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Competition of Gas Pipeline Projects: Game of Timing

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Abstract

The paper addresses the issue of optimal investments in innovations with strong long-term aftereffects. As an example, investments in the construction of gas pipelines are considered. The most sensible part of a gas pipeline project is the choice of the commercialization time, i.e., the time of finalizing the construction of the pipeline. If several projects compete for a gas market, the choices of the commercialization times determine the future structure of the market and thus become especially important. Rational decisions in this respect can be associated with Nash equilibria in a game between the projects. In this game, the total benefits gained during the pipelines' life periods act as payoffs and commercialization times as strategies. The goal of this paper is to characterize multiequilibria in this *game of timing*. The case of two players is studied in detail. A key point in the analysis is the observation that all player's best response commercialization times concentrate at two instants that are fixed in advance. This reduces decisionmaking to choosing between two fixed investment policies, "fast" and "slow", with the prescribed commercialization times. A description of a simple algorithm that finds all the Nash equilibria composed of "fast" and "slow" scenarios concludes the paper.

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Introduction

The present paper is motivated by [Klaassen, et. al., 2000] where a new mathematical model of operation of large-scale gas pipeline projects has been suggested. This model constructed on the basis of classical micro and macro patterns of mathematical economics (see [Arrow and Kurz, 1970], [Intriligator, 1971]) provides a macroeconomic tool for the analysis of future gas infrastructures (see, e.g., [Klaassen, et. al., 2001]). It comprises four microeconomic levels of optimization: assessment of the market of potential innovations, selection of innovation scenarios, regulation of the future supply and optimization of the current investments. In each level, the model is optimized using appropriate techniques of theory of optimal control and theory of differential games (see [Pontryagin, et. al., 1962], [Krasovskii and Subbotin, 1988]). If several gas pipeline projects compete for a gas market, the choices of the commercialization times, i.e., the times of finalizing the construction of the pipelines, determine the future structure of the market and thus become especially important. Accordingly, the choice of the commercialization times is the most sensible part of the model. In [Klaassen, et. al., 2000] rational commercialization times for the pipeline projects competing for the Turkey gas market have been presented. A methodology for numerical finding commercialization times via simulating the process of their mutual adaptation during the construction period has been suggested; the best reply dynamic adaptation principle widely used in applications of theory of evolutionary games (see [Hofbauer and Sigmund, 1988], [Friedman, 1991], [Kaniovski, et. al., 2000], [Kryazhimskii and Osipov, 1995], [Kryazhimskii, et. al., 2001], [Tarasyev, 1999]) has been utilized.

Rational choices of the commercialization times can be viewed as Nash equilibria in a game between the projects. The goal of the present paper is to study the structure of this game. Background in the analysis of problems of optimal timing (see [Barzel, 1968], [Tarasyev and Watanabe, 2001]) is employed. In order to make the model easily tractable in terms of game theory (see, e.g., [Basar, Olsder, 1982], [Vorobyev, 1985]), we introduce several simplifying assumptions, in particular, we reduce the number of competing projects to 2 (the analysis of the multi-agent game will be the next stage of research).

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The simplified model takes into account the stages of construction and exploitation of the pipelines. At the stage of exploitation, as gas supply policies compete on market, decisionmaking is relatively clear: the competitors search for an equilibrium supply at any instant. We focus, therefore, on the stage of construction, at which investment policies compete and decisionmaking is concerned with strong long-term aftereffects. The competitors interact through choosing their commercialization times. A proper individual choice is the best response to the choices of the other competitors. Therefore, a collection of commercialization times is suitable to every competitor if and only if the commercialization time of every competitor responds best to the commercialization times of the other competitors. Such situations constitute Nash equilibria in the game under consideration. In this game, the total benefits gained during the pipelines' life periods act as payoffs and commercialization times as strategies. Our goal is to characterize the equilibria in this game, which will further be referred to as the *game of timing*.

In section 1, we describe the general two-player game of timing, in which the cost and benefit functions determining the players' payoffs are not specified. We also introduce several natural assumptions.

In section 2, we find the Nash equilibria in the game. A key point in the analysis is the observation that all player's best response commercialization times concentrate at two instants that are fixed in advance. This reduces decisionmaking to choosing between two fixed investment policies, "fast" and "slow", with the prescribed commercialization times.

In section 3, we describe an algorithm that finds all the Nash equilibria in the game of timing.

In section 4, we study the game of timing for the model described in [Klaassen, et. al., 2000].

Section 5, the Appendix, contains the proves of the propositions formulated in section 4.

1 Game of Timing

In this section, we construct a game-theoretic model of competition of two gas pipeline projects. We call it the *game of timing*. The pipelines are expected to operate at the same market. We associate players 1 and 2 with the investors/managers of projects 1 and 2, respectively. Assuming that the starting time for making investments is 0, we consider "virtual" positive commercialization times of projects 1 and 2 (i.e., the final times of the construction of the pipelines), t_1 and t_2. Given a ("virtual") commercialization time, t_i, player i ($i = 1, 2$) can estimate the cost, $C_i(t_i)$, for finalizing project i at time t_i. The positive-valued *cost functions* $C_i(t_i)$ ($i = 1, 2$) are therefore defined on the positive half-axis. The following assumption will simplify our analysis.

Assumption 1.1 For each player, i, the cost function, $C_i(t_i)$, is smooth (continuously differentiable), monotonically decreasing and convex.

A formal interpretation of Assumption 1.1 is that the derivative $C_i'(t_i) = dC_i(t_i)/dt_i$ is negative and increasing. A substantial interpretation is that the cost of the project falls down as the project's commercialization period is prolonged; moreover, the longer is the commercialization period, the less sensitive, with respect to its prolongations, is the rate of cost reduction. In what follows, the *rate of cost reduction* for player i is understood as the positive-valued monotonically decreasing function

$$a_i(t_i) = -C_i'(t_i) \tag{1.1}$$

Let us argue for player 1 as the manager of pipeline 1. At any time $t > 0$, the price of gas and costs for extraction and transportation of gas determine the *benefit rate* of player 1, $b_1(t)$ (note that this benefit rate is "virtual" because t may precede the actual commercialization time of project 1). The costs for extraction and transportation of gas do not depend on the state of project 2, whereas the price of gas depends on the presence (absence) of player 2 on the marketplace. In the situation where both players operate on market, the price of gas should obviously be smaller compared to the situation where player 1 occupies market solely. Hence, the benefit rate $b_1(t)$ may take two values, $b_{11}(t)$ and $b_{12}(t)$,

$$b_{11}(t) > b_{12}(t) \tag{1.2}$$

We call $b_{11}(t)$ the *upper benefit rate* and $b_{12}(t)$ the *lower benefit rate* of player 1 at time t. At time t (which "virtually" follows the commercialization time of player 1), player 1 ("virtually") gets $b_{11}(t)$ if player 2 does not operate on market, and $b_{12}(t)$ if player 2 operates on market. Similarly, we introduce the *upper* and *lower benefit rates* of player 2 at time t, $b_{21}(t)$ and $b_{22}(t)$,

$$b_{21}(t) > b_{22}(t) \tag{1.3}$$

At time t player 2 gets $b_{21}(t)$ if player 1 does not operate on market, and $b_{22}(t)$ otherwise. We assume that the positive-valued upper and lower benefit rates $b_{i1}(t)$ and $b_{i2}(t)$ $(i = 1, 2)$ are continuous functions defined on the positive half-axis. We also introduce the following assumption.

Assumption 1.2 For every player i $(i = 1, 2)$, the graph of the rate of cost reduction, $a_i(t)$, intersects the graph of the upper benefit rate, $b_{i1}(t)$, from above at the unique point $t_i^- > 0$, and stays below it afterwards; similarly, the graph of $a_i(t)$ intersects the graph of $b_{i2}(t)$ from above at the unique point $t_i^+ > 0$, and stays below it afterwards; more accurately,

$$a_i(t) > b_{i1}(t) \text{ for } 0 < t < t_i^-, \quad a_i(t_i^-) = b_{i1}(t_i^-), \quad a_i(t) < b_{i1}(t) \text{ for } t > t_i^-, \tag{1.4}$$

$$a_i(t) > b_{i2}(t) \text{ for } 0 < t < t_i^+, \quad a_i(t_i^+) = b_{i2}(t_i^+), \quad a_i(t) < b_{i2}(t) \text{ for } t > t_i^+. \tag{1.5}$$

Remark 1.1 Assumption 1.2 implies in particular that if $t > 0$ is sufficiently small, the rate of cost reduction, $a_i(t)$, is greater than the upper benefit rate, $b_{i1}(t)$, and if $t > 0$ is sufficiently large, the rate of cost reduction, $a_i(t)$, is smaller than the lower benefit rate, $b_{i1}(t)$.

Remark 1.2 Since $a_i(t)$ is decreasing and $b_{i1}(t) > b_{i2}(t)$ (see (1.2) and (1.3)), we have

$$t_i^- < t_i^+ \tag{1.6}$$

The relations between the graph of the rate of cost reduction, $a_i(t)$, and the graphs of the and upper and lower benefit rates, $b_{i1}(t)$ and $b_{i2}(t)$, are shown schematically in Fig. 1.

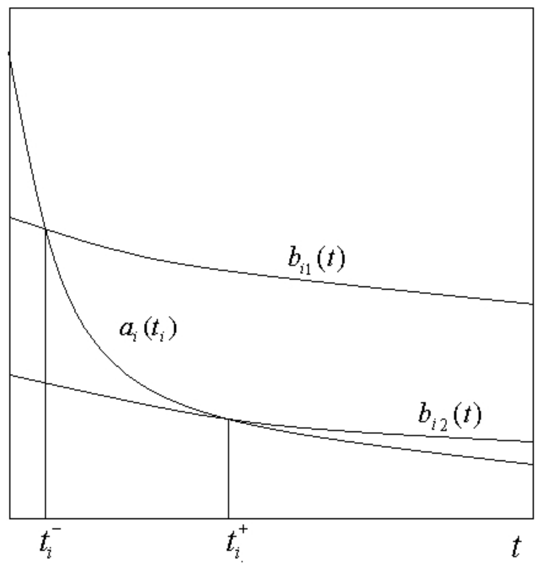


Fig 1.
The rate of cost reduction, $a_i(t)$, and
the upper and lower benefit rates, $b_{i1}(t)$ and $b_{i2}(t)$.

