldots, n$ and all $k = 0, 1, \ldots$ conditions (i) through (v) of subsection 3.1 are satisfied.

The next assertion follows easily from Lemma 5.1.

Lemma 5.2 *For every $k = 0, 1, \ldots$ and every $i = 1, \ldots, n$ the next statements hold true:*

(i) if $x_i^k \ge K_0$, then $r_i^k = 0$,

(ii) if $x_1^k, \ldots, x_n^k \le K_0$ and $x_i^k \le \kappa_0$, then $r_i^k = r_i^{k+}$.

Proof. Let $x_i^k \ge K_0$. Then by Lemma 5.1 (5.22) holds for $(x_1, \ldots, x_n) = (x_1^k, \ldots, x_n^k)$:

$$c_i'(x_i^k) > \sum_{j=1}^n f_j'(a_{j1}x_1^k + \ldots + a_{jn}x_n^k)a_{ji}.$$

Multiplying by the positive factor $(r^{k+} - r^{k-})\delta$ and using the notations (2.7) and (2.9), we represent the inequality as $p_i^k > q_{1i}^k + \ldots + q_{ni}^k$, or, recalling the notation (2.8), as $q_i^k < p_i^k$. The decisionamig rule (2.10) yields $r_i^k = 0$.

Let $x_1^k, \ldots, x_n^k \le K_0$ and $x_i^k \le \kappa_0$. Then by Lemma 5.1 (5.23) holds for $(x_1, \ldots, x_n) = (x_1^k, \ldots, x_n^k)$:

$$c_i'(x_i^k) < \sum_{j=1}^n f_j'(a_{j1}x_1^k + \ldots + a_{jn}x_n^k)a_{ji}.$$

Multiplying by the positive factor $(r^{k+} - r^{k-})\delta$ and using the notations (2.7), (2.8), (2.9), we represent the inequality as $q_i^k > p_i^k$. The decisionamig rule (2.10) yields $r_i^k = r_i^{k+}$. The lemma is proved.

Proof of Proposition 3.1. Fix $K > K_0$ and a positive $\kappa < \kappa_0$ and define A by (3.21). Choose $R > 0$ so that

$$R \ge \max\{\rho_i^{++}(x_i) : 0 \le x_i \le K\}$$

for all $i = 1, \dots n$. Let the time step δ satisfy

$$\delta < \min\left\{\frac{K - K_0}{R}, \frac{\kappa_0 - \kappa}{R}\right\}.$$

For the boundedly rational trajectory $(x_1^k, \dots, x_n^k)$, $k = 0, 1, \dots$, the transition formula (2.6) and estimates (3.17), (3.18) yield

$$|x_i^{k+1} - x_i^k| \le |r_i^k - r_i^{k-}|\delta \le R\delta < \min\{K - K_0, \kappa_0 - \kappa\} \tag{5.24}$$

if $x_i^k \le K$. Take some i between 1 and n. Assume that $x_i^0 > K$. Choose $r > 0$ so that

$$r \le \min\{\rho_i^{--}(x_i) : K_0 \le x_i \le x_i^0\};$$

r exists since ρ^{--} is continuous and takes positive values at positive arguments (see (3.18)). By Lemma 5.2, (i), $r_i^0 = 0$. Hence, the transition formula (2.6) and estimates (3.18) yield

$$x_i^1 = x_i^0 - r_i^{0-}\delta \le x_i^0 - \rho^{--}(x_i^0)\delta \le x_i^0 - r\delta.$$

