RM-76-49

SENSITIVITY ANALYSIS

Frans Willekens

June 1976

Preface

Interest in human settlement systems and policies has been a critical part of urban-related work at IIASA since its inception. Recently this interest has given rise to a concentrated research effort focusing on migration dynamics and settlement patterns. Four sub-tasks form the core of this research effort:

I. the study of spatial population dynamics;

II. the definition and elaboration of a new research area called demometrics and its application to migration analysis and spatial population forecasting;

III. the analysis and design of migration and settlement policy;

IV. a comparative study of national migration and settlement patterns and policies.

This paper, the eighth in the spatial population dynamics series, examines the dynamics of structural change in spatial demographic systems by extending the single-region formulas of mathematical demographers such as Goodman and Keyfitz to the multiregional case. It was written here at IIASA this past year as part of a doctoral dissertation submitted to Northwestern University and was financially supported by a research fellowship awarded to Willekens by the Institute.

Willeken's study illuminates an important aspect of our work in migration processes and settlement patterns. He uses matrix differentiation techniques to develop sensitivity functions which link changes in various age-specific rates to corresponding changes in important multiregional demographic parameters. In this way he is able to develop a uniform procedure for tracing through the impacts of changes in fertility, mortality, and migration.

Related papers in the spatial population dynamics series and other publications of the migration and settlement study are listed on the back page of this report.

A. Rogers
June 1976

Abstract

This paper studies the impact on major population characteristics of changes in structural demographic parameters. The parameters considered are age-specific fertility, mortality and migration rates. Applying the technique of matrix differentiation, sensitivity functions are derived which link changes in important multiregional demographic statistics, such as life-table statistics and population growth and stable population characteristics, to changes in age-specific rates. In addition it is shown how the discrete and continuous models of population growth may be reconciled.

Acknowledgements

This paper is part of my Ph.D. dissertation, entitled The Analytics of Multiregional Population Distribution Policy and submitted to the Graduate School of Northwestern University, Evanston, U.S.A. During the development of this study, as during my whole Ph.D. program, I have benefited from the close cooperation of Dr. A. Rogers, my adviser. His ideas and experience have been most valuable and I am extremely grateful to him.

I also would like to thank all the people who contributed, directly and indirectly, to this study. In particular, I am indebted to the other dissertation committee members: Professors J. Blin, G. Peterson and W. Pierskalla.

This study has been written at IIASA where I was a research assistant. The intellectual atmosphere and the scientific services at IIASA have largely stimulated my work.

The burden of typing the manuscript was borne by Linda Samide. She performed the difficult task of transforming my confusing handwriting into a final copy with great skill and good humour.

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CHAPTER 1

INTRODUCTION

The field of mathematical demography is concerned with the mathematical description of how fertility and mortality combine to determine the characteristics of population, and to shape their growth. Traditionally, demographers [e.g., Keyfitz (1968) and Coale (1972)] have restricted their attention to fertility and mortality, assuming in fact that populations are "closed" to migration, i.e., populations undisturbed by in- and outmigration. This is an unrealistic assumption, especially in population analysis at the subnational level. The introduction of migration into mathematical demography has been pioneered by Rogers (1975). He describes, in analytical terms, how fertility, mortality and migration combine to determine the features and the growth of multiregional population systems. The basic tool used is matrix algebra.

Mathematical demography demonstrates how various demographic characteristics may be expressed in terms of observed age-specific fertility, mortality and migration rates. The fundamental assumptions underlying the models is that the age-specific rates, i.e., the structural parameters, are known exactly and that they remain fixed over time. The implications of this are expressed by Keyfitz (1968; p. 27): "The object (of population projection) is to understand the past rather than to predict the future; apparently the way to think effectively about an observed set of birth and death rates is to ask what it would lead to if continued."

No one truly believes that fertility, mortality and migration schedules are measured without observation error and that they will remain unchanged for a prolonged period of time. However, variations in structural parameters have not been considered until recently (e.g., Keyfitz, 1971; Goodman, 1969, 1971b; Preston, 1974).

It is the purpose of this paper to contribute to a better understanding of the impact on the population system of changes in its structural parameters. The system considered is a multiregional demographic system, described in Rogers (1975). The parameters are the age-specific fertility, mortality and migration rates. In general terms, the problem is to find how sensitive stationary population characteristics, population projections, and stable population characteristics are to changes in age-specific rates.

The sensitivity of the stable characteristics of population systems undisturbed by migration have received most attention. That most effort has been devoted to the stable population becomes clear if one recalls that the stable population concept was developed as a device which displays the implications for age composition, birth rates, death rates, and growth rates of specified schedules of fertility and mortality, on the assumption that the schedules prevail long enough for other influences to be erased. In actual fact, however, the stable population is never achieved, since the basic schedules change through time. The question of the impact of such changes on the stable population therefore is principally one of theoretical rather than empirical importance.

Two approaches to impact analysis may be distinguished. The first is the simulation approach, or the arithmetic approach as Keyfitz (1971; p. 275) calls it. It is simply the computation of the population projection under the old and the new rates. The difference between the two in the ultimate age distribution and other features gives the impact of changing the rates. Suitable tools for the simulation approach are provided by the model life tables and model stable populations such as those developed by Coale and Demeny (1966) for a single-region demographic system and by Rogers (1975; Chapter 6) for a multiregional system. An illustration of this approach has been given by Rogers (1975; pp. 169-172) and Rogers and Willekens (1975; pp. 28-30). Besides its demanding character in terms of computer time, the approach tells us nothing about the complete set of parameters on which the changes in the final results depend. It will be found useful, however, for verifying the results of the second approach, which is the analytical approach. This procedure derives a general formula for assessing the impact of a particular change in terms of well-known population variables. Such a formula will be designated as a sensitivity function. Partial differentiation will be seen to be the basic ingredient in the analysis of such functions.

In this paper, impact analysis is performed using the analytical approach. It is assumed that all the functions are differentiable with respect to the variables in which the changes occur. Since multiregional demographic models are formulated in matrix terms, matrix differentiation techniques are applied. And because not much work has been done in the area of matrix calculus, the first section of the Appendix to this paper reviews several relevant topics of such a calculus[1].

In order to be able to study the sensitivity of the stable population characteristics, we need an additional piece of information. All stable population features may be expressed as functions of the stable population distribution, the growth ratio of the stable population, and the age-specific fertility, mortality and migration rates. Therefore, the prerequisite to impact analysis of the stable population is a knowledge of the sensitivity of the stable population distribution and the stable growth ratio to changes in the age-specific rates.

Rogers (1975; p. 128) has shown that the stable growth ratio is the dominant eigenvalue of the growth matrix, and that the stable population distribution is the associated right eigenvector. The problem may, therefore, be reformulated as finding the sensitivity of the dominant eigenvalue and eigenvector to changes in the growth matrix, and the sensitivity of the elements of the growth matrix to changes in the age-specific rates that are used to define it.

The problem of eigenvalue and eigenvector sensitivity has received some attention in the engineering literature (e.g., Cruz, 1970; Part III). An overview of the major

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[1]All major textbooks on matrix algebra lack a chapter on matrix calculus, although some scattered treatment may occur. The only unified treatment of matrix differentiation that we have found is by Dwyer and MacPhail (1948). A simplified and extended version appeared twenty years later in Dwyer (1967). The formulas given there are general enough to handle differentiation problems in life table functions and in the analysis of population projections over a finite time horizon.

relevant results of this literature is given in the second section of the Appendix. It is worth noting at this point that the application of this technique in population dynamics is not restricted to the stable population. This technique is relevant in every situation where the eigenvalues of a particular matrix have some demographic meaning. For instance, Rogers and Willekens (1975; p. 39) state that the dominant eigenvalue of the net reproduction matrix of a multiregional population system represents the net reproduction rate of the whole system. Hence examining the impact on the net reproduction rate of the United States of a change in the net reproduction rate of rural-born women living in urban areas, is a problem of eigenvalue sensitivity analysis.

CHAPTER 2

IMPACT OF CHANGES IN AGE-SPECIFIC RATES ON LIFE TABLE FUNCTIONS

The concept of a multiregional life table as developed by Rogers (1973 and 1975, Chapter 3) is a device for exhibiting the mortality and migration history of a set of regional cohorts as they age. It is assumed that the age-specific rates describing the mortality and mobility experience of an actual population remain constant, and that the system of regions is undisturbed by external migration.

The first part of this chapter sets out the life table functions. The cohorts we will consider are birth cohorts or radices. Their life history is of special interest because they provide the information required by population projection models. The life table statistics are given by place of birth. In the second part, we combine the life table functions with the matrix differentiation techniques described in the Appendix. This enables us to develop life table sensitivity functions.

2.1. THE MULTIREGIONAL LIFE TABLE

All the life table functions are derived from a set of age-specific death and out-migration rates. Let $\underset{\sim}{M}(x)$ denote the matrix of observed annual rates for the persons in the age interval from x to x + h. The length of the interval h is arbitrary. Without loss of generality, we will consider age intervals of five years. For a N-region system, $\underset{\sim}{M}(x)$ is

$$\underset{\sim}{M}(x) = \begin{bmatrix} \left(M_{1\delta}(x) + \sum_{j\neq 1}^{N} M_{1j}(x)\right) & -M_{21}(x) & -M_{31}(x) & \cdots \\ -M_{12}(x) & \left(M_{2\delta}(x) + \sum_{j\neq 2}^{N} M_{2j}(x)\right) & -M_{32}(x) & \cdots \\ -M_{13}(x) & -M_{23}(x) & \left(M_{3\delta}(x) + \sum_{j\neq 3}^{N} M_{3j}(x)\right) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{2.1}$$

where $M_{i\delta}(x)$ is the age-specific annual death rate in region i, and

$M_{ij}(x)$ is the age-specific annual out-migration rate from region i to region j. It is estimated by the annual number of out-migrants to j divided by the mid-year population of i.

Let $\underset{\sim}{P}(x)$ be the matrix of age-specific probabilities of dying and out-migrating:

$$\underset{\sim}{P}(x) = \begin{bmatrix} p_{11}(x) & p_{21}(x) & \cdots \\ p_{12}(x) & p_{22}(x) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \tag{2.2}$$

with $p_{ij}(x)$ being the probability that an individual in region i at exact age x will survive and be in region j at exact age x + 5. The diagonal element $p_{ii}(x)$ is the probability that an individual will survive and be in region i at the end of the interval. If $q_i(x)$ is the probability that an individual in region i at age x will die before reaching age x + 5, then the following relationship follows

$$p_{ii}(x) = 1 - q_i(x) - \sum_{j \neq i}^{N} p_{ij}(x) \quad . \tag{2.3}$$

If multiple transition between two states is allowed during a unit time interval, then $\underset{\sim}{P}(x)$ is given by (Schoen, 1975; Rogers and Ledent, 1976):

$$\underset{\sim}{P}(x) = [\underset{\sim}{I} + \frac{5}{2} \underset{\sim}{M}(x)]^{-1} [\underset{\sim}{I} - \frac{5}{2} \underset{\sim}{M}(x)] \quad . \tag{2.4}$$

The probability that an individual starting out in region j, i.e., born in j, will be in region i at exact age x is denoted by ${}_j\hat{\ell}_i(x)$. The matrix containing those probabilities is

$$\underset{\sim}{\hat{\ell}}(x) = \begin{bmatrix} {}_1\hat{\ell}_1(x) & {}_2\hat{\ell}_1(x) & \cdots \\ {}_1\hat{\ell}_2(x) & {}_2\hat{\ell}_2(x) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \quad . \tag{2.5}$$

By this definition we have that

$$\hat{\underset{\sim}{\ell}}(x) = \underset{\sim}{P}(x - 5)\ \underset{\sim}{P}(x - 10) \cdots \underset{\sim}{P}(0) = \underset{\sim}{P}(x - 5)\ \hat{\underset{\sim}{\ell}}(x - 5) \quad . \tag{2.6}$$

Define

$$\underset{\sim}{\ell}(x) = \hat{\underset{\sim}{\ell}}(x)\ \underset{\sim}{\ell}(0) \tag{2.7}$$

where $\underset{\sim}{\ell}(0)$ is a diagonal matrix of the cohorts of babies born in the N regions at a given instant in time. Typically, ${}_i\ell_i(0)$ is called the radix of region i and is set equal to some arbitrary constant such as 100,000. Then $\underset{\sim}{\ell}(x)$ is the matrix of the number of people at exact age x by place of residence and by place of birth.

Another life table function is the total number of people of age group x, i.e., aged x to x + 5, in each region by place of birth:

$$\underset{\sim}{L}(x) = \begin{bmatrix} {}_1L_1(x) & {}_2L_1(x) & \cdots \\ {}_1L_2(x) & {}_2L_2(x) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \tag{2.8}$$

with ${}_jL_i(x)$ being the number of people in region i in age group x who were born in region j. The element ${}_jL_i(x)$ can also be thought of as the total person-years lived in region i between ages x and x + 5, by the people born in region j. The matrix $\underset{\sim}{L}(x)$ is given by

$$\underset{\sim}{L}(x) = \int_0^5 \underset{\sim}{\ell}(x + t)\, dt = \left[\int_0^5 \hat{\underset{\sim}{\ell}}(x + 5)\, dt\right] \underset{\sim}{\ell}(0) \quad . \tag{2.9}$$

Assuming a uniform distribution of out-migrations and deaths over the 5-year age interval, we may obtain numerical values for $\underset{\sim}{L}(x)$ by the linear interpolation

$$\underset{\sim}{L}(x) = \frac{5}{2} [\underset{\sim}{\ell}(x) + \underset{\sim}{\ell}(x + 5)] \quad ,$$

or (2.10)

$$\underset{\sim}{L}(x) = \frac{5}{2} [\underset{\sim}{I} + \underset{\sim}{P}(x)] \underset{\sim}{\ell}(x) \quad .$$

Aggregating L(x) over various age groups, we define the expected total number of person-years remaining to the people at exact age x, as

$$\underset{\sim}{T}(x) = \sum_{y=x}^{z} \underset{\sim}{L}(y) \tag{2.11}$$

where z is the terminal age group. Expressing $\underset{\sim}{T}(x)$ per individual, we get the matrix of expectations of life of an individual at exact age x:

$$\underset{\sim}{e}(x) = \underset{\sim}{T}(x) \underset{\sim}{\ell}^{-1}(x) = \left[\sum_{y=x}^{z} \underset{\sim}{L}(y)\right] \underset{\sim}{\ell}^{-1}(x) \quad . \tag{2.12}$$

A very useful life table function is the survivorship matrix. It is an essential component of the population projection matrix. Rogers (1975; p. 79) has shown that the survivorship matrix

$$\underset{\sim}{S}(x) = \begin{bmatrix} s_{11}(x) & s_{21}(x) & \cdots \\ s_{12}(x) & s_{22}(x) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \tag{2.13}$$

is given by

$$\underset{\sim}{S}(x) = \underset{\sim}{L}(x + 5)\ \underset{\sim}{L}^{-1}(x) \quad . \tag{2.14}$$

The element $s_{ij}(x)$ denotes the proportion of individuals aged x to x + 4 in region i, who survive to be x + 5 to x + 9 years old five years later, and are then in region j.

We now have set up the important life table functions, and can proceed to the analysis of their sensitivities to changes in the underlying rates, i.e., in $\underset{\sim}{M}(x)$.

2.2 SENSITIVITY ANALYSIS OF LIFE TABLE FUNCTIONS

The fundamental question posed in this section is: what is the effect on the various life table statistics of a change in the observed age-specific rates? To resolve this question, the life table functions are combined with the matrix differentiation techniques of the appendix.

This section is divided into five parts. Each part starts out with a specific life table function.

The derivative of this function with respect to an element of the matrix of age-specific rates yields the corresponding sensitivity function.

a. Sensitivity of the probabilities of dying and out-migrating

Recall the estimating formula set out in (2.4):

$$\underset{\sim}{P}(x) = [\underset{\sim}{I} + \frac{5}{2}\underset{\sim}{M}(x)]^{-1} [\underset{\sim}{I} - \frac{5}{2}\underset{\sim}{M}(x)] \quad . \qquad (2.4)$$

In it $\underset{\sim}{P}(x)$ only depends on $\underset{\sim}{M}(x)$. Therefore, $\underset{\sim}{P}(a)$ is not affected by a change in $\underset{\sim}{M}(x)$ for $a \neq x$.

The derivative of $\underset{\sim}{P}(x)$ with respect to an arbitrary element of $\underset{\sim}{M}(x)$ is, by formulas (A.13) and (A.25) of the Appendix,

$$\frac{\delta \underset{\sim}{P}(x)}{\delta \langle \underset{\sim}{M}(x) \rangle} = \frac{\delta [\underset{\sim}{I} + \frac{5}{2}\underset{\sim}{M}(x)]^{-1}}{\delta \langle \underset{\sim}{M}(x) \rangle} [\underset{\sim}{I} - \frac{5}{2}\underset{\sim}{M}(x)]$$

$$+ [\underset{\sim}{I} + \frac{5}{2}\underset{\sim}{M}(x)]^{-1} \frac{\delta [\underset{\sim}{I} - \frac{5}{2}\underset{\sim}{M}(x)]}{\delta \langle \underset{\sim}{M}(x) \rangle}$$

$$= - [\underset{\sim}{I} + \frac{5}{2}\underset{\sim}{M}(x)]^{-1} \frac{\delta [\underset{\sim}{I} + \frac{5}{2}\underset{\sim}{M}(x)]}{\delta \langle \underset{\sim}{M}(x) \rangle} [\underset{\sim}{I} + \frac{5}{2}\underset{\sim}{M}(x)]^{-1}$$

$$[\underset{\sim}{I} - \frac{5}{2}\underset{\sim}{M}(x)] + [\underset{\sim}{I} + \frac{5}{2}\underset{\sim}{M}(x)]^{-1} \frac{\delta [\underset{\sim}{I} - \frac{5}{2}\underset{\sim}{M}(x)]}{\delta \langle \underset{\sim}{M}(x) \rangle}$$

$$= - [\underset{\sim}{I} + \frac{5}{2}\underset{\sim}{M}(x)]^{-1} [\frac{5}{2}\underset{\sim}{J}\underset{\sim}{P}(x) + \frac{5}{2}\underset{\sim}{J}]$$

where $\underset{\sim}{J}$ is a matrix of the dimension of $\underset{\sim}{M}(x)$ with all elements zero except for a one on the position of the arbitrary element $\langle \underset{\sim}{M}(x) \rangle$. (This notation is further explained in the Appendix.)

The sensitivity function for $\underset{\sim}{P}(x)$ therefore is

$$\frac{\delta \underset{\sim}{P}(x)}{\delta <\underset{\sim}{M}(x)>} = - \frac{5}{2} [\underset{\sim}{I} + \frac{5}{2} \underset{\sim}{M}(x)]^{-1} \underset{\sim}{J}[\underset{\sim}{P}(x) + \underset{\sim}{I}] \quad . \qquad (2.15)$$

After the transformation

$$[\underset{\sim}{P}(x) + \underset{\sim}{I}] = [\underset{\sim}{I} + \frac{5}{2} \underset{\sim}{M}(x)]^{-1} \left[[\underset{\sim}{I} - \frac{5}{2} \underset{\sim}{M}(x)] + [\underset{\sim}{I} + \frac{5}{2} \underset{\sim}{M}(x)] \right]$$

$$= [\underset{\sim}{I} + \frac{5}{2} \underset{\sim}{M}(x)]^{-1} [\underset{\sim}{I} + \underset{\sim}{I}] = 2[\underset{\sim}{I} + \frac{5}{2} \underset{\sim}{M}(x)]^{-1}$$

the sensitivity function becomes

$$\frac{\delta \underset{\sim}{P}(x)}{\delta <\underset{\sim}{M}(x)>} = - 5[\underset{\sim}{I} + \frac{5}{2} \underset{\sim}{M}(x)]^{-1} \underset{\sim}{J}[\underset{\sim}{I} + \frac{5}{2} \underset{\sim}{M}(x)]^{-1} \quad . \qquad (2.16)$$

b. Sensitivity of the number of people at exact age a

A change in $\underset{\sim}{M}(x)$ does not affect $\underset{\sim}{\ell}(a)$ for $a \le x$. Therefore we look only at the case $a > x$. Note that $\underset{\sim}{\ell}(a)$ may be written as

$$\underset{\sim}{\ell}(a) = \underset{\sim}{P}(a - 5) \underset{\sim}{P}(a - 10) \cdots \underset{\sim}{P}(x) \underset{\sim}{\ell}(x) \quad .$$

Recalling that $\underset{\sim}{M}(x)$ only affects $\underset{\sim}{P}(x)$, we write

$$\frac{\delta \underset{\sim}{\ell}(a)}{\delta <\underset{\sim}{M}(x)>} = \underset{\sim}{P}(a - 5) \underset{\sim}{P}(a - 10) \cdots \frac{\delta \underset{\sim}{P}(x)}{\delta <\underset{\sim}{M}(x)>} \underset{\sim}{\ell}(x)$$

$$= - \frac{5}{2} \left[\underset{\sim}{P}(a - 5) \underset{\sim}{P}(a - 10) \cdots \underset{\sim}{P}(x + 5) \right. \qquad (2.17)$$

$$\left. [\underset{\sim}{I} + \frac{5}{2} \underset{\sim}{M}(x)]^{-1} \underset{\sim}{J}[\underset{\sim}{P}(x) + \underset{\sim}{I}] \underset{\sim}{\ell}(x) \right] \quad .$$

Inserting

$$\underset{\sim}{I} = [\underset{\sim}{I} - \tfrac{5}{2}\underset{\sim}{M}(x)]\ \underset{\sim}{\ell}(x)\ \underset{\sim}{\ell}^{-1}(x)\ [\underset{\sim}{I} - \tfrac{5}{2}\underset{\sim}{M}(x)]^{-1}$$

in (2.17) and substituting for $\underset{\sim}{P}(x)$ yields

$$\frac{\delta\underset{\sim}{\ell}(a)}{\delta<\underset{\sim}{M}(x)>} = -\frac{5}{2}\Big[\underset{\sim}{P}(a-5)\ \underset{\sim}{P}(a-10)\cdots\underset{\sim}{P}(a+5)\ \underset{\sim}{P}(x)\ \underset{\sim}{\ell}(x)\ \underset{\sim}{\ell}^{-1}(x)$$

$$[\underset{\sim}{I} - \tfrac{5}{2}\underset{\sim}{M}(x)]^{-1}\ \underset{\sim}{J}[\underset{\sim}{P}(x) + \underset{\sim}{I}]\ \underset{\sim}{\ell}(x)\Big]$$

$$\frac{\delta\underset{\sim}{\ell}(a)}{\delta<\underset{\sim}{M}(x)>} = -\frac{5}{2}\ \underset{\sim}{\ell}(a)\ \underset{\sim}{\ell}^{-1}(x)\ [\underset{\sim}{I} - \tfrac{5}{2}\underset{\sim}{M}(x)]^{-1}\ \underset{\sim}{J}[\underset{\sim}{P}(x) + \underset{\sim}{I}]\ \underset{\sim}{\ell}(x) \tag{2.18}$$

or by (2.10)

$$\frac{\delta\underset{\sim}{\ell}(a)}{\delta<\underset{\sim}{M}(x)>} = -\ \underset{\sim}{\ell}(a)\ \underset{\sim}{\ell}^{-1}(x)\ [\underset{\sim}{I} - \tfrac{5}{2}\underset{\sim}{M}(x)]^{-1}\ \underset{\sim}{J}\underset{\sim}{L}(x)\ . \tag{2.19}$$

For a = x + 5, we have

$$\frac{\delta\underset{\sim}{\ell}(x+5)}{\delta<\underset{\sim}{M}(x)>} = -\ \underset{\sim}{P}(x)\ [\underset{\sim}{I} - \tfrac{5}{2}\underset{\sim}{M}(x)]^{-1}\ \underset{\sim}{J}\underset{\sim}{L}(x) = -\ [\underset{\sim}{I} + \tfrac{5}{2}\underset{\sim}{M}(x)]^{-1}\ \underset{\sim}{J}\underset{\sim}{L}(x) \tag{2.20}$$