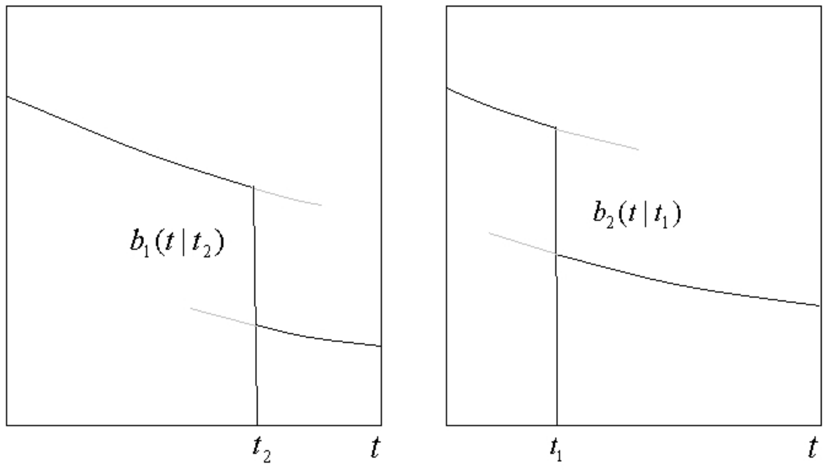
The fact that t_2 is the commercialization time of player 2 implies that player 2 does not operate on market at any time $t < t_2$ and operates on market at every time $t \geq t_2$. Accordingly, the benefit rate of player 1, $b_1(t)$, equals $b_{11}(t)$ for $t < t_2$ and equals $b_{12}(t)$ for $t \geq t_2$. We stress the dependence of $b_1(t)$ on t_2 and write $b_1(t|t_2)$ instead of $b_1(t)$. Thus, given a commercialization time t_2 of project 2, the benefit rate of player 1 is found as

$$b_1(t|t_2) = \begin{cases} b_{11}(t) & \text{if} \quad t < t_2, \\ b_{12}(t) & \text{if} \quad t \geq t_2 \end{cases} \tag{1.7}$$

Similarly, a commercialization time t_1 of project 1 determines the benefit rate of player 2 as

$$b_2(t|t_1) = \begin{cases} b_{21}(t) & \text{if} \quad t < t_1, \\ b_{22}(t) & \text{if} \quad t \geq t_1. \end{cases}$$

The graphs of the benefit rates $b_1(t|t_2)$ and $b_2(t|t_1)$ are shown schematically in Fig. 2.



(a)

(b)

Fig. 2.
(a) The benefit rate of player 1, $b_1(t|t_2)$
(project 2 commercialized at time t_2).
(b) The benefit rate of player 2, $b_2(t|t_1)$
(project 1 commercialized at time t_1).

Given a commercialization time of player 1, t_1, and a commercialization time of player 2, t_2, the *total benefits* of players 1 and 2 are represented by the integrals

$$B_1(t_1, t_2) = \int_{t_1}^{\infty} b_1(t|t_2)dt \tag{1.8}$$

and

$$B_2(t_1, t_2) = \int_{t_2}^{\infty} b_2(t|t_1)dt, \tag{1.9}$$

respectively. We make the following natural assumption.

Assumption 1.3 For every positive t_1 and every positive t_2 the integrals $B_1(t_1, t_2)$ and $B_2(t_1, t_2)$ are finite.

Remark 1.3 Assumption 1.3 is equivalent to the following: for every positive t_1 and every positive t_2 the integrals $\int_{t_2}^{\infty} b_{12}(t)dt$ and $\int_{t_1}^{\infty} b_{22}(t)dt$ are finite.

Given a commercialization time of player 1, t_1, and a commercialization time of player 2, t_2, the *total profit* of player i is defined as

$$P_i(t_1, t_2) = -C_i(t_i) + B_i(t_1, t_2). \tag{1.10}$$

We are ready to define the *game of timing* for players 1 and 2 in line with the standards of game theory (see, e.g., [Vorobyev, 1985]). In the game of timing, the strategies of player i ($i = 1, 2$) are the positive ("virtual") commercialization times, t_i, for project i, and the payoff to player i, thanks to strategies t_1 and t_2 of players 1 and 2, respectively, is the total profit $P_i(t_1, t_2)$.

2 Nash Equilibria

Let us focus on the game of timing. According to the standard terminology of game theory, a strategy t_1^* of player 1 is said to be *a best response* of player 1 to a strategy t_2 of player 2 if t_1^* maximizes the payoff to player 1, $P_1(t_1, t_2)$, over the set of all strategies of player 1, t_1:

$$P_1(t_1^*, t_2) = \max_{t_1>0} P_1(t_1, t_2)$$

Similarly, a strategy t_2^* of player 2 is said to be *a best response* of player 2 to a strategy t_1 of player 1 if t_2^* maximizes the payoff to player 2, $P_2(t_1, t_2)$, over the set of all strategies of player 2, t_2:

$$P_2(t_1, t_2^*) = \max_{t_2>0} P_2(t_1, t_2).$$

The pair (t_1^*, t_2^*), where t_1^* is a strategy of player 1 and t_2^* a strategy of player 2, is said to be a *Nash equilibrium* in the game of timing if t_1^* is a best response of player 1 to t_2^* and t_2^* is a best response of player 2 to t_1^*. Our goal is to characterize the Nash equilibria in the game of timing.

We start with a simple observation concerned with the dependence of the player's payoff on the strategy of the other player. Let us consider, for example, the payoff to player 1, $P_1(t_1, t_2)$. The differentiation of $P_1(t_1, t_2)$ with respect to t_1 yields

$$\begin{aligned} \frac{\partial P_1(t_1, t_2)}{\partial t_1} &= a_1(t_1) - b_1(t_1|t_2) \\ &= \begin{cases} a_1(t_1) - b_{11}(t_1) & \text{if } t_1 < t_2, \\ a_1(t_1) - b_{12}(t_1) & \text{if } t_1 > t_2 \end{cases} \end{aligned} \tag{2.1}$$

here we have used (1.10), (1.1), (1.8) and (1.7). Note that the above partial derivative exists and is continuous at any $t_1 > 0$ except for $t_1 = t_2$. Geometrically, (2.1) means that $P_1(t_1, t_2)$ grows in t_1 on the intervals where the graph of $a_1(t_1)$ lies above the graph of $b_1(t_1|t_2)$ and declines in t_1 on the intervals where the graph of $a_1(t_1)$ lies below the graph of $b_1(t_1|t_2)$.

Let us take two arbitrary strategies of player 2, t_{21} and $t_{22} > t_{21}$. As (2.1) shows,

$$\frac{\partial P_1(t_1, t_{22})}{\partial t_1} = \frac{\partial P_1(t_1, t_{21})}{\partial t_1}$$

for $t_1 < t_{21}$ and for $t_1 > t_{22}$, and

$$\frac{\partial P_1(t_1, t_{22})}{\partial t_1} = \frac{\partial P_1(t_1, t_{22})}{\partial t_1} - (b_{11}(t_1) - b_{12}(t_1))$$

for $t_{21} < t_1 < t_{22}$. Recall that $b_{11}(t_1) - b_{12}(t_1) > 0$ (see (1.2)). We have stated that beyond the time interval located between t_{21} and t_{22}, $P_1(t_1, t_{22})$ and $P_1(t_1, t_{21})$ have the same rate in t_1, and within this interval $P_1(t_1, t_{22})$ declines in t_1 faster than $P_1(t_1, t_{21})$. Thanks to (1.8) and (1.7) $P_1(t_1, t_{22}) = P_1(t_1, t_{21})$ for $t_1 \geq t_{22}$. Therefore, $P_1(t_1, t_{22}) > P_1(t_1, t_{21})$ for $t_1 < t_{22}$.

Let us sum up.

Proposition 2.1 *For every $t_1 > 0$, the payoff to player 1, $P_1(t_1, t_2)$, increases in t_2; moreover, given a $t_{21} > 0$ and a $t_{22} > t_{21}$, one has $P_1(t_1, t_{22}) = P_1(t_1, t_{21})$ for $t_1 \geq t_{22}$, and $P_1(t_1, t_{22}) > P_1(t_1, t_{21})$ for $t_1 < t_{22}$*

The graphs of $P_1(t_1, t_2)$ for $t_2 = t_{21}$ and $t_2 = t_{22} > t_{21}$ are shown in Fig. 3.

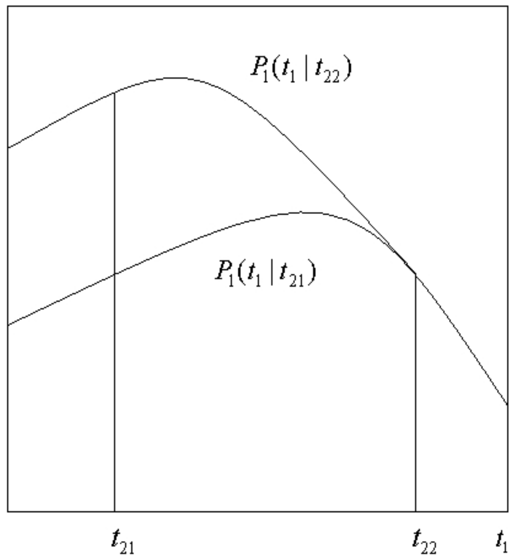


Fig. 3.
Payoff $P_1(t_1, t_2)$ for $t_2 = t_{21}$ and $t_2 = t_{22} > t_{21}$

A symmetric argument leads to a similar observation for player 2.

Proposition 2.2 *For every $t_2 > 0$, the payoff to player 2, $P_2(t_1, t_2)$, increases in t_1; moreover, given a $t_{11} > 0$ and a $t_{12} > t_{11}$, one has $P_2(t_{12}, t_2) = P_2(t_{11}, t_2)$ for $t_2 \geq t_{12}$, and $P_2(t_{12}, t_2) > P_2(t_{11}, t_2)$ for $t_2 < t_1^2$.*

REMARK P12

Remark 2.1 The fact stated in Propositions 2.1 and 2.2 is intuitively clear: for the investor/manager of a gas pipeline project, any prolongation of the commercialization period of the competing project is profitable.

Now let us find the best responses of player 1 to a given strategy, t_2, of player 2.

It is easy enough to identify the intervals of growth and decline of the payoff $P_1(t_1, t_2)$ as a function of t_1. We use formula (2.1) and refer to the points t_1^- and t_1^+, at which the graph of $a_1(t)$, intersects the graphs of $b_{11}(t)$ and $b_{12}(t)$ (see (1.4), (1.5) and Fig. 2).