If $x_i^1 > K$, we replace in the previous argument $k = 0$ sequentially by $k = 1, 2, \dots$ and, finally, arrive at a finite period k_i such that for all $k = 0, 1, \dots, k_{i-1}$ we have $x_i^{k+1} \le x_i^k - r\delta$ and $x_i^k > K$, and $x_i^{k_i} \le K$. If $x_i^0 \le K$, we set $k_i = 0$. Thus, we stated that there is k_i such that $x_i^{k_i} \le K$. Let us show that $x_i^k \le K$ for all $k \ge k_i$. Assume that, to the contrary, $x_i^{k+1} > K$ for some $k \ge k_i$. With no loss of generality assume $x_i^k \le K$. Due to (5.24)

$$x_i^k \ge x_i^{k+1} - (K - K_0) \ge K - (K - K_0) = K_0.$$

Then by Lemma 5.2, (i), $r_i^k = 0$, hence, by (2.6)

$$x_i^{k+1} = x_i^k - r_i^{k-}\delta \le x_i^k \le K.$$

The contradicuion with the assumption $x_i^{k+1} > K$ proves that $x_i^k \le K$ for all $k \ge k_i$. Let k^* be the maximum of $k_1, \dots, k_n$. Obviously, we have $x_i^k \le K$ for all $i = 1, \dots, n$ and all $k \ge k^*$.

Let us show that there is $k_* \ge k^*$ such that $x_i^k \ge \kappa$ for all $i = 1, \dots, n$ and all $k \ge k^*$. This will complete the proof. Take some i. Assume that $x_i^{k^*} < \kappa$. Choose $r > 0$ so that

$$r \le \min\{\rho_i^{+-}(x_i) - \rho_i^{-+}(x_i) : x_i^{k^*} \le x_i \le \kappa_0\};$$

r exists since ρ^{+-} and ρ^{-+} are continuous and their difference is positive at positive arguments (see (3.19)). By Lemma 5.2, (ii), $r_i^{k^*} = r_i^{k^*+}$. Hence, the transition formula (2.6) and estimates (3.18) yield

$$x_i^{k^*+1} = x_i^{k^*} + (r_i^{k^*+} - r_i^{k^*-})\delta \ge x_i^0 + (\rho^{-+}(x_i^{k^*}) - \rho^{-+}(x_i^{k^*}))\delta \ge x_i^0 + r\delta.$$

If $x_i^{k^*+1} < \kappa$, we replace in the previous argument $k = k^*$ sequentially by $k = 1, 2, \dots$ and, finally, arrive at a finite period s_i such that for all $k = k^*, k^* + 1, \dots, s_i - 1$ we have $x_i^{k+1} \ge x_i^k + r\delta$ and $x_i^k < \kappa$, and $x_i^{s_i} \ge \kappa$. If $x_i^{k^*} \ge \kappa$, we set $s_i = k^*$. Thus, we stated that there is s_i such that $x_i^{s_i} \ge \kappa$. Let us show that for each $k \ge s_i$ we have $x_i^k \geq \kappa$. Assume that, to the contrary, $x_i^{k+1} < \kappa$ for all $i = 1, \ldots, n$ and some $k \geq s_i$. With no loss of generality assume $x_i^k \geq \kappa$. Due to (5.24)

$$x_i^k \leq x_i^{k+1} - (\kappa_0 - \kappa) \leq \kappa - (\kappa_0 - \kappa) = \kappa_0.$$

Then by Lemma 5.2, (ii), $r_i^k = r_i^{k+}$, hence, by (2.6)

$$x_i^{k+1} = x_i^k + (r_i^{k+} - r_i^{k-})\delta \geq x_i^k \geq \kappa.$$

The contradiction with the assumption $x_i^{k+1} < \kappa$ proves that $x_i^k \geq \kappa$ for all $k \geq s_i$. Let k_* be the maximum of $s_1, \ldots, s_n$. Obviously, we have $x_i^k \geq \kappa$ for all $i = 1, \ldots, n$ and all $k \geq k_*$. The proposition is proved.

Proof of Proposition 3.2. Set $\sigma = \kappa_0$. Now the proposition follows from Lemma 5.2, (ii).