An interesting formulation of the sensitivity function follows from writing (2.18) as

$$\frac{\delta\underset{\sim}{\ell}(a)}{\delta<\underset{\sim}{M}(x)>} = \underset{\sim}{\ell}(a)\ [\underset{\sim}{P}(x)\ \underset{\sim}{\ell}(x)]^{-1}\ \frac{\delta\underset{\sim}{P}(x)}{\delta<\underset{\sim}{M}(x)>}\ \underset{\sim}{\ell}(x)$$

$$= \underset{\sim}{\ell}(a)\ \underset{\sim}{\ell}^{-1}(x)\ \underset{\sim}{P}^{-1}(x)\ \frac{\delta\underset{\sim}{P}(x)}{\delta<\underset{\sim}{M}(x)>}\ \underset{\sim}{\ell}(x)$$

or

$$\underset{\sim}{\ell}^{-1}(a)\ \frac{\delta \underset{\sim}{\ell}(a)}{\delta < \underset{\sim}{M}(x) >} = \underset{\sim}{\ell}^{-1}(x)\ \underset{\sim}{P}^{-1}(x)\ \frac{\delta \underset{\sim}{P}(x)}{\delta < \underset{\sim}{M}(x) >}\ \underset{\sim}{\ell}(x) \quad . \qquad (2.21)$$

This shows that the relative sensitivity of $\underset{\sim}{\ell}(a)$ to changes in $\underset{\sim}{M}(x)$ is a weighted average of the relative sensitivity of $\underset{\sim}{P}(x)$, and is independent of a. Consider the first age group and suppose that all regions have the same radices, i.e., $\underset{\sim}{\ell}(0)$ is a scalar matrix, i.e., a diagonal matrix with the same diagonal elements. The relative sensitivity of any $\underset{\sim}{\ell}(a)$ is then equal to the relative sensitivity of $\underset{\sim}{P}(0)$.

c. Sensitivity of the number of people in age group (a, a + 4)

What is the impact of a change in $\underset{\sim}{M}(x)$ on the number of people in age group (a, a + 4) and on their spatial distribution? It is clear that $\underset{\sim}{M}(x)$ does not affect $\underset{\sim}{L}(a)$ for $a < x$. Therefore, we consider here the case of $a \geq x$. Recall from (2.10) that

$$\underset{\sim}{L}(a) = \frac{5}{2}\ [\underset{\sim}{\ell}(a + 5) + \underset{\sim}{\ell}(a)] \quad .$$

Differentiating both sides gives

$$\frac{\delta \underset{\sim}{L}(a)}{\delta < \underset{\sim}{M}(x) >} = \frac{5}{2} \left[\frac{\delta \underset{\sim}{\ell}(a + 5)}{\delta < \underset{\sim}{M}(x) >} + \frac{\delta \underset{\sim}{\ell}(a)}{\delta < \underset{\sim}{M}(x) >} \right] \quad .$$

If $a = x$, then $\frac{\delta \underset{\sim}{\ell}(a)}{\delta < \underset{\sim}{M}(x) >} = \underset{\sim}{0}$ and we have

$$\frac{\delta \underset{\sim}{L}(x)}{\delta\langle \underset{\sim}{M}(x)\rangle} = \frac{5}{2}\,\frac{\delta \underset{\sim}{\ell}(x+5)}{\delta\langle \underset{\sim}{M}(x)\rangle} = \frac{5}{2}\,[\underset{\sim}{I} + \tfrac{5}{2}\,\underset{\sim}{M}(x)]^{-1}\;\underset{\sim}{J}\underset{\sim}{L}(x) \qquad (2.22)$$

which has the following alternative expressions:

$$\frac{\delta \underset{\sim}{L}(x)}{\delta\langle \underset{\sim}{M}(x)\rangle} = -\,5[\underset{\sim}{P}(x) + \underset{\sim}{I}]\;\underset{\sim}{J}\underset{\sim}{L}(x) \qquad (2.23)$$

$$= -\,\frac{5}{2}\,\underset{\sim}{P}(x)\;[\underset{\sim}{I} - \tfrac{5}{2}\,\underset{\sim}{M}(x)]^{-1}\;\underset{\sim}{J}\underset{\sim}{L}(x) \qquad (2.24)$$

$$= -\,\underset{\sim}{L}(x)\;\underset{\sim}{\ell}^{-1}(x)\;[\underset{\sim}{I} - \tfrac{5}{2}\,\underset{\sim}{M}(x)]^{-1}\;\underset{\sim}{J}\underset{\sim}{L}(x) + \frac{5}{2}\,[\underset{\sim}{I} - \tfrac{5}{2}\,\underset{\sim}{M}(x)]^{-1}\;\underset{\sim}{J}\underset{\sim}{L}(x)\;. \qquad (2.25)$$

If $a > x$, we know that $\underset{\sim}{P}(a)$ is independent of $\underset{\sim}{M}(x)$, and therefore

$$\frac{\delta \underset{\sim}{L}(a)}{\delta\langle \underset{\sim}{M}(x)\rangle} = \frac{5}{2}\,[\underset{\sim}{P}(a) + \underset{\sim}{I}]\;\frac{\delta \underset{\sim}{\ell}(a)}{\delta\langle \underset{\sim}{M}(x)\rangle}$$

$$= -\,\frac{5}{2}\,[\underset{\sim}{P}(a) + \underset{\sim}{I}]\;\underset{\sim}{\ell}(a)\;\underset{\sim}{\ell}^{-1}(x)\;[\underset{\sim}{I} - \tfrac{5}{2}\,\underset{\sim}{M}(x)]^{-1}\;\underset{\sim}{J}\underset{\sim}{L}(x)$$

$$= -\,\underset{\sim}{L}(a)\;\underset{\sim}{\ell}^{-1}(x)\;[\underset{\sim}{I} - \tfrac{5}{2}\,\underset{\sim}{M}(x)]^{-1}\;\underset{\sim}{J}\underset{\sim}{L}(x) \qquad (2.26)$$

which may also be written as

$$\frac{\delta \underset{\sim}{L}(a)}{\delta\langle \underset{\sim}{M}(x)\rangle} = \frac{5}{2}\,[\underset{\sim}{P}(a) + \underset{\sim}{I}]\;\underset{\sim}{\ell}(a)\;\underset{\sim}{\ell}^{-1}(a)\;\frac{\delta \underset{\sim}{\ell}(a)}{\delta\langle \underset{\sim}{M}(x)\rangle}$$

whence, since $\frac{5}{2}\ [\underset{\sim}{P}(a) + \underset{\sim}{I}]\ \underset{\sim}{\ell}(a) = \underset{\sim}{L}(a)$,

$$\underset{\sim}{L}^{-1}(a)\ \frac{\delta \underset{\sim}{L}(a)}{\delta <\underset{\sim}{M}(x)>} = \underset{\sim}{\ell}^{-1}(a)\ \frac{\delta \underset{\sim}{\ell}(a)}{\delta <\underset{\sim}{M}(x)>} \quad . \tag{2.27}$$

Equation (2.27) indicates that the relative sensitivity of the number of people in age group (a, a + 4) is equal to the relative sensitivity of the number of people at exact age a for a > x.

d. Sensitivity of the expectation of life at age a

We now proceed to deriving the sensitivity function of the most important life table statistic, namely the expectation of life. First consider the sensitivity of $\underset{\sim}{e}(x)$. Differentiating both sides of (2.12) yields

$$\frac{\delta \underset{\sim}{e}(x)}{\delta <\underset{\sim}{M}(x)>} = \frac{\delta \left[\sum_{y=x}^{z} \underset{\sim}{L}(y)\right]}{\delta <\underset{\sim}{M}(x)>}\ \underset{\sim}{\ell}^{-1}(x) + \left[\sum_{y=x}^{z} \underset{\sim}{L}(y)\right]\ \frac{\delta \underset{\sim}{\ell}^{-1}(x)}{\delta <\underset{\sim}{M}(x)>} \quad . \tag{2.28}$$

From (2.22) and (2.26), we see that

$$\frac{\delta \left[\sum_{y=x}^{z} \underset{\sim}{L}(y)\right]}{\delta <\underset{\sim}{M}(x)>} = - \left[\sum_{y=x+5}^{z} \underset{\sim}{L}(y)\right]\ \underset{\sim}{\ell}^{-1}(x)\ [\underset{\sim}{I} - \frac{5}{2} \underset{\sim}{M}(x)]^{-1}\ \underset{\sim}{J}\underset{\sim}{L}(x)$$

$$- \frac{5}{2}\ [\underset{\sim}{I} + \frac{5}{2} \underset{\sim}{M}(x)]^{-1}\ \underset{\sim}{J}\underset{\sim}{L}(x)$$

$$= - \left[\sum_{y=x+5}^{z} \underset{\sim}{L}(y)\ \underset{\sim}{\ell}^{-1}(x) + \frac{5}{2} \underset{\sim}{P}(x) + \frac{5}{2} \underset{\sim}{I} - \frac{5}{2} \underset{\sim}{I}\right]$$

$$\cdot\ [\underset{\sim}{I} - \frac{5}{2} \underset{\sim}{M}(x)]^{-1}\ \underset{\sim}{J}\underset{\sim}{L}(x)$$

$$= - \left[\sum_{y=x+5}^{z} \underset{\sim}{L}(y)\ \underset{\sim}{\ell}^{-1}(x) + \underset{\sim}{L}(x)\ \underset{\sim}{\ell}^{-1}(x) - \frac{5}{2}\underset{\sim}{I} \right]$$

$$\cdot\ [\underset{\sim}{I} - \frac{5}{2}\underset{\sim}{M}(x)]^{-1}\ \underset{\sim}{J}\underset{\sim}{L}(x)$$

$$= -\ [\underset{\sim}{e}(x) - \frac{5}{2}\underset{\sim}{I}]\ [\underset{\sim}{I} - \frac{5}{2}\underset{\sim}{M}(x)]^{-1}\ \underset{\sim}{J}\underset{\sim}{L}(x) \quad .$$

Since $\underset{\sim}{\ell}(x)$ is independent of $\underset{\sim}{M}(x)$, we may write (2.28) as follows

$$\frac{\delta\underset{\sim}{e}(x)}{\delta\langle\underset{\sim}{M}(x)\rangle} = -\ [\underset{\sim}{e}(x) - \frac{5}{2}\underset{\sim}{I}]\ [\underset{\sim}{I} - \frac{5}{2}\underset{\sim}{M}(x)]^{-1}\ \underset{\sim}{J}\underset{\sim}{L}(x)\ \underset{\sim}{\ell}^{-1}(x) \quad . \tag{2.29}$$

For $a < x$, we have

$$\frac{\delta\underset{\sim}{e}(a)}{\delta\langle\underset{\sim}{M}(x)\rangle} = \frac{\delta\left[\sum_{y=x+5}^{z} \underset{\sim}{L}(y) + \underset{\sim}{L}(x) + \sum_{y=a}^{x-5} \underset{\sim}{L}(y) \right] \underset{\sim}{\ell}^{-1}(a)}{\delta\langle\underset{\sim}{M}(x)\rangle} \quad .$$

We know that

$$\frac{\delta\underset{\sim}{L}(y)}{\delta\langle\underset{\sim}{M}(x)\rangle} = \underset{\sim}{0} \quad , \qquad \text{for } y < x$$

and

$$\frac{\delta\underset{\sim}{\ell}(a)}{\delta\langle\underset{\sim}{M}(x)\rangle} = \underset{\sim}{0} \quad , \qquad \text{for } a < x \quad .$$

Therefore

$$\frac{\delta \underset{\sim}{e}(a)}{\delta \langle \underset{\sim}{M}(x) \rangle} = \frac{\delta \left[\sum_{y=x}^{z} \underset{\sim}{L}(y) \right] \underset{\sim}{\ell}^{-1}(a)}{\delta \langle \underset{\sim}{M}(x) \rangle}$$

$$\frac{\delta \underset{\sim}{e}(a)}{\delta \langle \underset{\sim}{M}(x) \rangle} = \frac{\delta \left[\sum_{y=x}^{z} \underset{\sim}{L}(y) \right] \underset{\sim}{\ell}^{-1}(x) \; \underset{\sim}{\ell}(x) \; \underset{\sim}{\ell}^{-1}(a)}{\delta \langle \underset{\sim}{M}(x) \rangle}$$

$$\frac{\delta \underset{\sim}{e}(a)}{\delta \langle \underset{\sim}{M}(x) \rangle} = \frac{\delta \underset{\sim}{e}(a)}{\delta \langle \underset{\sim}{M}(x) \rangle} \underset{\sim}{\ell}(x) \; \underset{\sim}{\ell}^{-1}(a) \qquad (2.30)$$

$$\frac{\delta \underset{\sim}{e}(a)}{\delta \langle \underset{\sim}{M}(x) \rangle} = - \left[\underset{\sim}{e}(x) - \frac{5}{2} \underset{\sim}{I} \right] \left[\underset{\sim}{I} - \frac{5}{2} \underset{\sim}{M}(x) \right]^{-1} \underset{\sim}{J}\underset{\sim}{L}(x) \; \underset{\sim}{\ell}^{-1}(a) \qquad (2.31)$$

$$= - \underset{\sim}{e}(x) \left[\underset{\sim}{I} - \frac{5}{2} \underset{\sim}{M}(x) \right]^{-1} \underset{\sim}{J}\underset{\sim}{L}(x) \; \underset{\sim}{\ell}^{-1}(a)$$

$$+ \frac{5}{2} \left[\underset{\sim}{I} - \frac{5}{2} \underset{\sim}{M}(x) \right]^{-1} \underset{\sim}{J}\underset{\sim}{L}(x) \; \underset{\sim}{\ell}^{-1}(a) \quad . \qquad (2.32)$$

The second component of the sensitivity function is due to the linear approximation $\underset{\sim}{L}(x) = \frac{5}{2} [\underset{\sim}{\ell}(x + 5) + \underset{\sim}{\ell}(x)]$ of the continuous relationship

$$\underset{\sim}{L}(x) = \int_{x}^{x+5} \underset{\sim}{\ell}(t) \, dt \quad .$$

Consider the continuous definition of $\underset{\sim}{e}(a)$

$$\underset{\sim}{e}(a) = \left[\int_a^{\omega} \underset{\sim}{\ell}(t)\, dt \right] \underset{\sim}{\ell}^{-1}(a)$$

where ω is the terminal age. Differentiating yields

$$\frac{\delta \underset{\sim}{e}(a)}{\delta \langle \underset{\sim}{M}(x) \rangle} = \left[\int_a^{\omega} \frac{\delta \underset{\sim}{\ell}(t)}{\delta \langle \underset{\sim}{M}(x) \rangle}\, dt \right] \underset{\sim}{\ell}^{-1}(a) \qquad , \quad \text{for } a \leq x$$

$$= \left[\int_x^{\omega} - \underset{\sim}{\ell}(t)\ \underset{\sim}{\ell}^{-1}(x)\ [\underset{\sim}{I} - \tfrac{5}{2} \underset{\sim}{M}(x)]^{-1}\ \underset{\sim}{J}\underset{\sim}{\ell}(x)\ dt \right] \underset{\sim}{\ell}^{-1}(a)$$

Since $\underset{\sim}{\ell}(t)$ is independent of $\underset{\sim}{M}(x)$, if $t < x$

$$\frac{\delta \underset{\sim}{e}(a)}{\delta \langle \underset{\sim}{M}(x) \rangle} = - \left[\int_x^{\omega} \underset{\sim}{\ell}(t)\, dt \right] \underset{\sim}{\ell}^{-1}(x)\ [\underset{\sim}{I} - \tfrac{5}{2} \underset{\sim}{M}(x)]^{-1}\ \underset{\sim}{J}\underset{\sim}{\ell}(x)\ \underset{\sim}{\ell}^{-1}(a)$$

$$= - \underset{\sim}{e}(x)\ [\underset{\sim}{I} - \tfrac{5}{2} \underset{\sim}{M}(x)]^{-1}\ \underset{\sim}{J}\underset{\sim}{\ell}(x)\ \underset{\sim}{\ell}^{-1}(a) \qquad (2.33)$$

which is equivalent to the first term of (2.32) with the term $\underset{\sim}{\ell}(x)$ replaced by $\underset{\sim}{L}(x)$ in the discrete case. The expression (2.33), written in terms of differentials, is similar to the sensitivity function of the expectation of life, given by Keyfitz (1971, p. 276) for the single-region case

$$de(a) = - e(x)\ [dM(x)]\ \ell(x)\ \ell^{-1}(a) \quad ,$$

where $e(\cdot)$, $\ell(\cdot)$ and $M(\cdot)$ are scalars.

The term $[\underset{\sim}{I} - \frac{5}{2}\underset{\sim}{M}(x)]^{-1}$ in (2.33) is due to the fact that we consider observed rates where Keyfitz derived the formula using instantaneous rates. If $\underset{\sim}{M}(x)$ contained instantaneous rates, then $\underset{\sim}{M}(x) \doteq \underset{\sim}{0}$ and $[\underset{\sim}{I} - \frac{5}{2}\underset{\sim}{M}(x)] \doteq \underset{\sim}{I}$.

e. Sensitivity of the survivorship proportions

As in the proceeding sections, we treat separately $\underset{\sim}{S}(a)$ for a = x and for a > x. The survivorship matrix is given by (2.14) as

$$\underset{\sim}{S}(x) = \underset{\sim}{L}(x+5)\,\underset{\sim}{L}^{-1}(x) \quad ,$$

which may be reexpressed as

$$\begin{aligned}\underset{\sim}{S}(x) &= [\underset{\sim}{P}(x+5) + \underset{\sim}{I}]\,\underset{\sim}{P}(x)\,\underset{\sim}{\ell}(x)\,\underset{\sim}{\ell}^{-1}(x)\,[\underset{\sim}{P}(x) + \underset{\sim}{I}]^{-1} \\ &= [\underset{\sim}{P}(x+5) + \underset{\sim}{I}]\,\underset{\sim}{P}(x)\,[\underset{\sim}{P}(x) + \underset{\sim}{I}]^{-1} \quad .\end{aligned} \tag{2.34}$$

Differentiating with respect to $\langle\underset{\sim}{M}(x)\rangle$ yields

$$\begin{aligned}\frac{\delta\underset{\sim}{S}(x)}{\delta\langle\underset{\sim}{M}(x)\rangle} &= [\underset{\sim}{P}(x+5) + \underset{\sim}{I}]\,\frac{\delta\underset{\sim}{P}(x)}{\delta\langle\underset{\sim}{M}(x)\rangle}\,[\underset{\sim}{P}(x) + \underset{\sim}{I}]^{-1} \\ &\quad + [\underset{\sim}{P}(x+5) + \underset{\sim}{I}]\,\underset{\sim}{P}(x)\,\frac{\delta[\underset{\sim}{P}(x) + \underset{\sim}{I}]^{-1}}{\delta\langle\underset{\sim}{M}(x)\rangle} \\ &= [\underset{\sim}{P}(x+5) + \underset{\sim}{I}]\left[\frac{\delta\underset{\sim}{P}(x)}{\delta\langle\underset{\sim}{M}(x)\rangle} - \underset{\sim}{P}(x)[\underset{\sim}{P}(x) + \underset{\sim}{I}]^{-1}\,\frac{\delta\underset{\sim}{P}(x)}{\delta\langle\underset{\sim}{M}(x)\rangle}\right] \\ &\qquad \cdot\,[\underset{\sim}{P}(x) + \underset{\sim}{I}]^{-1}\end{aligned}$$

$$= [\underset{\sim}{P}(x+5) + \underset{\sim}{I}] \left[\underset{\sim}{I} - \underset{\sim}{P}(x) \; [\underset{\sim}{P}(x) + \underset{\sim}{I}]^{-1}\right] \frac{\delta \underset{\sim}{P}(x)}{\delta \langle \underset{\sim}{M}(x) \rangle}$$

$$\cdot \; [\underset{\sim}{P}(x) + \underset{\sim}{I}]^{-1}$$

$$= [\underset{\sim}{P}(x+5) + \underset{\sim}{I}] \left[\underset{\sim}{I} - \underset{\sim}{P}(x) \; [\underset{\sim}{P}(x) + \underset{\sim}{I}]^{-1}\right] \left[-\frac{5}{2}\right]$$

$$\cdot \; [\underset{\sim}{I} + \frac{5}{2}\underset{\sim}{M}(x)]^{-1} \; \underset{\sim}{J}[\underset{\sim}{P}(x) + \underset{\sim}{I}] \; [\underset{\sim}{P}(x) + \underset{\sim}{I}]^{-1}$$

$$= \frac{5}{2} \; [\underset{\sim}{P}(x+5) + \underset{\sim}{I}] \left[\underset{\sim}{P}(x) \; [\underset{\sim}{P}(x) + \underset{\sim}{I}]^{-1} \; \underset{\sim}{I}\right]$$

$$\cdot \; [\underset{\sim}{I} + \frac{5}{2}\underset{\sim}{M}(x)]^{-1} \; \underset{\sim}{J} \quad .$$

Substituting for $\underset{\sim}{S}(x)$ gives

$$\frac{\delta \underset{\sim}{S}(x)}{\delta \langle \underset{\sim}{M}(x) \rangle} = \frac{5}{2} \; [\underset{\sim}{S}(x) - \underset{\sim}{P}(x+5) + \underset{\sim}{I}] \; [\underset{\sim}{I} + \frac{5}{2}\underset{\sim}{M}(x)]^{-1} \; \underset{\sim}{J}$$

$$= \frac{5}{2} \; \underset{\sim}{S}(x) \; [\underset{\sim}{I} + \frac{5}{2}\underset{\sim}{M}(x)]^{-1} \; \underset{\sim}{J}$$

$$- \frac{5}{2} \; [\underset{\sim}{P}(x+5) + \underset{\sim}{I}] \; [\underset{\sim}{I} + \frac{5}{2}\underset{\sim}{M}(x)]^{-1} \; \underset{\sim}{J}$$

$$= \frac{5}{2} \; \underset{\sim}{S}(x) \; [\underset{\sim}{I} + \frac{5}{2}\underset{\sim}{M}(x)]^{-1} \; \underset{\sim}{J}$$

$$- \frac{5}{2} [\underset{\sim}{P}(x+5) + \underset{\sim}{I}] \; \underset{\sim}{P}(x) \; \underset{\sim}{\ell}(x) \; \underset{\sim}{\ell}^{-1}(x) \; \underset{\sim}{P}^{-1}(x) [\underset{\sim}{I} + \frac{5}{2}\underset{\sim}{M}(x)]^{-1} \; \underset{\sim}{J} \quad .$$

Since

$$\frac{5}{2}\,[\underset{\sim}{P}(x+5)+\underset{\sim}{I}]\;\underset{\sim}{P}(x)\;\underset{\sim}{\ell}(x) = \underset{\sim}{L}(x+5)$$

and

$$\underset{\sim}{\ell}^{-1}(x)\;\underset{\sim}{P}^{-1}(x)\;[\underset{\sim}{I}+\frac{5}{2}\underset{\sim}{M}(x)]^{-1} = \underset{\sim}{\ell}^{-1}(x)\;[\underset{\sim}{I}-\frac{5}{2}\underset{\sim}{M}(x)]^{-1} \quad ,$$

where $\underset{\sim}{\ell}^{-1}(x)$ may be written as

$$\underset{\sim}{\ell}^{-1}(x) = \left[\frac{5}{2}\right]\underset{\sim}{\ell}^{-1}(x)\;[\underset{\sim}{I}+\underset{\sim}{P}(x)]^{-1}\;\frac{5}{2}\;[\underset{\sim}{I}+\underset{\sim}{P}(x)]$$

$$= \underset{\sim}{L}^{-1}(x)\;\frac{5}{2}\;[\underset{\sim}{I}+\underset{\sim}{P}(x)] \quad ,$$

we have that

$$\frac{\delta\underset{\sim}{S}(x)}{\delta\langle\underset{\sim}{M}(x)\rangle} = \frac{5}{2}\,\underset{\sim}{S}(x)\;[\underset{\sim}{I}+\frac{5}{2}\underset{\sim}{M}(x)]^{-1}\;\underset{\sim}{J}$$

$$- \underset{\sim}{L}(x+5)\;\underset{\sim}{L}^{-1}(x)\;\frac{5}{2}\;[\underset{\sim}{I}+\underset{\sim}{P}(x)]\;[\underset{\sim}{I}-\frac{5}{2}\underset{\sim}{M}(x)]^{-1}\;\underset{\sim}{J}$$

$$= \frac{5}{2}\,\underset{\sim}{S}(x)\;[\underset{\sim}{I}+\frac{5}{2}\underset{\sim}{M}(x)]^{-1}\;\underset{\sim}{J} - \frac{5}{2}\,\underset{\sim}{S}(x)\;[\underset{\sim}{I}-\frac{5}{2}\underset{\sim}{M}(x)]^{-1}\;\underset{\sim}{J}$$

$$- \frac{5}{2}\,\underset{\sim}{S}(x)\;\underset{\sim}{P}(x)\;[\underset{\sim}{I}-\frac{5}{2}\underset{\sim}{M}(x)]^{-1}\;\underset{\sim}{J} \quad .$$

But

$$\underset{\sim}{P}(x)\;[\underset{\sim}{I}-\frac{5}{2}\underset{\sim}{M}(x)]^{-1} = [\underset{\sim}{I}+\frac{5}{2}\underset{\sim}{M}(x)]^{-1} \quad .$$

Therefore

$$\frac{\delta \underset{\sim}{S}(x)}{\delta < \underset{\sim}{M}(x) >} = - \frac{5}{2} \underset{\sim}{S}(x) \; [\underset{\sim}{I} - \frac{5}{2} \underset{\sim}{M}(x)]^{-1} \underset{\sim}{J} \quad . \tag{2.35}$$

To illustrate the dynamic relationship between the life table statistics, we may express the sensitivity of $\underset{\sim}{S}(x)$ in relation to the sensitivity of other statistics. For example, a combination of (2.35) with (2.26) yields

$$\underset{\sim}{S}^{-1}(x) \; \frac{\delta \underset{\sim}{S}(x)}{\delta < \underset{\sim}{M}(x) >} = \underset{\sim}{P}^{-1}(x) \; \frac{\delta \underset{\sim}{L}(x)}{\delta < \underset{\sim}{M}(x) >} \; \underset{\sim}{L}^{-1}(x)$$

and a combination of (2.35) with (2.19) gives

$$\underset{\sim}{S}^{-1}(x) \; \frac{\delta \underset{\sim}{S}(x)}{\delta < \underset{\sim}{M}(x) >} = - \frac{5}{2} \underset{\sim}{P}^{-1}(x) \; \frac{\delta \underset{\sim}{\ell}(x + 5)}{\delta < \underset{\sim}{M}(x) >} \; \underset{\sim}{L}^{-1}(x) \quad .$$

The relative sensitivity of $\underset{\sim}{S}(x)$ may be regarded as a weighted measure of the sensitivities of other statistics.