Assume, first, that $t_2 \leq t_1^-$; recall that $t_1^- < t_1^+$ (see (1.6)). Then, as (1.4), (1.5) and Fig. 2 show, the graph of $a_1(t_1)$ lies above the graph of $b_1(t_1|t_2)$ for $t_1 < t_1^+$ and lies below it for $t_1 > t_1^+$; at $t_1 = t_1^+$ the graphs intersect. Fig. 4, (a), illustrates the relations between the graphs.

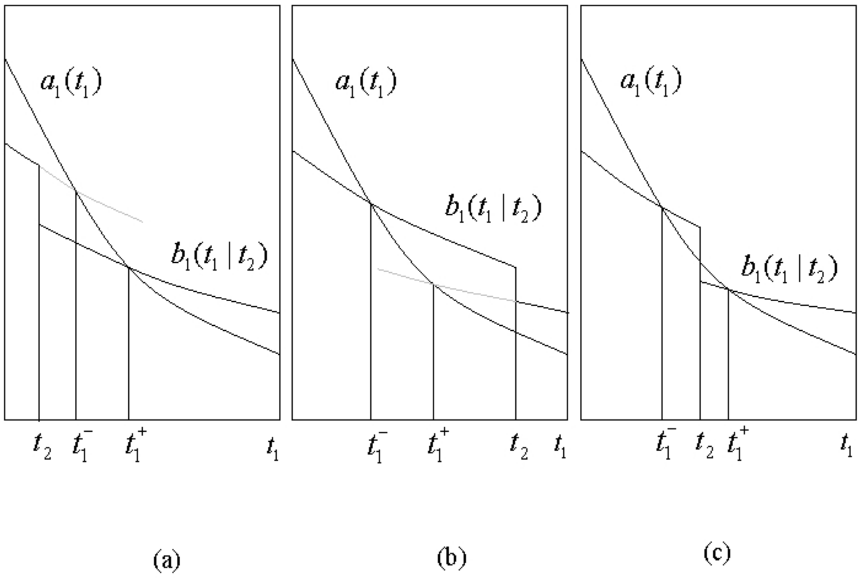


Fig. 4.
(a) $a_1(t_1)$ and $b_1(t|t_2)$ for $t_2 \le t_1^-$.
(b) $a_1(t_1)$ and $b_1(t|t_2)$ for $t_2 \ge t_1^+$.
(c) $a_1(t_1)$ and $b_1(t|t_2)$ for $t_1^- \le t_2 \le t_1^+$.

Due to (2.1), $\partial P_1(t_1,t_2)/\partial t_1$ is positive for $t_1 < t_1^+$ $(t_1 \ne t_2)$ and negative for $t_1 > t_1^+$. Therefore, $t_1 = t_1^+$ is the unique maximizer of $P_1(t_1,t_2)$ in the set of all positive t_1; in other words, t_1^+ is the single best response of player 1 to strategy t_2 of player 2.

Let us assume that $t_2 \ge t_1^+$. Then (1.4), (1.5) and Fig. 2 show that the graph of $a_1(t_1)$ lies above the graph of $b_1(t_1|t_2)$ for $t_1 < t_1^-$, and lies below it for $t_1 > t_1^-$; at $t_1 = t_1^+$ the graphs intersect. Fig. 4, (b), illustrates the relations between the graphs. Due to (2.1), $\partial P_1(t_1,t_2)/\partial t_1$ is positive for $t_1 < t_1^-$ and negative for $t_1 > t_1^-$ $(t_1 \ne t_2)$. Hence, $t_1 = t_1^-$ is the unique maximizer of $P_1(t_1,t_2)$ in the set of all positive t_1, i.e., t_1^- is the single best response of player 1 to t_2.

Now let t_2 lie in the interval $[t_1^-, t_1^+]$. Then (1.4), (1.5) and Fig. 2 show that the graph of $a_1(t_1)$ lies above the graph of $b_1(t_1|t_2)$ for $t_1 < t_1^-$, lies below it for $t_1^- < t_1 < t_2$, lies again above the graph of $b_1(t_1|t_2)$ for $t_2 < t_1 < t_1^+$ and again below it for $t_1 > t_1^+$. Fig. 4, (c), illustrates the relations between the graphs. Thanks to (2.1) we conclude that, $P_1(t_1,t_2)$, as a function of t_1, strictly increases on the interval $(0, t_1^-)$, strictly decreases on the interval (t_1^-, t_2), strictly increases on the interval (t_2, t_1^+), and strictly decreases on the interval (t_1^+, ∞). Therefore, the maximizers of $P_1(t_1,t_2)$ in the set of all positive t_1, i.e., the best responses of player 1 to t_2, are restricted to the two-element set $\{t_1^-, t_1^+\}$.

Let us identify the actual maximizers in this set. We refer to Proposition 2.1. Suppose $t_2 < t_1^+$. Set $t_1 = t_1^+$, $t_{21} = t_2$ and $t_{22} = t_1^+$. We see that $t_1 = t_{22} > t_{21}$. By Proposition 2.1 $P_1(t_1,t_{22}) = P_1(t_1,t_{21})$, or

$$P_1(t_1^+, t_1^+) = P_1(t_1^+, t_2) \tag{2.2}$$

Since $P_1(t_1^+, t_2)$ is continuous in t_2, (2.2) holds for $t_2 = t_1^+$ as well. Now we take arbitrary t_{21} and $t_{22} > t_{21}$ in the interval $[t_1^-, t_1^+]$. By Proposition 2.1 $P_1(t_1^-, t_{22}) > P_1(t_1^-, t_{21})$. Therefore, $P_1(t_1^-, t_2)$ strictly increases in t_2 on $[t_1^+, t_2^+]$. Consider the function

$$p(t_2) = P_1(t_1^-, t_2) - P_1(t_1^+, t_2) \tag{2.3}$$

defined on $[t_1^-, t_1^+]$. By (2.2) we have

$$p(t_2) = P_1(t_1^-, t_2) - P_1(t_1^+, t_1^+)$$

for all t_2 in the interval $[t_1^+, t_2^+]$. As long as $P_1(t_1^-, t_2)$ strictly increases in t_2 on $[t_1^-, t_1^+]$, $p(t_2)$ strictly increases on $[t_1^+, t_2^+]$. Earlier, we have stated that t_1^+ is the single best response of player 1 to any $t_2 \leq t_1^-$; this holds, in particular, for $t_2 = t_1^-$, i.e.,

$$P_1(t_1^+, t_1^-) > P_1(t_1^+, t_1^+)$$

Hence,

$$p(t_1^-) = P_1(t_1^-, t_1^-) - P_1(t_1^+, t_1^-) < 0$$

Earlier, we have stated that t_1^- is the single best response of player 1 to any $t_2 \geq t_1^+$; this holds, in particular, for $t_2 = t_1^+$, i.e.,

$$P_1(t_1^-, t_1^+) > P_1(t_1^+, t_1^+)$$

Hence,

$$p(t_1^+) = P_1(t_1^-, t_1^+) - P_1(t_1^+, t_1^+) > 0$$

We have found that $p(t_2)$ takes a negative value at the left end point of the interval $[t_1^-, t_1^+]$ and a positive value at the right end point of this interval. Since $p(t_2)$ is continuous, there exists a $\hat{t}_2$ in the interior of $[t_1^+, t_2^+]$, for which $p(\hat{t}_2) = 0$. The fact that $p(t_2)$ strictly increases on $[t_1^-, t_1^+]$ implies that the point $\hat{t}_2$ is unique, i.e., $p(t_2) < 0$ for $t_1^- \leq t_2 < \hat{t}_2$ and $p(t_2) > 0$ for $t_1^+ \geq t_2 > \hat{t}_2$. By the definition of $p(t_{12})$, (2.3), we have

$$P_1(t_1^-, \hat{t}_2) = P_1(t_1^+, \hat{t}_2)$$

$$P_1(t_1^-, t_2) < P_1(t_1^+, t_2) \quad \text{for} \quad t_1^- \leq t_2 < \hat{t}_2$$

$$P_1(t_1^-, t_2) < P_1(t_1^+, t_2) \quad \text{for} \quad t_1^+ \geq t_2 > \hat{t}_2$$

Earlier, we have stated that all the best responses of player 1 to t_2 lie in the two-element set $\{t_1^-, t_1^+\}$. Therefore, we conclude that if $t_2 = \hat{t}_2$, player 1 has two best responses, t_1^- and t_1^+, to t_2; if $t_1^- \leq t_2 < \hat{t}_2$, the unique best response of player 1 to t_2 is t_1^+; and if $t_1^+ \geq t_2 > \hat{t}_2$, the unique best response of player 1 to t_2 is t_1^-. Recall that the best response of player 1 to t_2 is t_1^+ if $t_2 < t_1^-$, and t_1^- if $t_2 > t_1^+$.

We summarize as follows.

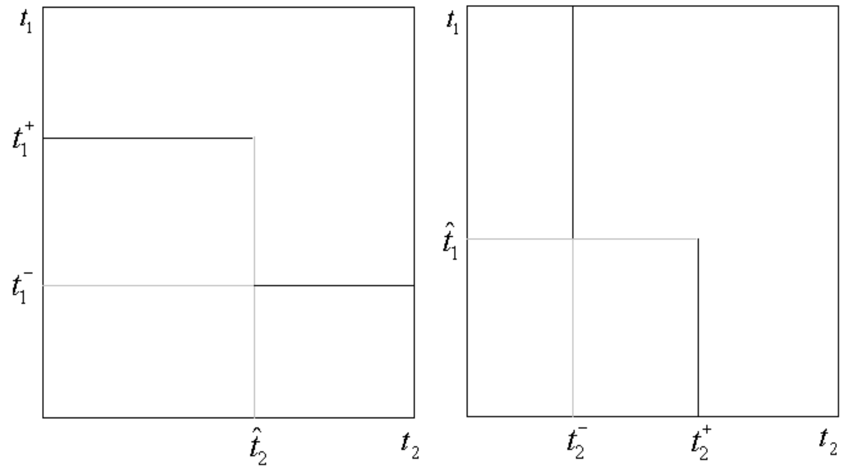
Proposition 2.3 *In the interval (t_1^-, t_1^+), there exists the unique point $\hat{t}_2$ such that*

$$P_1(t_1^-, \hat{t}_2) = P_1(t_1^+, \hat{t}_2) \tag{2.4}$$

The set of all best responses of player 1 to $\hat{t}_2$ is $\{t_1^-, t_1^+\}$. If $0 < t_2 < \hat{t}_2$, then the unique best response of player 1 to t_2 is t_1^+. If $t_2 > \hat{t}_2$, then the unique best response of player 1 to t_2 is t_1^-.