Proof of Proposition 3.3. By (1.5) and (1.4)

$$\frac{\partial u(x_1, \ldots, x_n)}{\partial x_i} = \sum_{j=1}^{n} f_j'(a_{j1}x_1 + \ldots + a_{jn}x_n)a_{ji} - c_i'(x_i). \tag{5.25}$$

By Lemma 5.1 for all $x_1, \ldots, x_n \geq K_0$ and all $i = 1, \ldots, n$ the inequality (5.22) holds. Hence, $\partial u(x_1, \ldots, x_n)/\partial x_i < 0$ for all $x_1, \ldots, x_n \geq K_0$ and all $i = 1, \ldots, n$. Consequently, for every $x_1, \ldots, x_n > K_0$ there are nonnegative $\bar{x}_1, \ldots, \bar{x}_n \leq K_0$ such that $u(\bar{x}_1, \ldots, \bar{x}_n) > u(x_1, \ldots, x_n)$. Therefore, u has a maximizer, $(\hat{x}_1, \ldots, \hat{x}_n)$, in the set of all $x_1, \ldots, x_n \geq 0$, and, moreover $\hat{x}_1, \ldots, \hat{x}_n \leq K_0$. Since u is strictly concave, the maximizer is unique. It remains to show that $\hat{x}_1, \ldots, \hat{x}_n > 0$. Assume that, to the contrary, $\hat{x}_i = 0$ for some i. Then by (3.15) and (5.25)

$$\begin{aligned}\frac{\partial u(\hat{x}_1, \ldots, \hat{x}_n)}{\partial x_i} &= f_1'(a_{11}\hat{x}_1 + \ldots + a_{1n}\hat{x}_n)a_{1i} + \ldots + \\ &\quad f_n'(a_{n1}\hat{x}_1 + \ldots + a_{nn}\hat{x}_n)a_{ni} > 0.\end{aligned}$$

Hence, for a sufficiently small $\varepsilon > 0$

$$\begin{aligned}u(\hat{x}_1, \ldots, \hat{x}_i + \varepsilon, \ldots, \hat{x}_n) &= u(\hat{x}_1, \ldots, \hat{x}_i, \ldots, \hat{x}_n) + \frac{\partial u(\hat{x}_1, \ldots, \hat{x}_n)}{\partial x_i}\varepsilon + o(\varepsilon) \\ &> u(\hat{x}_1, \ldots, \hat{x}_n),\end{aligned}$$

which is not possible since $(\hat{x}_1, \ldots, \hat{x}_n)$ is the maximizer for u. The proposition is proved.

Proof of Proposition 3.4. Below we consider only $k \geq k_*$, for which $(x_1^k, \ldots, x_n^k)$ lies in the domain A (see Proposition 3.1). Let us show that $u(x_1^k, \ldots, x_n^k)$ (1.5) increases as $k \geq k_*$ grows. Using (2.6), we get

$$\begin{aligned}&u(x_1^{k+1}, \ldots, x_n^{k+1}) - u(x_1^k, \ldots, x_n^k) \\ &= \sum_{i=1}^{n} \frac{\partial u(x_1^k, \ldots, x_n^k)}{\partial x_i}(x^{k+1} - x_i^k) + o^k(\delta) \\ &= \sum_{i=1}^{n} \frac{\partial u(x_1^k, \ldots, x_n^k)}{\partial x_i}(r_i^k - r_i^{k-})\delta + o^k(\delta);\end{aligned} \tag{5.26}$$

the fact that $(x_1^k, \ldots, x_n^k)$ lies in the bounded domain A together with the continuous differentiability of u, imply that $o^k(\delta)$ is second order in δ uniformly with respect to k; namely, for arbitrary $\mu > 0$ we have

$$\max_{k \geq k_*} |o^k(\delta)| \leq \mu\delta \tag{5.27}$$

provided δ is sufficiently small. According to (5.25) and the notations (2.7), (2.8), (2.9),

$$\frac{\partial u(x_1^k, \ldots, x_n^k)}{\partial x_i}(r_i^k - r_i^{k-})\delta = q_i^k - p_i^k.$$