We now turn to the sensitivity of $\underset{\sim}{S}(a)$ to changes in $\underset{\sim}{M}(x)$ for $a \neq x$. For $a > x$ and for $a < x - 5$, $\underset{\sim}{S}(a)$ is independent of a change in $\underset{\sim}{M}(x)$. This can easily be seen in equation (2.34) while noting that $\underset{\sim}{P}(a)$ is not affected by $\underset{\sim}{M}(x)$ if $a \neq x$. The sensitivity of $\underset{\sim}{S}(x - 5)$ to a change in $\underset{\sim}{M}(x)$ is derived next. We begin by writing (2.34) for $x - 5$

$$\underset{\sim}{S}(x-5) = [\underset{\sim}{P}(x) + \underset{\sim}{I}]\, \underset{\sim}{P}(x-5)\, [\underset{\sim}{P}(x-5) + \underset{\sim}{I}]^{-1}$$

$$\frac{\delta \underset{\sim}{S}(x-5)}{\delta\langle \underset{\sim}{M}(x)\rangle} = \frac{\delta[\underset{\sim}{P}(x) + \underset{\sim}{I}]}{\delta\langle \underset{\sim}{M}(x)\rangle}\, \underset{\sim}{P}(x-5)\, [\underset{\sim}{P}(x-5) + \underset{\sim}{I}]^{-1}$$

$$= -\frac{5}{2}\, [\underset{\sim}{I} + \frac{5}{2}\underset{\sim}{M}(x)]^{-1}\, \underset{\sim}{J}\, [\underset{\sim}{P}(x) + \underset{\sim}{I}]\, \underset{\sim}{P}(x-5) [\underset{\sim}{P}(x-5) + \underset{\sim}{I}]^{-1}$$

$$\frac{\delta \underset{\sim}{S}(x-5)}{\delta\langle \underset{\sim}{M}(x)\rangle} = -\frac{5}{2}\, [\underset{\sim}{I} + \frac{5}{2}\underset{\sim}{M}(x)]^{-1}\, \underset{\sim}{J}\, \underset{\sim}{S}(x-5) \qquad (2.36)$$

$$= -\frac{5}{2}\, \underset{\sim}{P}(x)\, [\underset{\sim}{I} - \frac{5}{2}\underset{\sim}{M}(x)]^{-1}\, \underset{\sim}{J}\, \underset{\sim}{S}\, (x-5) \quad . \qquad (2.37)$$

The relationship between the sensitivity of $\underset{\sim}{S}(x)$ and of $\underset{\sim}{S}(x-5)$ is

$$\frac{\delta \underset{\sim}{S}(x)}{\delta\langle \underset{\sim}{M}(x)\rangle} = -\frac{5}{2}\, \underset{\sim}{S}(x)\, \underset{\sim}{P}^{-1}(x)\, [\underset{\sim}{I} + \frac{5}{2}\underset{\sim}{M}(x)]^{-1}\, \underset{\sim}{J}\, \underset{\sim}{S}(x-5)\, \underset{\sim}{S}^{-1}(x-5)$$

$$\frac{\delta \underset{\sim}{S}(x)}{\delta\langle \underset{\sim}{M}(x)\rangle} = \underset{\sim}{S}(x)\, \underset{\sim}{P}^{-1}(x)\, \frac{\delta \underset{\sim}{S}(x-5)}{\delta\langle \underset{\sim}{M}(x)\rangle}\, \underset{\sim}{S}^{-1}(x-5) \qquad (2.38)$$

and

$$\frac{\delta \underset{\sim}{S}(x-5)}{\delta\langle \underset{\sim}{M}(x)\rangle} = \underset{\sim}{P}(x)\, \underset{\sim}{S}^{-1}(x)\, \frac{\delta \underset{\sim}{S}(x)}{\delta\langle \underset{\sim}{M}(x)\rangle}\, \underset{\sim}{S}(x-5) \quad . \qquad (2.39)$$

CHAPTER 3

IMPACT OF CHANGES IN AGE-SPECIFIC RATES ON THE POPULATION PROJECTION

Population projection is often carried out under the assumption that an observed population growth regime will remain constant. This implies that the observed age-specific rates will not change over the projection period. (This is a crude assumption and no demographer or planner considers it to be a realistic one. Nevertheless it produces a useful benchmark against which to compare other alternative projections.) In this chapter, we deal with the question of how sensitive population projections are to changes in age-specific rates. These variations may occur at any point in time. If they occur in the base year, they can be related to observation errors. The sensitivity functions we develop remain exactly the same, no matter what the causes of the variations are.

In the first part, the population growth model is set out as a system of first order linear homogenous difference equations with constant coefficients, as in Rogers (1975, Chapter 5). The second part studies the sensitivity of population growth to changes in observed age-specific rates.

3.1. THE DISCRETE MODEL OF MULTIREGIONAL DEMOGRAPHIC GROWTH

Population growth may be expressed in terms of the changing level of population or in terms of the variation of the number of births over time. In demography, it has been a custom to formulate the discrete model of population growth in terms of total population, while the continuous version describes the birth trajectory (Keyfitz, 1968; Rogers, 1975). A secondary objective of this and the next chapter is to contribute to the reconciliation of both growth models. We will formulate population growth in the discrete time domain. However, several particularities of the continuous model have a discrete counterpart. In this section, it will be shown how the population growth path relates to the trajectory of births.

a. The population model

A multiregional growth process may be described as a matrix multiplication (Rogers, 1975; p. 123):

$$\{\underset{\sim}{K}^{(t+1)}\} = \underset{\sim}{G}\{\underset{\sim}{K}^{(t)}\} \tag{3.1}$$

where the vector $\{\underset{\sim}{K}^{(t)}\}$ describes the regional age-specific population distribution at time t, with

$$\{\underset{\sim}{K}^{(t)}\} = \begin{bmatrix} \{\underset{\sim}{K}^{(t)}(0)\} \\ \{\underset{\sim}{K}^{(t)}(5)\} \\ \vdots \\ \cdot \\ \{\underset{\sim}{K}^{(t)}(z)\} \end{bmatrix} \text{ and } \{\underset{\sim}{K}^{(t)}(x)\} = \begin{bmatrix} K_1^{(t)}(x) \\ K_2^{(t)}(x) \\ \cdot \\ \cdot \\ K_N^{(t)}(x) \end{bmatrix}, \tag{3.2}$$

z being the terminal age interval and N the number of regions.

Each element $K_i^{(t)}(x)$ denotes the number of people in region i at time t, x to x + 4 years old. Note that t + 1 represents the next moment in time, i.e., 5 years later than t. We consider age-groups and time intervals of 5 years. The operator $\underset{\sim}{G}$ is the generalized Leslie matrix

$$\underset{\sim}{G} = \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{B}(\alpha - 5) & \cdots & \underset{\sim}{B}(\beta - 5) & \underset{\sim}{0} & \cdots & \underset{\sim}{0} \\ \underset{\sim}{S}(0) & \underset{\sim}{0} & & & & & & \\ \underset{\sim}{0} & \underset{\sim}{S}(5) & \ddots & & & & & \\ \vdots & \vdots & & & \ddots & & & \\ \underset{\sim}{0} & \underset{\sim}{0} & & & & & \underset{\sim}{S}(z - 5) & \underset{\sim}{0} \end{bmatrix} \quad (3.3)$$

with $\underset{\sim}{S}(x)$, the matrix of survivorship proportions, retaining the definition set out in the previous chapter. The first and last ages of childbearing may be denoted by α and β, respectively, and

$$\underset{\sim}{B}(x) = \begin{bmatrix} b_{11}(x) & b_{21}(x) & \cdots \\ b_{12}(x) & b_{22}(x) & \cdots \\ \vdots & \vdots & \end{bmatrix}$$

where an element $b_{ij}(x)$ denotes the average number of babies born during the unit time interval in region i and alive in region j at the end of that interval, per individual living in region i at the beginning of the interval and x to x + 4 years old. The off-diagonal elements of $\underset{\sim}{B}(x)$ are measures of the mobility of children 0 to 4 years old, who were born to a x to x + 4-year-old parent. It is clear that their mobility pattern is determined by the mobility pattern of the parents.

It can be shown that $\underset{\sim}{B}(x)$ obeys the relationship (Rogers, 1975; pp, 120-121):

$$B(x) = \frac{1}{2} L(0) \ \ell^{-1}(0) \ [F(x) + F(x+5) \ S(x)]$$

whence (3.4)

$$B(x) = \frac{5}{4} [P(0) + I] \ [F(x) + F(x+5) \ S(x)]$$

since

$$L(0) = \frac{5}{2} [\ell(5) + \ell(0)] = \frac{5}{2} [P(0) + I] \ \ell(0)$$

where $L(0)$, $\ell(0)$, $P(0)$ and $S(x)$ are defined in the previous chapter. Here $P(0)$ and $S(x)$ are given by the life table, and $F(x)$ is a diagonal matrix containing the annual regional birthrates of people aged x to x + 4. The number of births in year t from people aged x to x + 4 at t is $F(x)\{K^{(t)}(x)\}$. The number of births during a five year period starting at t, from people aged x to x + 4 at t, is

$$\frac{5}{2} \left[F(x)\{K^{(t)}(x)\} + F(x+5)\{K^{(t+1)}(x+5)\}\right]$$

$$= \frac{5}{2} [F(x) + F(x+5) \ S(x)] \ \{K^{(t)}(x)\} \quad .$$

Of these births, a proportion $L(0)[5\ell(0)]^{-1}$ will be surviving in the various regions at the end of the time interval. Because of the special structure of the generalized Leslie matrix, (3.1) may be written as two equation systems:

$$\{K^{(t+1)}(0)\} = \sum_{\alpha-5}^{\beta-5} B(x)\{K^{(t)}(x)\} \qquad (3.5)$$

$$\{\underset{\sim}{K}^{(t+1)}(x+5)\} = \underset{\sim}{S}(x)\{\underset{\sim}{K}^{(t)}(x)\} \quad , \tag{3.6}$$

$$\text{for } 5 \leq x \leq z - 5 \quad .$$

The vector $\{\underset{\sim}{K}^{(t)}(x)\}$ may be expressed in the form

$$\left\{\underset{\sim}{K}^{(t+\frac{x}{5})}(x)\right\} = [\underset{\sim}{S}(x-5)\ \underset{\sim}{S}(x-10) \cdots \underset{\sim}{S}(5)\ \underset{\sim}{S}(0)]\ \{\underset{\sim}{K}^{(t)}(0)\} \quad . \tag{3.7}$$

$$= \underset{\sim}{A}(x)\ \{\underset{\sim}{K}^{(t)}(0)\} \quad , \qquad \text{say}$$

where we define

$$\underset{\sim}{A}(x) = \left\{ \begin{array}{ll} \underset{\sim}{I} & \text{for } x = 0 \\ \prod_{y=x-5}^{0} \underset{\sim}{S}(y) & \text{for } x = 5,10\ldots z \end{array} \right\} = \underset{\sim}{L}(x)\ \underset{\sim}{L}^{-1}(0) \quad ,$$

with $\prod_{y=x-5}^{0} \underset{\sim}{S}(y) = \underset{\sim}{S}(x-5)\ \underset{\sim}{S}(x-10) \cdots \underset{\sim}{S}(5)\ \underset{\sim}{S}(0)$.

The element $a_{ij}(x)$ of $\underset{\sim}{A}(x)$ is the proportion of individuals aged 0 to 4 years in region i, who will survive to be x to x + 4 years old exactly x years later, and will at that time be in region j.

b. The birth model

The growth path of the births may easily be derived from the growth path of the population. Recall (3.5), and substitute (3.4) for $\underset{\sim}{B}(x)$. Then

$$\{\underset{\sim}{K}^{(t+1)}(0)\} = \sum_{\alpha-5}^{\beta-5} \frac{5}{4}\ [\underset{\sim}{I} + \underset{\sim}{P}(0)][\underset{\sim}{F}(x) + \underset{\sim}{F}(x+5)\ \underset{\sim}{S}(x)]\{\underset{\sim}{K}^{(t)}(x)\}$$

$$= \frac{1}{2} [\underset{\sim}{I} + \underset{\sim}{P}(0)] \sum_{\alpha-5}^{\beta-5} \frac{5}{2} [\underset{\sim}{F}(x) + \underset{\sim}{F}(x+5) \underset{\sim}{S}(x)]\{\underset{\sim}{K}^{(t)}(x)\}$$

$$= \frac{1}{2} [\underset{\sim}{I} + \underset{\sim}{P}(0)]\{\underset{\sim}{Q}^{(t+1,t)}\} \quad , \qquad (3.8)$$

where the regional distribution of births during a five-year period starting at t, is denoted by $\{\underset{\sim}{Q}^{(t+1,t)}\}$ and is defined as

$$\{\underset{\sim}{Q}^{(t+1,t)}\} = \sum_{\alpha-5}^{\beta-5} \frac{5}{2} [\underset{\sim}{F}(x) + \underset{\sim}{F}(x+5) \underset{\sim}{S}(x)]\{\underset{\sim}{K}^{(t)}(x)\} \quad . \qquad (3.9)$$

Note that

$$\{\underset{\sim}{K}^{(t+1)}(0)\} = \underset{\sim}{L}(0) \; \underset{\sim}{\ell}^{-1}(0) \; \{\underset{\sim}{Q}^{(t+1,t)}\} \qquad (3.10)$$

and

$$\{\underset{\sim}{Q}^{(t+1,t)}\} = \underset{\sim}{\ell}(0) \; \underset{\sim}{L}^{-1}(0) \; \{\underset{\sim}{K}^{(t+1)}(0)\}$$

$$= 2[\underset{\sim}{I} + \underset{\sim}{P}(0)]^{-1} \; \{\underset{\sim}{K}^{(t+1)}(0)\} \quad . \qquad (3.11)$$

Substituting

$$\underset{\sim}{K}^{(t)}(x) = \underset{\sim}{A}(x) \left\{\underset{\sim}{K}^{(t-\frac{x}{5})}(0)\right\} \quad , \qquad \text{for } t \geq \frac{x}{5} \quad ,$$

in (3.8), we have

$$\{\underset{\sim}{K}^{(t+1)}(0)\} = \sum_{\alpha-5}^{\beta-5} \frac{5}{4} [\underset{\sim}{I} + \underset{\sim}{P}(0)]$$

$$[\underset{\sim}{F}(x) + \underset{\sim}{F}(x+5) \underset{\sim}{S}(x)] \; \underset{\sim}{A}(x)\left\{\underset{\sim}{K}^{(t-\frac{x}{5})}(0)\right\} \quad , \qquad (3.12)$$

$$\text{for } t \geq \frac{x}{5} \quad ,$$

and, therefore, the growth path of the births may be related to the number of births that occurred some time ago. Substituting (3.10) into (3.12) gives:

$$\{\underset{\sim}{Q}^{(t+1,t)}\} = \sum_{\alpha-5}^{\beta-5} \frac{5}{4} [\underset{\sim}{F}(x) + \underset{\sim}{F}(x+5)\ \underset{\sim}{S}(x)]\ \underset{\sim}{A}(x)\ [\underset{\sim}{I} + \underset{\sim}{P}(0)] \cdot \left\{\underset{\sim}{Q}^{(t-\frac{x}{5},t-\frac{x}{5}-1)}\right\} ,$$

$$\text{for } t \geq \frac{x}{5} + 1 \quad ,$$

$$= \sum_{\alpha-5}^{\beta-5} \underset{\sim}{D}(x) \left\{\underset{\sim}{Q}^{(t-\frac{x}{5},t-\frac{x}{5}-1)}\right\} , \quad \text{say} , \qquad (3.13)$$

since

$$\{\underset{\sim}{Q}^{(t+1,t)}\} = 2[\underset{\sim}{I} + \underset{\sim}{P}(0)]^{-1} \{\underset{\sim}{K}^{(t+1)}(0)\}$$

and

$$\left\{\underset{\sim}{K}^{(t-\frac{x}{5})}(0)\right\} = \frac{1}{2} [\underset{\sim}{I} + \underset{\sim}{P}(0)] \left\{\underset{\sim}{Q}^{(t-\frac{x}{5},t-\frac{x}{5}-1)}\right\} .$$

Formula (3.13) expresses the growth path of the births, occurring during the period (t +1,t), five years say. The <u>annual</u> number of births is

$$\{\underset{\sim}{Q}^{(t)}\} = \sum_{\alpha-5}^{\beta-5} \underset{\sim}{F}(x)\{\underset{\sim}{K}^{(t)}(x)\}$$

$$= \sum_{\alpha-5}^{\beta-5} \underset{\sim}{F}(x)\ \underset{\sim}{A}(x)\left\{\underset{\sim}{K}^{(t-\frac{x}{5})}(0)\right\} . \qquad (3.14)$$

Assuming stationarity, we may express the number of people in the first age group as a function of the births, as in Equation (2.10)

$$\{\underset{\sim}{K}^{(t)}(0)\} = \frac{5}{2}\,[\underset{\sim}{I} + \underset{\sim}{P}(0)]\{\underset{\sim}{Q}^{(t)}\} \quad . \tag{3.15}$$

We have that

$$\{\underset{\sim}{Q}^{(t)}\} = \sum_{\alpha-5}^{\beta-5} \frac{5}{2}\,\underset{\sim}{F}(x)\ \underset{\sim}{A}(x)\ [\underset{\sim}{I} + \underset{\sim}{P}(0)]\left\{\underset{\sim}{Q}^{(t-\frac{x}{5})}\right\} \quad , \tag{3.16}$$

$$\text{for } t \geq \frac{x}{5}$$

which is equal to

$$\{\underset{\sim}{Q}^{(t)}\} = \sum_{\alpha-5}^{\beta-5} \underset{\sim}{F}(x)\ \underset{\sim}{L}(x)\ \left\{\underset{\sim}{Q}^{(t-\frac{x}{5})}\right\} \quad , \tag{3.17}$$

in which we once again relate the number of births at time t to the number that occurred some time ago.

The relation between (3.17) and (3.13) is implicit in expression (3.15). Substituting (3.8) into (3.15) gives:

$$\frac{1}{2}\,[\underset{\sim}{I} + \underset{\sim}{P}(0)]\{\underset{\sim}{Q}^{(t,t-1)}\} = \frac{5}{2}\,[\underset{\sim}{I} + \underset{\sim}{P}(0)]\{\underset{\sim}{Q}^{(t)}\}$$

or

$$\{\underset{\sim}{Q}^{(t)}\} = \frac{1}{5}\,\{\underset{\sim}{Q}^{(t,t-1)}\} \quad . \tag{3.18}$$

This implies that the annual number of births is a simple average of the births during the previous period. Equation (3.17) is an (β-5)-th order difference equation. To derive a birth growth model analogue to (3.1), we replace (3.17) by a system of (β-5) first order difference equations:

$$\begin{bmatrix} \{\underset{\sim}{Q}^{(t)}\} \\ \{\underset{\sim}{Q}^{(t-1)}\} \\ \{\underset{\sim}{Q}^{(t-\frac{\alpha-5}{5})}\} \\ \vdots \\ \{\underset{\sim}{Q}^{(t-\frac{\beta-10}{5})}\} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} & \cdots & \left(\underset{\sim}{F}(\alpha-5)\ \underset{\sim}{L}(\alpha-5)\right) & \cdots & \left(\underset{\sim}{F}(\beta-5)\ \underset{\sim}{L}(\beta-5)\right) \\ \underset{\sim}{I} & & & & & \underset{\sim}{0} \\ \vdots & \underset{\sim}{I} & \ddots & & & \vdots \\ & & & \ddots & & \vdots \\ \underset{\sim}{0} & \cdots & \cdots & \cdots & \underset{\sim}{I} & \underset{\sim}{0} \end{bmatrix} \begin{bmatrix} \{\underset{\sim}{Q}^{(t-1)}\} \\ \{\underset{\sim}{Q}^{(t-2)}\} \\ \{\underset{\sim}{Q}^{(t-\frac{\alpha}{5})}\} \\ \vdots \\ \{\underset{\sim}{Q}^{(t-\frac{\beta-5}{5})}\} \end{bmatrix} \tag{3.19}$$

or, in condensed form,

$$\{\hat{\underset{\sim}{Q}}^{(t)}\} = \underset{\sim}{H}\{\hat{\underset{\sim}{Q}}^{(t-1)}\} \quad . \tag{3.20}$$

Equation (3.20) relates the births at time t to the births at t-1. Once the birth trajectory is known, the trajectory of the population distribution may be computed by (3.15) and (3.8).

3.2. SENSITIVITY ANALYSIS OF THE POPULATION PROJECTION

Recall the population growth model defined in (3.1):

$$\{\underset{\sim}{K}^{(t+1)}\} = \underset{\sim}{G}\{\underset{\sim}{K}^{(t)}\} \quad . \tag{3.1}$$

The assessment of the sensitivity of $\{\underset{\sim}{K}^{(t+1)}\}$ to changes in age-specific rates $\underset{\sim}{M}(x)$, may be analyzed by means of a two-step process. The first step considers the sensitivity of the growth matrix to changes in age-specific rates. The second step derives a sensitivity function which describes the impact on the population distribution of a change in the growth matrix. In our sensitivity analysis of life table statistics, we were not concerned with the time when the change in $\underset{\sim}{M}(x)$ occurred. The time consideration was irrelevant, since the life table is a static model. For the sensitivity analysis of the population growth, however, it is important to know not only the age group <u>where</u> a change in $\underset{\sim}{M}(x)$ occurs, but also the time <u>when</u> the change occurs. We will denote this time by t_0. The time at which the change in the population distribution is measured will be denoted by t_1.