We call t_1^- the *fast choice* of player 1 and t_1^+ the *slow choice* of player 1. Proposition 2.3 claims that the slow choice of player 1 is the best response of player 1 to all "fast" strategies, t_2, of player 2, namely, those satisfying $t_2 < \hat{t}_2$, and the fast choice of player 1 is the best response of player 1 all "slow" strategies, t_2, of player 2, namely, those satisfying $t_2 > \hat{t}_2$; finally, both fast and slow choices of player 1 respond best to $t_2 = \hat{t}_2$. We call $\hat{t}_2$ the *switch point* for player 1.

Let us consider the function that associates to each strategy, t_2, of player 2 the set of all best responses of player 1 to t_2; we call it the *best response function* of player 1. The graph of the best response function of player 1 is shown in Fig. 5, (a). It consists of the horizontal segment located strictly above the segment $(0, \hat{t}_2]$ on the t_2 - axis at level t_1^+, and the unbounded horizontal segment located strictly above the segment $[\hat{t}_2, \infty)$ on the t_2-axis at level t_1^-. Points $(t_1^+, \hat{t}_2)$ and $(t_1^-, \hat{t}_2)$ lie on the graph.



(a) (b)

Fig. 5.
(a) The best response function of player 1.
(b) The best response function of player 2.

A symmetric argument leads to a similar characterization of the best responses of player 1.

Proposition 2.4 *In the interval (t_2^-, t_2^+), there exists the unique point $\hat{t}_1$ such that*

$$P_2(\hat{t}_1, t_2^-) = P_1(\hat{t}_1, t_2^+) \tag{2.5}$$

The set of all best responses of player 2 to $\hat{t}_1$ is $\{t_2^-, t_2^+\}$. If $0 < t_1 < \hat{t}_1$, then the unique best response of player 2 to t_1 is t_2^+. If $t_1 > \hat{t}_1$, then the unique best response of player 2 to t_1 is t_2^-.

We call t_2^- the *fast choice* of player 2, t_2^+ the *slow choice* of player 2, and $\hat{t}_2$ the *switch point* for player 2. We also introduce the *best response function* of player 2, which associates to each strategy, t_1, of player 1 the set of all best responses of player 1 to t_1. The graph of the best response function of player 2 is shown in Fig. 5, (b). Here, the independent variable, t_1, is shown on the vertical axis, and the best responses of player 2 are located on the horizontal axis. The graph of the best response function of player 2 consists of the vertical segment located to the right of the segment $(0, \hat{t}_1]$ on the t_1 - axis at distance t_2^+, and the unbounded vertical segment located to the right of the segment $[\hat{t}_1, \infty)$ on the t_1 - axis at distance t_2^-. Points $(\hat{t}_1, t_2^+)$ and $(\hat{t}_1, t_2^-)$ lie on the graph.

Now we recall the definition of a Nash equilibrium and easily find that a strategy pair (t_1^*, t_2^*) is a Nash equilibrium if and only if the point (t_1^*, t_2^*) belongs to the intersection of the graphs of the best response functions of players 1 and 2. Fig. 5 shows that the graphs necessarily intersect. Fig. 6 gives an example of the intersection.

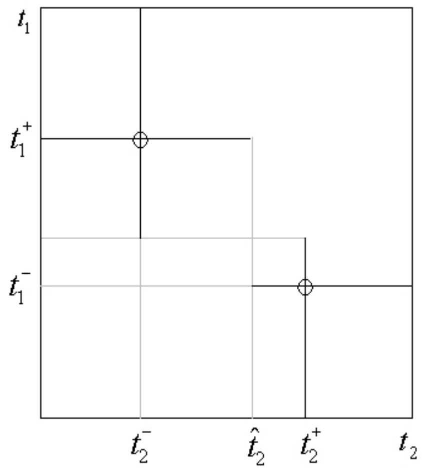


Fig. 6.
The intersection of the graphs of the best response functions
of players 1 and 2 (example).

For each intersection point, i.e., each Nash equilibrium, (t_1^*, t_2^*), point t_1^* is the fast or slow choice of player 1, and point t_2^* is the fast or slow choice of player 2. In case t_1^* is the fast choice of player 1 and t_2^* the slow choice of player 2, we call (t_1^*, t_2^*), the *fast-slow* Nash equilibrium; similarly, we define the *slow-fast*, *fast-fast* and *slow-slow* Nash equilibria.

Nash equilibria of different types arise under different relations between the players' fast and slow choices and the switch points of their rivals. The list of all admissible cases is as follows:

$$\hat{t}_2 \geq t_2^+, \quad \hat{t}_1 < t_1^+ \tag{2.6}$$

$$\hat{t}_2 \geq t_2^+, \quad t_1^- < \hat{t}_1 < t_1^+ \tag{2.7}$$

$$\hat{t}_2 \leq t_2^-, \quad t_1^- < \hat{t}_1 < t_1^+ \tag{2.8}$$

$$t_2^- \leq \hat{t}_2 < t_2^+, \quad t_1^- < \hat{t}_1 \leq t_1^+ \tag{2.9}$$

$$t_2^- < \hat{t}_2 \leq t_2^+, \quad t_1^- \leq \hat{t}_1 < t_1^+ \tag{2.10}$$

$$t_2^- < \hat{t}_2 < t_2^+, \quad \hat{t}_1 \leq t_1^- \tag{2.11}$$

$$t_2^- < \hat{t}_2 < t_2^+, \quad \hat{t}_1 \geq t_1^+ \tag{2.12}$$

$$\hat{t}_2 < t_2^-, \quad \hat{t}_1 \geq t_1^+ \tag{2.13}$$

An elementary analysis in the spirit of Fig. 6 leads to the full classification of the Nash equilibria in the game of timing.

Proposition 2.5 *In cases (2.6), (2.7) and (2.11) the unique Nash equilibrium is slow-fast, (t_1^-, t_2^+) (Fig. 7, (a), (b), (c)). In cases (2.8), (2.12) and (2.13) the unique Nash equilibrium is fast-slow, (t_1^+, t_2^-) (Fig. 7, (d), (e), (f)). In cases (2.9) and (2.10) the game of timing has precisely two Nash equilibria, fast-slow, (t_1^-, t_2^+), and slow-fast, (t_1^+, t_2^-) (Fig. 7, (g)).*

Remark 2.2 Proposition 2.1 shows that the game of timing admits fast-slow and slow-fast equilibria only.

Let us consider in more detail the most interesting situation where the game of timing has two Nash equilibria, fast-slow and slow-fast, i.e., (2.9) or (2.10) holds (Fig. 7, (g)). By Proposition 2.1 and due to the inequalities $t_1^- < \hat{t}^2 \leq t_2^+$ we have

$$P_1(t_1^-, t_2^+) \geq P_1(t_1^-, \hat{t}_2)$$

moreover, the inequality is strict if and only if $\hat{t}^2 < t_2^+$. Using equality (2.4), Proposition 2.1 and the inequalities $t_1^+ > \hat{t}_2 \geq t_2^-$, we transform the right hand side as follows:

$$P_1(t_1^-, \hat{t}_2) = P_1(t_1^+, \hat{t}_2) = P_1(t_1^+, t_2^-)$$

Thus, for the fast-slow and slow-fast equilibria, (t_1^-, t_2^+) and (t_1^+, t_2^-), we have

$$P_1(t_1^-, t_2^+) \geq P_1(t_1^+, t_2^-)$$

more over, the inequality is strict if $\hat{t}_2 < t_2^+$. If this is so, player 1 prefers the fast-slow equilibrium; otherwise, the fast-slow and slow-fast equilibria are equivalent for this player. Similarly, we state that if $\hat{t}_1 < t_1^+$, player 2 prefers the slow-fast equilibrium; otherwise, the equilibria are equivalent for this player. Thus, each player, generally, prefers his "fast" equilibrium.

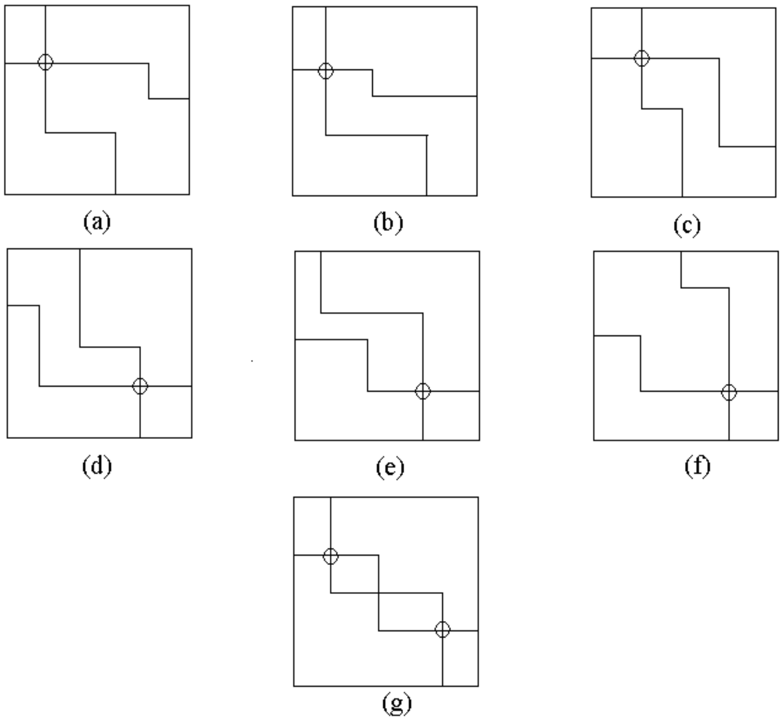


Fig. 7.
(a) One equilibrium, slow-fast ($\hat{t}_1 > t_1^+$, $\hat{t}_2 > t_2^+$).
(b) One equilibrium, slow-fast ($\hat{t}_1 < t_1^-$, $t_2^- < \hat{t}_2 < t_2^+$).
(c) One equilibrium, slow-fast ($t_1^- < \hat{t}_1 < t_1^+$, $\hat{t}_2 > t_2^+$).
(d) One equilibrium, fast-slow ($t_1^- < \hat{t}_1 < t_1^+$, $\hat{t}_2 < t_2^-$).
(e) One equilibrium, fast-slow ($\hat{t}_1 > t_1^+$, $t_2^- < \hat{t}_2 < t_2^+$).
(f) One equilibrium, fast-slow ($\hat{t}_1 < t_1^-$, $\hat{t}_2 < t_2^-$).
(g) Two equilibria, fast-slow and slow-fast ($t_1^- < \hat{t}_1 \le t_1^+$, $t_2^- \le \hat{t}_2 < t_2^+$, or $t_1^- \le \hat{t}_1 < t_1^+$ $t_2^- < \hat{t}_2 \le t_2^+$).