Since $(r_i^k - r_i^{k-})\delta > 0$, the sign of $\partial u(x_1^k, \ldots, x_n^k)/\partial x_i$ concides with the sign of $q_i^k - p_i^k$. Hence, the decisionmaking rule (2.10) prescribes

$$r_i^k = \begin{cases} r_i^{k+} & \text{if } \dfrac{\partial u(x_1^k, \ldots, x_n^k)}{\partial x_i} \geq 0, \\ 0 & \text{if } \dfrac{\partial u(x_1^k, \ldots, x_n^k)}{\partial x_i} < 0, \end{cases}$$

and we get

$$\frac{\partial u(x_1^k, \ldots, x_n^k)}{\partial x_i}(r_i^k - r_i^{k-})\delta = \begin{cases} \left|\dfrac{\partial u(x_1^k, \ldots, x_n^k)}{\partial x_i}\right| |r_i^{k+} - r_i^{k-}|\delta & \text{if } \dfrac{\partial u(x_1^k, \ldots, x_n^k)}{\partial x_i} \geq 0, \\ \left|\dfrac{\partial u(x_1^k, \ldots, x_n^k)}{\partial x_i}\right| |r_i^{k-}|\delta & \text{if } \dfrac{\partial u(x_1^k, \ldots, x_n^k)}{\partial x_i} < 0, \end{cases}$$

Substituting in (5.26), we find:

$$u(x_1^{k+1}, \ldots, x_n^{k+1}) - u(x_1^k, \ldots, x_n^k) \geq \sum_{i=1}^n |\frac{\partial u(x_1^k, \ldots, x_n^k)}{\partial x_i}|\zeta_i^k \delta + o^k(\delta)$$

where

$$\zeta_i^k = \min\{|r_i^{k+} - r_i^{k-}|, |r_i^{k-}|\}.$$

Choose $r_1 > 0$ and $r_2 > 0$ so that

$$r_1 \leq \min\{\rho_i^{+-}(x_i) - \rho_i^{-+}(x_i) : \kappa \leq x_i \leq K\},$$

$$r_2 \leq \min\{\rho_i^{--}(x_i) : \kappa \leq x_i \leq K\}$$

for all $i = 1, \ldots, n$; r_1 and r_2 exist since ρ^{--} and $(\rho^{+-} - \rho^{-+})$ are continuous and take positive values at positive arguments (see (3.18) and (3.19)). Then $\zeta_i^k \geq r = \min\{r_1, r_2\}$, and we have

$$u(x_1^{k+1}, \ldots, x_n^{k+1}) - u(x_1^k, \ldots, x_n^k) \geq \sum_{i=1}^n \left|\frac{\partial u(x_1^k, \ldots, x_n^k)}{\partial x_i}\right| r\delta + o^k(\delta). \tag{5.28}$$

This estimate shows that if δ is sufficientl small, then, in a region where not all partial derivatives of u are close to zero, the growth rate of $u(x_1^k, \ldots, x_n^k)$ is bounded from below by a positive constant. The region, in which all partial derivatives of u

are close to zero, is located in a neighborhood of $(\hat{x}_1, \dots, \hat{x}_n)$, the maximizer of u. Since $u(x_1^k, \dots, x_n^k)$ cannot grow to infinity, it must approach $(\hat{x}_1, \dots, \hat{x}_n)$.

This is the logical pattern of the rest of our proof. Now we argue accurately.

The continuous function u is bounded on the bounded domain A. Therefore, there is $M > 0$ such that

$$|u(x_1^k, \dots, x_n^k)| \le M \tag{5.29}$$

for all $k \ge k_*$. For any $\gamma > 0$ let A_γ^+ denote the set of all $(x_1^k, \dots, x_n^k)$ from A such that

$$\sum_{i=1}^n \left| \frac{\partial u(x_1^k, \dots, x_n^k)}{\partial x_i} \right| > \gamma$$

and A_γ^- denote its complement in A, that is, the set of all $(x_1^k, \dots, x_n^k)$ from A such that

$$\sum_{i=1}^n \left| \frac{\partial u(x_1^k, \dots, x_n^k)}{\partial x_i} \right| \le \gamma.$$