Besides the change in $\{\underset{\sim}{K}^{(t_1)}\}$ due to a change in the age-specific rates at t_0, one may also consider the problem of how a unique change in $\{\underset{\sim}{K}^{(t_0)}\}$ affects $\{\underset{\sim}{K}^{(t_1)}\}$. These are two separate sensitivity problems. In the first, the parameter changes at t_0 and remains at his new level thereafter. The second problem, however, is equivalent to a parameter change at t_0 only. These two sensitivity problems will be treated separately.

a. Sensitivity of the growth matrix

The growth matrix $\underset{\sim}{G}$ is composed of two types of submatrices, $\underset{\sim}{S}(x)$ and $\underset{\sim}{B}(x)$. The sensitivity on $\underset{\sim}{S}(x)$ of changes in $\underset{\sim}{M}(x)$, as given in Section 2.2, appears only in the two age groups, x and x-5:

$$\frac{\delta \underset{\sim}{S}(x)}{\delta < \underset{\sim}{M}(x) >} = - \frac{5}{2} \underset{\sim}{S}(x) [\underset{\sim}{I} - \frac{5}{2} \underset{\sim}{M}(x)]^{-1} \underset{\sim}{J} \qquad (2.35)$$

$$\frac{\delta \underset{\sim}{S}(x - 5)}{\delta < \underset{\sim}{M}(x) >} = - \frac{5}{2} \underset{\sim}{P}(x) [\underset{\sim}{I} - \frac{5}{2} \underset{\sim}{M}(x)]^{-1} \underset{\sim}{J} \underset{\sim}{S}(x - 5) \qquad (2.37)$$

$$\frac{\delta \underset{\sim}{S}(a)}{\delta <\underset{\sim}{M}(x)>} = \underset{\sim}{0} \qquad \text{for } a > x \text{ , or}$$
$$\text{for } a < x - 5 \text{ .}$$

The sensitivity function of $\underset{\sim}{B}(x)$ remains to be derived. Recall from (3.4) that

$$\underset{\sim}{B}(x) = \frac{5}{4} [\underset{\sim}{P}(0) + \underset{\sim}{I}][\underset{\sim}{F}(x) + \underset{\sim}{F}(x + 5) \ \underset{\sim}{S}(x)] \quad , \qquad (3.4)$$

where $\underset{\sim}{B}(x)$ depends on the age-specific death and out-migration rates through $\underset{\sim}{S}(x)$ and $\underset{\sim}{P}(0)$, and on the age-specific fertility rates $\underset{\sim}{F}(x)$ and $\underset{\sim}{F}(x + 5)$. Consider the partial derivative of $\underset{\sim}{B}(x)$ with respect to $\underset{\sim}{M}(x)$:

$$\frac{\delta \underset{\sim}{B}(x)}{\delta <\underset{\sim}{M}(x)>} = \frac{5}{4} \frac{\delta [\underset{\sim}{P}(0) + \underset{\sim}{I}]}{\delta <\underset{\sim}{M}(x)>} \underset{\sim}{F}(x) + \frac{5}{4} \frac{\delta [\underset{\sim}{P}(0) + \underset{\sim}{I}]}{\delta <\underset{\sim}{M}(x)>} \underset{\sim}{F}(x + 5) \ \underset{\sim}{S}(x)$$
$$+ \frac{5}{4} [\underset{\sim}{P}(0) + \underset{\sim}{I}] \ \underset{\sim}{F}(x + 5) \frac{\delta \underset{\sim}{S}(x)}{\delta <\underset{\sim}{M}(x)>} \quad . \qquad (3.21)$$

Since $\underset{\sim}{P}(0)$ is affected by a change in $\underset{\sim}{M}(x)$ only if $x = 0$, and because for this case $\underset{\sim}{F}(x)$ and $\underset{\sim}{F}(x + 5)$ are $\underset{\sim}{0}$, (3.21) reduces to

$$\frac{\delta \underset{\sim}{B}(x)}{\delta <\underset{\sim}{M}(x)>} = \frac{5}{4} [\underset{\sim}{P}(0) + \underset{\sim}{I}] \ \underset{\sim}{F}(x + 5) \frac{\delta \underset{\sim}{S}(x)}{\delta <\underset{\sim}{M}(x)>} \qquad (3.22)$$

which, by (2.35), is

$$\frac{\delta \underset{\sim}{B}(x)}{\delta \langle \underset{\sim}{M}(x) \rangle} = - \frac{5}{4} [\underset{\sim}{P}(0) + \underset{\sim}{I}] \; \underset{\sim}{F}(x+5) \; \frac{5}{2} \; \underset{\sim}{S}(x) [\underset{\sim}{I} - \frac{5}{2} \underset{\sim}{M}(x)]^{-1} \; \underset{\sim}{J}$$

(3.23)

$$= - \frac{25}{8} [\underset{\sim}{P}(0) + \underset{\sim}{I}] \; \underset{\sim}{F}(x+5) \; \underset{\sim}{S}(x) [\underset{\sim}{I} - \frac{5}{2} \underset{\sim}{M}(x)]^{-1} \; \underset{\sim}{J} \quad .$$

Since a change of $\underset{\sim}{M}(x)$ affects $\underset{\sim}{S}(x - 5)$, it also affects $\underset{\sim}{B}(x - 5)$

$$\frac{\delta \underset{\sim}{B}(x-5)}{\delta \langle \underset{\sim}{M}(x) \rangle} = - \frac{25}{8} [\underset{\sim}{P}(0) + \underset{\sim}{I}] \; \underset{\sim}{F}(x) \; \underset{\sim}{P}(x) [\underset{\sim}{I} - \frac{5}{2} \underset{\sim}{M}(x)]^{-1} \; \underset{\sim}{J} \; \underset{\sim}{S}(x-5) \qquad (3.24)$$

$$= - \frac{25}{8} [\underset{\sim}{P}(0) + \underset{\sim}{I}] \; \underset{\sim}{F}(x) [\underset{\sim}{P}(x) + \underset{\sim}{I}] \; \underset{\sim}{J} \; \underset{\sim}{S}(x-5) \quad . \qquad (3.25)$$

The sensitivity of $\underset{\sim}{B}(x)$ with respect to $\underset{\sim}{F}(x)$ and $\underset{\sim}{F}(x + 5)$ also may be derived easily:

$$\frac{\delta \underset{\sim}{B}(x)}{\delta \langle \underset{\sim}{F}(x) \rangle} = \frac{5}{4} [\underset{\sim}{P}(0) + \underset{\sim}{I}] \; \underset{\sim}{J}$$

(3.26)

$$= \frac{5}{2} \underset{\sim}{L}(0) [5 \underset{\sim}{\ell}(0)]^{-1} \; \underset{\sim}{J}$$

and

$$\frac{\delta \underset{\sim}{B}(x-5)}{\delta \langle \underset{\sim}{F}(x) \rangle} = \frac{5}{4} [\underset{\sim}{P}(0) + \underset{\sim}{I}] \; \underset{\sim}{J} \; \underset{\sim}{S}(x-5) \quad . \qquad (3.27)$$

Thus the impact of a unit change in the fertility matrix $\underset{\sim}{F}(x)$ on the element $\underset{\sim}{B}(x)$ is $\frac{5}{2}$ times the proportion of new-born babies that will be alive at the end of the time interval.

Having derived sensitivity functions for the elements of the growth matrix, we now can proceed to the question of how changes in the growth matrix affect the growth of the population. This is sometimes called trajectory sensitivity.

b. Sensitivity of the population trajectory

Recall the population growth equation

$$\{\underset{\sim}{K}^{(t+1)}\} = \underset{\sim}{G}\{\underset{\sim}{K}^{(t)}\} \quad . \tag{3.1}$$

Since $\underset{\sim}{G}$ is assumed to be constant over time, the population distribution at time t_1 is given by

$$\left\{\underset{\sim}{K}^{(t_1)}\right\} = \underset{\sim}{G}^{t_1-t_0}\left\{\underset{\sim}{K}^{(t_0)}\right\} \quad .$$

We assume that the change in the growth matrix occurs at t_0. Without loss of generality, we may set t_0 equal to zero, and t_1 equal to t. Then

$$\{\underset{\sim}{K}^{(t)}\} = \underset{\sim}{G}^{t}\{\underset{\sim}{K}^{(0)}\} \quad .$$

The sensitivity of $\{\underset{\sim}{K}^{(t)}\}$ to a change in $\underset{\sim}{G}$ is

$$\frac{\delta\{\underset{\sim}{K}^{(t)}\}}{\delta\langle\underset{\sim}{G}\rangle} = \frac{\delta[\underset{\sim}{G}^{t}]}{\delta\langle\underset{\sim}{G}\rangle}\{K^{(0)}\} \quad .$$

The sensitivity of $\underset{\sim}{G}^t$ to a change in $<\underset{\sim}{G}>$ is given by (A.24) of the Appendix. Applying this result, yields:

$$\frac{\delta\{\underset{\sim}{K}^{(t)}\}}{\delta<\underset{\sim}{G}>} = \sum_{i=0}^{t-1} \underset{\sim}{G}^i \underset{\sim}{J}\underset{\sim}{G}^{t-1-i} \{\underset{\sim}{K}^{(0)}\} \quad . \tag{3.28}$$

A related problem might come up in policy making. Under the growth model (3.1), the population distribution which yields a specified distribution at time t is given by

$$\{\underset{\sim}{K}^{(0)}\} = [\underset{\sim}{G}^t]^{-1} \{\underset{\sim}{K}^{(t)}\} \quad .$$

If $\{\underset{\sim}{K}^{(0)}\}$ deviates much from the actual population distribution, the policy maker may consider changing some elements of the growth matrix through policy measures. The impact on $\{\underset{\sim}{K}^{(0)}\}$ is

$$\begin{aligned}\frac{\delta\{\underset{\sim}{K}^{(0)}\}}{\delta<\underset{\sim}{G}>} &= - [\underset{\sim}{G}^t]^{-1} \frac{\delta \underset{\sim}{G}^t}{\delta<\underset{\sim}{G}>} [\underset{\sim}{G}^t]^{-1} \{\underset{\sim}{K}^{(t)}\} \\ &= - [\underset{\sim}{G}^t]^{-1} \left[\sum_{i=0}^{t-1} \underset{\sim}{G}^i \underset{\sim}{J}\underset{\sim}{G}^{t-1-i}\right] [\underset{\sim}{G}^t]^{-1} \{\underset{\sim}{K}^{(0)}\} \\ &= - \sum_{i=0}^{t-1} [\underset{\sim}{G}^{(t-i)}]^{-1} \underset{\sim}{J}[\underset{\sim}{G}^{(i+1)}]^{-1} \{\underset{\sim}{K}^{(0)}\} \quad .\end{aligned} \tag{3.29}$$

If, by some means, an optimal growth matrix is defined which leads a population $\{\underset{\sim}{K}^{(0)}\}$ to a desired $\{\underset{\sim}{K}^{(t)}\}$, the next problem is to find out under what conditions variations in $\underset{\sim}{G}$ do not affect $\{\underset{\sim}{K}^{(t)}\}$. Such specific conditions are derived by Tomović and Vukobratović (1972; p. 138). They will not be discussed here. This and similar problems of trajectory insensitivity or invariance are receiving an increasing attention in system theory and optimal control theory. For a review of some applications in the social sciences, see Erickson and Norton (1973).

The next section addresses the topic of the sensitivity of population growth to changes in the population distribution at a certain point in time. This will be called the analysis of small perturbations around the growth path.

c. Perturbations around the population growth path

The impact on $\{\underset{\sim}{K}^{(t)}\}$ of a change in $\{\underset{\sim}{K}^{(0)}\}$ is very simple in the time-invariant equation system (3.1). Applying the results of vector differentiation of the Appendix gives:

$$\frac{\delta\{\underset{\sim}{K}^{(t)}\}}{\delta\{K^{(0)}\}'} = \frac{\delta[\underset{\sim}{G}^{t}]\{\underset{\sim}{K}^{(0)}\}}{\delta\{\underset{\sim}{K}^{(0)}\}'} = \underset{\sim}{G}^{t} \quad . \tag{3.30}$$

where $\{\underset{\sim}{K}^{(0)}\}'$ is the transpose of $\{\underset{\sim}{K}^{(0)}\}$.

Equation (3.30) relates changes in the state vector at time t to changes in the state vector at time zero. If the growth matrix is time-dependent, then this problem cannot be solved analytically, and one must rely on simulation. An illustration of such a situation is when the model incorporates a feedback loop, i.e., the growth matrix at time t depends on the state vector at time t. An application of feedback models to urban analysis is given by Forrester (1969). Nelson and Kern (1971) have simulated the impact of small perturbations around the trajectory for a Forrester-type of urban model.

d. Sensitivity of the sequence of births

The sensitivity analysis of the growth matrix of the system trajectory and of perturbations around the trajectory could be repeated with the growth model (3.20). There are no real differences in methodology. The growth matrix now is simpler, and the state vector is the spatial distribution of the births. We will only consider the impact on the births sequence of a change in births at time zero where the birth sequence is described by

$$\{\hat{Q}^{(t)}\} = H^t\{\hat{Q}^{(0)}\} \quad , \tag{3.31}$$

with H given by (3.20).

Suppose that a change occurs in the first sub-vector of $\{\hat{Q}^{(0)}\}$, and that the impact is measured on the first sub-vector of $\{\hat{Q}^{(t)}\}$, then the sensitivity coefficients are given by the submatrix $[H^t]_{11}$. Since new-born babies only affect the births sequence if they reach the reproductive ages, $[H^t]_{11}$ is 0 for $t \leq \frac{\alpha-5}{5}$.

Another approach to sensitivity analysis of the births sequence may be more convenient, especially if, at the same time, one is interested in the sensitivity of the growth path of the whole population. This approach is based on the relationship

$$\{Q^{(t)}\} = F\{K^{(t)}\} = FG^t\{K^{(0)}\} \tag{3.32}$$

where F is the matrix of age-specific fertility rates

$$F = [0 \quad 0 \quad F(\alpha) \cdots F(\beta-5) \quad 0 \cdots] \quad .$$

A change in the growth matrix $\underset{\sim}{G}$ affects $\{\underset{\sim}{Q}^{(t)}\}$ in the following sense

$$\frac{\delta\{\underset{\sim}{Q}^{(t)}\}}{\delta<\underset{\sim}{G}>} = \frac{\delta\underset{\sim}{F}}{\delta<\underset{\sim}{G}>}\{\underset{\sim}{K}^{(t)}\} + \underset{\sim}{F}\frac{\delta\{\underset{\sim}{K}^{(t)}\}}{\delta<\underset{\sim}{G}>} \quad .$$

If the change occurs in the mortality or migration, but not in the fertility, then

$$\frac{\delta\{\underset{\sim}{Q}^{(t)}\}}{\delta<\underset{\sim}{G}>} = \underset{\sim}{F}\frac{\delta\{\underset{\sim}{K}^{(t)}\}}{\delta<\underset{\sim}{G}>} = \underset{\sim}{F}\sum_{i=0}^{t-1}\underset{\sim}{G}^{i}\,\underset{\sim}{J}\underset{\sim}{G}^{t-1-i}\,\{\underset{\sim}{K}^{(0)}\} \quad . \quad (3.33)$$

This chapter dealt with the sensitivity analysis of demographic growth. It has been shown that demographic growth may be expressed equally well in terms of births as in terms of population. This analogy will be extended in the next chapter while discussing the sensitivity of stable population characteristics.

CHAPTER 4

IMPACT OF CHANGES IN AGE-SPECIFIC RATES ON STABLE POPULATION CHARACTERISTICS

The stable population concept provides a major framework for analysis in mathematical demography. It has proved to be a helpful device in understanding how age compositions and regional distributions of populations are determined. The premise upon which the concept is based is the property that a human population tends to "forget" its past. This property is called ergodicity. The regional age compositions and regional shares of a closed multiregional population are completely determined by the recent history of fertility, mortality and migration to which the population has been subject. It is not necessary to know anything about the history of a population more than a century or two ago in order to account for its present demographic characteristics (Lopez, 1961). In fact, the regional shares, the age compositions and the sequence of births can be calculated from no more than a specified sequence of fertility, mortality and migration schedules over a moderate time interval.

Therefore, a particularly useful way to understand how the age and spatial structure of a population are formed and its vital rates determined, is to imagine them as describing a population which has been subjected to constant fertility, mortality and migration schedules for an extended period of time. The population that develops under such circumstances is called a stable multiregional population. Its principal characteristics are: constant regional age compositions and regional shares; constant regional annual rates of birth, death and migration; and a fixed multiregional annual rate of growth that also is the annual growth rate in each region. Such multiregional stable populations have been studied by Rogers (1973, 1974, 1975).

The first section of this chapter is an exposition of the major characteristics of stable populations. It is customary in mathematical demography to distinguish between a discrete and a continuous model of population growth, and the stable populations associated with them. The reason is mainly historical. The discrete model, which expresses the population growth as a matrix multiplication using a discrete time-variable and a discrete age-scale, derives largely from the work of Leslie (1945). The Leslie model is, in fact, a system of homogenous first-order difference equations, similar to (3.1). The continuous model uses a continuous time-variable and a continuous age-scale, and in its modern form originates from the work of Lotka (1907) and Sharpe and Lotka (1911). Lotka's work starts out with the population growth equation provided by Malthus (1798), which is, in fact, a homogenous first-order differential equation. Although in the literature the formulations of the continuous and the discrete model of growth seem very different, they are closely related. Goodman (1967) and Keyfitz (1968) have provided insights in the reconciliation of both growth models.

We focus in this chapter on the discrete model of population growth. However, we shall frequently refer to aspects of the continuous model that can be developed as well for the discrete case.

The second part of this chapter deals with the sensitivity analysis of the most important stable population statistics: the stable population distribution and the stable growth ratio. Demetrius (1969), Keyfitz (1971), Goodman (1971), Coale (1972) and Preston (1974), among others, have addressed this problem for a single region population without migration. Most take the continuous version of the stable population as a vehicle for sensitivity analysis. Demetrius and Goodman, however, use the discrete version. Their approach is our starting point for the sensitivity analysis. However, there are fundamental differences between the forumulation of a single region and a multiregion stable population which necessitate other tools for analysis. One such tool is the eigenvalue and eigenvector analysis derived in the Appendix. An alternative approach, which starts out from the characteristic equation as in Keyfitz (1971), is also provided. This enables us to derive sensitivity functions that are similar to their single-region counterparts.

4.1. THE MULTIREGIONAL STABLE POPULATION

As in the previous chapter, we distinguish between the population model and the birth model. They are two equivalent formulations for population dynamics.

a. The population model

Recall the discrete model of population growth that was set out in (3.1). It may be written as

$$\{\underset{\sim}{K}^{(t)}\} = \underset{\sim}{G}^{t}\{\underset{\sim}{K}^{(0)}\} \quad . \tag{4.1}$$

Consider the asymptotic properties of (4.1) when t gets large. Such properties have been studied by Keyfitz (1968), Sykes (1969), Feeney (1973), Le Bras (1973) and Pollard (1973; pp. 39-46), among others. Rogers (1975; pp. 124-129) extends the arguments of Le Bras, Feeney, and Sykes to a multiregional system. The key element in the analysis is the Perron-Frobenius theorem. It establishes that any nonnegative, indecomposable, primitive square matrix has a unique, real, positive eigenvalue, λ_1 say, that is larger in absolute value than any other eigenvalue of that matrix. With this dominant eigenvalue are associated a right and left eigenvector, both with only positive elements. The growth operator is nonnegative and decomposable. However, $\underset{\sim}{G}$ may be partitioned, yielding a square submatrix, $\underset{\sim}{W}$ say, which is indecomposable and which is similar to $\underset{\sim}{G}$, and which therefore has the same eigenvalues. The matrix $\underset{\sim}{W}$ is primitive if the fertility of two adjacent age groups are positive in each and every region, i.e., if in (3.3) two consecutive matrices, $\underset{\sim}{B}(x)$ are positive (e.g., see Rogers (1975; pp. 124-129)). The dominant eigenvalue and the two associated eigenvectors have demographic meaning. The dominant eigenvalue of $\underset{\sim}{G}$ represents the stable growth ratio of the population. The associated right eigenvector gives the stable age- and region-specific population distribution, while the corresponding left eigenvector gives the spatial reproductive values. Therefore, the sensitivity of the growth ratio of the stable population to changes in the growth matrix is a problem of eigenvalue sensitivity. The sensitivity of the stable population distribution may be translated into eigenvector sensitivity.

We have seen, in the previous chapter, that because of the particular structure of $\underset{\sim}{G}$, the growth equation may be written as:

$$\{\underset{\sim}{K}^{(t+1)}(0)\} = \sum_{\alpha-5}^{\beta-5} \underset{\sim}{B}(x) \; \{\underset{\sim}{K}^{(t)}(x)\} \tag{3.5}$$

$$\{\underset{\sim}{K}^{(t+1)}(x+5)\} = \underset{\sim}{S}(x) \; \{\underset{\sim}{K}^{(t)}(x)\} \quad . \tag{3.6}$$

At stability, the characteristic value equation holds. Thus

$$\{\underset{\sim}{K}^{(t+1)}\} = \underset{\sim}{G}\{\underset{\sim}{K}^{(t)}\} = \lambda\{\underset{\sim}{K}^{(t)}\} \quad , \tag{4.2}$$

where λ is the dominant eigenvalue of $\underset{\sim}{G}$. Therefore,

$$\{\underset{\sim}{K}^{(t+1)}(x+5)\} = \underset{\sim}{S}(x) \; \{\underset{\sim}{K}^{(t)}(x)\} = \lambda\{\underset{\sim}{K}^{(t)}(x+5)\} \quad ; \tag{4.3}$$

hence

$$\{\underset{\sim}{K}^{(t)}(x+5)\} = \frac{1}{\lambda} \underset{\sim}{S}(x) \; \{\underset{\sim}{K}^{(t)}(x)\} \quad . \tag{4.4}$$

Combining (4.4) with (3.6), we have

$$\{\underset{\sim}{K}^{(t)}(x)\} = \lambda^{-\frac{x}{5}} \underset{\sim}{A}(x) \; \{\underset{\sim}{K}^{(t)}(0)\} \tag{4.5}$$

where $\underset{\sim}{A}(x)$ is defined by (3.6).

The single-region analogue to (4.5) may be found in Goodman (1967; p. 543, and 1971; p. 340), Demetrius (1969; p. 133) and Cull and Vogt (1973; p. 647), among others.

Equation (4.3) gives the number of people in each age group and region in terms of the regional distribution of the people in the first age group. Now we derive an expression for the stable growth path of the population in the first age group. By (4.3) and (3.5) we may write:

$$\{\underset{\sim}{K}^{(t+1)}(0)\} = \lambda\{\underset{\sim}{K}^{(t)}(0)\}$$

$$= \sum_{\alpha-5}^{\beta-5} \underset{\sim}{B}(x)\ \{\underset{\sim}{K}^{(t)}(x)\} \quad .$$

Substituting for (4.5) and deleting the superscript, gives

$$\lambda\{\underset{\sim}{K}(0)\} = \sum_{\alpha-5}^{\beta-5} \underset{\sim}{B}(x)\ \lambda^{-\frac{x}{5}}\ \underset{\sim}{A}(x)\ \{\underset{\sim}{K}(0)\} \quad , \tag{4.6}$$

which is the expression given by Rogers (1975; p. 140). It may be replaced by

$$\left[\sum_{\alpha-5}^{\beta-5} \lambda^{-(\frac{x}{5}+1)}\ \underset{\sim}{B}(x)\ \underset{\sim}{A}(x) - \underset{\sim}{I}\right] \{\underset{\sim}{K}(0)\} = \{\underset{\sim}{0}\} \quad . \tag{4.7}$$

Equation (4.7) is the discrete version of equation (4.7) in Rogers (1975; p. 93).

The matrix

$$\underset{\sim}{\bar{\Phi}}(x) = \underset{\sim}{B}(x)\ \underset{\sim}{A}(x) \tag{4.8}$$

is the discrete formulation of the multiregional net maternity function, and

$$\underset{\sim}{\bar{\Psi}}(\lambda) = \sum_{\alpha-5}^{\beta-5} \lambda^{-(\frac{x}{5}+1)}\ \underset{\sim}{\bar{\Phi}}(x) \tag{4.9}$$

is the corresponding discrete multiregional characteristic matrix.