Let us give an exact formulation.

Proposition 2.6 *Let the game of timing have two Nash equilibria, i.e., (2.9) or (2.10) hold. Then*
(i) $P_1(t_1^-, t_2^+) \ge P_1(t_1^+, t_2^-)$, *moreover, the inequality is strict if and only if* $\hat{t}_2 < t_2^+$;
(ii) $P_2(t_1^-, t_2^+) \ge P_2(t_1^+, t_2^-)$ *moreover, the inequality is strict if and only if* $\hat{t}_1 < t_1^+$.

Remark 2.3 Let the game have two equilibria (i.e., (2.9) or (2.10) holds). Assume that the fast-slow and slow-fast equilibria are equivalent to player 1, i.e., $P_1(t_1^-, t_2^+) = P_1(t_1^+, t_2^-)$. Then, by Proposition 2.6, (i), $\hat{t}_2 \ge t_2^+$. As (2.9), (2.10) show, we actually have $\hat{t}_2 = t_2^+$, which is an exceptional situation for the case of two equilibria. Hence, $\hat{t}_1 < t_2^+ \le \hat{t}_2 < t_1^+$. By Proposition 2.6, (ii), $P_2(t_1^-, t_2^+) > P_2(t_1^+, t_2^-)$. In other words, the slow-fast equilibrium is strictly preferable for player 2. In the symmetric case where the fast-slow and slow-fast equilibria are equivalent to player 2, i.e., $P_1(t_1^-, t_2^+) = P_1(t_1^+, t_2^-)$, we find similarly that the fast-slow equilibrium is strictly preferable for player 1. Thus, in those exceptional cases where one of the plyers has no preferences in choosing an equilibrium, the other player strictly prefers his "fast" equilibrium.

Remark 2.4 Let us assume that the parameters of projects 1 and 2 are identical, i.e., $C_1(t) = C_2(t)$ and $B_1(t, s) = B_2(s, t)$ for all positive t and s. Then the game of timing takes a symmetric form. The players have the same fast and slow choices and switch times, $t_1^- = t_2^-$, $t_1^+ = t_2^+$, $\hat{t}_2 = \hat{t}_1$. Hence, (2.9) and (2.10) hold. By Proposition 2.5 the game of timing has the fast-slow and slow-fast equilibria. The inequality $\hat{t}_2 < t_2^+$ is equivalent to $\hat{t}_2 < t_1^+$ which hold trivially (see (1.6)). By Proposition 2.6 we conclude that $P_1(t_1^-, t_2^+) > P_1(t_1^+, t_2^-)$. Similarly, we find that $P_2(t_1^-, t_2^+) > P_2(t_1^+, t_2^-)$. Thus, in the symmetric game of timing, player 1 prefers the fast-slow equilibrium, and player 2 prefers the slow-fast equilibrium. Obviously, the situation does not change if the parameters of projects 1 and 2 are sufficiently close to each other. The question of a practical choice of an equilibrium in the case where the players have different preferences arises. Here, we do not argue on this; we note only that game theory does not provide any clear recommendations in this respect.

3 Solution Algorithm

For convenience, we represent the obtained classification of the Nash equilibria in a table form.

Case	Number of equilibria	Types of equilibria	Notation
$\hat{t}_1 < t_1^-$ $\hat{t}_2 \geq t_2^+$	1	slow-fast	(t_1^+, t_2^+)
$t_1^- < \hat{t}_1 < t_1^+$ $\hat{t}_2 \geq t_2^+$	1	slow-fast	(t_1^+, t_2^-)
$t_1^- < \hat{t}_1 < t_1^+$ $\hat{t}_2 \leq t_2^-$	1	fast-slow	(t_1^-, t_2^+)
$t_1^- < \hat{t}_1 \leq t_1^+$ $t_2^- \leq \hat{t}_2 < t_2^+$	2	fast-slow slow-fast	(t_1^-, t_2^+) (t_1^+, t_2^-)
$t_1^- \leq \hat{t}_1 < t_1^+$ $t_2^- < \hat{t}_2 \leq t_2^+$	2	fast-slow slow-fast	(t_1^-, t_2^+) (t_1^+, t_2^-)
$\hat{t}_1 \leq t_1^-$ $t_2^- < \hat{t}_2 < t_2^+$	1	slow-fast	(t_1^+, t_2^-)
$\hat{t}_1 \geq t_1^+$ $t_2^- < \hat{t}_2 < t_2^+$	1	fast-slow	(t_1^-, t_2^+)
$\hat{t}_1 \geq t_1^+$ $\hat{t}_2 < t_2^-$	1	fast-slow	(t_1^-, t_2^-)

Table 1.
Classification of Nash equilibria in the game of timing
(a table form of Proposition 2.5).

We conclude the general part of our study with the description of an algorithm that finds the Nash equilibria in the game of timing. The algorithm refers to the definitions of the players' fast and slow choices, t_i^-, t_i^+ $(i = 1, 2)$, players' switch times, $\hat{t}_i$ $(i = 1, 2)$, and Table 1.

Solution algorithm.

Step 1. Use definitions (1.4) and (1.5) for finding the players' fast and slow choices, t_i^-, t_2^+ $(i = 1, 2)$.

Step 2. Use definitions (2.4) and (2.5) for finding the players' switch times, $\hat{t}_i$ $(i = 1, 2)$.

Step 3. Use Table 1 for identifying the Nash equilibria.

4 Gas Pipeline Game

In this section, we apply the suggested solution method to a model described in [Klaassen, Roehrl, Tarasyev, 2000]. Wishing to demonstrate a clear analytic result, we consider a simplified version of the model. Namely, we eliminate the price of liquid natural gas, which acts as an upper bound for the price of gas in the original model; we do not introduce the upper bounds for the rates of supply, or pipelines' capacities; we assume that the costs for extraction and transportation of gas are functions of time only; finally, we analyze competition of two pipeline projects (as our theory prescribes).

The model is as follows.

The cost for finalizing the construction of pipeline i ($i = 1, 2$) at time t_i, $C_i(t_i)$, is defined to be the minimum of the *integral investment*

$$I_i(r_i) = \int_0^{t_i} e^{-\lambda t} r_i(t) dt$$

here λ is a positive discount. The minimum is taken over all admissible *open-loop investment strategies*, $r_i(t)$, of player i. An admissible open-loop investment strategy of player i (for a commercialization time t_i) is modeled as an integrable control function,

$$r_i(t) > 0 \tag{4.1}$$

that brings the accumulated investment, $x_i(t)$, from 0 to the prescribed commercialization level $\bar{x}_i > 0$ at time t_i. Thus, for the initial and final values of the accumulated investment we have

$$x_i(0) = 0, \quad x_i(t_i) = \bar{x}_i \tag{4.2}$$

The dynamics of $x_i(t)$ is modeled as

$$\dot{x}_i(t) = -\sigma x_i(t) + r_i^{\gamma}(t) \tag{4.3}$$

here σ is a positive obsolescence coefficient and γ located strictly between 0 and 1 is a delay parameter. In terminology of control theory ([Pontryagin, et., al., 1969]), the cost $C_i(t_i)$ is defined to be the optimal value in the problem of minimizing the performance index $I_i(r_i)$ for the control system (4.3), (4.1) subject to the boundary constraints (4.2).

The upper and lower benefit rates, $b_{i1}(t)$ and $b_{i2}(t)$, for player i at time $t > 0$ are found as equilibrium payoffs in the static *supply game* modeling the instantaneous gas market. In the supply game arising at time t, the strategies of player i are nonnegative *rates of supply*, y_i, and the payoff to player i is defined as

$$p_i(y_1, y_2|t) = e^{-\lambda t}(\pi(t, y) - c_i(t))y_i \tag{4.4}$$

here y is the total rate of supply, $\pi(t, y)$ the price of gas and $c_i(t) > 0$ the cost for extraction and transportation of gas for player i. The price of gas is modeled as

$$\pi(y|t) = \left(\frac{g(t)}{y}\right)^{\beta}$$

where $g(t) > 0$ is the consumer's GDP at time t and β the inverse to the price elasticity of gas demand; we have

$$0 < \beta < 1$$

The total supply, y, equals y_i if player i occupies market solely and equals $y_1 + y_2$ if both players operate on market.

The next proposition gives expressions for the costs, $C_i(t_i)$, rates of cost reduction, $a_i(t_i)$, and upper and lower benefit rates, $b_{i1}(t_i)$ and $b_{i2}(t_i)$ ($i = 1, 2$). We need, however, the following assumption.

Assumption 4.4 It holds that

$$1 - \frac{(2-\beta)c_i(t)}{c_1(t)+c_2(t)} > 0 \quad (i = 1, 2). \tag{4.5}$$

Remark 4.1 Condition (4.5) implies that the costs $c_1(t)$ and $c_2(t)$ are relatively close to each other. Indeed, in the extremal case where $c_1(t) = c_2(t) = c(t)$ (4.5) is equivalent to the trivial inequality $\beta > 0$. Another interpretation of condition (4.5) is that β is close to 1. Indeed, in the limit case where $\beta = 1$ (4.5) is equivalent to the trivial inequality

$$1 - \frac{c_i(t)}{c_1(t)+c_2(t)} > 0$$

Proposition 4.1 *For player i $(i = 1, 2)$ the following formulas hold.*

1. The cost, $C_i(t_i)$, is given by

$$C_i(t_i) = \rho^{\alpha-1} \frac{e^{-\lambda t_i} \bar{x}_i^{\alpha}}{(1-e^{-\rho t_i})^{\alpha-1}} \tag{4.6}$$

where

$$\alpha = \frac{1}{\gamma}, \quad \rho = \frac{\alpha\sigma + \lambda}{\alpha - 1} \tag{4.7}$$

2. The rate of cost reduction, $a_i(t_i)$, is given by

$$a_i(t) = \rho^{\alpha-1} \bar{x}_i^{\alpha} \frac{e^{-\lambda t}(\lambda + \nu e^{-\rho t})}{(1-e^{-\rho t})^{\alpha}} \tag{4.8}$$

where

$$\nu = \alpha\sigma \tag{4.9}$$

3. The upper benefit rate, $b_{i1}(t_i)$, is given by

$$b_{i1}(t) = e^{-\lambda t}(1-\beta)^{1/\beta-1} \frac{g(t)}{c_i^{1/\beta-1}(t)}. \tag{4.10}$$

4. If Assumption 4.4 holds, then the lower benefit rate, $b_{i2}(t_i)$, is given by

$$b_{i2}(t) = e^{-\lambda t}(2-\beta)^{1/\beta-1} \left(1 - \frac{(2-\beta)c_i(t)}{c_1(t)+c_2(t)}\right)^2 \frac{g(t)}{(c_1(t)+c_2(t))^{1/\beta-1}}. \tag{4.11}$$

5. Under Assumption 4.4,

$$b_{i1}(t) > b_{i2}(t) \tag{4.12}$$

(see (1.2) and (1.3)).