Take an arbitrarily small $\gamma > 0$. Let us show that if δ is small enough, then there is $k^- \ge k_*$ such that $(x_1^{k^-}, \dots, x_n^{k^-})$ lies in A_γ^-. We choose δ so small that (5.27) holds with $\mu = r/2$. If $(x_1^{k_*}, \dots, x_n^{k_*})$ lies in A_γ^-, we set $k^- = k_*$. Assume that $(x_1^{k_*}, \dots, x_n^{k_*})$ does not lie in A_γ^-, that is, belongs to A_γ^+. Suppose that the desired k^- does not exist. Then $(x_1^k, \dots, x_n^k)$ lies in A_γ^+ for all $k \ge k_*$. Hence, by (5.28)

$$u(x_1^{k+1}, \dots, x_n^{k+1}) - u(x_1^k, \dots, x_n^k) \ge r\delta\gamma + o(\delta) \ge \frac{r\gamma}{2}$$

for all $k \ge k_*$. Then

$$\lim_{k \to \infty} u(x_1^{k+1}, \dots, x_n^{k+1}) = \infty$$

which contradicts (5.29). The contradiction shows that there is $k^- \ge k_*$ such that $(x_1^k, \dots, x_n^k)$ lies in A_γ^-. By Proposition 3.3 $(\hat{x}_1, \dots, \hat{x}_n)$ is the unique maximizer of u in the positive orthant O^+. This fact and the strict concavity of u imply that the point $(\hat{x}_1, \dots, \hat{x}_n)$ is uniquely determined by the condition that at this point all partial derivatives of u vanish. Hence, choosing arbitrary $\nu > 0$, we find that if $\gamma > 0$ is small enough, the set A_γ^- is contained in the ν-neighborhood of $(\hat{x}_1, \dots, \hat{x}_n)$. We assume that γ is chosen this way in advance. The choice of ν will be specified in the next paragraph.

Denote $\hat{u} = (\hat{x}_1, \dots, \hat{x}_n)$. Take arbitrary $\varepsilon > 0$. The strict concavity of u implies that there exists $\alpha > 0$ such that all $(x_1, \dots, x_n)$, for which $u(x_1, \dots, x_n) \ge \hat{u} - \alpha$, lie in the ε-neighborhood of $(\hat{x}_1, \dots, \hat{x}_n)$. We assume that ν is so small that for all $(x_1, \dots, x_n)$ from the ν-neighborhood of $(\hat{x}_1, \dots, \hat{x}_n)$ the inequality $u(x_1, \dots, x_n) \ge \hat{u} - \alpha/2$ holds. In particular, we have $u(x_1^{k^-}, \dots, x_n^{k^-}) \ge \hat{u} - \alpha/2$.

Now we will show that δ can be chosen so small that

$$u(x_1^k, \dots, x_n^k) \ge \hat{u} - \alpha \tag{5.30}$$

for all $k \ge k^-$. Then by the definition of α we will have $(x_1^k, \dots, x_n^k)$ in the ε-heighborhood of $(\hat{x}_1, \dots, \hat{x}_n)$ for all $k \ge k^-$, which will complete the proof.

Since the partial derivatives of u are uniformly continuous on A, for sufficienty small δ we have

$$\sum_{i=1}^{n}\left(\left|\frac{\partial u(x_1^k,\ldots,x_n^k)}{\partial x_i}\right| - \left|\frac{\partial u(x_1^k,\ldots,x_n^k)}{\partial x_i}\right|\right) < \frac{\omega}{2} \tag{5.31}$$

for all $k \geq k_*$. Let δ satisfy this constraint. Since $(\hat{x}_1,\ldots,\hat{x}_n)$ is the unique maximizer of u, there is $\omega > 0$ such that all $(x_1,\ldots,x_n)$ satisfying $u(x_1,\ldots,x_n) < \hat{u} - \alpha$ lie in A_ω^+, that is,

$$\sum_{i=1}^{n}\left|\frac{\partial u(x_1,\ldots,x_n)}{\partial x_i}\right| > \omega.$$

We assume that δ is so small that (5.27) holds with $\mu = \omega/4$, that is,

$$o^k(\delta) \leq \frac{\omega\delta}{4}. \tag{5.32}$$

Now suppose that (5.30) does not hold for all $k \geq k^-$, and, therefore,

$$u(x_1^{m+1},\ldots,x_n^{m+1}) < \hat{u} - \alpha \tag{5.33}$$

for some $m \geq k^-$. Without loss of generality we assume

$$u(x_1^m,\ldots,x_n^m) \geq \hat{u} - \alpha. \tag{5.34}$$

The assumption (5.33) implies

$$\sum_{i=1}^{n}\left|\frac{\partial u(x_1^{m+1},\ldots,x_n^{m+1})}{\partial x_i}\right| > \omega.$$