The stable growth ratio λ is the number that gives $\bar{\Psi}(\lambda)$ a characteristic root of unity. The vector $\{K(0)\}$ is the associated eigenvector. An equivalent formulation is

$$|\bar{\Psi}(\lambda) - I| = 0 \quad . \tag{4.10}$$

Condition (4.10) may also be derived in a different way. The idea is to reduce the growth matrix G to its generalized companion form. The notion of companion form of a matrix occupies a central place in system theory. See, for example, Wolovich (1974; p. 79) and Barnett (1974; p. 671). Kalman (1969; p. 44) considers several companion forms. Two commonly used forms are

$$M = \begin{bmatrix} m_1 & m_2 & m_3 & \cdots & m_{Z-1} & m_Z \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 & 0 \end{bmatrix} \quad \text{and } N = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & 1 \\ m_Z & m_{Z-1} & \cdots & \cdots & \cdots & m_1 \end{bmatrix}$$

The companion form arises when a dynamic system is written as a linear differential or difference equation of the Z-th order. The elements of the first row of M or last row of N, respectively, are the coefficients of the characteristic equation. Recall that the growth equation (3.1) is a system of Z linear first-order difference equations, where Z is the number of age groups. Each system of linear first-order difference equations may be transformed into one linear difference equation of the Z-th order, and vice versa. This transformation corresponds to a change in the coordinate system. For example, (3.19) is a companion form, arising from the (β-5)-th order difference equation (3.17). Instead of scalar elements, (3.19) has submatrices as elements. Barnett (1973; p. 6) has called this form a generalized companion matrix. A transformation of a single region population growth matrix into a companion matrix of form $\mathbf{M}$ is given by Pielou (1969; p. 37). Wu (1972) sets up a transformation to both forms $\mathbf{M}$ and $\mathbf{N}$. In fact

$$\mathbf{E}\mathbf{M}\mathbf{E} = \mathbf{N} \quad , \tag{4.11}$$

where

$$\mathbf{E} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix} .$$

The transformation of the multiregional growth matrix $\mathbf{G}$ into a generalized companion matrix $\hat{\mathbf{G}}$ may be expressed as

$$\hat{\mathbf{G}} = \mathbf{H}\mathbf{G}\mathbf{H}^{-1} \tag{4.12}$$

where

$$\underset{\sim}{H} = \begin{bmatrix} \underset{\sim}{A}(Z) & \underset{\sim}{0} \cdots\cdots & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{A}(Z-5) & \vdots \\ & \ddots & \vdots \\ 0 & \cdots\cdots\cdots & \underset{\sim}{A}(0) \end{bmatrix}$$

with $\underset{\sim}{A}(x)$ as defined by (3.6), and where

$$\hat{\underset{\sim}{G}} = \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} & [\underset{\sim}{B}(10)\ \underset{\sim}{A}(10)] & [\underset{\sim}{B}(15)\ \underset{\sim}{A}(15)] & \cdots & [\underset{\sim}{B}(\beta-5)\ \underset{\sim}{A}(\beta-5)] & \cdots & & \underset{\sim}{0} \\ \underset{\sim}{I} & \underset{\sim}{0} & & & & & & & \bullet \\ \underset{\sim}{0} & \underset{\sim}{I} & & & & & & & \bullet \\ \vdots & & \bullet & \bullet & & \vdots & & & \vdots \\ \underset{\sim}{0} & & \bullet & \bullet & \cdots & \bullet & & \underset{\sim}{I} & \underset{\sim}{0} \end{bmatrix}$$

(4.13)

Since (4.12) is a similarity transformation, it implies that $\underset{\sim}{G}$ and $\hat{\underset{\sim}{G}}$ have the same eigenvalues. They may be found by solving

$$|\underset{\sim}{G} - \lambda \underset{\sim}{I}| = 0 \tag{4.14}$$

or

$$|\hat{\underset{\sim}{G}} - \lambda \underset{\sim}{I}| = 0 \quad . \tag{4.15}$$

Kenkel (1974; pp. 319-322) shows that (4.15) may be reduced:

$$|\hat{\underset{\sim}{G}} - \lambda \underset{\sim}{I}| = |\lambda^{\frac{Z}{5}+1} \underset{\sim}{I} - \lambda^{\frac{Z}{5}} \underset{\sim}{B}(0)\underset{\sim}{A}(0) - \lambda^{\frac{Z}{5}-1} \underset{\sim}{B}(5)\underset{\sim}{A}(5) \ \ldots \tag{4.16}$$

$$- \lambda \underset{\sim}{B}(Z-5)\ \underset{\sim}{A}(Z-5) - \underset{\sim}{B}(Z)\underset{\sim}{A}(Z)| \quad .$$

Dividing by $\lambda^{\frac{Z}{5}+1}$, and since $\underset{\sim}{B}(x) = \underset{\sim}{0}$ for $x < \alpha - 5$ and for $x > \beta - 5$, we have that

$$|\hat{\underset{\sim}{G}} - \lambda \underset{\sim}{I}| = \left| \sum_{\alpha-5}^{\beta-5} \lambda^{-(\frac{x}{5}+1)} \underset{\sim}{B}(x) \underset{\sim}{A}(x) - \underset{\sim}{I} \right| \qquad (4.17)$$

which is condition (4.10). Wilkinson (1965; p. 432) labels (4.17) as the generalized eigenvalue problem.

The generalized companion matrix provides a mathematical tool to link (4.10) to (4.14). Since (4.10) is the discrete version of the condition in the continuous model that the stable growth rate must give the characteristic matrix an eigenvalue of unity, the companion matrix has a role in the reconciliation of the discrete and the continuous models of demographic growth.

The eigenvector of $\hat{\underset{\sim}{G}}$ and $\underset{\sim}{G}$ are related as

$$\{\hat{\underset{\sim}{K}}\} = \underset{\sim}{H}\{\underset{\sim}{K}\} \quad . \qquad (4.18)$$

b. The birth model

The birth trajectory may be described by (3.20):

$$\{\hat{\underset{\sim}{Q}}^{(t)}\} = \underset{\sim}{H}\{\hat{\underset{\sim}{Q}}^{(t-1)}\} \quad . \qquad (3.20)$$

Since all the elements of $\underset{\sim}{H}$ are nonnegative, we may apply the Perron-Frobenius theorem and derive expressions for λ analogue to (4.10) and (4.14). However, there is a third formulation of the condition that λ must satisfy. It draws on the relationship between $\{\underset{\sim}{K}(0)\}$ and $\{\underset{\sim}{Q}\}$, the births in the stable population:

$$\{\underset{\sim}{K}(0)\} = \lambda^{\frac{1}{2}} \frac{5}{2} [\underset{\sim}{I} + \underset{\sim}{P}(0)]\{\underset{\sim}{Q}\} \quad , \qquad (4.19)$$

which has its origin in (3.15). Substituting this into (4.6) and introducing $\underset{\sim}{B}(x)$ yields

$$\sum_{\alpha-5}^{\beta-5} \lambda^{-(\frac{x}{5}+1)} \frac{5}{4} [\underset{\sim}{I} + \underset{\sim}{P}(0)][\underset{\sim}{F}(x) + \underset{\sim}{F}(x+5) \; \underset{\sim}{S}(x) \; \underset{\sim}{A}(x) \; \frac{5}{2} [\underset{\sim}{I} + \underset{\sim}{P}(0)]\{\underset{\sim}{Q}\}$$

$$= \lambda^{\frac{1}{2}} \frac{5}{2} [\underset{\sim}{I} + \underset{\sim}{P}(0)]\{\underset{\sim}{Q}\} \quad . \qquad (4.20)$$

Multiplying both sides by $\lambda^{\frac{1}{2}} \frac{2}{5} [\underset{\sim}{I} + \underset{\sim}{P}(0)]^{-1}$ gives

$$\sum_{\alpha-5}^{\beta-5} \lambda^{-(\frac{x}{5}+\frac{1}{2})} \frac{1}{2} [\underset{\sim}{F}(x) + \underset{\sim}{F}(x+5) \; \underset{\sim}{S}(x)] \; \underset{\sim}{A}(x) \; \frac{5}{2} [\underset{\sim}{I} + \underset{\sim}{P}(0)]\{\underset{\sim}{Q}\} = \{\underset{\sim}{Q}\} \quad . \qquad (4.21)$$

But

$$\frac{5}{2} [\underset{\sim}{I} + \underset{\sim}{P}(0)] = \underset{\sim}{L}(0)$$

and

$$\underset{\sim}{A}(x) \; \underset{\sim}{L}(0) = \underset{\sim}{L}(x)$$

where $\underset{\sim}{L}(x)$ is the number of years lived in the age group x to x + 4 by unit regional radices. Therefore (4.21) becomes

$$\left[\sum_{\alpha-5}^{\beta-5} \lambda^{-(\frac{x}{5}+\frac{1}{2})} \frac{1}{2} [\underset{\sim}{F}(x) + \underset{\sim}{F}(x+5) \; \underset{\sim}{S}(x)] \; \underset{\sim}{L}(x)\right] \{\underset{\sim}{Q}\} = \{\underset{\sim}{Q}\} \quad . \qquad (4.22)$$

The matrix

$$\hat{\underset{\sim}{\Psi}}(\lambda) = \sum_{\alpha-5}^{\beta-5} \lambda^{-(\frac{x}{5}+\frac{1}{2})} \frac{1}{2} [\underset{\sim}{F}(x) + \underset{\sim}{F}(x+5) \underset{\sim}{S}(x)] \underset{\sim}{L}(x) \tag{4.23}$$

is very close to the numerical approximation of the continuous characteristic matrix, given by Rogers (1975; p. 100):

$$\underset{\sim}{\Psi}(r) = \sum_{\alpha-5}^{\beta-5} e^{-r(x+2.5)} \underset{\sim}{F}(x) \underset{\sim}{L}(x) \quad , \tag{4.24}$$

where $\lambda = e^{5r}$ and $\underset{\sim}{F}(x) \doteq \frac{1}{2} [\underset{\sim}{F}(x) + \underset{\sim}{F}(x+5) \underset{\sim}{S}(x)]$. The stable growth rate λ is the solution of

$$\left| \sum_{\alpha-5}^{\beta-5} \lambda^{-(\frac{x}{5}+\frac{1}{2})} \frac{1}{2} [\underset{\sim}{F}(x) + \underset{\sim}{F}(x+5) \underset{\sim}{S}(x)] \underset{\sim}{L}(x) - \underset{\sim}{I} \right| = 0 \quad .$$

Once the stable distribution of births is known, the stable population distribution can be computed by means of (4.19) and (4.5).

4.2. SENSITIVITY ANALYSIS OF THE STABLE POPULATION

To perform a sensitivity analysis of the stable population, we may apply the eigenvalue and eigenvector sensitivity functions, derived in the Appendix, directly to the growth matrix. Another approach starts out from the generalized eigenvalue problem, expressed in (4.17) and (4.22). This approach is more related to the sensitivity analysis in the single-region case. There is a crucial difference, however. For a single-region growth matrix, the companion form is composed of scalars. The elements of the first row are the coefficients of the characteristic equation, a scalar polynomial. The characteristic equation of the multiregional growth matrix is a matrix polynomial. Its analysis is much more complicated. Both approaches will be discussed here.

a. Sensitivity analysis with the whole growth matrix

The sensitivity of the eigenvalue to changes in the matrix is given in the Appendix by (A.56):

$$d\lambda_i = [\{\underset{\sim}{\xi}\}_i \{\underset{\sim}{\nu}\}_i'] * d\underset{\sim}{A} \qquad (A.56)$$

where $\{\underset{\sim}{\xi}\}_i$ and $\{\underset{\sim}{\nu}\}_i$ are the right and left normalized eigenvector of $\underset{\sim}{A}$, respectively, associated with the root λ_i. Let $\underset{\sim}{A} = \underset{\sim}{G}$, the multiregional growth matrix, and denote the eigenvectors by $\{\underset{\sim}{K}\}$ and $\{\underset{\sim}{\nu}\}$, respectively. When the eigenvectors are not normalized, the formula becomes

$$d\lambda = \frac{1}{\{\underset{\sim}{\nu}\}'\{\underset{\sim}{K}\}} [\{\underset{\sim}{K}\}\{\underset{\sim}{\nu}\}'] * d\underset{\sim}{G} \qquad (4.25)$$

where

$$\{\underset{\sim}{K}\} = \begin{bmatrix} \{\underset{\sim}{K}(0)\} \\ \{\underset{\sim}{K}(5)\} \\ \vdots \\ \{\underset{\sim}{K}(Z)\} \end{bmatrix} \qquad \{\underset{\sim}{\nu}\} = \begin{bmatrix} \{\underset{\sim}{\nu}(0)\} \\ \{\underset{\sim}{\nu}(5)\} \\ \vdots \\ \{\underset{\sim}{\nu}(Z)\} \end{bmatrix} .$$

The inner product is

$$\{\underset{\sim}{\nu}\}'\{\underset{\sim}{K}\} = \sum_{x=0}^{Z} \{\underset{\sim}{\nu}(x)\}'\{\underset{\sim}{K}(x)\} .$$

In the single-region case, the inner product

$$V = \{\underset{\sim}{\nu}\}' \{\underset{\sim}{K}\} = \sum_{x=0}^{Z} \nu(x)\ K(x)$$

is the total reproductive value of the stable population. If the eigenvectors are normalized, then $\{\underset{\sim}{\nu}\}' \{\underset{\sim}{K}\} = 1$, and $\nu(x)\ K(x)$ is the reproductive value of age group x, as a fraction of the total reproductive value.

If one applies formula (A.59), other useful relationships may be derived

$$d\lambda = [\text{tr}\ \underset{\sim}{R}(\lambda)]\ \underset{\sim}{R}(\lambda) * d\underset{\sim}{G} \qquad (A.59)$$

where $\underset{\sim}{R}(\lambda)$ is the adjoint matrix of $[\underset{\sim}{G} - \lambda\underset{\sim}{I}]$ and $\underset{\sim}{G}$ is the growth matrix. The single-region analogue of (A.59) is derived by Demetrius (1969; p. 134). Morgan (1966; p. 198) has shown that tr $\underset{\sim}{R}(\lambda)$ is equal to the first derivative of the characteristic equation of $\underset{\sim}{G}$. Based on this result, it can be shown that for the single-region case, the following equality holds:

$$\frac{\delta g(\lambda)}{\delta \lambda} = \text{tr}\ \underset{\sim}{R}(\lambda) = \frac{1}{\lambda} A \qquad (4.26)$$

where A is the mean age of childbearing of the stable population and $g(\lambda)$ is the characteristic equation of $\underset{\sim}{G}$. This result is similar to the one derived by Goodman (1971; p. 346) and Keyfitz (1968; p. 100).

Formula (4.25) and (A.59) are particularly useful to study the interaction of the population distribution and the distribution of the reproductive values. Goodman (1971) and Demetrius (1969) illustrate this for a single-region system. Consider, for example (4.25), and let $t = \{\underset{\sim}{\nu}\}'\{\underset{\sim}{K}\}$. Written in component terms, (4.25) is

$$d\lambda = \frac{1}{t}\left[\begin{bmatrix}\{\underset{\sim}{K}(0)\} \\ \{\underset{\sim}{K}(5)\} \\ \vdots \\ \{\underset{\sim}{K}(Z)\}\end{bmatrix} [\{\underset{\sim}{\nu}(0)\}' \cdots \{\underset{\sim}{\nu}(Z)\}']\right] * \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} & d\underset{\sim}{B}(10) & \cdots \\ d\underset{\sim}{S}(0) & & & \\ & d\underset{\sim}{S}(5) & & \\ & & \ddots & \\ & & d\underset{\sim}{S}(Z) & \underset{\sim}{0} \end{bmatrix} \tag{4.27}$$

The impact on λ of a change in $\underset{\sim}{B}(x)$ is

$$d\lambda = \frac{1}{t}\ [\{\underset{\sim}{K}(x)\}\{\underset{\sim}{\nu}(0)\}']\ d\underset{\sim}{B}(x) \quad . \tag{4.28}$$

The impact of a change in $\underset{\sim}{S}(x)$ is

$$d\lambda = \frac{1}{t}\ [\{\underset{\sim}{K}(x)\}\{\underset{\sim}{\nu}(x+5)\}']\ d\underset{\sim}{S}(x) \quad . \tag{4.29}$$

From (4.28) and (4.29), we see that a change in $\underset{\sim}{B}(x)$ is equivalent to a change in $\underset{\sim}{S}(x)$ if

$$[\{\underset{\sim}{K}(x)\}\{\underset{\sim}{\nu}(0)\}']\ d\underset{\sim}{B}(x) = [\{\underset{\sim}{K}(x)\}\{\underset{\sim}{\nu}(x+5)\}']\ d\underset{\sim}{S}(x)$$

or

$$d\underset{\sim}{B}(x) = [\{\underset{\sim}{K}(x)\}\{\underset{\sim}{\nu}(0)\}']^{-1}\ [\{\underset{\sim}{K}(x)\}\{\underset{\sim}{\nu}(x+5)\}']\ d\underset{\sim}{S}(x)$$

if the inverse exists.

since

$$\{\underset{\sim}{K}(x)\} = \lambda^{-\frac{x}{5}} \underset{\sim}{A}(x)\{\underset{\sim}{K}(0)\} \quad ,$$

we have

$$d\underset{\sim}{B}(x) = [\{\underset{\sim}{K}(0)\}\{\underset{\sim}{\nu}(0)\}']^{-1} \left[\lambda^{-\frac{x}{5}} \underset{\sim}{A}(x)^{-1} \quad \lambda^{-\frac{x}{5}} \underset{\sim}{A}(x)\right]$$

$$[\{\underset{\sim}{K}(0)\}\{\underset{\sim}{\nu}(x + 5)\}'] \; d\underset{\sim}{S}(x)$$

$$d\underset{\sim}{B}(x) = [\{\underset{\sim}{K}(0)\}\{\underset{\sim}{\nu}(0)\}']^{-1} \; [\{\underset{\sim}{K}(0)\}\{\underset{\sim}{\nu}(x + 5)\}'] \; d\underset{\sim}{S}(x) \quad . \qquad (4.30)$$

Equation (4.30) shows that a change in $\underset{\sim}{B}(x)$ may be translated into a change in $\underset{\sim}{S}(x)$, having the same impact on the growth ratio. It formulates, therefore, a trade-off between fertility change and mortality and migration change. The change in $\underset{\sim}{S}(x)$ to have the same effect as $d\underset{\sim}{B}(x)$ must be smaller the greater are the reproductive values of the people aged x + 5 to x + 9, i.e., $\{\underset{\sim}{\nu}(x + 5)\}$.

It should be noted that the equivalence only holds for the growth ratio, and not for the stable population distribution and other stable characteristics. The stable populations which result from applying $d\underset{\sim}{S}(x)$ or $d\underset{\sim}{B}(x)$ given by (4.30) have the same growth ratio, but all other characteristics are different.

b. Sensitivity analysis with the characteristic matrix

The discrete multiregional characteristic matrix is (4.9)

$$\underset{\sim}{\bar{\Psi}}(\lambda) = \sum_{\alpha-5}^{\beta-5} \lambda^{-(\frac{x}{5}+1)} \underset{\sim}{\bar{\Phi}}(x) \quad , \qquad (4.9)$$

where the stable growth ratio λ is the solution of

$$|\bar{\underset{\sim}{\Psi}}(\lambda) - \underset{\sim}{I}| = 0 \quad . \tag{4.10}$$

What effect does a change in an element of the growth matrix have on λ? As in the previous section, we distinguish between a change in fertility, as expressed by $\underset{\sim}{B}(x)$, and a change in mortality and migration, as expressed by $\underset{\sim}{S}(x)$. This approach is equally valid to trace through the impact of changing fertility, mortality and migration patterns in the continuous model of demographic growth. Instead of using $\bar{\underset{\sim}{\Psi}}(\lambda)$, one then uses its continuous counterpart, given by Rogers (1975; p. 93),

$$\underset{\sim}{\Psi}(r) = \int_{\alpha}^{\beta} e^{-rx} \underset{\sim}{\Phi}(x)\, dx \tag{4.31}$$

where r is the intrinsic growth rate.

The impact on λ of a changing element of $\bar{\underset{\sim}{\Psi}}(\lambda)$ is such that the determinant $|\bar{\underset{\sim}{\Psi}}(\lambda) - \underset{\sim}{I}|$ remains zero. We treat the impact on λ of a change in $\underset{\sim}{B}(x)$ and $\underset{\sim}{S}(x)$ separately.

b.1. Sensitivity of the growth ratio to changes in fertility

Consider first the derivative of the determinant with respect to an element of $\underset{\sim}{B}(x)$, denoted by $<\underset{\sim}{B}(x)>$. Applying the chain rule of matrix differentiation, given in the Appendix by (A.30), we get

$$\frac{\delta|\bar{\underset{\sim}{\Psi}}(\lambda) - \underset{\sim}{I}|}{\delta<\underset{\sim}{B}(x)>} = \text{tr}\left[\frac{\delta|\bar{\underset{\sim}{\Psi}}(\lambda) - \underset{\sim}{I}|}{\delta\bar{\underset{\sim}{\Psi}}(\lambda)} \cdot \frac{\delta[\bar{\underset{\sim}{\Psi}}(\lambda)]'}{\delta<\underset{\sim}{B}(x)>}\right] = 0 \quad . \tag{4.32}$$

By (A.35)

$$\frac{\delta\,|\bar{\underset{\sim}{\Psi}}(\lambda) - \underset{\sim}{I}|}{\delta\bar{\underset{\sim}{\Psi}}(\lambda)} = \text{cof}\ [\bar{\underset{\sim}{\Psi}}(\lambda) - \underset{\sim}{I}] \quad . \tag{4.33}$$

The derivative of the transpose of the characteristic matrix with respect to $<\underset{\sim}{B}(x)>$ is

$$\frac{\delta\,[\bar{\underset{\sim}{\Psi}}(\lambda)]'}{\delta<\underset{\sim}{B}(x)>} = \frac{\delta\left[\sum_{\alpha-5}^{\beta-5} \lambda^{-(\frac{x}{5}+1)}\ \underset{\sim}{B}(x)\ \underset{\sim}{A}(x)\right]'}{\delta<\underset{\sim}{B}(x)>} \quad .$$

Assume that the change in $\underset{\sim}{B}(x)$ is due to a fertility change, then

$$\frac{\delta\left[\sum_{\alpha-5}^{\beta-5} \lambda^{-(\frac{x}{5}+1)}\ [A(x)]'[B(x)]'\right]}{\delta<\underset{\sim}{B}(x)>} =$$

$$= \sum_{\alpha-5}^{\beta-5} [\underset{\sim}{A}(x)]'\ [\underset{\sim}{B}(x)]'\ \frac{\delta\lambda^{-(\frac{x}{5}+1)}}{\delta<\underset{\sim}{B}(x)>} + \lambda^{-(\frac{x}{5}+1)}\ [\underset{\sim}{A}(x)]'\ \frac{\delta\,[\underset{\sim}{B}(x)]'}{\delta<\underset{\sim}{B}(x)>}$$

where

$$\frac{\delta\lambda^{-(\frac{x}{5}+1)}}{\delta<\underset{\sim}{B}(x)>} = \frac{\delta\lambda^{-(\frac{x}{5}+1)}}{\delta\lambda} \cdot \frac{\delta\lambda}{\delta<\underset{\sim}{B}(x)>}$$

$$= -\ (\frac{x}{5} + 1)\ \lambda^{-(\frac{x}{5}+2)}\ \frac{\delta\lambda}{\delta<\underset{\sim}{B}(x)>}$$

and

$$\frac{\delta [\underset{\sim}{B}(x)]'}{\delta \langle \underset{\sim}{B}(x) \rangle} = \underset{\sim}{J}' \quad .$$

Therefore

$$\frac{\delta [\bar{\underset{\sim}{\Psi}}(\lambda)]'}{\delta \langle \underset{\sim}{B}(x) \rangle} = - \left[\frac{1}{\lambda} \sum_{\alpha-5}^{\beta-5} (\frac{x}{5} + 1) \lambda^{-(\frac{x}{5}+1)} [\underset{\sim}{B}(x) \underset{\sim}{A}(x)]' \right] \frac{\delta \lambda}{\delta \langle \underset{\sim}{B}(x) \rangle}$$

$$+ \lambda^{-(\frac{x}{5}+1)} \underset{\sim}{A}'(x) \underset{\sim}{J}' \quad . \tag{4.34}$$

Let

$$\sum_{\alpha-5}^{\beta-5} (\frac{x}{5} + 1) \lambda^{-(\frac{x}{5}+1)} [\underset{\sim}{B}(x) \underset{\sim}{A}(x)]' = [\underset{\sim}{V}(0)]^{-1} \text{ [2]} . \tag{4.35}$$

Generalizing the idea of Goodman, $[\underset{\sim}{V}(0)]^{-1}$ is the matrix of the average age of mothers of children who are in the 0-th age group in the stable population. It is the discrete approximation of the mean age of childbearing. The matrix $\underset{\sim}{V}(0)$ represents the eventual reproductive value of a female in the 0-th age group in the stable population.