In what follows, we assume that $c_i(t)$ $(i = 1, 2)$ and $g(t)$ are defined on the positive half-axis and are continuous. We also fix the functions described in Proposition 4.1 and introduce the next assumption.

Assumption 4.5 For $i = 1, 2$, the functions

$$h_{i1}(t) = \frac{g(t)}{c_i(t)^{1/\beta-1}}, \quad h_{i2}(t) = \left(1 - \frac{(2-\beta)c_i(t)}{c_1(t)+c_2(t)}\right)^2 \frac{g(t)}{(c_1(t)+c_2(t))^{1/\beta-1}} \tag{4.13}$$

$(t > 0)$ increase and tend to infinity as t tends to infinity, and the integral $\int_0^\infty e^{-\lambda t} h_{i1}(t)dt$ is finite.

Remark 4.2 Assumption 4.5 holds if the consumer's GDP, $g(t)$, and costs $c_i(t)$ grow exponentially,

$$g(t) = g^0 e^{\zeta t}, \quad c_i(t) = c_1^0 e^{\omega t} \quad (i = 1, 2) \tag{4.14}$$

(ζ and ω are nonnegative), and

$$0 < \kappa < \lambda \tag{4.15}$$

where

$$\kappa = \zeta - \left(\frac{1}{\beta} - 1\right)\omega \tag{4.16}$$

Note that g_0 is the consumer's GDP at time 0, and c_i^0 is the cost for transportation and extraction for player i at time 0.

The theory described earlier for the general case is applicable for the considered model. Namely, the following is true.

Proposition 4.2 *Let Assumptions 4.4 and 4.5 hold. Then Assumptions 1.1 and 1.2 hold. Moreover, the fast choice, t_i^-, of player i ($i = 1, 2$) is the unique solution of the algebraic equation*

$$\frac{\rho^{\alpha-1}\bar{x}_i^{\alpha}}{(1-\beta)^{1/\beta-1}} = \frac{(1-e^{-\rho t})^{\alpha}}{\lambda + \nu e^{-\rho t}} h_{i1}(t) \tag{4.17}$$

and the slow choice, t_i^+, of player i is the unique solution of the algebraic equation

$$\frac{\rho^{\alpha-1}\bar{x}_i^{\alpha}}{(2-\beta)^{1/\beta-1}} = \frac{(1-e^{-\rho t})^{\alpha}}{\lambda + \nu e^{-\rho t}} h_{i2}(t) \tag{4.18}$$

Thus, under Assumptions 4.4 and 4.5, the general algorithm for the resolution of the game of timing (see section 3) is specified as follows.

Solution algorithm.

Step 1. Solve equations (4.17) and (4.18) for finding the players' fast and slow choices, t_i^- and t_2^+, respectively ($i = 1, 2$).

Step 2. Use equalities (2.4) and (2.5) for finding the players' switch times, $\hat{t}_i$ ($i = 1, 2$).

Step 3. Use Table 1 for identifying the Nash equilibria in the game of timing.

As a specific example, let us consider the case described in Remark 4.2. Thus, in what follows, we assume that $g(t)$ and $c_i(t)$ ($i = 1, 2$) are given by (4.14) and inequality (4.15) is satisfied. Formulas (4.10) and (4.11) for $b_{i1}(t)$ and $b_{i2}(t)$ are specified as

$$b_{i1}(t) = b_{i1}^0 e^{-\psi t}, \quad b_{i2}(t) = b_{i2}^0 e^{-\psi t}$$

where

$$\psi = \lambda - \kappa$$

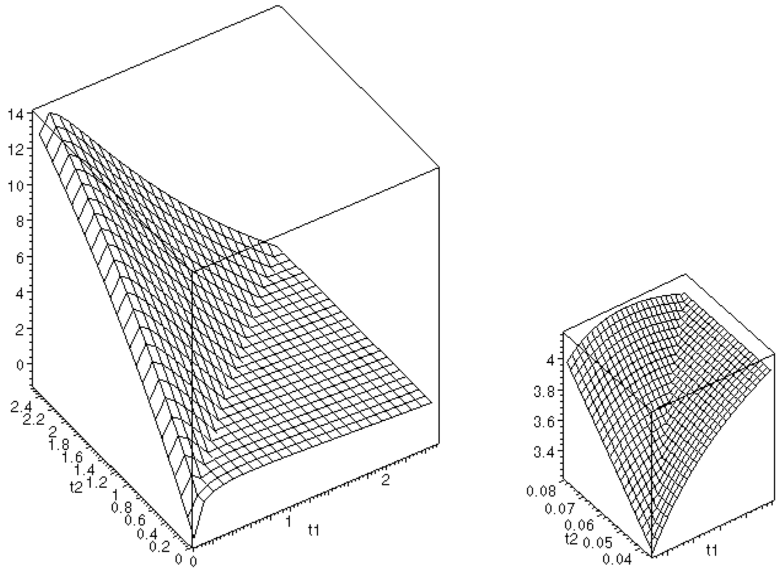
$$b_{i1}^0 = (1-\beta)^{1/\beta-1} \frac{g^0}{(c_i^0)^{1/\beta-1}}$$

$$b_{i2}^0 = (2-\beta)^{1/\beta-1} \left(1 - \frac{(2-\beta)c_i^0}{c_1^0 + c_i^0}\right)^2 \frac{g^0}{(c_1^0 + c_2^0)^{1/\beta-1}}$$

Using the definition of the total benefit, $B_i(t_1, t_2)$, of player i (see (1.8) and (1.9)) and expression (4.6) for cost $C_i(t_i)$, we find an explicit formula for the total profit, $P_i(t_1, t_2)$, (1.10) of player i, which is determined by player's strategies t_1 and t_2. We have

$$\begin{aligned} P_1(t_1,t_2) &= -\rho^{\alpha-1}\frac{e^{-\lambda t_1}\bar{x}_1^{\alpha}}{(1-e^{-\rho t_1})^{\alpha-1}} + \begin{cases} \frac{b^0_{11}e^{-\psi t_1}}{\psi} + \frac{(b^0_{12}-b^0_{11})e^{-\psi t_2}}{\psi} & \text{if } t_1 \le t_2,\\ \frac{b^0_{12}e^{-\psi t_1}}{\psi} & \text{if } t_1 \ge t_2, \end{cases} \\ P_2(t_1,t_2) &= -\rho^{\alpha-1}\frac{e^{-\lambda t_2}\bar{x}_2^{\alpha}}{(1-e^{-\rho t_2})^{\alpha-1}} + \begin{cases} \frac{b^0_{21}e^{-\psi t_1}}{\psi} + \frac{(b^0_{22}-b^0_{21})e^{-\psi t_1}}{\psi} & \text{if } t_2 \le t_1,\\ \frac{b^0_{22}e^{-\psi t_2}}{\psi} & \text{if } t_2 \ge t_1. \end{cases} \end{aligned} \tag{4.19}$$

Fig. 8 shows the Maple-simulated landscape of $P_1(t_1, t_2)$ ($\alpha = 1.5$, $\lambda = 0.3$, $\sigma + 0.3$, $g^0 = 3.5$, $\bar{x}_1 = 0.7$, $\beta = 0.5$, $c_1^0 = c_2^0 = 0.2$).



(a) (b)

Fig 8.
Payoff landscape for player 1: (a) for large t_2 the fast choice, t_1^-, replies best;
(a) for small t_2 the slow choice, t_1^+, replies best.

Recall that by Proposition 2.3 the critical points $\hat{t}_2$ and $\hat{t}_1$ needed for the identification of the type of the equilibria in the game of timing (see Table 1) are found from the equalities $P_1(t_1^-, \hat{t}_2) = P_1(t_1^+, \hat{t}_2)$ and $P_1(\hat{t}_1, t_2^-) = P_1(\hat{t}_1, t_2^+)$, respectively. In the situation considered now the critical points are given explicitly. The next proposition is true.

Proposition 4.3 *For $i = 1, 2$ we have*

$$\hat{t}_i = -\frac{1}{\psi}\log\left(\frac{\psi G_i}{b^0_{i2} - b^0_{i1}}\right) \tag{4.20}$$

where

$$G_i = -\frac{\rho^{\alpha-1}e^{-\lambda t_i^+}\bar{x}_1^{\alpha}}{(1-e^{-\rho t_1^+})^{\alpha-1}} + \frac{b^0_{i2}e^{-\psi t_1^+}}{\psi} + \frac{\rho^{\alpha-1}e^{\lambda t_i^-}\bar{x}_1^{\alpha}}{(1-e^{-\rho t_i^-})^{\alpha-1}} - \frac{b^0_{i1}e^{-\psi t_i^-}}{\psi} \tag{4.21}$$

The next proposition specifies Proposition 4.2.

Proposition 4.4 *Let $g(t)$ and $c_i(t)$ $(i = 1, 2)$ be given by (4.14), and inequality (4.15) be satisfied. Then for every player i $(i = 1, 2)$ the following assertions hold.*

1. The fast choice, t_i^-, of player i is the unique solution of the algebraic equation

$$l_i w_i = \frac{e^{\kappa t}(1 - e^{-\rho t})^\alpha}{\lambda + \nu e^{-\rho t}} \tag{4.22}$$

where

$$l_i = \frac{\rho^{\alpha-1}}{(1-\beta)^{1/\beta-1} g^0}, \tag{4.23}$$

$$w_i = \bar{x}_i^\alpha (c_i^0)^{1/\beta-1} \tag{4.24}$$

2. The slow choice, t_i^+, of player i is the unique solution of the algebraic equation

$$l_i z_i = \frac{e^{\kappa t}(1 - e^{-\rho t})^\alpha}{\lambda + \nu e^{-\rho t}} \tag{4.25}$$

where l_i is defined by (4.23) and

$$z_i = \frac{\bar{x}_i^\alpha (c_1^0 + c_2^0)^{1/\beta-1}}{\left(1 - \frac{(2-\beta)c_i^0}{c_1^0 + c_2^0}\right)^2} \tag{4.26}$$

Thus, under assumptions of Remark 4.2 the suggested solution algorithm for the game of timing (section 3) takes the following form.