Then by (5.31)

$$\sum_{i=1}^{n}\left|\frac{\partial u(x_1^m,\ldots,x_n^m)}{\partial x_i}\right| > \frac{\omega}{2}.$$

In other words, $(x_1^m,\ldots,x_n^m)$ lies in $A_{\omega/2}^+$. Now, arguing as in the proof of (5.28) (repacing k by m and r by $\omega/2$) and using (5.32) and (5.34), we find that

$$\begin{aligned} u(x_1^{m+1},\ldots,x_n^{m+1}) &\geq u(x_1^m,\ldots,x_n^m) + \sum_{i=1}^{n}\left|\frac{\partial u(x_1^m,\ldots,x_n^m)}{\partial x_i}\right|\delta + o(\delta) \\ &\geq u(x_1^m,\ldots,x_n^m) + \frac{\omega\delta}{2} - \frac{\omega\delta}{4} > u(x_1^m,\ldots,x_n^m) \\ &\geq \hat{u} - \alpha, \end{aligned}$$

which contradicts (5.33). The proof is completed.

References

1. Arrow, K.L., and Kurz, M., 1970, Public Investment, the Rate of Return and Optimal Fiscal Policy, John Hopkins University Press, Baltimore.

2. Axelrod, R., 1984, The Evolution of Cooperation, Basic Books, New York.

3. Basar, T., and Olsder, G.J., 1982, Dynamic Noncooperative Game Theory, Academic Press, London.

4. Borisov, V.F., Hutschenreiter, G., and Kryazhimskii, A.V., 1999, Asymptotic growth rates in knowledge-exchanging economies, to appear in Annals of Operations Research.

5. Fudenberg, D., and Krebs, D.M., 1993, Learning mixed equilibria, Games and Econ. Behavior, Vol. 5, 320-367.

6. Germeyer, Yu.B., 1976, Games with Nonantagonistic Interests, Nauka, Moscow (in Russian).

7. Grossman, G.M., and Helpman, E., 1991, Innovation and Growth in the Global Economy, M.I.T. Press, Cambridge.

8. Hutschenreiter, G., Kaniovski, Yu.M., and Kryazhimskii, A.V., 1995, Endogenous growth, absorptive capacities, and international R&D spillovers, Working Paper WP-95-92, International Institute for Applied Systems Anlysis, Laxenburg, Austria, 1995.

9. Intriligator, M., 1971, Mathematical Optimization and Economic Theory, Prentice-Hall, New York.

10. Jaffe, A.B., 1986, Technological opportunity and spillovers of R&D: evidence from firm's patents, profits and market value, The American Economic Review, Vol. 76, No 5, 984-1001.

11. Kaniovski, Yu.M., and Young, H.P., 1995, Learning dynamics in games with stochastic perturbations, Games and Econ. Behavior, Vol. 11, 330-363.

12. Nowak, M., and Sigmund, K., 1992, The evolution of stochastic strategies in the Prisoner's Dilemma, Acta Applic. Math., Vol. 20, 247-265.

13. Smale, S., 1980, The Prisoner's Dilemma and dynamical systems associated to non-cooperative games, Econometrica, Vol. 48, 1617-1634.

14. Vorobyov, N.N., 1985, Games Theory, Nauka, Moscow (in Russian).

15. Watanabe, C., 1992, Trends in substitution of production factors to technology – empirical analysis of the inducing impact of the energy crises on Japaneese industrial technology, Research Policy, Vol. 21, 481-505.

16. Watanabe, C., Zhu, B., and Asgari, B., 1998, Analysis on global technology spillover and its impact to technology management, Abstract of Annual Conference of the Japan Society for Science Policy and Research Management (Tokyo, 1998), 369-374.