Substituting (4.33), and (4.34) in (4.32) gives

$$\text{tr cof } [\bar{\underset{\sim}{\Psi}}(\lambda) - \underset{\sim}{I}] \left[- \frac{1}{\lambda} [\underset{\sim}{V}(0)]^{-1} \frac{\delta \lambda}{\delta \langle \underset{\sim}{B}(x) \rangle} + \lambda^{-(\frac{x}{5}+1)} \underset{\sim}{A}'(x) \underset{\sim}{J}' \right] = 0 \tag{4.36}$$

[2] The single region counterpart of (4.35) is given by Goodman (1971; p. 346).

which may be written as

$$\frac{1}{\lambda} \text{cof } [\bar{\Psi}(\lambda) - I] * V^{-1}(0) \frac{\delta\lambda}{\delta<B(x)>} = \lambda^{-(\frac{x}{5}+1)} \text{cof } [\bar{\Psi}(\lambda) - I] * [A'(x) J'] \quad .$$

Pre-multiplying both sides with $\left[\text{cof } [\bar{\Psi}(\lambda) - I]\right]'^{-1}$ yields

$$\frac{1}{\lambda} I * [V^{-1}(0)] \frac{\delta\lambda}{\delta<B(x)>} = \lambda^{-(\frac{x}{5}+1)} I * A'(x) J' \quad .$$

But $I * [V^{-1}(0)]$ is nothing else than tr $[V^{-1}(0)]$. Therefore, we have

$$\frac{\delta\lambda}{\delta<B(x)>} = [\text{tr } V^{-1}(0)]^{-1} \lambda^{-\frac{x}{5}} \text{tr } [A'(x) J'] \quad . \tag{4.37}$$

By (A.32) of the Appendix,

$$\frac{\delta\lambda}{\delta B(x)} = [\text{tr } V^{-1}(0)]^{-1} \lambda^{-\frac{x}{5}} \sum_{k\ell} \text{tr } [A'(x) J_{\ell k}]$$

$$= [\text{tr } V^{-1}(0)]^{-1} \lambda^{-\frac{x}{5}} A'(x) \quad . \tag{4.38}$$

In a single-region system, (4.38) reduces to

$$\frac{\delta\lambda}{\delta b(x)} = v(0) \lambda^{-\frac{x}{5}} a(x) \tag{4.39}$$

where b(x), v(0) and a(x) are scalars. Formula (4.39) is identical to the sensitivity function given by Goodman (1971;

p. 346), and equivalent to the ones derived by Demetrius (1969; p. 134), Keyfitz (1971; p. 277), Emlen (1970) and others. Note that $\lambda^{-\frac{x}{5}} \underset{\sim}{A}(x)$ is the eventual expected number of people in age group x to x + 4, per individual in the 0 - 4 age group. In other words, $\lambda^{-\frac{x}{5}} \underset{\sim}{A}(x)$ describes the age composition of the stable population.

b.2. Sensitivity of the growth ratio to changes in mortality and migration

The impact on λ of a change in $\underset{\sim}{S}(x)$ may be derived in a way similar to the above arguments. First, note that

$$\frac{\delta|\bar{\underset{\sim}{\Psi}}(\lambda) - \underset{\sim}{I}|}{\delta<\underset{\sim}{S}(x)>} = \text{tr}\left[\frac{\delta|\bar{\underset{\sim}{\Psi}}(\lambda) - \underset{\sim}{I}|}{\delta\bar{\underset{\sim}{\Psi}}(\lambda)} \cdot \frac{\delta[\bar{\underset{\sim}{\Psi}}(\lambda)]'}{\delta<\underset{\sim}{S}(x)>}\right] = 0 \quad . \tag{4.40}$$

The derivative of $[\bar{\underset{\sim}{\Psi}}(\lambda)]'$ with respect to an element of $\underset{\sim}{S}(x)$ is

$$\frac{\delta[\bar{\underset{\sim}{\Psi}}(\lambda)]'}{\delta<\underset{\sim}{S}(x)>} = \frac{\delta\left[\sum_{\alpha-5}^{\beta-5} \lambda^{-(\frac{x}{5}+1)} \underset{\sim}{B}(x)\ \underset{\sim}{A}(x)\right]'}{\delta<\underset{\sim}{S}(x)>} \tag{4.41}$$

$$= \sum_{\alpha-5}^{\beta-5} [\underset{\sim}{B}(x)\ \underset{\sim}{A}(x)]' \frac{\delta\lambda^{-(\frac{x}{5}+1)}}{\delta<\underset{\sim}{S}(x)>} + \sum_{\alpha-5}^{\beta-5} \lambda^{-(\frac{x}{5}+1)} \frac{\delta\underset{\sim}{A}'(x)}{\delta<\underset{\sim}{S}(x)>} \underset{\sim}{B}'(x)$$

$$+ \sum_{\alpha-5}^{\beta-5} \lambda^{-(\frac{x}{5}+1)} \underset{\sim}{A}'(x) \frac{\delta\underset{\sim}{B}'(x)}{\delta<\underset{\sim}{S}(x)>} \quad . \tag{4.42}$$

The derivatives are

$$\frac{\delta\lambda^{-(\frac{x}{5}+1)}}{\delta\langle\underset{\sim}{S}(x)\rangle} = \frac{\delta\lambda^{-(\frac{x}{5}+1)}}{\delta\lambda} \cdot \frac{\delta\lambda}{\delta\langle\underset{\sim}{S}(x)\rangle}$$

$$= -(\frac{x}{5}+1)\ \lambda^{-(\frac{x}{5}+2)} \frac{\delta\lambda}{\delta\langle\underset{\sim}{S}(x)\rangle} \quad . \tag{4.43}$$

To derive an expression for

$$\sum_{\alpha-5}^{\beta-5} \lambda^{-(\frac{x}{5}+1)} \frac{\delta\underset{\sim}{A}'(x)}{\delta\langle\underset{\sim}{S}(x)\rangle} \underset{\sim}{B}'(x) \tag{4.44}$$

recall that

$$\underset{\sim}{A}'(x) = \underset{\sim}{S}'(0)\ \underset{\sim}{S}'(5)\ \ldots\ \underset{\sim}{S}'(x-5) \quad .$$

Therefore, a change in $\underset{\sim}{S}(x)$ affects $\underset{\sim}{A}'(y)$ if $y > x$. For example,

$$\frac{\delta\underset{\sim}{A}'(y)}{\delta\langle\underset{\sim}{S}(x)\rangle} = \underset{\sim}{S}'(0)\ \underset{\sim}{S}'(5)\ \ldots\ \underset{\sim}{S}'(x-5)\ \underset{\sim}{J}'\underset{\sim}{S}'(x+5)\ \ldots\ \underset{\sim}{S}'(y-5)$$

$$= \underset{\sim}{A}'(x)\ \underset{\sim}{J}'[\underset{\sim}{A}'(x+5)]^{-1}\ \underset{\sim}{A}'(y) \quad . \tag{4.45}$$

Applying this result, (4.44) reduces to

$$\sum_{y=x+5}^{\beta-5} \lambda^{-(\frac{y}{5}+1)}\ \underset{\sim}{A}'(x)\ \underset{\sim}{J}'[\underset{\sim}{A}'(x+5)]^{-1}\ \underset{\sim}{A}'(y)\ \underset{\sim}{B}'(y) \quad . \tag{4.46}$$

To compute the third element of (4.41), we need

$$\frac{\delta \underset{\sim}{B}'(x)}{\delta \langle \underset{\sim}{S}(x) \rangle} = \frac{5}{4} \frac{\delta \underset{\sim}{S}'(x)}{\delta \langle \underset{\sim}{S}(x) \rangle} \underset{\sim}{F}'(x+5) \ [\underset{\sim}{P}(0) + \underset{\sim}{I}]' \qquad (4.47)$$

$$= \frac{5}{4} \underset{\sim}{J}'\underset{\sim}{F}'(x+5) \ [\underset{\sim}{P}(0) + \underset{\sim}{I}]' \ .$$

Therefore (4.42) becomes

$$\frac{\delta [\underset{\sim}{\bar{\Psi}}(\lambda)]'}{\delta \langle \underset{\sim}{S}(x) \rangle} = \left[- \sum_{\alpha-5}^{\beta-5} \left(\frac{x}{5}+1\right) \lambda^{-\left(\frac{x}{5}+2\right)} [\underset{\sim}{B}(x) \ \underset{\sim}{A}(x)]' \right] \frac{\delta\lambda}{\delta \langle \underset{\sim}{S}(x) \rangle} \qquad (4.48)$$

$$+ \frac{5}{4} \lambda^{-\left(\frac{x}{5}+1\right)} \underset{\sim}{A}'(x) \ \underset{\sim}{J}'\underset{\sim}{F}'(x+5) \ [\underset{\sim}{P}(0) + \underset{\sim}{I}]'$$

where by (4.35)

$$\sum_{\alpha-5}^{\beta-5} \left(\frac{x}{5}+1\right) \lambda^{-\left(\frac{x}{5}+1\right)} [\underset{\sim}{B}(x) \ \underset{\sim}{A}(x)]' = [\underset{\sim}{V}(0)]^{-1} \ .$$

Substituting (4.48) in (4.40) gives

$$\frac{\delta |\underset{\sim}{\bar{\Psi}}(\lambda) - \underset{\sim}{I}|}{\delta \langle \underset{\sim}{S}(x) \rangle} = \text{tr cof } [\underset{\sim}{\bar{\Psi}}(\lambda) - \underset{\sim}{I}] \left[- \frac{1}{\lambda} [\underset{\sim}{V}(0)]^{-1} \frac{\delta\lambda}{\delta \langle \underset{\sim}{S}(x) \rangle} \right.$$

$$+ \sum_{y=x+5}^{\beta-5} \lambda^{-\left(\frac{y}{5}+1\right)} \underset{\sim}{A}'(x) \ \underset{\sim}{J}'[\underset{\sim}{A}'(x+5)]^{-1} \underset{\sim}{A}'(y) \ \underset{\sim}{B}'(y)$$

$$\left. + \frac{5}{4} \lambda^{-\left(\frac{x}{5}+1\right)} \underset{\sim}{A}'(x) \ \underset{\sim}{J}'\underset{\sim}{F}'(x+5) \ [\underset{\sim}{P}(0) + \underset{\sim}{I}]' \right] = 0$$

which is equivalent to

$$\left[\frac{1}{\lambda}\,\mathrm{tr}\,[\underset{\sim}{V}(0)]^{-1}\right]\frac{\delta\lambda}{\delta<\underset{\sim}{S}(x)>} = \sum_{y=x+5}^{\beta-5} \lambda^{-(\frac{y}{5}+1)}\,\mathrm{tr}\left[\underset{\sim}{A}'(x)\ \underset{\sim}{J}'[\underset{\sim}{A}'(x+5)]^{-1}\right.$$

$$\left.\underset{\sim}{A}'(y)\ \underset{\sim}{B}'(y)\right] + \frac{5}{4}\lambda^{-(\frac{x}{5}+1)}\ \mathrm{tr}\ [\underset{\sim}{A}'(x)\ \underset{\sim}{J}'\underset{\sim}{F}'(x+5)[\underset{\sim}{P}(0)+\underset{\sim}{I}]'$$

or

$$\frac{\delta\lambda}{\delta<\underset{\sim}{S}(x)>} = [\mathrm{tr}\ \underset{\sim}{V}^{-1}(0)]^{-1}\left[\sum_{y=x+5}^{\beta-5}\lambda^{-\frac{y}{5}}\ \mathrm{tr}\left[\underset{\sim}{B}(y)\ \underset{\sim}{A}(y)[\underset{\sim}{A}(x+5)]^{-1}\ \underset{\sim}{J}\underset{\sim}{A}(x)\right]\right.$$

$$\left. + \frac{5}{4}\lambda^{-\frac{x}{5}}\ \mathrm{tr}\left[[\underset{\sim}{P}(0)+\underset{\sim}{I}]\ \underset{\sim}{F}(x+5)\ \underset{\sim}{J}\underset{\sim}{A}(x)\right]\right] \tag{4.49}$$

and

$$\frac{\delta\lambda}{\delta\underset{\sim}{S}(x)} = [\mathrm{tr}\ \underset{\sim}{V}^{-1}(0)]^{-1}\left[\sum_{y=x+5}^{\beta-5}\lambda^{-\frac{y}{5}}\ \underset{\sim}{B}(y)\ \underset{\sim}{A}(y)[\underset{\sim}{A}(x+5)]^{-1}\ \underset{\sim}{A}(x)\right.$$

$$\left. + \frac{5}{4}\lambda^{-\frac{x}{5}}\ [\underset{\sim}{P}(0)+\underset{\sim}{I}]\ \underset{\sim}{F}(x+5)\ \underset{\sim}{A}(x)\right] \ . \tag{4.50}$$

If the effect on $\underset{\sim}{B}(x)$ of $\underset{\sim}{S}(x)$ is negligible, as Goodman (1971; p. 343) assumes, and which is so in the continuous case, then

$$\frac{\delta\lambda}{\delta\underset{\sim}{S}(x)} = [\mathrm{tr}\ \underset{\sim}{V}^{-1}(0)]^{-1}\left[\sum_{y=x+5}^{\beta-5}\lambda^{-\frac{y}{5}}\ \underset{\sim}{B}(y)\ \underset{\sim}{A}(y)\ \underset{\sim}{A}^{-1}(x)\ \underset{\sim}{S}^{-1}(x)\ \underset{\sim}{A}(x)\right] \ . \tag{4.51}$$

The single-region analogue of (4.51) is

$$\frac{\delta\lambda}{\delta s(x)} = \frac{v(0)}{s(x)} \left[\sum_{y=x+5}^{\beta-5} \lambda^{-\frac{y}{5}} b(y)\ a(y) \right] \qquad (4.52)$$

which is identical to formula (35) of Goodman (1971; p. 346), and equivalent to expressions provided by other authors. The expression

$$v(0) \sum_{y=x}^{\beta-5} \lambda^{-\frac{y}{5}} b(y)\ a(y) = v(x) \qquad (4.53)$$

is defined by Goodman as the eventual reproductive value of an individual in the x, x + 4 age interval. Generalizing this concept to the multiregional case, we define the matrix of eventual reproductive values per individual in the x, x + 4 age group, by place of birth and by place of residence, to be

$$\underset{\sim}{V}(x) = \sum_{y=x}^{\beta-5} \lambda^{-\frac{y}{5}} \underset{\sim}{B}(y)\ \underset{\sim}{A}(y) \quad . \qquad (4.54)$$

The sensitivity function (4.51) becomes

$$\frac{\delta\lambda}{\delta \underset{\sim}{S}(x)} = [\mathrm{tr}\ \underset{\sim}{V}^{-1}(0)]^{-1}\ \underset{\sim}{V}(x+5)\ \underset{\sim}{A}^{-1}(x)\ \underset{\sim}{S}^{-1}(x)\ \underset{\sim}{A}(x) \quad . \qquad (4.55)$$

CHAPTER 5

CONCLUSION

This paper has been devoted to the problem of sensitivity analysis in multiregional demographic systems. From mathematical demography, we know that demographic change may be traced back to changes in age-specific fertility, mortality and migration rates. To show how the mechanism works has been the subject of this paper.

We derived a set of sensitivity functions relating a change in demographic characteristics to a change in the vital rates. The primary purpose was to contribute to the knowledge of spatial population dynamics by presenting a unifying technique of impact assessments. In the single-region mathematical demography, ordinary differential calculus is used to perform sensitivity analysis. In multiregional demography, where we deal with matrix and vector functions, the application of ordinary calculus is very complicated. Instead, matrix differentiation techniques prove to be very useful. A review of these techniques has been given in the Appendix. These mathematical tools have been applied to derive analytical expressions for multiregional demographic features, such as life table statistics, population projection, and stable population characteristics, representing the impacts of changes in vital rates. The sensitivity functions reveal how each spatial demographic characteristic depends on the age-specific rates and how it reacts to changes in those rates. Matrix differentiation techniques form a powerful tool for the analysis of structural change in multiregional systems.

A secondary objective of this paper was to contribute to the reconciliation of the discrete and continuous models of demographic growth. Traditionally, there has been a sharp distinction between the discrete model and the continuous model of population growth. It is our belief that the reason is mainly historical. We have attempted to show that the results derived for the continuous model, may easily be extended to the discrete model. Therefore, the discrete and continuous models of demographic growth are equivalent tools for the analysis of population dynamics.

APPENDIX

MATRIX DIFFERENTIATION TECHNIQUES

The purpose of this appendix is to provide the necessary mathematical tools to perform sensitivity analysis of structural change in multiregional demographic systems. The basic notion is that of matrix differentiation. Neudecker (1969; p. 953) defines matrix differentiation as the procedure of finding partial derivatives of the elements of a matrix function with respect to the elements of the argument matrix. Although not much has been written on matrix differentiation and the technique is not covered in most textbooks on matrix algebra, this appendix does not intend to be complete. It only covers the techniques applied in this study.

The appendix is divided into two parts. The first part deals with the derivatives of matrix functions. It is mainly based on the work of Dwyer and MacPhail (1948) and Dwyer (1967). The second part develops several expressions for the sensitivity of the eigenvalues and the eigenvectors of a matrix with respect to change in its elements. The behavior of the eigenvalues under perturbations of the elements of a matrix has been studied by Lancaster (1969; Chapter 7), among others, under the heading of perturbation theory. In this theory, qualitative measures of eigenvalue sensitivity are developed, in the sense that upper and lower bounds to eigenvalue changes are formulated. Perturbation theory, however, does not provide us with sensitivity functions defining the exact change of eigenvalues and eigenvectors under changing matrix elements.

An eigenvalue sensitivity function was derived by Jacobi in 1846 and has been applied and extended in the systems theory and design literature.

A.1. DIFFERENTIATION OF FUNCTIONS OF MATRICES

Let $\underset{\sim}{Y}$ be an P x Q matrix with elements y_{ij}, and let $\underset{\sim}{X}$ be an M x N matrix with elements $x_{k\ell}$. Dwyer makes a distinction between the position of an element in the matrix and its value. The symbol $\langle \underset{\sim}{X} \rangle_{k\ell}$ is used to indicate a specific k, ℓ-element of $\underset{\sim}{X}$. Its scalar value is $x_{k\ell}$. Less formally, $\langle \underset{\sim}{X} \rangle_{k\ell}$ may be replaced by $\langle \underset{\sim}{X} \rangle$. Therefore, $\langle \underset{\sim}{X} \rangle$ is an arbitrary element of the matrix $\underset{\sim}{X}$. As in conventional notation $\underset{\sim}{X}'$ denotes the transpose of $\underset{\sim}{X}$ and $\underset{\sim}{X}^{-1}$ is the inverse of $\underset{\sim}{X}$.

The relevant results of matrix calculus are given below. To introduce some notation, we start out with the differentiation of a matrix with respect to its elements. We follow this with the differentiation of a matrix with respect to a scalar, and the differentiation of a scalar function with respect to a matrix. The most important scalar function is the determinant. The tools provided in the section on the differentiation of matrix products are frequently used in performing sensitivity analysis of multiregional systems. Also of great importance is the derivative of the inverse. The next section gives some chain rules of matrix differentiation. Vector calculus and matrix calculus are closely related, since a vector is a matrix with only one row or one column. The formulas for vector differentiation, however, have a different appearance and are less complex. Therefore, a separate section will be devoted to vector differentiation.

A.1.1. Differentiation of a matrix with respect to its elements

The derivative of a matrix $\underset{\sim}{X}$ with respect to the element $\langle \underset{\sim}{X} \rangle_{k\ell}$ is

$$\frac{\delta \underset{\sim}{X}}{\delta \langle \underset{\sim}{X} \rangle_{k\ell}} = \underset{\sim}{J}_{k\ell} \qquad \text{(A.1)}$$

where $\underset{\sim}{J}_{k\ell}$ denotes an M x N matrix with zero elements everywhere except for a unit element in the k-th row and ℓ-th column.

Similarly

$$\frac{\delta \underset{\sim}{X}'}{\delta \langle \underset{\sim}{X} \rangle_{k\ell}} = \underset{\sim}{J}'_{\ell k} \qquad \text{(A.2)}$$

where $\underset{\sim}{J}'_{k\ell}$ is an N x M matrix with all elements zero except for a unit element in the ℓ-th row and k-th column.

Instead of considering the derivative of a matrix with respect to an element, one may also consider the derivative of a matrix-element with respect to the matrix.

$$\frac{\delta \langle \underset{\sim}{Y} \rangle_{ij}}{\delta \underset{\sim}{Y}} = \underset{\sim}{K}_{ij} \qquad \text{(A.3)}$$

where K_{ij} is a P x Q matrix with zeroes everywhere except for a unit element in the i-th row and j-th column.

Similarly

$$\frac{\delta \langle Y \rangle_{ij}}{\delta Y'} = K'_{ji} \quad . \tag{A.4}$$

For convenience, the subscripts will be dropped. For example, $\langle X \rangle$ will denote an arbitrary element of X and J a matrix with all elements zero except a unit element on the appropriate place determined by the location of $\langle X \rangle$.

A.1.2. Differentiation of a matrix with respect to a scalar and of a scalar with respect to a matrix

Let $Y(a)$ be a matrix function of the scalar a. The derivative

$$\frac{\delta Y(a)}{\delta a} \tag{A.5}$$

is a matrix with elements $\frac{\delta y_{ij}}{\delta a}$. Each element of $Y(a)$ is differentiated.

The derivative of a matrix function with respect to a matrix is denoted by

$$\frac{\delta f(X)}{\delta X} \tag{A.6}$$

and is a matrix with elements

$$\frac{\delta f(\underset{\sim}{X})}{\delta \langle \underset{\sim}{X} \rangle_{ij}} \quad . \tag{A.7}$$

Two important matrix functions are considered: the determinant and the trace. We begin with the assumption that $\underset{\sim}{X}$ is a square matrix.

a. Determinant

The determinant of the square matrix $\underset{\sim}{X}$ can be evaluated in terms of the cofactors of the elements of the i-th row (Rogers, 1971; p. 81):

$$|\underset{\sim}{X}| = x_{i1}X^c_{i1} + x_{i2}X^c_{i2} + \cdots + x_{iN}X^c_{iN} \quad .$$

It can easily be seen that

$$\frac{\delta |\underset{\sim}{X}|}{\delta \langle \underset{\sim}{X} \rangle_{ij}} = X^c_{ij} \tag{A.8}$$

where X^c_{ij} is the cofactor of the element $|\underset{\sim}{X}|_{ij}$. And

$$\frac{\delta |\underset{\sim}{X}|}{\delta \underset{\sim}{X}} = \text{cof } \underset{\sim}{X} = [\text{adj } \underset{\sim}{X}]'$$

where cof $\underset{\sim}{X}$ is the matrix of cofactors, and adj $\underset{\sim}{X}$ is the adjoint matrix of the matrix $\underset{\sim}{X}$. But if $\underset{\sim}{X}$ is nonsingular,

$$\text{cof } \underset{\sim}{X} = |\underset{\sim}{X}| [\underset{\sim}{X}']^{-1} \quad . \tag{A.9}$$

Equation (A.8) may be written as

$$\frac{\delta|\underset{\sim}{X}|}{\delta\underset{\sim}{X}} = |X|[X']^{-1} \quad . \tag{A.10}$$

This formula is well known in matrix theory and can also be found in Bellman (1970; p. 182).