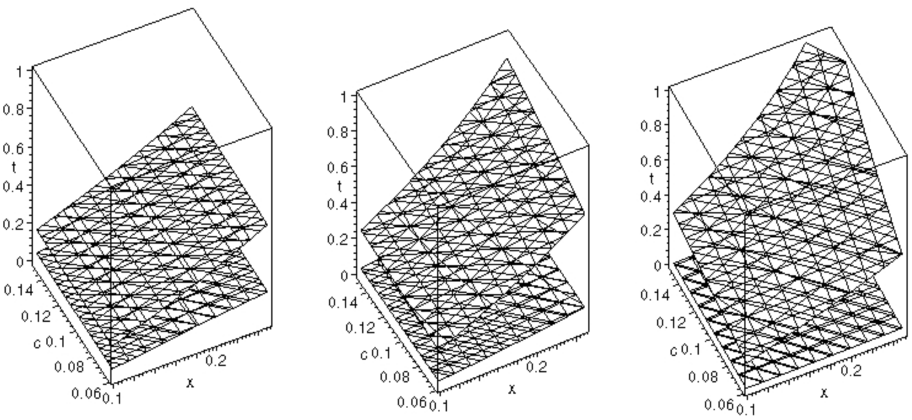
Solution algorithm.

Step 1. Solve equations (4.22) and (4.25), for finding the players' fast and slow choices, t_i^- and t_i^+, respectively, $(i = 1, 2)$.

Step 2. Use formula (4.20) for finding the players' switch times, $\hat{t}_i$ $(i = 1, 2)$.

Step 3. Use Table 1 for identifying the Nash equilibria in the game of timing.

Fig. 9 shows the Maple-simulated graphs of the fast choice, t_1^-, and slow choice, t_1^+, of player 1 as functions of $\bar{x}_1$ and $c_1^0 = c_2^0$ for different values of β ($\alpha = 1.5$, $\lambda = 0.3$, $\sigma = 0.3$, $g^0 = 3.5$).



(a) (b) (c)

Fig 9.
The graphs of the fast choice, t_1^-, and slow choice, t_1^+,
of player 1 as functions of $\bar{x}_1$ and c_1^0:
(a) $\beta = 0.97$; (b) $\beta = 0.75$; (c) $\beta = 0.5$.

5 Appendix: Proves of the Main Results

Here we prove Propositions 4.1, 4.2, 4.3, and 4.4.

Proof of Proposition 4.1.
1. Formula (4.6) was obtained in [Tarasyev, Watanabe, 2000].
2. The differentiation of (4.6) gives

$$\begin{aligned} C_i'(t_i) &= \rho^{\alpha-1}\frac{-\lambda e^{-\lambda t_i}\bar{x}_i^\alpha(1-e^{-\rho t_i})^{\alpha-1} - e^{-\lambda t_i}\bar{x}_i^\alpha(\alpha-1)(1-e^{-\rho t_i})^{\alpha-2}\rho e^{-\rho t_i}}{(1-e^{-\rho t_i})^{2\alpha-2}} \\ &= \frac{\rho^{\alpha-1}e^{-\lambda t_i}\bar{x}_i^\alpha(1-e^{-\rho t_i})^{\alpha-2}}{(1-e^{-\rho t_i})^{2\alpha-2}}\Big[-\lambda(1-e^{-\rho t_i}) - (\alpha-1)\rho e^{-\rho t_i}\Big] \\ &= -\rho^{\alpha-1}\bar{x}_i^\alpha\frac{e^{-\lambda t_i}}{(1-e^{-\rho t_i})^\alpha}\Big[\lambda(1-e^{-\rho t_i}) + (\alpha-1)\rho e^{-\rho t_i}\Big] \\ &= -\rho^{\alpha-1}\frac{e^{-\lambda t_i}\bar{x}_i^\alpha}{(1-e^{-\rho t_i})^\alpha}\Big[\lambda + (\rho(\alpha-1)-\lambda)e^{-\rho t_i}\Big] \\ &= -\rho^{\alpha-1}\frac{e^{-\lambda t_i}\bar{x}_i^\alpha(\lambda+\nu e^{-\rho t_i})}{(1-e^{-\rho t_i})^\alpha} \qquad (5.1) \end{aligned}$$

for the last transformation we have used the equality $\rho(\alpha-1) - \lambda = \alpha\sigma$ following from (4.7) and notation (4.9). For $a_i(t_i) = -C_i'(t_i)$ (see (1.1)) we have (4.8).

3. Assume that player i occupies market solely. Then the price is given by

$$\pi(y|t) = \left(\frac{g(t)}{y_i}\right)^\beta$$

and the payoff to player i, $p_i(y_1, y_2|t)$, equals

$$p_i(y_i|t) = e^{-\lambda t}[g(t)^\beta y_i^{1-\beta} - c_i(t)y_i] \tag{5.2}$$

The supply game is reduced to an optimization problem, and $b_{i1}(t)$ is found as the maximum of $p_i(y_i|t)$ over all positive y_i. Since $p_i(y_i|t)$ is strictly concave in y_i, its maximum is reached at the unique point $y_i(t) > 0$ such that

$$\frac{dp_i(y_i(t)|t)}{dy_i} = e^{-\lambda t}g(t)^\beta[(1-\beta)y_i^{-\beta}(t) - c_i(t)] = 0$$

Hence,

$$y_i(t) = \frac{g(t)}{c_i(t)^{1/\beta}}(1-\beta)^{1/\beta}$$

Recall that $b_{i1}(t) = p_i(y_i(t)|t)$ and substitute $y_i = y_i(t)$ into (5.2). We get

$$\begin{aligned} b_{i1}(t) &= e^{-\lambda t}\left[\frac{g^\beta(t)}{y_i^\beta} - c_i(t)\right] y_i(t) \\ &= e^{-\lambda t}\left[\frac{c_i(t)}{1-\beta} - c_i(t)\right] \frac{g(t)}{c_i(t)^{1/\beta}}(1-\beta)^{1/\beta} \end{aligned}$$

and, finally,

$$b_{i1}(t) = e^{-\lambda t}\beta(1-\beta)^{1/\beta-1}\frac{g(t)}{c_i^{1/\beta-1}(t)}$$

i.e. (4.10) holds.

4. Now let Assumption 4.4 hold and both players operate on market. Then

$$\pi(y|t) = \left(\frac{g(t)}{y_1+y_2}\right)^\beta$$

and for the payoff to player i, we have

$$p_i(y_1, y_2|t) = e^{-\lambda t}\left[\frac{g(t)^\beta y_i}{(y_1+y_2)^\beta} - c_i(t)y_i\right], \tag{5.3}$$

Let us show that the instantaneous supply game has the unique Nash equilibrium under Assumption 4.4.

Since $p_i(y_1, y_2|t)$ $(i = 1, 2)$ is strictly concave in y_i, a point (y_1, y_2) is a Nash equilibrium if and only if

$$\frac{\partial p_i(y_1, y_2|t)}{\partial y_i} = 0 \tag{5.4}$$

or, explicitly,

$$\frac{g^\beta(t)}{y^\beta} - \frac{\beta g^\beta(t)y_i}{y^{\beta+1}} - c_i(t) = 0 \tag{5.5}$$

Here, as above, $y = y_1 + y_2$. For the sum of the left hand sides for $i = 1, 2$, we have

$$2\frac{g^\beta(t)}{y^\beta} - \frac{\beta g^\beta(t)}{y^\beta} - (c_1(t) + c_2(t)) = 0$$

Hence,

$$(2-\beta)g^\beta(t) = (c_1(t) + c_2(t))y^\beta$$

and

$$y^\beta = (2-\beta)\frac{g^\beta(t)}{c_1(t)+c_2(t)} \tag{5.6}$$

Rewriting (5.5) as

$$\beta g^\beta(t) y_i = g^\beta(t) y - c_i(t) y^{\beta+1}$$

and using (5.6), we get

$$\begin{aligned} y_i &= \frac{y}{\beta g^\beta(t)}(g^\beta(t) - c_i(t)y^\beta) \\ &= \frac{y}{\beta g^\beta(t)}\left(g^\beta(t) - \frac{(2-\beta)c_i(t)}{c_1(t)+c_2(t)}g^\beta(t)\right) \\ &= \left((2-\beta)\frac{g^\beta(t)}{c_1(t)+c_2(t)}\right)^{1/\beta}\frac{1}{\beta}\left(1-\frac{(2-\beta)c_i(t)}{c_1(t)+c_2(t)}\right) \\ &= \frac{(2-\beta)^{1/\beta}}{\beta}\left(1-\frac{(2-\beta)c_i(t)}{c_1(t)+c_2(t)}\right)\frac{g(t)}{(c_1(t)+c_2(t))^{1/\beta}} \end{aligned} \tag{5.7}$$

The latter is necessary for (y_1, y_2) to be a Nash equilibrium in the supply game. Hence, if the Nash equilibrium exists, it is unique. Point (y_1, y_2) given by (5.7) has positive components due to Assumption 4.4 (see (4.5)). Moreover, (y_1, y_2) satisfies (5.5), where $y = y_1 + y_2$, which is equivalent to (5.4). Hence, (y_1, y_2) is the Nash equilibrium. We have stated that the unique Nash equilibrium exists. Denote it $(y_1(t), y_2(t))$. By (5.7)

$$y_i(t) = \frac{(2-\beta)^{1/\beta}}{\beta}\left(1-\frac{(2-\beta)c_i(t)}{c_1(t)+c_2(t)}\right)\frac{g(t)}{(c_1(t)+c_2(t))^{1/\beta}}$$

By definition $b_{i2}(t) = p_i(y_1(t), y_2(t)|t)$. Substituting $y_i = y_i(t)$ $(i = 1, 2)$ into (5.3) and noticing that $y = y_1(t) + y_2(t)$ is given by (5.6), we get

$$\begin{aligned} &b_{i2}(t) = e^{-\lambda t}\left[\frac{g^\beta(t)}{y^\beta} - c_i(t)\right] y_i(t) = \\ &e^{-\lambda t}\left[\frac{c_1(t)+c_2(t)}{2-\beta} - c_i(t)\right] y_i(t) = \\ &e^{-\lambda t}\frac{c_1(t)+c_2(t)}{2-\beta}\left(1-\frac{(2-\beta)c_i(t)}{c_1(t)+c_2(t)}\right)\frac{(2-\beta)^{1/\beta}}{\beta}\times \\ &\left(1-\frac{(2-\beta)c_i(t)}{c_1(t)+c_2(t)}\right)\frac{g(t)}{(c_1(t)+c_2(t))^{1/\beta}} \end{aligned}$$

and, finally,

$$b_{i2}(t) = e^{-\lambda t}(2-\beta)^{1/\beta-1}\left(1-\frac{(2-\beta)c_i(t)}{c_1(t)+c_2(t)}\right)^2\frac{g(t)}{(c_1(t)+c_2(t))^{1/\beta-1}}$$

Formula (4.11) is proved.

5. By definition

$$\begin{aligned} b_{i2}(t) &= p_i(y_1(t), y_2(t)|t) \\ &= e^{-\lambda t}(\pi(t, y_1(t)+y_2(t)) - c_i(t))y_i(t) \\ &= \left(\frac{g(t)}{y_1(t)+y_2(t)}\right)^\beta y_i(t) - c_i(t)y_i(t) \end{aligned}$$

$$
\begin{aligned}
&< \left(\frac{g(t)}{y_i(t)}\right)^\beta y_i(t) - c_i(t) y_i(t) \\
&\le \sup_{y_i>0} \left[\left(\frac{g(t)}{y_i}\right)^\beta y_i - c_i(t) y_i\right] \\
&= b_{i1}(t)
\end{aligned}
$$

Inequality (4.12) is stated. Proposition 4.1 is proved.

Proof of Proposition 4.2.

Let us check Assumption 1.1. Function $C_i(t_i)$ (4.6) is continuously differentiable. Expression (5.1) for $C_i'(t)$ shows that $C_i'(t) < 0$. Hence, $C_i(t)$ is monotonically decreasing. Consider the ratio in the right hand side. The numerator, $e^{-\lambda t_i}\bar{x}_i$, decreases in t_i and denominator, $(1-e^{-\rho t_i})^\alpha$ increases in t_i. Hence, the ratio decreases in t_i. Since the square bracket decreases in t_i, its product with the ratio decreases in t_i. As a result, we conclude that $C_i'(t_i)$ increases in t_i. We have shown that Assumption 1.1 is satisfied.

Let us turn to Assumption 1.2. For the rate of cost reduction we have expression (4.8) whose denominator tends to 0 when t approaches 0. Hence, $a_i(t)$ tends to infinity as t approaches 0. Therefore, for all $t > 0$ sufficiently small, we have

$$a_i(t) > b_{i1}(t) > b_{i2}(t)$$

The expressions for $a_i(t)$ and $b_{i2}(t)$ (see (4.11)) show that $b_{i2}(t)/a_i(t) = h_0(t)h_{i2}(t)$ where $h_{i2}(t)$ is given in (4.13) and $h_0(t)$ is such that for some $\tau > 0$ and $\varepsilon > 0$ the lower bound $\inf_{t\ge\tau} h_0(t) > \varepsilon$ holds. By Assumption 4.5 $h(t)$ tends to infinity as t tends to infinity. Therefore, for all t sufficiently large we have

$$b_{i1}(t) > b_{i2}(t) > a_i(t).$$

Since functions $a_i(t)$, $b_{i1}(t)$ and $b_{i2}(t)$ are continuous, there exist a $t_i^- > 0$ that solves the equation

$$a_i(t) = b_{i1}(t) \tag{5.8}$$

and a $t_i^+ > 0$ that solves the equation

$$a_i(t) = b_{i2}(t) \tag{5.9}$$

In order to state that Assumption 1.2 holds, it is now sufficient to show that t_i^- and t_i^+ are unique. We specify equation (5.8) by substituting the expressions for $a_i(t)$ and $b_{i1}(t)$ (see (4.8) and (4.10)). We get

$$\rho^{\alpha-1}\bar{x}_i^\alpha \frac{e^{-\lambda t}(\lambda + \nu\bar{x}_i^\alpha e^{-\rho t})}{(1-e^{-\rho t})^\alpha} = e^{-\lambda t}(1-\beta)^{1/\beta-1}\frac{g(t)}{c_i^{1/\beta-1}(t)}$$

Cancelling $e^{-\lambda t}$ and using the definition of $h_{i1}(t)$ (see (4.13)) we arrive at equation (4.17). The right hand side (4.17) strictly increases in t due to Assumption 4.5. Hence, equation (5.8) has the unique root, t_i^-.

For equation (5.9) we argue similarly. Specify (5.9) by substituting the expressions for $a_i(t)$ and $b_{i1}(t)$ (see (4.8) and (4.11)). We get

$$\rho^{\alpha-1}\bar{x}_i^\alpha \frac{e^{-\lambda t}(\lambda + \nu e^{-\rho t})}{(1-e^{-\rho t})^\alpha} = e^{-\lambda t}(2-\beta)^{1/\beta-1}\left(1 - \frac{(2-\beta)c_i(t)}{c_1(t)+c_2(t)}\right)^2 \frac{g(t)}{(c_1(t)+c_2(t))^{1/\beta-1}}$$

Using the definition of $h_{i2}(t)$ (see (4.13)) we arrive at equation (4.18). The right hand side of (4.18) strictly increases in t due to Assumption 4.5. Hence, equation (5.9) has the unique root, t_i^+.

Proposition 4.2 is proved.

Proof of Proposition 4.3.

Let $i = 1$ (for $i = 2$ the argument is similar). Using formula (4.20) for $P_1(t_1, t_2)$ and taking into account that $\hat{t}_2$ lies between t_1^- and t_1^+ (see Proposition 2.3) we specify the equality $P_1(t_1^-, \hat{t}_2) = P_1(t_1^+, \hat{t}_2)$ into

$$-\frac{\rho^{\alpha-1}e^{-\lambda t_1^-}\bar{x}_1^\alpha}{(1-e^{-\rho t_1^-})^{\alpha-1}} + \frac{b_{11}^0 e^{-\psi t_1^-}}{\psi} + \frac{(b_{12}^0 - b_{11}^0)e^{-\psi \hat{t}_2}}{\psi} = -\frac{\rho^{\alpha-1}e^{-\lambda t_1^+}\bar{x}_1^\alpha}{(1-e^{-\rho t_1^+})^{\alpha-1}} + \frac{b_{12}^0 e^{-\psi t_1^+}}{\psi}$$

Resolving with respect to $\hat{t}_2$, we get

$$\frac{(b_{12}^0 - b_{11}^0)e^{-\psi \hat{t}_2}}{\psi} = -\frac{\rho^{\alpha-1}e^{-\lambda t_1^+}\bar{x}_1^\alpha}{(1-e^{-\rho t_1^+})^{\alpha-1}} + \frac{b_{12}^0 e^{-\psi t_1^+}}{\psi} + \frac{\rho^{\alpha-1}e^{-\lambda t_1^-}\bar{x}_1^\alpha}{(1-e^{-\rho t_1^-})^{\alpha-1}} - \frac{b_{11}^0 e^{-\psi t_1^-}}{\psi}$$

or

$$\hat{t}_2 = -\frac{1}{\psi}\log\left(\frac{\psi G_1}{b_{12}^0 - b_{11}^0}\right)$$

where

$$G_1 = -\frac{\rho^{\alpha-1}e^{-\lambda t_1^+}\bar{x}_1^\alpha}{(1-e^{-\rho t_1^+})^{\alpha-1}} + \frac{b_{12}^0 e^{-\psi t_1^+}}{\psi} + \frac{\rho^{\alpha-1}e^{-\lambda t_1^-}\bar{x}_1^\alpha}{(1-e^{-\rho t_1^-})^{\alpha-1}} - \frac{b_{11}^0 e^{-\psi t_1^-}}{\psi}.$$

Representation (4.20), (4.21) is stated.

Proof of Proposition 4.4.

1. Due to the form of $g(t)$ and $c_i(t)$ (see (4.14)) equation (4.17), which determines the fast choice, t_i^-, of player i, is specified as

$$\frac{\rho^{\alpha-1}\bar{x}_i^\alpha}{(1-\beta)^{1/\beta-1}} = \frac{(1-e^{-\rho t})^\alpha}{\lambda + \nu e^{-\rho t}} \frac{g^0 e^{\kappa t}}{(c_i^0)^{1/\beta-1}}$$

or

$$\frac{\rho^{\alpha-1}\bar{x}_i^\alpha (c_i^0)^{1/\beta-1}}{(1-\beta)^{1/\beta-1} g^0} = \frac{e^{\kappa t}(1-e^{-\rho t})^\alpha}{\lambda + \nu e^{-\rho t}}$$

Using notations (4.23) and (4.24), we arrive at equation (4.22).

2. Due to (4.14) equation (4.18) determining t_i^+ is specified as

$$\frac{\rho^{\alpha-1}\bar{x}_i^\alpha}{(1-\beta)^{1/\beta-1}} = \frac{(1-e^{-\rho t})^\alpha}{\lambda + \nu e^{-\rho t}} \left(1 - \frac{(2-\beta)c_i^0}{c_1^0 + c_2^0}\right)^2 \frac{g^0 e^{\kappa t}}{(c_1^0 + c_2^0)^{1/\beta-1}}$$

Using notations (4.23) and (4.26), we arrive at equation (4.25).

Conclusion

The paper is devoted to the analysis of a two-player game, in which the players' strategies are times of terminating individual dynamical processes. The formal setting is related to management of large-scale innovation projects, whose key feature is that the profits gained through the implementation of the projects are highly sensitive to the projects' commercialization times. The basic reason for that is that the price formation mechanism rapidly changes the price as a new project is commercialized and supply sharply increases. This situation is analyzed in the context of competition of two projects on the construction of gas pipelines. In the game between the projects the total profits gained during the pipelines' life periods act as payoffs and commercialization times as strategies. The reduction of project management to choices of the commercialization times is justified by the assumption that the individual regulation mechanisms comprising investments into the construction of the gas pipelines and regulation of supply work optimally provided the commercialization times are given. The analysis of the game leads to the restriction of player's rational choices to no more than two prescribed combinations of commercialization times, which constitute the Nash equilibria in the game. Typically, two Nash equilibria arise and the projects compete for a fast commercialization scenario; its complement, a slow commercialization scenario, is less profitable, representing the best response to the fast scenario of the competitor. A simple algorithm for finding the Nash equilibria is described.

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