It should be noted that if $\underset{\sim}{X}$ is symmetric

$$\begin{aligned} \frac{\delta|\underset{\sim}{X}|}{\delta\langle X\rangle_{ij}} &= 2\underset{\sim}{X}^{c}_{ij} && \text{for } i \neq j \\ &= \underset{\sim}{X}^{c}_{ij} && \text{for } i = j \end{aligned} \tag{A.11}$$

b. Trace

The trace of the square matrix $\underset{\sim}{X}$ is the sum of its diagonal elements, and

$$\begin{aligned} \frac{\delta \mathrm{tr}(\underset{\sim}{X})}{\delta\langle\underset{\sim}{X}\rangle_{ij}} &= 1 && \text{for } i = j \\ &= 0 && \text{for } i \neq j \end{aligned} \tag{A.12}$$

with

$$\frac{\delta \mathrm{tr}(\underset{\sim}{X})}{\delta\underset{\sim}{X}} = \underset{\sim}{I}$$

where $\underset{\sim}{I}$ is the identity matrix.

A.1.3. Differentiation of matrix products

Let $\underset{\sim}{U}$ and $\underset{\sim}{V}$ be two matrix functions of the matrix $\underset{\sim}{X}$. The derivative of their product $\underset{\sim}{Y} = \underset{\sim}{U}\underset{\sim}{V}$ with respect to $\langle\underset{\sim}{X}\rangle$ is

$$\frac{\delta \underset{\sim}{Y}}{\delta \langle\underset{\sim}{X}\rangle} = \frac{\delta [\underset{\sim}{U}\underset{\sim}{V}]}{\delta \langle\underset{\sim}{X}\rangle} = \frac{\delta \underset{\sim}{U}}{\delta \langle\underset{\sim}{X}\rangle} V + U \frac{\delta \underset{\sim}{V}}{\delta \langle\underset{\sim}{X}\rangle} \quad . \tag{A.13}$$

The derivative of a product of three matrices is

$$\frac{\delta \underset{\sim}{Y}}{\delta \langle\underset{\sim}{X}\rangle} = \frac{\delta [\underset{\sim}{U}\underset{\sim}{V}\underset{\sim}{W}]}{\delta \langle\underset{\sim}{X}\rangle} = \frac{\delta \underset{\sim}{U}}{\delta \langle\underset{\sim}{X}\rangle} VW + U \frac{\delta \underset{\sim}{V}}{\delta \langle\underset{\sim}{X}\rangle} W + UV \frac{\delta \underset{\sim}{W}}{\delta \langle\underset{\sim}{X}\rangle} \quad . \tag{A.14}$$

These general formulas may be applied to various cases. Some cases of interest are listed below. The matrices $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are constant, i.e. independent of $\underset{\sim}{X}$. The matrices $\underset{\sim}{J}$, and $\underset{\sim}{K}$ are as defined in A.1.1.

$\underset{\sim}{Y}$	$\frac{\delta \underset{\sim}{Y}}{\delta \langle\underset{\sim}{X}\rangle}$	
$\underset{\sim}{A}\underset{\sim}{X}$	$\underset{\sim}{A}\underset{\sim}{J}$	(A.15)
$\underset{\sim}{X}\underset{\sim}{B}$	$\underset{\sim}{J}\underset{\sim}{B}$	(A.16)
$\underset{\sim}{X}'\underset{\sim}{B}$	$\underset{\sim}{J}'\underset{\sim}{B}$	(A.17)
$\underset{\sim}{X}\underset{\sim}{X}$	$\underset{\sim}{J}\underset{\sim}{X} + \underset{\sim}{X}\underset{\sim}{J}$	(A.18)
$\underset{\sim}{X}'\underset{\sim}{X}$	$\underset{\sim}{J}'\underset{\sim}{X} + \underset{\sim}{X}'\underset{\sim}{J}$	(A.19)

$\underset{\sim}{A}\underset{\sim}{X}\underset{\sim}{B}$	$\underset{\sim}{A}\underset{\sim}{J}\underset{\sim}{B}$	(A.20)
$\underset{\sim}{X}\underset{\sim}{X}\underset{\sim}{X}$	$\underset{\sim}{J}\underset{\sim}{X}\underset{\sim}{X} + \underset{\sim}{X}\underset{\sim}{J}\underset{\sim}{X} + \underset{\sim}{X}\underset{\sim}{X}\underset{\sim}{J}$	(A.21)
$\underset{\sim}{A}\underset{\sim}{X}\underset{\sim}{A}^{-1}$	$\underset{\sim}{A}\underset{\sim}{J}\underset{\sim}{A}^{-1}$	(A.22)

The derivative of the power of a square matrix can readily be computed using these formulas

$$\frac{\delta[\underset{\sim}{X}^n]}{\delta\langle\underset{\sim}{X}\rangle} = \underset{\sim}{J}\underset{\sim}{X}^{n-1} + \sum_{s=1}^{n-2} \underset{\sim}{X}^s\underset{\sim}{J}\underset{\sim}{X}^{n-1-s} + \underset{\sim}{X}^{n-1}\underset{\sim}{J} \qquad (A.23)$$

or, if we write $\underset{\sim}{X}^0 = \underset{\sim}{I}$, then

$$\frac{\delta[\underset{\sim}{X}^n]}{\delta\langle X\rangle} = \sum_{s=0}^{n-1} \underset{\sim}{X}^s\underset{\sim}{J}\underset{\sim}{X}^{n-1-s} \quad . \qquad (A.24)$$

The derivative of an inverse follows. By definition

$$\underset{\sim}{X}\underset{\sim}{X}^{-1} = \underset{\sim}{I} \quad .$$

Therefore

$$\frac{\delta[\underset{\sim}{X}\underset{\sim}{X}^{-1}]}{\delta\langle\underset{\sim}{X}\rangle} = \frac{\delta\underset{\sim}{I}}{\delta\langle\underset{\sim}{X}\rangle} = \underset{\sim}{0} \quad ,$$

but

$$\frac{\delta[\underset{\sim}{X}\underset{\sim}{X}^{-1}]}{\delta\langle\underset{\sim}{X}\rangle} = \frac{\delta\underset{\sim}{X}}{\delta\langle\underset{\sim}{X}\rangle}\underset{\sim}{X}^{-1} + \underset{\sim}{X}\frac{\delta[\underset{\sim}{X}^{-1}]}{\delta\langle\underset{\sim}{X}\rangle} \quad .$$

It follows that

$$\frac{\delta[\underset{\sim}{X}^{-1}]}{\delta\langle\underset{\sim}{X}\rangle} = -\underset{\sim}{X}^{-1}\underset{\sim}{J}\underset{\sim}{X}^{-1} \quad . \tag{A.25}$$

An application of this result is

$$\frac{\delta\underset{\sim}{X}\underset{\sim}{A}\underset{\sim}{X}^{-1}}{\delta\langle\underset{\sim}{X}\rangle} = \underset{\sim}{J}\underset{\sim}{A}\underset{\sim}{X}^{-1} - \underset{\sim}{X}\underset{\sim}{A}\underset{\sim}{X}^{-1}\underset{\sim}{J}\underset{\sim}{X}^{-1} \quad . \tag{A.26}$$

So far we have considered the derivative $\frac{\delta\underset{\sim}{Y}}{\delta\langle X\rangle}$ where $\underset{\sim}{Y}$ is a matrix product and $\langle\underset{\sim}{X}\rangle$ is an arbitrary element of $\underset{\sim}{X}$. The result is a matrix of partial derivatives. But what is the formula for $\frac{\delta\underset{\sim}{Y}}{\delta X}$, where $\underset{\sim}{X}$ represents the full matrix? This question has been studied by Neudecker (1969). Its solution involves the transformation of a matrix into a vector and the use of Kronecker products. For example, let $Y = \underset{\sim}{A}\underset{\sim}{X}\underset{\sim}{B}$ and one is interested in the derivative of $\underset{\sim}{Y}$ with respect to $\underset{\sim}{X}$.

If $\underset{\sim}{Y}$ is of order P x Q, define the PQ column vector vec $\underset{\sim}{Y}$ (denoted this way to distinguish it from the vector $\{\underset{\sim}{y}\}$) where

$$\text{vec } \underset{\sim}{Y} = \begin{bmatrix} \{\underset{\sim}{Y}_{\cdot 1}\} \\ \{\underset{\sim}{Y}_{\cdot 2}\} \\ \vdots \\ \{\underset{\sim}{Y}_{\cdot Q}\} \end{bmatrix} \quad .$$

In a similar way, one can construct vec $\underset{\sim}{X}$. Neudecker shows that

$$\text{vec } (\underset{\sim}{A}\underset{\sim}{X}\underset{\sim}{B}) = [\underset{\sim}{B}' \otimes \underset{\sim}{A}] \text{ vec } \underset{\sim}{X} \tag{A.27}$$

where ⊗ denotes the Kronecker product. Equation (A.27) may be differentiated using the formulas for vector differentiation:

$$\frac{\delta \text{ vec } [\underset{\sim}{A}\underset{\sim}{X}\underset{\sim}{B}]}{\delta \text{ vec } \underset{\sim}{X}} = [\underset{\sim}{B}' \otimes \underset{\sim}{A}]' \quad .$$

Since the transpose of a Kronecker product is the Kronecker product of the transposes, we have[3]

$$\frac{\delta \text{ vec } [\underset{\sim}{A}\underset{\sim}{X}\underset{\sim}{B}]}{\delta \text{ vec } \underset{\sim}{X}} = \underset{\sim}{B} \otimes \underset{\sim}{A}' \quad . \tag{A.28}$$

We will not explore the various formulas for $\frac{\delta \underset{\sim}{Y}}{\delta \underset{\sim}{X}}$ further since they are not explicitly used in this study.

A.1.4. <u>Chain rules of differentiation</u>

Let $f(\underset{\sim}{Y})$ be a scalar function of $\underset{\sim}{Y}$ and let $\underset{\sim}{Y}$ be a matrix function of $\underset{\sim}{X}$.

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[3] For an exposition of the properties of Kronecker products or direct products, see Lancaster (1969; pp. 256-259).

Then

$$\frac{\delta f(\underset{\sim}{Y})}{\delta\langle\underset{\sim}{X}\rangle} = \sum_{k\ell} \frac{\delta f(\underset{\sim}{Y})}{\delta\langle\underset{\sim}{Y}\rangle_{k\ell}} \cdot \frac{\delta\langle\underset{\sim}{Y}\rangle_{k\ell}}{\delta\langle\underset{\sim}{X}\rangle} \tag{A.29}$$

$$\frac{\delta f(\underset{\sim}{Y})}{\delta\langle\underset{\sim}{X}\rangle} = \mathrm{tr}\left[\frac{\delta f(\underset{\sim}{Y})}{\delta\underset{\sim}{Y}} \cdot \frac{\delta\underset{\sim}{Y}'}{\delta\langle\underset{\sim}{X}\rangle}\right] . \tag{A.30}$$

If $\underset{\sim}{Y}$ is a matrix function of a scalar a, i.e. $\underset{\sim}{Y}(a)$, the formula becomes

$$\frac{\delta f(\underset{\sim}{Y})}{\delta a} = \mathrm{tr}\left[\frac{\delta f(\underset{\sim}{Y})}{\delta\underset{\sim}{Y}} \cdot \frac{\delta\underset{\sim}{Y}'}{\delta a}\right] . \tag{A.31}$$

Consider also the derivative

$$\frac{\delta f(\underset{\sim}{Y})}{\delta\underset{\sim}{X}} = \sum_{k\ell} \frac{\delta f(\underset{\sim}{Y})}{\delta\langle\underset{\sim}{Y}\rangle_{k\ell}} \cdot \frac{\delta\langle\underset{\sim}{Y}\rangle_{k\ell}}{\delta\underset{\sim}{X}} . \tag{A.32}$$

Several interesting applications arise. For example, let $f(\underset{\sim}{Y}) = |\underset{\sim}{X} - \lambda\underset{\sim}{I}|$, where $\underset{\sim}{X}$ may be the population growth matrix. Then

$$\frac{\delta|\underset{\sim}{X} - \lambda\underset{\sim}{I}|}{\delta\langle\underset{\sim}{X}\rangle} = \mathrm{tr}\left[\frac{\delta|\underset{\sim}{X} - \lambda\underset{\sim}{I}|}{\delta[\underset{\sim}{X} - \lambda\underset{\sim}{I}]} \cdot \frac{\delta[\underset{\sim}{X} - \lambda\underset{\sim}{I}]'}{\delta\langle\underset{\sim}{X}\rangle}\right]$$

$$\frac{\delta|\underset{\sim}{X} - \lambda\underset{\sim}{I}|}{\delta\langle\underset{\sim}{X}\rangle} = |\underset{\sim}{X} - \lambda\underset{\sim}{I}|\ \mathrm{tr}\left(\left[[\underset{\sim}{X} - \lambda\underset{\sim}{I}]'\right]^{-1} \underset{\sim}{J}'\right) \tag{A.33}$$

$$= \mathrm{tr}\left[[\mathrm{cof}(\underset{\sim}{X} - \lambda\underset{\sim}{I})]\ \underset{\sim}{J}'\right]$$

and

$$\frac{\delta|\underset{\sim}{X} - \lambda\underset{\sim}{I}|}{\delta\underset{\sim}{X}} = \sum_{k\ell} \frac{\delta|\underset{\sim}{X} - \lambda\underset{\sim}{I}|}{\delta<[\underset{\sim}{X} - \lambda\underset{\sim}{I}]>_{k\ell}} \cdot \frac{\delta<[\underset{\sim}{X} - \lambda\underset{\sim}{I}]'>_{k\ell}}{\delta\underset{\sim}{X}}$$

$$= \sum_{k\ell} |\underset{\sim}{X} - \lambda\underset{\sim}{I}|\ \ [\underset{\sim}{X} - \lambda\underset{\sim}{I}]'^{-1}_{\ k\ell}\ \ \underset{\sim}{J}'_{k\ell} \qquad \text{(A.34)}$$

$$\frac{\delta|\underset{\sim}{X} - \lambda\underset{\sim}{I}|}{\delta\underset{\sim}{X}} = |\underset{\sim}{X} - \lambda\underset{\sim}{I}|\ \ [\underset{\sim}{X} - \lambda\underset{\sim}{I}]'^{\ -1} = \mathrm{cof}\ [\underset{\sim}{X} - \lambda\underset{\sim}{I}] \qquad \text{(A.35)}$$

where cof $[\underset{\sim}{X} - \lambda\underset{\sim}{I}]$ is the cofactor matrix of $[\underset{\sim}{X} - \lambda\underset{\sim}{I}]$.

If $\underset{\sim}{Y}(r)$ is a function of the scalar r, then

$$\frac{\delta|\underset{\sim}{Y}(r)|}{\delta r} = \mathrm{tr}\left[\frac{\delta|\underset{\sim}{Y}(r)|}{\delta[\underset{\sim}{Y}(r)]} \cdot \frac{\delta[\underset{\sim}{Y}(r)]'}{\delta r}\right]$$

$$= \mathrm{tr}\left[|Y(r)|\left[[\underset{\sim}{Y}(r)]'\right]^{-1} \cdot \frac{\delta[Y(r)]'}{\delta r}\right]$$

and since tr $\underset{\sim}{A}\underset{\sim}{B}$ = tr$[\underset{\sim}{A}\underset{\sim}{B}]'$ = tr $\underset{\sim}{B}'\underset{\sim}{A}'$

$$\frac{\delta|\underset{\sim}{Y}(r)|}{\delta r} = |\underset{\sim}{Y}(r)|\ \mathrm{tr}\left[\frac{\delta[\underset{\sim}{Y}(r)]}{\delta r}\ [\underset{\sim}{Y}(r)]^{-1}\right] \qquad \text{(A.36)}$$

Formula (A.36) is not only of interest in a study of the sensitivity of the determinant of a polynomial matrix, but is also useful in order to compute the determinant, as shown by Emre and Hüseyin (1975; p. 136). An application of (A.36) which is relevant is

$$\frac{\delta|\underset{\sim}{A} - \lambda\underset{\sim}{I}|}{\delta\lambda} = - |\underset{\sim}{A} - \lambda\underset{\sim}{I}| \; tr[\underset{\sim}{A} - \lambda\underset{\sim}{I}]^{-1} \quad . \qquad (A.37)$$

This formula can also be found in Newbery (1974; p. 1016). Finally, consider the application, where $f(\underset{\sim}{Y}) = tr[\underset{\sim}{A}\underset{\sim}{X}\underset{\sim}{B}]$, whence

$$\frac{\delta f(\underset{\sim}{Y})}{\delta \underset{\sim}{X}} = \frac{\delta \; tr[\underset{\sim}{A}\underset{\sim}{X}\underset{\sim}{B}]}{\delta \underset{\sim}{X}} = \underset{\sim}{A}'\underset{\sim}{B}' \quad . \qquad (A.38)$$

A.1.5. Vector differentiation

Vectors may be considered as matrices with only one row or one column, and the rules for matrix differentiation may be applied. But the derivative of a vector or of a vector equation has a simpler form than the matrix analogue. It is, therefore, worthwhile to list the formulas for vector differentiation separately. Two cases are considered: the derivative of a scalar function with respect to a vector and the derivative of a vector function with respect to a vector.

a. Differentiation of a scalar function with respect to a vector

Consider the general scalar function $f(\{\underset{\sim}{x}\})$, where $\{\underset{\sim}{x}\}$ is the argument vector. Some relevant formulations of $f(\{\underset{\sim}{x}\})$ and their derivatives are listed below.

$f(\{\underset{\sim}{x}\})$	$\dfrac{\delta f(\{\underset{\sim}{x}\})}{\delta\{\underset{\sim}{x}\}}$	
$\{\underset{\sim}{a}\}'\{\underset{\sim}{x}\}$	$\{\underset{\sim}{a}\}$	(A.39)
$\{\underset{\sim}{x}\}'\{\underset{\sim}{x}\}$	$2\{\underset{\sim}{x}\}$	(A.40)
$\{\underset{\sim}{x}\}'\ \underset{\sim}{A}\{\underset{\sim}{x}\}$	$\underset{\sim}{A}\{\underset{\sim}{x}\} + \underset{\sim}{A}'\{\underset{\sim}{x}\}$	(A.41)

b. Differentiation of a vector function with respect to a vector

Let $\{f(\{\underset{\sim}{x}\})\}$ denote a column vector of scalar functions $f_i(\{\underset{\sim}{x}\})$, where $\{\underset{\sim}{x}\}$ is the argument vector and $\{\underset{\sim}{f}(\{\underset{\sim}{x}\})\}$ represents a system of equations. For example, let $\{\underset{\sim}{f}(\{\underset{\sim}{x}\})\}$ be a system of linear equations in $\{\underset{\sim}{x}\}$, then

$$\frac{\delta \underset{\sim}{A}\{\underset{\sim}{x}\}}{\delta<\{\underset{\sim}{x}\}>_i} = \{\underset{\sim}{a}_i\} \qquad \text{(A.42)}$$

where $\{\underset{\sim}{a}_i\}$ is the i-th column of $\underset{\sim}{A}$.

The derivatives of $\{\underset{\sim}{f}(\{\underset{\sim}{x}\})\}$ with respect to all the elements of the argument vector form a matrix if the argument vector is a row vector. For example

$$\frac{\delta \underset{\sim}{A}\{\underset{\sim}{x}\}}{\delta\{\underset{\sim}{x}\}'} = \underset{\sim}{A} \qquad \text{(A.43)}$$

The determinant $\left|\frac{\delta\{\underset{\sim}{f}(\{\underset{\sim}{x}\})\}}{\delta\{\underset{\sim}{x}\}'}\right|$ is known as the Jacobian or functional determinant.

Corresponding to the chain rule of matrix differentiation, one may formulate the chain rule of vector differentiation. Let $\{\underset{\sim}{y}\}$, $\{\underset{\sim}{x}\}$ and $\{\underset{\sim}{z}\}$ be vectors. It can be shown that

$$\frac{\delta\{\underset{\sim}{y}\}}{\delta\{\underset{\sim}{x}\}'} = \frac{\delta\{\underset{\sim}{y}\}}{\delta\{\underset{\sim}{z}\}'} \cdot \frac{\delta\{\underset{\sim}{z}\}}{\delta\{\underset{\sim}{x}\}'} \quad . \qquad \text{(A.44)}$$

A.2. DIFFERENTIATION OF EIGENVALUES AND EIGENVECTORS OF MATRICES

The topic of eigenvalue sensitivity has received most attention in the engineering literature. The design engineer is interested in identifying the impact of changes in the parameters of a system on the system's performance. There is a vast literature on sensitivity analysis in design[4]. Although most of this literature is not related to the problem in this study, some relevant elements are repeated here. We will separate the eigenvalue sensitivity problem and the eigenvector

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[4]See Cruz (1973) and Tomović and Vukobratović (1972) for example.

sensitivity problem. The former has received considerable attention, while the latter has been very much neglected.

A.2.1. Differentiation of the eigenvalue with respect to the matrix elements

The method which follows is described by Faddeev and Faddeeva (1963; p. 229) and can also be found in Van Ness et al. (1973; p. 100) and in Tomović and Vukobratović (1972; pp. 196-197). The assumption underlying the method is that all the eigenvalues of the matrix are distinct. Let $\underset{\sim}{A}$ be such a matrix. Consider the equation

$$\underset{\sim}{A}\{\underset{\sim}{\xi}\}_i = \lambda_i\{\underset{\sim}{\xi}\}_i \tag{A.45}$$

where λ_i is the i-th eigenvalue of $\underset{\sim}{A}$ and $\{\underset{\sim}{\xi}\}_i$ is the right eigenvector associated with λ_i.

Taking the partial derivatives of both sides with respect to an element of $\underset{\sim}{A}$, $<\underset{\sim}{A}>$ say, gives

$$\frac{\delta \underset{\sim}{A}}{\delta<\underset{\sim}{A}>}\{\underset{\sim}{\xi}\}_i + \underset{\sim}{A}\frac{\delta\{\underset{\sim}{\xi}\}_i}{\delta<\underset{\sim}{A}>} = \frac{\delta\lambda_i}{\delta<\underset{\sim}{A}>}\{\underset{\sim}{\xi}\}_i + \lambda_i\frac{\delta\{\underset{\sim}{\xi}\}_i}{\delta<\underset{\sim}{A}>} \tag{A.46}$$

If the real matrix $\underset{\sim}{A}$ is transposed, the eigenvalues will not change. However, a new set of eigenvectors will be formed: the left eigenvectors, denoted by $\{\underset{\sim}{\nu}\}_j$. The scalar product of each of the terms of (A.46) with $\{\underset{\sim}{\nu}\}_j$ is:

$$\left(\underset{\sim}{J}\{\underset{\sim}{\xi}\}_i \;,\; \{\underset{\sim}{\nu}\}_j\right) + \left(\underset{\sim}{A}\,\frac{\delta\{\underset{\sim}{\xi}\}_i}{\delta<\underset{\sim}{A}>} \;,\; \{\underset{\sim}{\nu}\}_j\right) =$$

$$\lambda_i\left(\frac{\delta\{\underset{\sim}{\xi}\}_i}{\delta<\underset{\sim}{A}>} \;,\; \{\underset{\sim}{\nu}\}_j\right) + \frac{\delta\lambda_i}{\delta<\underset{\sim}{A}>}\left(\{\underset{\sim}{\xi}\}_i \;,\; \{\underset{\sim}{\nu}\}_j\right) \tag{A.47}$$

where $\underset{\sim}{J}$ has the same meaning as in section A.1. If i is taken equal to j, and use is made of the relationship

$$\underset{\sim}{A}'\{\underset{\sim}{\nu}\}_j = \lambda_j\{\underset{\sim}{\nu}\}_j \tag{A.48}$$

then (A.47) becomes

$$\left(\underset{\sim}{J}\{\underset{\sim}{\xi}\}_i \;,\; \{\underset{\sim}{\nu}\}_i\right) + \left(\underset{\sim}{A}\,\frac{\delta\{\underset{\sim}{\xi}\}_i}{\delta<\underset{\sim}{A}>} \;,\; \{\underset{\sim}{\nu}\}_i\right) =$$

$$\left(\frac{\delta\{\underset{\sim}{\xi}\}_i}{\delta<\underset{\sim}{A}>} \;,\; \underset{\sim}{A}'\{\underset{\sim}{\nu}\}_i\right) + \frac{\delta\lambda_i}{\delta<\underset{\sim}{A}>}\left(\{\underset{\sim}{\xi}\}_i \;,\; \{\underset{\sim}{\nu}\}_i\right) . \tag{A.49}$$

Since

$$\left(\underset{\sim}{A}\,\frac{\delta\{\underset{\sim}{\xi}\}_i}{\delta<\underset{\sim}{A}>} \;,\; \{\underset{\sim}{\nu}\}_i\right) = \left(\frac{\delta\{\underset{\sim}{\xi}\}_i}{\delta<\underset{\sim}{A}>} \;,\; \underset{\sim}{A}'\{\underset{\sim}{\nu}\}_i\right) \;,$$

we may write

$$\frac{\delta\lambda_i}{\delta\langle \underset{\sim}{A}\rangle} = \frac{\left(\underset{\sim}{J}\{\underset{\sim}{\xi}\}_i\ ,\ \{\underset{\sim}{\nu}\}_i\right)}{\left(\{\underset{\sim}{\xi}\}_i\ ,\ \{\underset{\sim}{\nu}\}_i\right)} \qquad \text{(A.50)}$$

Expression (A.50) represents the sensitivity of the eigenvalues of $\underset{\sim}{A}$ with respect to an element of $\underset{\sim}{A}$.

If the eigenvectors are normalized such that their inner product is unity, i.e.

$$\left(\{\underset{\sim}{\xi}\}_i\ ,\ \{\underset{\sim}{\nu}\}_i\right) = 1$$

then

$$\frac{\delta\lambda_i}{\delta\langle \underset{\sim}{A}\rangle} = \left(\underset{\sim}{J}\{\underset{\sim}{\xi}\}_i\ ,\ \{\underset{\sim}{\nu}\}_i\right) = \left(\{\underset{\sim}{\nu}\}_i\ ,\ \underset{\sim}{J}\{\underset{\sim}{\xi}\}_i\right). \qquad \text{(A.51)}$$

It can be shown that (A.51) is equivalent to

$$\frac{\delta\lambda_i}{\delta\langle \underset{\sim}{A}\rangle} = \text{tr}\left[[\{\underset{\sim}{\xi}\}_i\{\underset{\sim}{\nu}\}_i'] \ \underset{\sim}{J}\right] \qquad \text{(A.52)}$$

or

$$\frac{\delta\lambda_i}{\delta\langle \underset{\sim}{A}\rangle} = [\{\underset{\sim}{\xi}\}_i\{\underset{\sim}{\nu}\}_i'] * \underset{\sim}{J} \qquad \text{(A.53)}$$

where * denotes the inner product of two matrices[5].

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[5]The inner product $\underset{\sim}{A} * \underset{\sim}{B}$ is defined as $\sum_i \sum_k a_{ik}b_{ki}$. The result is equal to tr[AB].

The structure of (A.52) is very similar to (A.33) of the previous section. The derivative of λ_i with respect to the whole matrix $\underset{\sim}{A}$ is

$$\frac{\delta\lambda_i}{\delta\underset{\sim}{A}} = \{\underset{\sim}{\xi}\}_i\{\underset{\sim}{\nu}\}_i' \quad . \tag{A.54}$$

The matrix $\{\underset{\sim}{\xi}\}_i\{\underset{\sim}{\nu}\}_i'$ is the adjoint matrix of $[\underset{\sim}{A} - \lambda\underset{\sim}{I}]$, normalized such that the trace is equal to one[6]. The sensitivity of the eigenvalue is sometimes expressed in terms of differentials

$$d\lambda_i = \{\underset{\sim}{\nu}\}_i' \; [d\underset{\sim}{A}]\{\underset{\sim}{\xi}\} \tag{A.55}$$

or

$$d\lambda_i = \left[\{\underset{\sim}{\xi}\}_i\{\underset{\sim}{\nu}\}_i'\right] * d\underset{\sim}{A} \quad . \tag{A.56}$$

The computation of the sensitivity of λ_i requires that the left and right eigenvectors be known.

If the eigenvectors are not normalized, the sensitivity function is

$$\frac{\delta\lambda_i}{\delta\underset{\sim}{A}} = \frac{1}{\{\underset{\sim}{\nu}\}_i'\{\underset{\sim}{\xi}\}_i} \left[\{\underset{\sim}{\xi}\}_i\{\underset{\sim}{\nu}\}_i'\right] \tag{A.57}$$

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[6] $tr[\{\underset{\sim}{\xi}\}_i\{\underset{\sim}{\nu}\}_i']$ is equal to $\{\underset{\sim}{\nu}\}_i'\{\underset{\sim}{\xi}\}_i$ which is equal to one for normalized ν eigenvectors.

where $\left[\{\underset{\sim}{\xi}\}_i \{\underset{\sim}{\nu}\}_i'\right]$ is the adjoint matrix of $[\underset{\sim}{A} - \lambda \underset{\sim}{I}]$. Denoting the adjoint matrix by $\underset{\sim}{R}(\lambda_i)$, (A. 51) may be written as

$$\frac{\delta\lambda_i}{\delta \underset{\sim}{A}} = [\mathrm{tr}\ \underset{\sim}{R}(\lambda_i)]^{-1}\ \underset{\sim}{R}(\lambda_i) \tag{A.58}$$

and (A.56) becomes

$$d\lambda_i = [\mathrm{tr}\ \underset{\sim}{R}(\lambda_i)]^{-1}\ \underset{\sim}{R}(\lambda_i) * d\underset{\sim}{A} \quad . \tag{A.59}$$

Equation (A.59) is exactly the sensitivity formula given by Morgan (1973; p. 76). The matrix $\underset{\sim}{R}(\lambda_i)$ can be efficiently computed by means of the Leverrier algorithm, described by Faddeev and Faddeeva (1963; p. 260) and Morgan (1973; p. 76). This is particularly interesting since the rows of $\underset{\sim}{R}(\lambda_i)$ are left eigenvectors and the columns are right eigenvectors. For a formal proof that (A.59) is identical to (A.56), see Mac Farlane (1970; pp. 413-419).

Formulas (A.54) and (A.58) have the benefit that they are easily computed. For analytical purposes, however, it would be beneficial to have an expression linking the change in the eigenvalue directly to a change in $\underset{\sim}{A}$, and to the original value of $\underset{\sim}{A}$ and of the eigenvalues. Such an expression is derived by Rosenbrock (1965; p. 278):

$$d\lambda_i = \frac{\mathrm{tr}\left[\prod_{r\neq i} [\underset{\sim}{A} - \lambda_i \underset{\sim}{I}]\ d\underset{\sim}{A}\right]}{\prod_{r\neq i} (\lambda_i - \lambda_r)} \quad . \tag{A.60}$$

A.2.2. Differentiation of the eigenvector with respect to the matrix elements

Recall equation (A.47):

$$\left(\underset{\sim}{J}\{\underset{\sim}{\xi}\}_i \ , \ \{\underset{\sim}{\nu}\}_j\right) + \left(\underset{\sim}{A}\ \frac{\delta\{\underset{\sim}{\xi}\}_i}{\delta<\underset{\sim}{A}>} \ , \ \{\underset{\sim}{\nu}\}_j\right) = \tag{A.47}$$

$$\lambda_i \left(\frac{\delta\{\underset{\sim}{\xi}\}_i}{\delta<\underset{\sim}{A}>} \ , \ \{\underset{\sim}{\nu}\}_j\right) + \frac{\delta\lambda_i}{\delta<\underset{\sim}{A}>} \left(\{\underset{\sim}{\xi}\}_i \ , \ \{\underset{\sim}{\nu}\}_j\right)$$

For $i \neq j$, we have

$$\left(\{\underset{\sim}{\xi}\}_i \ , \ \{\underset{\sim}{\nu}\}_j\right) = 0 \quad .$$

We have also that

$$\left(\underset{\sim}{A}\ \frac{\delta\{\underset{\sim}{\xi}\}_i}{\delta<\underset{\sim}{A}>} \ , \ \{\underset{\sim}{\nu}\}_j\right) = \left(\frac{\delta\{\underset{\sim}{\xi}\}_i}{\delta<\underset{\sim}{A}>} \ , \ \underset{\sim}{A}'\{\underset{\sim}{\nu}\}_j\right) = \lambda_j \left(\frac{\delta\{\underset{\sim}{\xi}\}_i}{\delta<\underset{\sim}{A}>} \ , \ \{\underset{\sim}{\nu}\}_j\right)$$

Equation (A.47) may be rewritten as

$$\left(\underset{\sim}{J}\{\underset{\sim}{\xi}\}_i \ , \ \{\underset{\sim}{\nu}\}_j\right) = (\lambda_i - \lambda_j) \left[\left(\frac{\delta\{\underset{\sim}{\xi}\}_i}{\delta<\underset{\sim}{A}>} \ , \ \{\underset{\sim}{\nu}\}_j\right)\right]$$

$$\left(\frac{\delta\{\underset{\sim}{\xi}\}_i}{\delta<\underset{\sim}{A}>} \ , \ \{\underset{\sim}{\nu}\}_j\right) = \frac{\left(\underset{\sim}{J}\{\underset{\sim}{\xi}\}_i \ , \ \{\underset{\sim}{\nu}\}_j\right)}{(\lambda_i - \lambda_j)} \quad . \tag{A.61}$$

Let $$\frac{\delta\{\tilde{\xi}\}_i}{\delta\langle\tilde{A}\rangle} = \sum_{j=1}^{N} c_{ij}\{\tilde{\xi}\}_j \tag{A.62}$$

then

$$\left(\frac{\delta\{\tilde{\xi}\}_i}{\delta\langle\tilde{A}\rangle} \ , \ \{\tilde{\nu}\}_j\right) = c_{ij}\left(\{\tilde{\xi}\}_j \ , \ \{\tilde{\nu}\}_j\right)$$

and consequently, for normalized eigenvectors

$$c_{ij} = \frac{\left(\tilde{J}\{\tilde{\xi}\}_i \ , \ \{\tilde{\nu}\}_j\right)}{\lambda_i - \lambda_j} \qquad \text{for } i \neq j \tag{A.63}$$

The element c_{ii} remains undefined in view of the non-uniqueness of the eigenvector. We may assume that $c_{ii} = 0$ without loss of generality.

The computation of the sensitivity of the eigenvector by (A.62) has a disadvantage, since it requires the knowledge of all the eigenvalues and eigenvectors. Another approach that relates the change in a specific eigenvector to the change in $\tilde{A}$ and to the change in the associated eigenvalue, is given below. Consider the homogeneous equation

$$[\tilde{A} - \lambda_i \tilde{I}]\{\xi\}_i = \{\tilde{0}\} \quad . \tag{A.64}$$

Assume that all the eigenvalues of $\tilde{A}$ are distinct, and let the first element of $\{\tilde{\xi}\}_i$, i.e. ξ_{1i}, be equal to 1. We may now delete the first equation of (A.64). The resulting set forms a linearly independent system of non-homogenous equations of order N-1.

$$\begin{bmatrix} a_{21} \\ a_{31} \\ \vdots \\ a_{N1} \end{bmatrix} + \begin{bmatrix} a_{22} - \lambda_i & a_{23} & & a_{N1} \\ a_{32} & a_{33} - \lambda_i & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{N2} & a_{N3} & & a_{NN} - \lambda_i \end{bmatrix} \begin{bmatrix} \xi_{2i} \\ \xi_{3i} \\ \vdots \\ \xi_{Ni} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or in matrix notation

$$\{\bar{\underset{\sim}{a}}_1\} + [\bar{\underset{\sim}{A}} - \lambda_i \bar{\underset{\sim}{I}}]\{\bar{\underset{\sim}{\xi}}\}_i = \{\bar{\underset{\sim}{0}}\} \tag{A.65}$$

where the bar denotes the order N-1. Because of the non-singularity of $[\bar{\underset{\sim}{A}} - \lambda_i \bar{\underset{\sim}{I}}]$, we have

$$\{\bar{\underset{\sim}{\xi}}\}_i = - [\bar{\underset{\sim}{A}} - \lambda_i \bar{\underset{\sim}{I}}]^{-1} \{\bar{\underset{\sim}{a}}_1\} \quad . \tag{A.66}$$

Applying formula (A.13) of section A.1. to (A.66) gives

$$\frac{\delta\{\bar{\underset{\sim}{\xi}}\}_i}{\delta\langle\underset{\sim}{A}\rangle} = - \frac{\delta[\bar{\underset{\sim}{A}} - \lambda_i \bar{\underset{\sim}{I}}]^{-1}}{\delta\langle\underset{\sim}{A}\rangle} \{\bar{\underset{\sim}{a}}_1\} - [\bar{\underset{\sim}{A}} - \lambda_i \bar{\underset{\sim}{I}}]^{-1} \frac{\delta\{\bar{\underset{\sim}{a}}_1\}}{\delta\langle\underset{\sim}{A}\rangle}$$

$$= [\bar{\underset{\sim}{A}} - \lambda_i \bar{\underset{\sim}{I}}]^{-1} \frac{\delta[\bar{\underset{\sim}{A}} - \lambda_i \bar{\underset{\sim}{I}}]}{\delta\langle\underset{\sim}{A}\rangle} [\bar{\underset{\sim}{A}} - \lambda_i \bar{\underset{\sim}{I}}]^{-1} \{\bar{\underset{\sim}{a}}_1\}$$

$$- [\bar{\underset{\sim}{A}} - \lambda_i \bar{\underset{\sim}{I}}] \frac{\delta\{\bar{a}_1\}}{\delta\langle\underset{\sim}{A}\rangle}$$

Substituting for $\{\bar{\underset{\sim}{\xi}}\}_i$ and differentiating $[\bar{\underset{\sim}{A}} - \lambda_i \bar{\underset{\sim}{I}}]$ yields

$$\frac{\delta\{\bar{\underset{\sim}{\xi}}\}_i}{\delta<\underset{\sim}{A}>} = - [\underset{\sim}{A} - \lambda_i \underset{\sim}{I}] \left[\left[\frac{\delta \underset{\sim}{A}}{\delta<\underset{\sim}{A}>} - \frac{\delta\lambda_i}{\delta<\underset{\sim}{A}>} \bar{\underset{\sim}{I}} \right] \{\bar{\underset{\sim}{\xi}}\}_i + \frac{\delta\{\bar{\underset{\sim}{a}}_1\}}{\delta<\underset{\sim}{A}>} \right] \tag{A.67}$$

where $\frac{\delta\lambda_i}{\delta<\underset{\sim}{A}>}$ is computed using (A.51) or an equivalent formula.

Some special cases now may be considered.

a. If the change in $\underset{\sim}{A}$ occurs in the first row, this change has no direct impact on the eigenvector, since $\bar{\underset{\sim}{A}}$ and $\{\bar{\underset{\sim}{a}}_1\}$ do not include elements of the first row of $\underset{\sim}{A}$. There is an indirect effect on $\{\bar{\underset{\sim}{\xi}}\}_i$, however, through the change in the eigenvalue.

b. If the change in $\underset{\sim}{A}$ occurs in the first column, i.e. in $\{\bar{\underset{\sim}{a}}_1\}$, then

$$\frac{\delta\bar{\underset{\sim}{A}}}{\delta<\underset{\sim}{A}>} = \underset{\sim}{0} \quad .$$

c. If the change in $\underset{\sim}{A}$ occurs not in the first column nor in the first row, then

$$\frac{\delta\{\bar{\underset{\sim}{a}}_1\}}{\delta<\underset{\sim}{A}>} = \underset{\sim}{0} \quad .$$

Besides (A.62) and (A.67), a third method to compute the eigenvector sensitivity may be derived. It is based on the fact that the columns of the adjoint matrix are right eigenvectors and that the rows are left eigenvectors. This technique will not be discussed here.

References

[1] Barnett, S. "Some Topics in Algebraic Systems Theory: a Survey." International Journal of Control, 19 (1974), pp. 669-688.

[2] Barnett, S. "Matrices, Polynomials, and Linear Time-Invariant Systems." IEEE-Transactions on Automatic Control, AC-18 (1973), pp. 1-10.

[3] Bellman, R. Introduction to Matrix Analysis, Second Edition. New York, McGraw-Hill Book Co., 1970.

[4] Coale, A.J. The Growth and Structure of Human Populations: A Mathematical Investigation. Princeton, N.J., Princeton University Press, 1972.

[5] Coale, A.J. and P. Demeny. Regional Model Life Tables and Stable Populations. Princeton, N.J., Princeton University Press, 1966.

[6] Crane, R.N. and A.R. Stubberud. "Closed Loop Formulations of Optimal Control Problems for Minimum Sensitivity." In Leondes, C.T., Ed. Control and Dynamic Systems, 9. New York, Academic Press, 1973, pp. 375-505.

[7] Cruz, J.B., Ed. System Sensitivity Analysis. Stroudsburg, Pa., Dowden, Hutchinson and Ross, Inc., 1973.

[8] Cull, P. and A. Vogt. "Mathematical Analysis of the Asymptotic Behavior of the Leslie Population Matrix Model." Bulletin of Mathematical Biology, 35 (1973), pp. 645-661.

[9] Demetrius, L. "The Sensitivity of Population Growth Rate to Perturbations in the Life Cycle Components." Mathematical Biosciences, 4 (1969), pp. 129-136.

[10] Dwyer, P.S. "Some Applications of Matrix Derivatives in Multivariate Analysis." Journal of the American Statistical Association, 62 (1967), pp. 607-625.

[11] Dwyer, P.S. and M.S. MacPhail. "Symbolic Matrix Derivatives." Annals of Mathematical Studies, 16 (1948), pp. 517-534.

[12] Emlen, J.M. "Age Specificity and Ecological Theory." Ecology, 51 (1970), pp. 588-601.

[13] Emre, E. and Ö. Hüseyin. "Generalization of Leverrier's Algorithm to Polynomial Matrices of Arbitrary Degree." IEEE Transactions on Automatic Control, AC-20 (1975), p. 136.

[14] Erickson, D.L. and F.E. Norton. "Application of Sensitivity Constrained Optimal Control to National Economic Policy Formulation." In Leondes, C.T., Ed. Control and Dynamic Systems. Advances in Theory and Applications. New York, Academic Press, Inc., 1973, pp. 132-237.

[15] Faddeev, D.K. and V.N. Faddeeva. Computational Methods of Linear Algebra. San Francisco, W.H. Freeman and Co., 1963.

[16] Feeney, G.M. "Stable Age by Region Distributions." Demography, 6 (1970), pp. 341-348.

[17] Forrester, J.W. Urban Dynamics. Cambridge, Mass., MIT Press, 1969.

[18] Goodman, L.A. "On the Reconciliation of Mathematical Theories of Population Growth." Journal of the Royal Statistical Society, A-130 (1967), pp. 541-553.

[19] Goodman, L.A. "The Analysis of Population Growth When the Birth and Death Rates Depend upon Several Factors." Biometrics, 25 (1969), pp. 659-681.

[20] Goodman, L.A. "On the Sensitivity of the Intrinsic Growth Rate to Changes in the Age-Specific Birth and Death Rates." Theoretical Population Biology, 2 (1971), pp. 339-354.

[21] Kalman, R.E. "Elementary Control Theory from the Modern Point of View." In Kalman, R.D., P.L. Falb and M.A. Arbib, Topics in Mathematical System Theory. New York, McGraw-Hill Book Co., 1969, pp. 24-66.

[22] Kenkel, J.L. Dynamic Linear Economic Models. London, Gordon and Breach, 1974.

[23] Keyfitz, N. Introduction to the Mathematics of Population. Reading, Mass., Addison-Wesley, 1968.

[24] Keyfitz, N. "Linkages of Intrinsic to Age-Specific Rates." Journal of the American Statistical Association, 66 (1971), pp. 275-281.

[25] Lancaster, P. Theory of Matrices. New York, Academic Press, 1969.

[26] LeBras, H. "Equilibre et croissance de populations soumises à des migrations." Theoretical Population Biology, 2 (1971), pp. 100-121.

[27] Leslie, P.H. "On the Use of Matrices in Certain Population Mathematics." Biometrika, 33 (1945), pp. 183-212.

[28] Lopez, A. Problems in Stable Population Theory. Princeton, N.J., Office of Population Research, 1961.

[29] Lotka, A.J. "Mode of Growth of Material Aggregates." American Journal of Science, 24 (1907), pp. 199-216.

[30] MacFarlane, A.G. Dynamical System Models. London, George G. Harrap and Co., Ltd., 1970.

[31] Malthus, T.R. An Essay on the Principles of Population. Printed for J. Johnson in St. Paul's Churchyard, London, 1798.

[32] Morgan, B.S. "Computational Procedure for the Sensitivity of an Eigenvalue." Electronics Letters, 2 (1966), pp. 197-198.

[33] Morgan, B.S., Jr. "Sensitivity Analysis and Synthesis of Multivariable Systems." In Cruz, J.B., Ed. System Sensitivity Analysis. Stroudsburg, Pa., Dowden, Hutchinson and Ross, Inc., 1973, pp. 75-81.

[34] Nelson, N. and F. Kern. "Perturbation and Sensitivity Analysis of an Urban Model." In J. Tou, Ed. Conference on Decision and Control. Department of Electrical Engineering, University of Florida, 1971, pp. 283-288.

[35] Neudecker, N. "Some Theorems on Matrix Differentiation with Special Reference to Kronecker Matrix Products." Journal of the American Statistical Association, 64 (1969), pp. 953-963.

[36] Newbery, A.C. "Numerical Analysis." In Pearson, C.E., Ed. Handbook of Applied Mathematics. New York, Van Nostrand Reinhold Co., 1974, pp. 1002-1057.

[37] Pielou, E.C. An Introduction to Mathematical Ecology. New York, John Wiley and Sons, 1969.

[38] Pollard, J.H. Mathematical Models for the Growth of Human Populations. London, Cambridge University Press, 1973.

[39] Preston, S.H. "Effect of Mortality Change on Stable Population Parameters." Demography, 11 (1974), pp. 119-130.

[40] Rogers, A. "The Mathematics of Multiregional Demographic Growth." Environment and Planning, 5 (1973), pp. 3-29.

[41] Rogers, A. "The Multiregional Net Maternity Function and Multiregional Stable Growth." Demography, 11 (1974), pp. 473-481.

[42] Rogers, A. Introduction to Multiregional Mathematical Demography. New York, John Wiley and Sons, 1975.

[43] Rogers, A. and J. Ledent. "Increment-Decrement Life Tables: A Comment." Forthcoming in Demography, 1976.

[44] Rogers, A. and F. Willekens. "Spatial Population Dynamics." IIASA Research Report 75-24. Laxenburg, Austria, International Institute for Applied Systems Analysis, 1975.

[45] Rosenbrock, H.H. "Sensitivity of an Eigenvalue to Changes in the Matrix." Electronics Letters, 1 (1965), p. 278.

[46] Schoen, R. "Constructing Increment-Decrement Life Tables." Demography, 12 (1975), pp. 313-324.

[47] Sharpe, F.R. and J.J. Lotka. "A Problem in Age Distribution." Philosophical Magazine, 21 (1911), pp. 435-438.

[48] Sykes, Z.M. "On Discrete Stable Population Theory." Biometrics, 25 (1969), pp. 285-293.

[49] Tomović, R. and M. Vukobratović. General Sensitivity Theory. New York, American Elsevier Publishing Co., Inc., 1972.

[50] Van Ness, J.E., J.M. Boyle and F.P. Imad. "Sensitivities in Large, Multi-Loop Control Systems." In Cruz, J.B., Ed. System Sensitivity Analysis. Stroudsburg, Pa., Dowden, Hutchinson and Ross, Inc., 1973, pp. 98-105.

[51] Wilkinson, J.H. The Algebraic Eigenvalue Problem. London, Oxford University Press, 1965.

[52] Wolovich, W.A. Linear Multivariable Systems. New York, Springer Verlag, 1974.

[53] Wu, M.Y. "A Note on Matrix Transformation." International Journal of Systems Science, 3 (1972), pp. 287-291.

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